THE ASYMPTOTIC FORMULA IN WARING'S PROBLEM

TREVOR D. WOOLEY*

ABSTRACT. We derive a new minor arc bound, suitable for applications associated with Waring's problem, from Vinogradov's mean value theorem. In this way, the conjectured asymptotic formula in Waring's problem is established for sums of s kth powers of natural numbers when $k \geqslant 6$ and $s \geqslant 2k^2 - 11$.

1. Introduction

A brief review of the progress achieved in nearly a century of development of the Hardy-Littlewood (circle) method reveals that a substantial part has originated in work devoted to the challenge of establishing the asymptotic formula in Waring's problem. Estimates associated with the latter problem have very recently been significantly improved, as a consequence of the author's work [13] concerning Vinogradov's mean value theorem. As we remarked in that paper, the scale of recent improvements opens new avenues for investigation, and it is the goal of this paper to indicate what may nowadays be achieved. Along the way we establish further improvements in the number of variables required to establish the asymptotic formula in Waring's problem.

When s and k are natural numbers, denote by $R_{s,k}(n)$ the number of representations of a positive integer n as the sum of s positive integral kth powers. Motivated by a heuristic application of the circle method, one expects that when $k \geqslant 3$ and $s \geqslant k+1$, then

$$R_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}), \tag{1.1}$$

where

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(q^{-1} \sum_{r=1}^{q} e(ar^k/q) \Big)^s e(-na/q), \tag{1.2}$$

and $e(z)$ denotes $e^{2\pi i z}$. It is worth noting at this point that, subject to modest congruence conditions on n, one has $1 \ll \mathfrak{S}_{s,k}(n) \ll n^{\varepsilon}$ (see [10, Chapter 4]). We take $\widetilde{G}(k)$ to be the least integer t with the property that, for all $s \geqslant t$, and all sufficiently large natural numbers n, one has the asymptotic formula (1.1). In §3 of this paper, we establish the upper bound for $\widetilde{G}(k)$ recorded in the following theorem.

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Theorem 1.1. *Let k be a natural number with $k \geqslant 2$. Then one has*

$$\widetilde{G}(k) \leqslant 2k^2 + 1 - \max_{\substack{1\leqslant j\leqslant k-1\\ 2^j\leqslant k(2k+1)}} \left\lceil \frac{2kj-2^j}{k+1-j} \right\rceil . \tag{1.3}$$

Here, as usual, we write $\lceil \theta \rceil$ for the smallest integer no smaller than θ.

Corollary 1.2. *One has $\widetilde{G}(7) \leqslant 86$, and when $k \geqslant 8$ one has $\widetilde{G}(k) \leqslant 2k^2 - 11$. Moreover, for all natural numbers k with $k \geqslant 2$, one has*

$$\widetilde{G}(k) \leqslant 2k^2 - 2\left\lfloor \frac{\log k}{\log 2} \right\rfloor .$$

For comparison, we note that Vaughan [8], [9] has shown that $\widetilde{G}(k) \leqslant 2^k$ for $k \geqslant 3$, and that Boklan [1] has established the bound $\widetilde{G}(k) \leqslant \frac{7}{8}2^k$ ($k \geqslant 6$). In addition, Theorem 1.4 of the author's very recent work [13] shows that $\widetilde{G}(k) \leqslant 2k^2 + 2k - 3$ for every natural number k with $k \geqslant 2$. The conclusion of Corollary 1.2 supersedes these bounds when $k \geqslant 7$. We note that when $k = 6$, the conclusion of Theorem 1.1 shows that $\widetilde{G}(6) \leqslant 59$, and comes within ε of the bound $\widetilde{G}(6) \leqslant 58$, whereas [1] shows that $\widetilde{G}(6) \leqslant 56$. Work predating the advances of [13], meanwhile, delivers weaker bounds. Thus, the aforementioned work of Boklan [1] shows that $\widetilde{G}(7) \leqslant 112$, $\widetilde{G}(8) \leqslant 224$, and Ford [4] established the bounds $\widetilde{G}(9) \leqslant 393$, $\widetilde{G}(10) \leqslant 551$, ..., $\widetilde{G}(20) \leqslant 2703$, and $\widetilde{G}(k) \leqslant k^2(\log k + \log\log k + O(1))$ when k is large. These bounds have been improved slightly in work of Parsell [7], and Boklan and Wooley [2] have obtained further modest improvements, so that one has the bounds $\widetilde{G}(9) \leqslant 365$, $\widetilde{G}(10) \leqslant 497$, ..., $\widetilde{G}(20) \leqslant 2534$, with the same asymptotic bound for $\widetilde{G}(k)$ when k is large. The conclusion of Corollary 1.2 yields the superior bounds $\widetilde{G}(7) \leqslant 86$, $\widetilde{G}(8) \leqslant 117$, $\widetilde{G}(9) \leqslant 151$, $\widetilde{G}(10) \leqslant 189$, ..., $\widetilde{G}(20) \leqslant 789$ and $\widetilde{G}(k) \leqslant 2k^2 + 2 - 2(\log k)/(\log 2)$.

Our proof of Theorem 1.1 in §3 makes use of an auxiliary mean value estimate of independent interest, and this we establish in §2. When X is a large positive number, define the exponential sum $g(\alpha) = g_k(\alpha; X)$ by

$$g_k(\alpha; X) = \sum_{1\leqslant x\leqslant X} e(\alpha x^k). \tag{1.4}$$

Also, define the set of minor arcs $\mathfrak{m} = \mathfrak{m}_k$ to be the set of real numbers $\alpha \in [0,1)$ satisfying the property that, whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$ and $|q\alpha - a| \leqslant (2k)^{-1}X^{1-k}$, then $q > (2k)^{-1}X$.

