

Translation invariance, exponential sums, and Waring's problem

Trevor D. Wooley

Abstract. We describe mean value estimates for exponential sums of degree exceeding 2 that approach those conjectured to be best possible. The vehicle for this recent progress is the *efficient congruencing method*, which iteratively exploits the translation invariance of associated systems of Diophantine equations to derive powerful congruence constraints on the underlying variables. There are applications to Weyl sums, the distribution of polynomials modulo 1, and other Diophantine problems such as Waring's problem.

Mathematics Subject Classification (2010). Primary 11L15; Secondary 11P05.

Keywords. Exponential sums, Waring's problem, Hardy-Littlewood method.

1. Introduction

Although pivotal to the development of vast swathes of analytic number theory in the twentieth century, the differencing methods devised by Weyl [54] and van der Corput [15] are in many respects unsatisfactory. In particular, they improve on the trivial estimate for an exponential sum by a margin exponentially small in terms of its degree. The method introduced by Vinogradov [50, 51] in 1935, based on mean values, is rightly celebrated as a great leap forward, replacing this exponentially weak margin by one polynomial in the degree. Nonetheless, Vinogradov's methods yield bounds removed from the sharpest conjectured to hold by a margin at least logarithmic in the degree, a defect that has endured for six decades since the era in which these ideas were comprehensively analysed. In this report, we describe progress since 2010 that eliminates this defect, placing us within a whisker of establishing in full the main conjecture of the subject.

When $k, s \in \mathbb{N}$ and $\alpha \in \mathbb{R}^k$, consider the exponential sum

$$f_k(\alpha; X) = \sum_{1 \leq x \leq X} e(\alpha_1 x + \dots + \alpha_k x^k) \quad (1.1)$$

and the mean value

$$J_{s,k}(X) = \oint |f_k(\alpha; X)|^{2s} d\alpha. \quad (1.2)$$

Here, as usual, we write $e(z)$ for $e^{2\pi iz}$. Also, to save clutter, when $G : [0, 1]^k \rightarrow \mathbb{C}$ is integrable, we write $\oint G(\alpha) d\alpha = \int_{[0,1]^k} G(\alpha) d\alpha$. By orthogonality, one sees

that $J_{s,k}(X)$ counts the number of integral solutions of the system of equations

$$x_1^j + \dots + x_s^j = y_1^j + \dots + y_s^j \quad (1 \leq j \leq k), \quad (1.3)$$

with $1 \leq x_i, y_i \leq X$ ($1 \leq i \leq s$). Upper bounds for $J_{s,k}(X)$ are known collectively as *Vinogradov's mean value theorem*. We now focus discussion by recording the classical version of this theorem that emerged from the first half-century of refinements following Vinogradov's seminal paper [50] (see in particular [27, 33, 51]), culminating in the papers of Karatsuba [29] and Stechkin [42].

Theorem 1.1. *There is an absolute constant $A > 0$ having the property that, whenever s, r and k are natural numbers with $s \geq rk$, then*

$$J_{s,k}(X) \leq C(k, r) X^{2s-k(k+1)/2+\Delta_{s,k}}, \quad (1.4)$$

where $\Delta_{s,k} = \frac{1}{2}k^2(1 - 1/k)^r$ and $C(k, r) = \min\{k^{Ask}, k^{Ak^3}\}$.

We will not concern ourselves with the dependence on s and k of constants such as $C(k, r)$ appearing in bounds for $J_{s,k}(X)$ and its allies (but see [57] for improvements in this direction). Although significant in applications to the zero-free region of the Riemann zeta function, this is not relevant to those central to this paper. Thus, implicit constants in the notation of Landau and Vinogradov will depend at most on s, k and ε , unless otherwise indicated¹.

When $k \geq 2$, the exponent $\Delta_{s,k}$ of Theorem 1.1 satisfies $\Delta_{s,k} \leq k^2 e^{-s/k^2}$, and so $\Delta_{s,k} = O(1/\log k)$ for $s \geq k^2(2 \log k + \log \log k)$. One can refine (1.4) to obtain an asymptotic formula when s is slightly larger (see [2, Theorem 3.9], for example).

Theorem 1.2. *Let $k, s \in \mathbb{N}$ and suppose that $s \geq k^2(2 \log k + \log \log k + 5)$. Then there exists a positive number $\mathfrak{C}(s, k)$ with $J_{s,k}(X) \sim \mathfrak{C}(s, k) X^{2s-k(k+1)/2}$.*

With these theorems in hand, we consider the motivation for investigating the sums $f_k(\alpha; X)$. Many number-theoretic functions may be estimated in terms of such sums. Thus, when $\operatorname{Re}(s)$ is close to 1, estimates for the Riemann zeta function $\zeta(s)$ stem from partial summation and Taylor expansions for $\log(1 + x/N)$, since

$$\sum_{N < n \leq N+X} n^{-it} = N^{-it} \sum_{1 \leq x \leq X} e\left(-\frac{t}{2\pi} \log(1 + x/N)\right).$$

On the other hand, specialisations of $f_k(\alpha; X)$ arise naturally in applications of interest. Indeed, work on the asymptotic formula in Waring's problem depends on the sum obtained by setting $\alpha_1 = \dots = \alpha_{k-1} = 0$ and $\alpha_k = \beta$, namely

$$g_k(\beta; X) = \sum_{1 \leq x \leq X} e(\beta x^k). \quad (1.5)$$

¹Given a complex-valued function $f(t)$ and positive function $g(t)$, we use Vinogradov's notation $f(t) \ll g(t)$, or Landau's notation $f(t) = O(g(t))$, to mean that there is a positive number C for which $f(t) \leq Cg(t)$ for all large enough values of t . Also, we write $f(t) \gg g(t)$ when $g(t) \ll f(t)$. If C depends on certain parameters, then we indicate this by appending these as subscripts to the notation. Also, we write $f(t) = o(g(t))$ when $f(t)/g(t) \rightarrow 0$ as $t \rightarrow \infty$. Finally, we use the convention that whenever ε occurs in a statement, then the statement holds for each fixed $\varepsilon > 0$.

Writing $R_{s,k}(n)$ for the number of representations of n as the sum of s positive integral k th powers, one finds by orthogonality that

$$R_{s,k}(n) = \int_0^1 g_k(\beta; n^{1/k})^s e(-\beta n) d\beta.$$

The uninitiated reader will wonder why one should focus on estimates for the mean value $J_{s,k}(X)$ when many applications depend on pointwise estimates for $f_k(\alpha; X)$. Vinogradov observed that mean value estimates suffice to obtain useful pointwise estimates for $f_k(\alpha; X)$. To see why this is the case, note first that $|f_k(\beta; X)|$ differs little from $|f_k(\alpha; X)|$ provided that the latter is large, and in addition $|\beta_j - \alpha_j|$ is rather smaller than X^{-j} for each j , so that β lies in a small neighbourhood of α having measure of order $X^{-k(k+1)/2}$. Second, one sees from (1.1) that for each integer h the sum $f_k(\alpha; X)$ may be rewritten in the form

$$f_k(\alpha; X) = \sum_{1-h \leq x \leq X-h} e(\alpha_1(x+h) + \dots + \alpha_k(x+h)^k).$$

By estimating the tails of this sum and applying the binomial theorem to identify the coefficient of each monomial x^j , one obtains new k -tuples $\alpha^{(h)}$ for which $f_k(\alpha; X) = f_k(\alpha^{(h)}; X) + O(|h|)$. These ideas combine to show that one large value $|f_k(\alpha; X)|$ generates a collection of neighbourhoods $\mathfrak{B}(h)$, with the property that whenever $\beta \in \mathfrak{B}(h)$, then $|f_k(\beta; X)|$ is almost as large as $|f_k(\alpha; X)|$. Given N disjoint such neighbourhoods over which to integrate $|f_k(\beta; X)|^{2s}$, non-trivial estimates for $|f_k(\alpha; X)|$ follow from the relation $NX^{-k(k+1)/2}|f_k(\alpha; X)|^{2s} \ll J_{s,k}(X)$. This circle of ideas leads to the following theorem (see [11] and [47, Theorem 5.2]).