Theorem 1.3. *Suppose that $s \geqslant k(k+1)$. Then for each $\varepsilon > 0$, one has*

$$\int_{\mathfrak{m}} |g_k(\alpha; X)|^{2s}\, d\alpha \ll X^{2s-k-1+\varepsilon}. \tag{1.5}$$

Hitherto, in order to establish a bound of the shape (1.5), one would first seek an estimate of the form

$$\int_0^1 |g_k(\alpha;X)|^{2t}\,\mathrm{d}\alpha \ll X^{2t-k+\varepsilon}, \tag{1.6}$$

for some natural number t, and then apply a variant of Weyl's inequality via the trivial estimate

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^{2t+2u}\,\mathrm{d}\alpha \leqslant \Bigl(\sup_{\alpha\in\mathfrak{m}}|g_k(\alpha;X)|\Bigr)^{2u}\int_0^1 |g_k(\alpha;X)|^{2t}\,\mathrm{d}\alpha. \tag{1.7}$$

For example, if one applies [13, Theorem 1.5], then one finds that

$$\sup_{\alpha\in\mathfrak{m}}|g_k(\alpha;X)| \ll X^{1-\sigma(k)+\varepsilon},$$

with $\sigma(k)^{-1} = 2k(k-1)$. In addition, the bound (1.6) is supplied by [13, Corollary 10.2] for $t \geqslant k^2+k-2$. From (1.7) one would therefore obtain the desired conclusion (1.5) with $s=t+u$ provided that $t\geqslant k^2+k-2$ and $u\geqslant k(k-1)$, whence it suffices to take $s\geqslant 2k^2-2$. In this way one sees that Theorem 1.3 is unexpectedly strong. Indeed, in terms of the condition imposed on s, the conclusion of Theorem 1.3 is superior to all bounds available hitherto for $k\geqslant 6$. The relative strength of Theorem 1.3 is of utility in the author's work joint with Vaughan [11] concerning higher order terms in the asymptotic formula in Waring's problem.

A second consequence of the estimate supplied by Theorem 1.3 concerns slim exceptional sets for the asymptotic formula in Waring's problem, and this topic we explore in §5. We measure the frequency with which the anticipated formula (1.1) fails as follows. When $\psi(t)$ is a function of a positive variable t, we denote by $\widetilde{E}_{s,k}(N;\psi)$ the number of integers n, with $1\leqslant n\leqslant N$, for which

$$\Bigl|R_{s,k}(n)-\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}_{s,k}(n)n^{s/k-1}\Bigr| > n^{s/k-1}\psi(n)^{-1}. \tag{1.8}$$

It is convenient here, and in what follows, to refer to a function $\psi(t)$ as being a *sedately increasing function* when $\psi(t)$ is a function of a positive variable t, increasing monotonically to infinity, and satisfying the condition that when t is large, one has $\psi(t)=O(t^\delta)$ for a positive number δ sufficiently small in the ambient context. Finally, we define $\widetilde{G}^+(k)$ to be the least positive integer s for which $\widetilde{E}_{s,k}(N;\psi)=o(N)$ for some function $\psi(t)$ increasing to infinity with t. In §5 we establish the following theorem.

Theorem 1.4. *One has $\widetilde{G}^+(k)\leqslant k^2-[(\log k)/(\log 2)]$. Furthermore, one has $\widetilde{G}^+(7)\leqslant 43$, and when $k\geqslant 8$ one has $\widetilde{G}^+(k)\leqslant k^2-5$. Finally, suppose that $k\geqslant 3$ and that $\psi(t)$ is a sedately increasing function. Then for each $\varepsilon>0$, one has*

$$\widetilde{E}_{s,k}(N;\psi)\ll N^{1-1/k+\varepsilon}\psi(N)^2,\quad \text{for } s\geqslant k(k+1), \tag{1.9}$$

and

$$\widetilde{E}_{s,k}(N;\psi)\ll N^{1-2/k+\varepsilon}\psi(N)^4,\quad \text{for } s\geqslant k(k+1)+2^{k-3}. \tag{1.10}$$

Sharper conclusions than those reported in (1.9) and (1.10) may be obtained when $s > k(k+1)$, and $s > k(k+1) + 2^{k-3}$, respectively. However, we have chosen simplicity of exposition over excessive detail here. These conclusions are superior to those described in the discussion associated with Kawada and Wooley [6, Theorems 1.4, 1.5 and 1.6]. It follows from the latter that $\widetilde{E}_{s,k}(N;\psi) \ll N^{1-1/k+\varepsilon}\psi(N)^4$ for $s \geqslant s_0(k)$, where $s_0(6) = 43$, $s_0(7) = 84$, $s_0(8) = 164$. In contrast, the estimate (1.9) of Theorem 1.4 delivers such conclusions with $s_0(6) = 42$, $s_0(7) = 56$ and $s_0(8) = 72$. Similarly, one finds from the above-cited discussion of Kawada and Wooley that $\widetilde{E}_{s,k}(N;\psi) \ll N^{1-2/k+\varepsilon}\psi(N)^4$ for $s \geqslant s_1(k)$, where $s_1(6) = 51$, $s_1(7) = 100$, $s_1(8) = 196$. Here, the estimate (1.10) of Theorem 1.4 provides such conclusions with $s_1(6) = 50$, $s_1(7) = 72$, $s_1(8) = 104$. The estimates supplied by [13, Theorem 1.5 and Corollary 10.2], meanwhile, lead in (1.7) to the permissible exponent $s_1(k) = 2k^2$, and this proves superior to the conclusion (1.10) supplied by Theorem 1.4 for $k \geqslant 10$.

Throughout the letter ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, unless otherwise indicated. In an effort to simplify our analysis, we adopt the convention that whenever ε appears in a statement, then we are implicitly asserting that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, from time to time, we make use of vector notation in a slightly unconventional manner. Thus, we may write $a \leqslant \mathbf{z} \leqslant b$ to denote that $a \leqslant z_i \leqslant b$ for $1 \leqslant i \leqslant t$, and $\mathbf{z} - a$ to denote the t-tuple $(z_1 - a, \dots, z_t - a)$. Confusion should not arise if the reader interprets similar statements in like manner.