Theorem 1.3. *Let k be an integer with $k \geq 2$, and let $\alpha \in \mathbb{R}^k$. Suppose that there exists a natural number j with $2 \leq j \leq k$ such that, for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, one has $|\alpha_j - a/q| \leq q^{-2}$. Then one has*

$$f_k(\alpha; X) \ll \left(X^{k(k-1)/2} J_{s,k-1}(2X)(q^{-1} + X^{-1} + qX^{-j}) \right)^{1/(2s)} \log(2X).$$

To illustrate the power of this theorem, suppose that k is large, and β satisfies the condition that, whenever $b \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(b, q) = 1$ and $|q\beta - b| \leq X^{1-k}$, then $q > X$. By substituting the conclusion of Theorem 1.1 into Theorem 1.3, one finds that $g_k(\beta; X) \ll X^{1-\sigma(k)}$, where $\sigma(k)^{-1} = (4 + o(1))k^2 \log k$.

2. Translation invariance and a congruencing idea

A key feature of the system of equations (1.3) is *translation-dilation invariance*. Thus, the pair $\mathbf{x}, \mathbf{y}$ is an integral solution of the system

$$x_1^j + \dots + x_t^j = y_1^j + \dots + y_t^j \quad (1 \leq j \leq k), \quad (2.1)$$

if and only if, for any $\xi \in \mathbb{Z}$ and $q \in \mathbb{N}$, the pair $\mathbf{x}, \mathbf{y}$ satisfies the system

$$(qx_1 + \xi)^j + \dots + (qx_t + \xi)^j = (qy_1 + \xi)^j + \dots + (qy_t + \xi)^j \quad (1 \leq j \leq k). \quad (2.2)$$

This property ensures that $J_t(X)$ is homogeneous with respect to restriction to arithmetic progressions². Let $M = X^\theta$ be a parameter to be chosen later, consider a set $\mathcal{P}$ of $\lceil k^2/\theta \rceil$ primes p with $M < p \leq 2M$, and fix some $p \in \mathcal{P}$. Also, define

$$f_c(\alpha; \xi) = \sum_{\substack{1 \leq x \leq X \\ x \equiv \xi \pmod{p^c}}} e(\alpha_1 x + \dots + \alpha_k x^k).$$

Since $\oint |f_c(\alpha; \xi)|^{2t} d\alpha$ counts the number of solutions of (2.2), with $q = p^c$, for which³ $(1 - \xi)/q \leq \mathbf{x}, \mathbf{y} \leq (X - \xi)/q$, by translation-dilation invariance, it counts solutions of (2.1) under the same conditions on $\mathbf{x}$ and $\mathbf{y}$. Thus

$$\max_{1 \leq \xi \leq p^c} \oint |f_c(\alpha; \xi)|^{2t} d\alpha \ll 1 + J_t(X/M^c). \quad (2.3)$$

Translation invariance also generates useful auxiliary congruences. Let $t = s + k$, and consider the solutions of (2.1) with $1 \leq \mathbf{x}, \mathbf{y} \leq X$. The number of solutions T_0 in which $x_i = x_j$ for some $1 \leq i < j \leq k$ may be bounded via orthogonality and Hölder's inequality, giving $T_0 \ll J_t(X)^{1-1/(2t)}$. Given a *conditioned* solution with $x_i \neq x_j$ for $1 \leq i < j \leq k$, there exists a prime $p \in \mathcal{P}$ with $x_i \not\equiv x_j \pmod{p}$ for $1 \leq i < j \leq k$. Let $\Xi_c(\xi)$ denote the set of k -tuples $(\xi_1, \dots, \xi_k)$, with $1 \leq \xi \leq p^{c+1}$ and $\xi \equiv \xi \pmod{p^c}$, and satisfying the property that $\xi_i \not\equiv \xi_j \pmod{p^{c+1}}$ for $i \neq j$. Also, put

$$f_c(\alpha; \xi) = \sum_{\xi \in \Xi_c(\xi)} f_{c+1}(\alpha; \xi_1) \dots f_{c+1}(\alpha; \xi_k),$$

and define

$$I_{a,b}(X) = \max_{\xi, \eta} \oint |\mathfrak{f}_a(\alpha; \xi)^2 \mathfrak{f}_b(\alpha; \eta)^{2s}| d\alpha. \quad (2.4)$$