2. A minor arc estimate

Our goal in this section is to establish the minor arc estimate recorded in Theorem 1.3. Our strategy involves adapting the argument of §10 of [13] to an analogous treatment restricted to minor arcs only. We must first introduce some notation. Define the exponential sums $f(\boldsymbol{\alpha}) = f_k(\boldsymbol{\alpha};X)$ and $F(\boldsymbol{\beta},\theta) = F_k(\boldsymbol{\beta},\theta;X)$ by

$$f_k(\boldsymbol{\alpha};X) = \sum_{1\leqslant x\leqslant X} e(\alpha_1 x + \ldots + \alpha_k x^k)$$

and

$$F_k(\boldsymbol{\beta},\theta;X) = \sum_{1\leqslant x\leqslant X} e(\beta_1 x + \ldots + \beta_{k-2}x^{k-2} + \theta x^k).$$

Recall also the definition (1.4) of the exponential sum $g(\alpha) = g_k(\alpha;X)$. In addition, define the mean value $J_{s,k}(X)$ by

$$J_{s,k}(X) = \oint |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}. \tag{2.1}$$

We will omit the subscript k in what follows, since implicit will be the convention that all exponential sums have arguments of degree k. Here and elsewhere we adopt the convention that, given a measurable function $H : [0,1)^r \to \mathbb{C}$, for some natural number r, then

$$\oint H(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = \int_{[0,1)^r} H(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}.$$

Finally, it is convenient to define

$$\sigma_{s,j}(\mathbf{x}) = \sum_{i=1}^{s} (x_i^j - x_{s+i}^j) \quad (1 \leqslant j \leqslant k).$$

Theorem 2.1. *One has*

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^{2s}\,\mathrm{d}\alpha \ll X^{\frac{1}{2}k(k-1)-1}(\log X)^{2s+1} J_{s,k}(2X).$$

Proof. We begin by reinterpreting the mean value involving $g(\alpha)$ in terms of an analogous one involving $F(\boldsymbol{\beta},\theta)$. Observe first that when $\mathbf{h} \in \mathbb{Z}^{k-2}$, one has

$$\int_{\mathfrak{m}} \oint |F(\boldsymbol{\beta},\theta)|^{2s} e(-\beta_1 h_1 - \ldots - \beta_{k-2}h_{k-2})\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta \\ = \sum_{1 \leqslant \mathbf{x} \leqslant X} \delta(\mathbf{x},\mathbf{h}) \int_{\mathfrak{m}} e(\theta \sigma_{s,k}(\mathbf{x}))\,\mathrm{d}\theta,$$

where

$$\delta(\mathbf{x},\mathbf{h}) = \prod_{j=1}^{k-2} \Big(\int_0^1 e(\beta_j(\sigma_{s,j}(\mathbf{x}) - h_j))\,\mathrm{d}\beta_j \Big).$$

By orthogonality, one has

$$\int_0^1 e(\beta_j(\sigma_{s,j}(\mathbf{x}) - h_j))\,\mathrm{d}\beta_j = \begin{cases} 1, & \text{when } \sigma_{s,j}(\mathbf{x}) = h_j, \\ 0, & \text{when } \sigma_{s,j}(\mathbf{x}) \neq h_j. \end{cases}$$

When $1 \leqslant \mathbf{x} \leqslant X$, moreover, one has $|\sigma_{s,j}(\mathbf{x})| \leqslant sX^j$ $(1 \leqslant j \leqslant k-2)$, and so

$$\sum_{|h_1| \leqslant sX} \cdots \sum_{|h_{k-2}| \leqslant sX^{k-2}} \delta(\mathbf{x},\mathbf{h}) = 1.$$

Consequently, on noting that

$$\sum_{1 \leqslant \mathbf{x} \leqslant X} e(\theta \sigma_{s,k}(\mathbf{x})) = |g(\theta)|^{2s},$$

we deduce that

$$\begin{aligned}\sum_{|h_1|\leqslant sX}\cdots\sum_{|h_{k-2}|\leqslant sX^{k-2}}\int_{\mathfrak{m}}\oint |F(\boldsymbol{\beta},\theta)|^{2s}e(-\beta_1h_1-\ldots-\beta_{k-2}h_{k-2})\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta\\ =\int_{\mathfrak{m}}\sum_{1\leqslant \mathbf{x}\leqslant X}\Bigl(\sum_{\mathbf{h}}\delta(\mathbf{x},\mathbf{h})\Bigr)e(\theta\sigma_{s,k}(\mathbf{x}))\,\mathrm{d}\theta\\ =\int_{\mathfrak{m}}|g(\theta)|^{2s}\,\mathrm{d}\theta.\end{aligned}$$

It therefore follows from the triangle inequality that

$$\begin{aligned}\int_{\mathfrak{m}}|g(\alpha)|^{2s}\,\mathrm{d}\alpha &\leqslant \sum_{|h_1|\leqslant sX}\cdots\sum_{|h_{k-2}|\leqslant sX^{k-2}}\int_{\mathfrak{m}}\oint |F(\boldsymbol{\beta},\theta)|^{2s}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta\\ &\ll X^{\frac{1}{2}(k-1)(k-2)}\int_{\mathfrak{m}}\oint |F(\boldsymbol{\beta},\theta)|^{2s}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta. \qquad (2.2)\end{aligned}$$

An argument similar to that employed in the last paragraph permits us to show in addition that

$$\int_{\mathfrak{m}}\oint |F(\boldsymbol{\beta};\theta)|^{2s}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta=\sum_{|h|\leqslant sX^{k-1}}\int_{\mathfrak{m}}\oint |f(\boldsymbol{\alpha},\theta)|^{2s}e(-\alpha_{k-1}h)\,\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\theta. \qquad (2.3)$$

Here, for the sake of clarity, we note that

$$f(\boldsymbol{\alpha},\theta)=f(\alpha_1,\alpha_2,\ldots,\alpha_{k-1},\theta).$$

Observe next that by shifting the variable of summation, for each integer y one has

$$f(\boldsymbol{\alpha})=\sum_{1+y\leqslant x\leqslant X+y}e(\psi(x-y;\boldsymbol{\alpha})), \qquad (2.4)$$

where

$$\psi(z;\boldsymbol{\alpha})=\alpha_1z+\ldots+\alpha_kz^k.$$

But as a consequence of the Binomial Theorem, if we adopt the convention that $\alpha_0=0$, then we may write $\psi(x-y;\boldsymbol{\alpha})$ in the shape

$$\psi(x-y;\boldsymbol{\alpha})=\sum_{i=0}^k\beta_ix^i,$$

where

$$\beta_i=\sum_{j=i}^k\binom{j}{i}(-y)^{j-i}\alpha_j\quad (0\leqslant i\leqslant k).$$

Write

$$K(\gamma)=\sum_{1\leqslant z\leqslant X}e(-\gamma z). \qquad (2.5)$$

Then we deduce from (2.4) that when $1\leqslant y\leqslant X$, one has

$$f(\boldsymbol{\alpha})=\int_0^1\mathfrak{f}_y(\boldsymbol{\alpha};\gamma)K(\gamma)\,\mathrm{d}\gamma, \qquad (2.6)$$

in which we have written

$$\mathfrak{f}_y(\boldsymbol{\alpha};\gamma)=\sum_{1\leqslant x\leqslant 2X} e(\psi(x-y;\boldsymbol{\alpha})+\gamma(x-y)). \tag{2.7}$$

Define

$$\mathcal{F}_y(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})=\prod_{i=1}^s \mathfrak{f}_y(\boldsymbol{\alpha},\theta;\gamma_i)\mathfrak{f}_y(-\boldsymbol{\alpha},-\theta;-\gamma_{s+i}).$$

Then on substituting (2.6) into (2.3), we deduce that when $1\leqslant y\leqslant X$, one has

$$\int_{\mathfrak{m}}\oint |F(\boldsymbol{\beta};\theta)|^{2s}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta=\sum_{|h|\leqslant sX^{k-1}}\oint I_h(\boldsymbol{\gamma},y)\widetilde{K}(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma}, \tag{2.8}$$

where

$$I_h(\boldsymbol{\gamma},y)=\int_{\mathfrak{m}}\oint \mathcal{F}_y(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})e(-\alpha_{k-1}h)\,\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\theta \tag{2.9}$$

and

$$\widetilde{K}(\boldsymbol{\gamma})=\prod_{i=1}^s K(\gamma_i)K(-\gamma_{s+i}).$$

By orthogonality, one finds that

$$\oint \mathcal{F}_y(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})e(-\alpha_{k-1}h)\,\mathrm{d}\boldsymbol{\alpha}=\sum_{1\leqslant \mathbf{x}\leqslant 2X}\Delta(\theta,\boldsymbol{\gamma},h,y), \tag{2.10}$$

where $\Delta(\theta,\boldsymbol{\gamma},h,y)$ is equal to

$$e(\theta\sigma_{s,k}(\mathbf{x}-y)+\gamma_1(x_1-y)+\ldots+\gamma_s(x_s-y)-\gamma_{s+1}(x_{s+1}-y)-\ldots-\gamma_{2s}(x_{2s}-y)),$$

when

$$\begin{aligned}\sum_{i=1}^s((x_i-y)^j-(x_{s+i}-y)^j)&=0\quad (1\leqslant j\leqslant k-2)\\ \sum_{i=1}^s((x_i-y)^{k-1}-(x_{s+i}-y)^{k-1})&=h,\end{aligned} \tag{2.11}$$

and otherwise $\Delta(\theta,\boldsymbol{\gamma},h,y)$ is equal to 0.

By applying the Binomial Theorem, one discerns that whenever the system (2.11) is satisfied for the $2s$-tuple $\mathbf{x}$, then

$$\begin{aligned}\sum_{i=1}^s(x_i^j-x_{s+i}^j)&=0\quad (1\leqslant j\leqslant k-2)\\ \sum_{i=1}^s(x_i^{k-1}-x_{s+i}^{k-1})&=h,\end{aligned}$$

and hence

$$\sigma_{s,k}(\mathbf{x}-y)=\sum_{i=1}^s((x_i-y)^k-(x_{s+i}-y)^k)=\sigma_{s,k}(\mathbf{x})-khy.$$

Then it follows from (2.10) that

$$\oint \mathcal{F}_y(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})e(-\alpha_{k-1}h)\,\mathrm{d}\boldsymbol{\alpha}=\omega_{y,\boldsymbol{\gamma}}\oint \mathcal{F}_0(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})e(-khy\theta-h\alpha_{k-1})\,\mathrm{d}\boldsymbol{\alpha},$$

where $\omega_{y,\boldsymbol{\gamma}}=e(-(\gamma_1+\ldots+\gamma_s-\gamma_{s+1}-\ldots-\gamma_{2s})y)$. From here, we are led via (2.9) to the relation

$$\begin{aligned}\sum_{|h|\leqslant sX^{k-1}} I_h(\boldsymbol{\gamma};y)&=\omega_{y,\boldsymbol{\gamma}}\int_{\mathfrak{m}}\oint \mathcal{F}_0(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})\sum_{|h|\leqslant sX^{k-1}} e(-khy\theta-h\alpha_{k-1})\,\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\theta\\ &\ll \int_{\mathfrak{m}}\oint |\mathcal{F}_0(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})|\min\{X^{k-1},\|ky\theta+\alpha_{k-1}\|^{-1}\}\,\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\theta.\end{aligned}$$