Then for some $p \in \mathcal{P}$, which we now fix, the number T_1 of conditioned solutions satisfies

$$T_1 \ll \oint \mathfrak{f}_0(\alpha; 0) f(\alpha; X)^s f(-\alpha; X)^{s+k} d\alpha. \quad (2.5)$$

Thus, by Schwarz's inequality and orthogonality, one has $T_1 \ll I_{0,0}(X)^{1/2} J_t(X)^{1/2}$. By combining the above estimates for T_0 and T_1 , we derive the upper bound $J_t(X) \ll J_t(X)^{1-1/(2t)} + I_{0,0}(X)^{1/2} J_t(X)^{1/2}$, whence $J_t(X) \ll I_{0,0}(X)$.

By Hölder's inequality, one finds that

$$|f(\alpha; X)|^{2s} = \left| \sum_{\eta=1}^p \sum_{\substack{1 \leq x \leq X \\ x \equiv \eta \pmod{p}}} e(\alpha_1 x + \dots + \alpha_k x^k) \right|^{2s} \leq p^{2s-1} \sum_{\eta=1}^p |f_1(\alpha; \eta)|^{2s}.$$

Thus, on noting the trivial relation $f(\alpha; X) = f_0(\alpha; \eta)$, one sees from (2.4) that

$$J_t(X) \ll I_{0,0}(X) \ll M^{2s} \max_{\xi, \eta} \oint |\mathfrak{f}_0(\alpha; \xi)^2 \mathfrak{f}_1(\alpha; \eta)^{2s}| d\alpha = M^{2s} I_{0,1}(X). \quad (2.6)$$

²In this section we consider k to be fixed, and hence we drop mention of k from our notations.

³Here we make use of slightly unconventional vector notation. Thus, we write $\mathbf{z} \equiv \xi \pmod{q}$ when $z_i \equiv \xi \pmod{q}$ for $1 \leq i \leq t$, or $a \leq \mathbf{z} \leq b$ when $a \leq z_i \leq b$ ($1 \leq i \leq t$), and so on.

The mean value underlying $I_{0,1}(X)$ counts the number of integral solutions of

$$\sum_{i=1}^k (x_i^j - y_i^j) = \sum_{l=1}^s ((pu_l + \eta)^j - (pv_l + \eta)^j) \quad (1 \leq j \leq k),$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq X$ and $1 \leq p\mathbf{u} + \eta, p\mathbf{v} + \eta \leq X$, in which $x_i \not\equiv x_j \pmod{p}$ for $i \neq j$, and similarly for $\mathbf{y}$. Translation invariance leads from these equations to

$$\sum_{i=1}^k ((x_i - \eta)^j - (y_i - \eta)^j) = p^j \sum_{l=1}^s (u_l^j - v_l^j) \quad (1 \leq j \leq k),$$

and hence

$$(x_1 - \eta)^j + \dots + (x_k - \eta)^j \equiv (y_1 - \eta)^j + \dots + (y_k - \eta)^j \pmod{p^j} \quad (1 \leq j \leq k). \quad (2.7)$$

Since the x_i are distinct modulo p , Hensel's lemma shows that, for each fixed choice of $\mathbf{y}$, there are at most $k!p^{k(k-1)/2}$ choices for $\mathbf{x} \pmod{p^k}$ satisfying (2.7). An application of Cauchy's inequality shows from here that

$$I_{0,1}(X) \ll M^{k(k-1)/2} \max_{\eta} \oint \left(\sum_{\nu=1}^{p^k} |f_k(\alpha; \nu)|^2 \right)^k |f_1(\alpha; \eta)|^{2s} d\alpha. \quad (2.8)$$

Although our notation has been crafted for later discussion of efficient congruencing, the classical approach remains visible. One applies (2.8) with $\theta = 1/k$, so that $p^k > X$. Thus $|f_k(\alpha; \nu)| \leq 1$, and it follows from (2.6) and (2.8) that

$$J_t(X) \ll M^{2s} I_{0,1}(X) \ll M^{2s+k(k-1)/2} (M^k)^k \max_{\eta} \oint |f_1(\alpha; \eta)|^{2s} d\alpha.$$

It therefore follows from (2.3) that

$$J_{s+k}(X) \ll M^{2s+k(k-1)/2} X^k J_s(X/M) \ll X^{2k} (X^{1/k})^{2s-k(k+1)/2} J_s(X^{1-1/k}).$$

This iterative relation leads from the bound $J_{k,k}(X) \ll X^k$ to the estimate presented in Theorem 1.1. Early authors, such as Vinogradov and Hua, made use of short real intervals in place of congruences, the modern shift to congruences merely adjusting the point of view from the infinite place to a finite place.