We may consequently conclude thus far that

$$X^{-1}\sum_{1\leqslant y\leqslant X}\sum_{|h|\leqslant sX^{k-1}} I_h(\boldsymbol{\gamma};y)\ll \int_{\mathfrak{m}}\oint |\mathcal{F}_0(\boldsymbol{\alpha},\theta;\boldsymbol{\gamma})|\Psi(\theta,\alpha_{k-1})\,\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\theta, \tag{2.12}$$

where

$$\Psi(\theta,\alpha_{k-1})=X^{-1}\sum_{1\leqslant y\leqslant X}\min\{X^{k-1},\|ky\theta+\alpha_{k-1}\|^{-1}\}.$$

Suppose that $\theta\in\mathbb{R}$, and that $b\in\mathbb{Z}$ and $r\in\mathbb{N}$ satisfy $(b,r)=1$ and $|\theta-b/r|\leqslant r^{-2}$. Then as in the discussion leading to [13, equation (10.6)], one finds that

$$\Psi(\theta,\alpha_{k-1})\ll X^{k-1}(X^{-1}+r^{-1}+rX^{-k})(\log(2r)). \tag{2.13}$$

By Dirichlet's approximation theorem, given $\theta\in\mathfrak{m}$, one may find $b\in\mathbb{Z}$ and $r\in\mathbb{N}$ with $(b,r)=1$, $|r\theta-b|\leqslant (2k)^{-1}X^{1-k}$ and $r\leqslant 2kX^{k-1}$. The definition of $\mathfrak{m}$ ensures that $r>(2k)^{-1}X$, and hence it follows from (2.13) that

$$\sup_{\theta\in\mathfrak{m}}\Psi(\theta,\alpha_{k-1})\ll X^{k-2}\log X.$$

On recalling (2.7) and substituting this estimate into (2.12), and then applying Hölder's inequality, one finds that

$$\begin{aligned}X^{-1}&\sum_{1\leqslant y\leqslant X}\sum_{|h|\leqslant sX^{k-1}} I_h(\boldsymbol{\gamma};y)\\ &\ll X^{k-2}(\log X)\prod_{i=1}^{2s}\Bigl(\int_{\mathfrak{m}}\oint |\mathfrak{f}_0(\boldsymbol{\alpha},\theta;\gamma_i)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\theta\Bigr)^{1/(2s)}\\ &\leqslant X^{k-2}(\log X)\sup_{\gamma\in[0,1)}\int_0^1\oint |\mathfrak{f}_0(\boldsymbol{\alpha},\theta;\gamma)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\theta\\ &= X^{k-2}(\log X)\oint |f_k(\boldsymbol{\alpha};2X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}.\end{aligned}$$

We therefore deduce from (2.1) and (2.8) that

$$\int_{\mathfrak{m}} \oint |F(\boldsymbol{\beta};\theta)|^{2s}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta = X^{-1} \sum_{1\leqslant y\leqslant X} \int_{\mathfrak{m}} \oint |F(\boldsymbol{\beta};\theta)|^{2s}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta$$
$$\ll X^{k-2}(\log X) J_{s,k}(2X) \oint |\widetilde{K}(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma}. \qquad (2.14)$$

On recalling (2.5), we find that

$$\int_0^1 |K(\gamma)|\,\mathrm{d}\gamma \leqslant \int_0^1 \min\{X, \|\gamma\|^{-1}\}\,\mathrm{d}\gamma \ll \log X,$$

and hence

$$\oint |\widetilde{K}(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma} \ll (\log X)^{2s}.$$

On substituting this estimate into (2.14), we conclude that

$$\int_{\mathfrak{m}} \oint |F(\boldsymbol{\beta};\theta)|^{2s}\,\mathrm{d}\boldsymbol{\beta}\,\mathrm{d}\theta \ll X^{k-2}(\log X)^{2s+1} J_{s,k}(2X),$$

whence (2.2) delivers the upper bound

$$\int_{\mathfrak{m}} |g(\alpha)|^{2s}\,\mathrm{d}\alpha \ll X^{\frac{1}{2}(k+1)(k-2)}(\log X)^{2s+1} J_{s,k}(2X).$$

The conclusion of the theorem is now immediate. □

Theorem 1.1 of [13] shows that when $s \geqslant k(k+1)$, one has

$$J_{s,k}(2X) \ll X^{2s-\frac{1}{2}k(k+1)+\varepsilon}.$$

Then under the same condition on s, it follows from Theorem 2.1 that

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^{2s}\,\mathrm{d}\alpha \ll X^{\frac{1}{2}k(k-1)-1+\varepsilon}(X^{2s-\frac{1}{2}k(k+1)+\varepsilon}) \ll X^{2s-k-1+\varepsilon}.$$

This establishes the conclusion of Theorem 1.3.

3. The asymptotic formula

Equipped with the upper bound supplied by Theorem 1.3, the argument leading to Theorem 1.1 is essentially routine. We begin by deriving a less precocious minor arc estimate. In this context, for each natural number k, we define the positive integer $s_0(j) = s_0(k,j)$ by means of the relation

$$s_0(k,j) = 2k^2 + 1 - \left\lceil \frac{2kj - 2^j}{k+1-j} \right\rceil.$$

We then put

$$s_1(k) = \min_{\substack{1\leqslant j\leqslant k-1\\ 2^j\leqslant k(2k+1)}} s_0(k,j).$$

Lemma 3.1. *Suppose that s is a natural number with $s \geqslant s_1(k)$. Then there exists a positive number δ with the property that*

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^s \, d\alpha \ll X^{s-k-\delta}.$$

Proof. Let l be a natural number with $1 \leqslant l \leqslant k$. Then by Hua's lemma (see [10, Lemma 2.5]), one has

$$\int_0^1 |g(\alpha)|^{2^l} \, d\alpha \ll X^{2^l-l+\varepsilon}. \tag{3.1}$$

Let j be a natural number with $1 \leqslant j \leqslant k-1$ and $2^j \leqslant k(2k+1)$, and write

$$t_0(j) = 2k^2 - \frac{2kj - 2^j}{k+1-j} \quad \text{and} \quad \delta(j) = s_0(j) - t_0(j).$$

Then we find that $0 < \delta(j) \leqslant 1$. By Hölder's inequality, we have

$$\int_{\mathfrak{m}} |g(\alpha)|^{s_0(j)} \, d\alpha \leqslant \Big(\int_{\mathfrak{m}} |g(\alpha)|^{2k(k+1)} \, d\alpha \Big)^a \Big(\int_0^1 |g(\alpha)|^{2^j} \, d\alpha \Big)^b,$$

where

$$a = \frac{k-j}{k+1-j} + \frac{\delta(j)}{2k(k+1)-2^j}$$

and

$$b = \frac{1}{k+1-j} - \frac{\delta(j)}{2k(k+1)-2^j}.$$

Note, in particular, that the hypothesis $2^j \leqslant k(2k+1)$ ensures that $b \geqslant 0$. Then by (3.1) and the conclusion of Theorem 1.3, we obtain the bound

$$\int_{\mathfrak{m}} |g(\alpha)|^{s_0(j)} \, d\alpha \ll X^\varepsilon (X^{2k(k+1)-k-1})^a (X^{2^j-j})^b \ll X^{s_0(j)-k-\eta+\varepsilon},$$

where

$$\eta = a - (k-j)b = \Big(\frac{k+1-j}{2k(k+1)-2^j} \Big) \delta(j) > 0.$$

From the definition of $s_1(k)$, we may therefore conclude that there is a positive number η for which

$$\int_{\mathfrak{m}} |g(\alpha)|^{s_1(k)} \, d\alpha \ll X^{s_1(k)-k-\eta+\varepsilon},$$

and the conclusion of the lemma follows on making use of the trivial estimate $|g(\alpha)| \leqslant X$. □

We are now equipped to establish Theorem 1.1. Consider a large integer n, and put $X = [n^{1/k}]$. Recall the definition of the minor arcs $\mathfrak{m}$ from the preamble to Theorem 1.3. The complement $\mathfrak{M} = \mathfrak{M}_k$ of the minor arcs $\mathfrak{m}_k$ within the unit interval $[0,1)$ is given by the union of the arcs

$$\mathfrak{M}_k(q,a) = \{ \alpha \in [0,1) : |q\alpha - a| \leqslant (2k)^{-1} X^{1-k} \},$$

with $0 \leqslant a \leqslant q \leqslant (2k)^{-1}X$ and $(a,q)=1$. We take $s \geqslant s_1(k)$, so that

$$s \geqslant 2k^2+1-\max_{\substack{1\leqslant j\leqslant k-1\\ 2^j\leqslant k(2k+1)}} \left\lceil \frac{2kj-2^j}{k+1-j} \right\rceil .$$

Then by Lemma 3.1 and the triangle inequality, we have

$$\int_{\mathfrak{m}} g(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha \leqslant X^{s-s_1(k)} \int_{\mathfrak{m}} |g(\alpha)|^{s_1(k)}\,\mathrm{d}\alpha \ll X^{s-k-\delta},$$

for some positive number δ. Meanwhile, the methods of [10, §4.4] show that under the same conditions on s one has

$$\int_{\mathfrak{M}} g(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}),$$

where the singular series $\mathfrak{S}_{s,k}(n)$ is defined as in (1.2). Thus we deduce that for $s \geqslant s_1(k)$, one has

$$\begin{aligned} R_{s,k}(n) &= \int_{\mathfrak{M}} g(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha + \int_{\mathfrak{m}} g(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha \\ &= \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}). \end{aligned}$$

We may therefore conclude that $\widetilde{G}(k) \leqslant s_1(k)$, and this completes the proof of Theorem 1.1.

The corollary to Theorem 1.1 is easily established by direct computation. When $k=7$ one obtains the estimate $\widetilde{G}(7) \leqslant 86$ by noting that the maximum in (1.3) corresponds to the choice $j=5$. Meanwhile, on taking $j=6$, we find that for $k \geqslant 10$ one has

$$\frac{2kj-2^j}{k+1-j} = \frac{12k-64}{k-5} = 12 - \frac{4}{k-5} > 11.$$

The same lower bound follows by direct computation for $k=8$ and 9 by setting $j=5$. Thus, when $k \geqslant 8$, one has the upper bound

$$\widetilde{G}(k) \leqslant (2k^2+1) - 12 = 2k^2 - 11.$$

Finally, on taking $j = [(\log k)/(\log 2)] + 1$, we find that $2^{j-1} \leqslant k$, and hence

$$\frac{2kj-2^j}{k+1-j} = 2j - \left(\frac{2^j+2j-2j^2}{k+1-j} \right) > 2j-2,$$

since one has $2^{j-1}+j-j^2 \leqslant k+j-j^2 < k+1-j$ for $j \geqslant 2$. We therefore find from Theorem 1.1 that with this choice of j, one has

$$\widetilde{G}(k) \leqslant 2k^2+1-(2j-1) = 2k^2 - 2\left[\frac{\log k}{\log 2}\right].$$

This completes the proof of Corollary 1.2.

4. Further consequences of the new minor arc bounds

The mean value $J_{s,k}(X)$ defined in (2.1) is the central object of attention associated with Vinogradov's mean value theorem. A consideration of diagonal solutions, together with the product of local densities, suggests that when $k \geqslant 3$ one should have

$$J_{s,k}(X) \ll X^s + X^{2s-\frac{1}{2}k(k+1)}. \tag{4.1}$$

For well over half a century, progress towards the conjectured estimate (4.1) was so glacial that a serious consideration of its consequences for upper bounds for such quantities as $\widetilde{G}(k)$ would have seemed premature. However, with the author's very recent work [13], the landscape has been transformed. The estimate (4.1) is now known to hold for $s \geqslant k^2+k+1$, only a factor 2 away from the smallest exponent s for which the non-diagonal contribution would dominate the diagonal. Previously, such a bound was accessible only for $s \geqslant (1+o(1))k^2\log k$. With such progress in the air, one may feel optimistic enough to hope that the conjecture (4.1) is within reach. We therefore feel justified in noting the consequences of the conjectured bound (4.1) for $\widetilde{G}(k)$.