3. Lower bounds and the Main Conjecture

Write $T_s(X)$ for the number of diagonal solutions of (1.3) with $1 \leq \mathbf{x}, \mathbf{y} \leq X$ and $\{x_1, \dots, x_s\} = \{y_1, \dots, y_s\}$. Then $J_{s,k}(X) \geq T_s(X) = s!X^s + O_s(X^{s-1})$. Meanwhile, when $1 \leq \mathbf{x}, \mathbf{y} \leq X$, one has $|(x_1^j - y_1^j) + \dots + (x_s^j - y_s^j)| \leq sX^j$. Hence

$$[X]^{2s} = \sum_{|h_1| \leq sX} \dots \sum_{|h_k| \leq sX^k} \oint |f_k(\alpha; X)|^{2s} e(-\alpha_1 h_1 - \dots - \alpha_k h_k) d\alpha,$$

and we deduce from the triangle inequality in combination with (1.2) that

$$X^{2s} \ll \sum_{|h_1| \leq sX} \dots \sum_{|h_k| \leq sX^k} J_{s,k}(X) \ll X^{k(k+1)/2} J_{s,k}(X).$$

Thus we conclude that $J_{s,k}(X) \gg X^s + X^{2s-k(k+1)/2}$, a lower bound that guides a heuristic application of the circle method towards the following conjecture.

Conjecture 3.1 (The Main Conjecture). *Suppose that s and k are natural numbers. Then for each $\varepsilon > 0$, one has $J_{s,k}(X) \ll X^\varepsilon (X^s + X^{2s-k(k+1)/2})$.*

We emphasise that the implicit constant here may depend on ε , s and k . The critical case of the Main Conjecture with $s = k(k+1)/2$ has special significance.

Conjecture 3.2. *When $k \in \mathbb{N}$ and $\varepsilon > 0$, one has $J_{k(k+1)/2,k}(X) \ll X^{k(k+1)/2+\varepsilon}$.*

Suppose temporarily that this critical case of the Main Conjecture holds. Then, when $s \geq k(k+1)/2$, one may apply a trivial estimate for $f_k(\alpha; X)$ to show that

$$J_{s,k}(X) \leq X^{2s-k(k+1)} \oint |f_k(\alpha; X)|^{k(k+1)} d\alpha \ll X^{2s-k(k+1)/2+\varepsilon},$$

and when $s < k(k+1)/2$, one may instead apply Hölder's inequality to obtain

$$J_{s,k}(X) \leq \left(\oint |f_k(\alpha; X)|^{k(k+1)} d\alpha \right)^{\frac{2s}{k(k+1)}} \ll X^{s+\varepsilon}.$$

In both cases, therefore, the Main Conjecture is recovered from the critical case.

Until 2014, the critical case of the Main Conjecture was known to hold in only two cases. The case $k = 1$ is trivial. The case $k = 2$, on the other hand, depends on bounds for the number of integral solutions of the simultaneous equations

$$\left. \begin{aligned} x_1^2 + x_2^2 + x_3^2 &= y_1^2 + y_2^2 + y_3^2 \\ x_1 + x_2 + x_3 &= y_1 + y_2 + y_3 \end{aligned} \right\}, \quad (3.1)$$

with $1 \leq x_i, y_i \leq X$. From the identity $(a+b-c)^2 - (a^2 + b^2 - c^2) = 2(a-c)(b-c)$, one finds that the solutions of (3.1) satisfy $(x_1 - y_3)(x_2 - y_3) = (y_1 - x_3)(y_2 - x_3)$. From here, elementary estimates for the divisor function convey us to the bound $J_{3,2}(X) \ll X^{3+\varepsilon}$, so that Conjecture 3.2 and the Main Conjecture hold when $k = 2$. In fact, improving on earlier work of Rogovskaya [41], it was shown by Blomer and Brüdern [10] that

$$J_{3,2}(X) = \frac{18}{\pi^2} X^3 \log X + \frac{3}{\pi^2} \left(12\gamma - 6 \frac{\zeta'(2)}{\zeta(2)} - 5 \right) X^3 + O(X^{5/2} \log X).$$