Theorem 4.1. *Suppose that $k \geqslant 3$ and the upper bound (4.1) holds. Then one has*

$$\widetilde{G}(k) \leqslant k^2 + 1 - \max_{\substack{1\leqslant j\leqslant k-1\\ 2^j\leqslant k^2}} \left\lceil \frac{kj-2^j}{k+1-j} \right\rceil. \tag{4.2}$$

In particular, one has $\widetilde{G}(4) \leqslant 15$, $\widetilde{G}(k) \leqslant k^2-2$ $(k \geqslant 5)$, $\widetilde{G}(k) \leqslant k^2-3$ $(k \geqslant 8)$, $\widetilde{G}(k) \leqslant k^2-4$ $(k \geqslant 17)$, and

$$\widetilde{G}(k) \leqslant k^2 + 1 - \left[\frac{\log k}{\log 2}\right] \quad (k \geqslant 3).$$

Proof. We imitate the proof of Lemma 3.1, and the discussion that follows, noting that in view of the estimate (4.1), it follows from Theorem 2.1 that when $s \geqslant \frac{1}{2}k(k+1)$, one has the conditional upper bound

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^{2s}\,\mathrm{d}\alpha \ll X^{2s-k-1+\varepsilon}.$$

The computations implicit in the consequences of (4.2) are easily completed by imitating those described at the end of §3. □

The conclusions of Theorem 4.1 are superior to the unconditional estimates currently available whenever $k \geqslant 4$. The main competitor for small values of k are the bounds $\widetilde{G}(k) \leqslant 2^k$ due to Vaughan [8, 9]. While the latter bound is superseded by the conditional bounds $\widetilde{G}(4) \leqslant 15$ and $\widetilde{G}(5) \leqslant 23$ of Theorem 4.1, the upper bound $\widetilde{G}(3) \leqslant 8$ of Vaughan remains unchallenged.

A further consequence of Theorem 1.3 concerns the density of integral solutions of diagonal Diophantine equations. When s and k are natural numbers, and a_i $(1 \leqslant i \leqslant s)$ are integers, we write

$$\phi(\mathbf{x}) = \sum_{i=1}^{s} a_i x_i^k,$$

and we consider the Diophantine equation $\phi(\mathbf{x}) = 0$. We write $N(B)$ for the number of integral solutions of the equation $\phi(\mathbf{x}) = 0$ with $|\mathbf{x}| \leqslant B$. Associated with this equation are the (formal) real and p-adic densities. When $L > 0$, define

$$\lambda_L(\eta) = \begin{cases} L(1 - L|\eta|), & \text{when } |\eta| \leqslant L^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

The limit

$$\sigma_\infty = \lim_{L\to\infty} \int_{|\boldsymbol{\xi}|\leqslant 1} \lambda_L(\phi(\boldsymbol{\xi}))\,\mathrm{d}\boldsymbol{\xi},$$

when it exists, is called the *real density*. Meanwhile, given a natural number q, we write

$$M(q) = \mathrm{card}\{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^s : \phi(\mathbf{x}) \equiv 0 \pmod q\}.$$

For each prime number p, we then put

$$\sigma_p = \lim_{H\to\infty} p^{H(1-s)} M(p^H),$$

provided that this limit exists, and refer to σ_p as the *p-adic density*.

Theorem 4.2. *Let s and k be natural numbers with $k \geqslant 3$ and*

$$s \geqslant 2k^2 + 1 - \max_{\substack{1\leqslant j\leqslant k-1\\ 2^j\leqslant k(2k+1)}} \left\lceil \frac{2kj - 2^j}{k+1-j} \right\rceil.$$

Suppose that a_i $(1 \leqslant i \leqslant s)$ are non-zero integers. Then one has

$$N(B) \sim \sigma_\infty \left(\prod_p \sigma_p \right) B^{s-k}.$$

Note that the p-adic solubility of the equation $\phi(\mathbf{x}) = 0$ is assured by the work of Davenport and Lewis [3] for $s \geqslant k^2 + 1$. The proof of Theorem 4.2 is routine from the upper bound presented in Lemma 3.1. One may verify that the methods of [10, Chapters 2 and 4] suffice in combination with Lemma 3.1 to deliver the desired asymptotic formula. Indeed, such a strategy is sketched in [10, §§9.1 and 9.2]. Conditional improvements along the lines of Theorem 4.1 follow without any great effort.

5. Exceptional sets and the asymptotic formula

Our approach to the problem of bounding exceptional sets is modelled on that applied in our recent work with Kawada [6]. We consider integers k and s with $k \geqslant 2$ and $s > k + 1$. Suppose that N is a large positive integer, and let $\psi(t)$ be a sedately increasing function. We denote by $\mathcal{Z}_{s,k}(N)$ the set of integers n with $N/2 < n \leqslant N$ for which the inequality (1.8) holds, and we abbreviate card($\mathcal{Z}_{s,k}(N)$) to $Z = Z_{s,k}$. Next, write $X = X_k$ for $[N^{1/k}]$, and recall the definition (1.4) of the exponential sum $g(\alpha) = g_k(\alpha; X)$. Also, let $\mathfrak{m}_k$ and $\mathfrak{M}_k$ be defined, respectively, as in the preamble to the statement of Theorem 1.3, and that of the proof of Theorem 1.1 in §3. Then the argument of [12] leading to equation (2.5) of that paper shows that there exist complex numbers $\eta_n = \eta_n(s,k)$, with $|\eta_n| = 1$, satisfying the condition that, with the exponential sum $K(\alpha) = K_{s,k}(\alpha)$ defined by

$$K_{s,k}(\alpha) = \sum_{n\in\mathcal{Z}_{s,k}(N)} \eta_n(s,k)e(n\alpha),$$

one has

$$\int_{\mathfrak{m}} |g(\alpha)^s K(\alpha)|\,d\alpha \gg N^{s/k-1}\psi(N)^{-1}Z. \tag{5.1}$$