In particular, the factor X^ε cannot be removed from the statements of Conjectures 3.1 and 3.2. However, a careful heuristic analysis of the circle method reveals that when $(s, k) \neq (3, 2)$, the Main Conjecture should hold with $\varepsilon = 0$. See [47, equation (7.5)] for a discussion that records precisely such a conjecture.

The classical picture of the Main Conjecture splits naturally into two parts: small s and large s . When $1 \leq s \leq k$, the relation $J_{s,k}(X) = T_s(X) \sim s!X^s$ is immediate from Newton's formulae concerning roots of polynomials. Identities analogous to that above yield multiplicative relations amongst variables in the system (1.3) when $s = k + 1$. In this way, Hua [26] confirmed the Main Conjecture for $s \leq k + 1$ by obtaining the bound $J_{k+1,k}(X) \ll X^{k+1+\varepsilon}$. Vaughan and Wooley have since obtained the asymptotic formula $J_{k+1,k}(X) = T_{k+1}(X) + O(X^{\theta_k+\varepsilon})$, where $\theta_3 = \frac{10}{3}$ [48, Theorem 1.5] and $\theta_k = \sqrt{4k+5}$ ($k \geq 4$) [49, Theorem 1]. Approximations to the Main Conjecture of the type $J_{s,k}(X) \ll X^{s+\delta_{s,k}}$, with $\delta_{s,k}$ small, can be obtained for larger values of s . Thus, on writing $\gamma = s/k$, the work of Arkhipov and Karatsuba [3] shows that permissible exponents $\delta_{s,k}$ exist with $\delta_{s,k} \ll \gamma^{3/2}k^{1/2}$, Tyrina [43] gets $\delta_{s,k} \ll \gamma^2$, and Wooley [58, Theorem 1] obtains $\delta_{s,k} = \exp(-Ak/\gamma^2)$, when $s \leq k^{3/2}(\log k)^{-1}$, for a certain positive constant A .

We turn next to large values of s . When $k \in \mathbb{N}$, denote by $H(k)$ the least integer for which the Main Conjecture for $J_{s,k}(X)$ holds whenever $s \geq H(k)$. Theorem 1.2 gives $H(k) \leq (2 + o(1))k^2 \log k$, a consequence of the classical estimate (1.4) with permissible exponent $\Delta_{s,k} = k^2 e^{-s/k^2}$. In 1992, the author [56] found a means of combining Vinogradov's methods with the *efficient differencing method* (see [55], and the author's previous ICM lecture [61]), obtaining $\Delta_{s,k} \approx k^2 e^{-2s/k^2}$. This yields $H(k) \leq (1 + o(1))k^2 \log k$ (see [60]), halving the previous bound. Meanwhile, Hua [26, Theorem 7] has applied Weyl differencing to bound $H(k)$ for small k . We summarise the classical status of the Main Conjecture in the following theorem.

Theorem 3.3. *The Main Conjecture holds for $J_{s,k}(X)$ when:*

- (i) $k = 1$ and 2 ;
- (ii) $k \geq 2$ and $1 \leq s \leq k + 1$;
- (iii) $s \geq H(k)$, where $H(3) = 8$, $H(4) = 23$, $H(5) = 55$, $H(6) = 120$, ..., and $H(k) = k^2(\log k + 2 \log \log k + O(1))$.

4. The advent of efficient congruencing

The introduction of the *efficient congruencing method* [62] at the end of 2010 has transformed our understanding of the Main Conjecture. Incorporating subsequent developments [20, 64], and the multigrade enhancement of the method [65, 66, 67], we can summarise the current state of affairs in the form of a theorem.

Theorem 4.1. *The Main Conjecture holds for $J_{s,k}(X)$ when:*

- (i) $k = 1, 2$ and 3 ;
- (ii) $1 \leq s \leq D(k)$, where $D(4) = 8$, $D(5) = 10$, $D(6) = 17$, $D(7) = 20$, ..., and $D(k) = \frac{1}{2}k(k+1) - \frac{1}{3}k + O(k^{2/3})$;
- (iii) $k \geq 3$ and $s \geq H(k)$, where $H(k) = k(k-1)$.