We begin by bounding $\widetilde{G}^+(k)$. Define the natural number $u_0(j) = u_0(k,j)$ by means of the relation

$$u_0(k,j) = k^2 + 1 - \left\lceil \frac{kj - 2^{j-1}}{k+1-j} \right\rceil,$$

and then put

$$u_1(k) = \min_{\substack{1\leqslant j\leqslant k-1\\ 2^j\leqslant k(2k+1)}} u_0(k,j).$$

Suppose that $s \geqslant u_1(k)$. Then an application of Schwarz's inequality leads from (5.1) to the upper bound

$$N^{s/k-1}\psi(N)^{-1}Z \ll \Bigl(\int_{\mathfrak{m}} |g(\alpha)|^{2s}\,\mathrm{d}\alpha\Bigr)^{1/2}\Bigl(\int_0^1 |K(\alpha)|^2\,\mathrm{d}\alpha\Bigr)^{1/2}. \tag{5.2}$$

But Parseval's identity shows that

$$\int_0^1 |K(\alpha)|^2\,\mathrm{d}\alpha = \sum_{n\in\mathcal{Z}_{s,k}(N)} 1 \leqslant Z. \tag{5.3}$$

Furthermore, since

$$2u_0(k,j) > 2k^2 - \frac{2kj - 2^j}{k+1-j},$$

one finds just as in the argument of the proof of Lemma 3.1 that there exists a positive number δ with the property that

$$\int_{\mathfrak{m}} |g(\alpha)|^{2s}\,\mathrm{d}\alpha \ll X^{2s-k-\delta}.$$

Thus, on recalling that $X = [N^{1/k}]$, we deduce that

$$N^{s/k-1}\psi(N)^{-1}Z \ll (N^{2s/k-1-\delta/k})^{1/2}Z^{1/2},$$

whence $Z \ll N^{1-\delta/k}\psi(N)^2$. Suppose that $\psi(N)$ grows sufficiently slowly, say $\psi(N) = O(N^{\delta/(3k)})$, as we are at liberty to assume. Then one concludes by summing over dyadic intervals for N that

$$\widetilde{E}_{s,k}(N;\psi) \ll N^{1-\delta/(4k)} = o(N),$$

and hence $\widetilde{G}^+(k) \leqslant u_1(k)$. The remaining conclusions of Theorem 1.4 concerning the quantity $\widetilde{G}^+(k)$ follow with some computation much as those concerning $\widetilde{G}(k)$ in §3.

Suppose next that $k \geqslant 3$ and $s \geqslant k(k+1)$. Then it follows from Theorem 1.3 that

$$\int_{\mathfrak{m}} |g(\alpha)|^{2s}\, \mathrm{d}\alpha \ll X^{2s-k-1+\varepsilon}.$$

On substituting this estimate along with (5.3) into (5.2), we find on this occasion that

$$N^{s/k-1}\psi(N)^{-1}Z \ll (N^{(2s-1)/k-1+\varepsilon})^{1/2}Z^{1/2},$$

whence

$$Z \ll N^{1-1/k+\varepsilon}\psi(N)^2.$$

The estimate (1.9) is confirmed by summing over dyadic intervals.

Finally, we consider the situation with $k \geqslant 3$ and $s \geqslant k(k+1) + 2^{k-3}$. In this situation we apply Schwarz's inequality in an alternate manner to (5.1), obtaining the upper bound

$$N^{s/k-1}\psi(N)^{-1}Z \ll \Big(\int_{\mathfrak{m}} |g(\alpha)|^{2s-2^{k-2}}\, \mathrm{d}\alpha\Big)^{1/2}\Big(\int_0^1 |g(\alpha)^{2^{k-2}}K(\alpha)^2|\, \mathrm{d}\alpha\Big)^{1/2}.$$

Since $2s - 2^{k-2} \geqslant 2k(k+1)$, we may again apply Theorem 1.3 to obtain

$$\int_{\mathfrak{m}} |g(\alpha)|^{2s-2^{k-2}}\, \mathrm{d}\alpha \ll X^{2s-2^{k-2}-k-1+\varepsilon}.$$

Meanwhile, as an immediate consequence of [5, Lemma 6.1], one finds that

$$\int_0^1 |g(\alpha)^{2^{k-2}}K(\alpha)^2|\, \mathrm{d}\alpha \ll X^{2^{k-2}-1}Z + X^{2^{k-2}-k/2+\varepsilon}Z^{3/2}.$$

Hence we deduce that

$$\begin{aligned} N^{s/k-1}\psi(N)^{-1}Z &\ll N^{\varepsilon}(N^{2s/k-1-(2^{k-2}+1)/k})^{1/2} \\ &\quad \times (N^{(2^{k-2}-1)/k}Z + N^{2^{k-2}/k-1/2}Z^{3/2})^{1/2} \\ &\ll N^{s/k-1+\varepsilon}((N^{1-2/k})^{1/2}Z^{1/2} + (N^{1-2/k})^{1/4}Z^{3/4}), \end{aligned}$$

whence

$$Z \ll N^{1-2/k+\varepsilon}\psi(N)^4.$$

Thus the final conclusion (1.10) of Theorem 1.4 has been established, and this completes the proof of the theorem.

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School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, United Kingdom

E-mail address: matdw@bristol.ac.uk