As compared to the classical situation, there are three principal advances:

(a) First, the Main Conjecture holds for $J_{s,k}(X)$ in the cubic case $k = 3$ (see [67, Theorem 1.1]), so that $J_{s,3}(X) \ll X^\varepsilon(X^s + X^{2s-6})$. This is the first occasion, for any polynomial Weyl sum of degree exceeding 2, that the conjectural mean value estimates have been established in full, even if the underlying variables are restricted to lie in such special sets as the smooth numbers.

(b) Second, the Main Conjecture holds in the form $J_{s,k}(X) \ll X^{s+\varepsilon}$ provided that $1 \leq s \leq \frac{1}{2}k(k+1) - \frac{1}{3}k + O(k^{2/3})$, which as $k \rightarrow \infty$ represents 100% of the critical interval $1 \leq s \leq k(k+1)/2$ (see [66, Theorem 1.3]). The classical result reported in Theorem 3.3(ii) only provides such a conclusion for $1 \leq s \leq k+1$, amounting to 0% of the critical interval. Here, the first substantial advance was achieved by Ford and Wooley [20, Theorem 1.1], giving the Main Conjecture for $1 \leq s \leq \frac{1}{4}(k+1)^2$. Although Theorem 4.1(ii) comes within $(\frac{1}{3} + o(1))k$ variables of proving the critical case of the Main Conjecture, it seems that a new idea is required to replace this defect by $(c + o(1))k$, for some real number c with $c < \frac{1}{3}$.

(c) Third, the Main Conjecture holds in the form $J_{s,k}(X) \ll X^{2s-k(k+1)/2+\varepsilon}$ for $s \geq k(k-1)$. The classical result reported in Theorem 3.3(iii) provides such a conclusion for $s \geq (1 + o(1))k^2 \log k$, a constraint weaker by a factor $\log k$. So far as applications are concerned, this is by far the most significant advance thus far captured by the efficient congruencing method. The initial progress [62, Theorem 1.1] shows that the Main Conjecture holds for $s \geq k(k+1)$, already within a factor 2 of the critical exponent $s = k(k+1)/2$. Subsequently, this constraint was improved first to $s \geq k^2 - 1$, and then to $s \geq k^2 - k + 1$ (see [64, Theorem 1.1] and [65, Corollary 1.2]). The further modest progress reported in Theorem 4.1(iii) was announced in [67, Theorem 1.2], and will appear in a forthcoming paper.

Prior to the advent of efficient congruencing, much effort had been spent on refining estimates of the shape $J_{s,k}(X) \ll X^{2s-k(k+1)/2+\Delta_{s,k}}$, with the permissible exponent $\Delta_{s,k}$ as small as possible (see [9, 19, 56, 58]). Of great significance for applications, efficient congruencing permits substantially sharper bounds to be obtained for such exponents than were hitherto available. Such ideas feature in [64, Theorem 1.4], and the discussion following [20, Theorem 1.2] shows that when $\frac{1}{4} \leq \alpha \leq 1$ and $s = \alpha k^2$, then the exponent $\Delta_{s,k} = (1 - \sqrt{\alpha})^2 k^2 + O(k)$ is permissible. Thus, in particular, the critical exponent $\Delta_{k(k+1)/2,k} = (\frac{3}{2} - \sqrt{2})k^2$ is permissible. By combining [66, Theorem 1.5] and the discussion following [65, Corollary 1.2], one arrives at the following improvement.

Theorem 4.2. *When k is large, there is a positive number $C(s) \leq \frac{1}{3}$ for which*

$$J_{s,k}(X) \ll X^{(C(s)+o(1))k} (X^s + X^{2s-k(k+1)/2}).$$

When $\alpha \in [\frac{5}{8}, 1]$, moreover, one may take $C(\alpha k^2) \leq (2 - 3\alpha + (2\alpha - 1)^{3/2})/(3\alpha)$.

We finish this section by noting that Theorem 4.1(iii) permits a substantial improvement in the conclusion of Theorem 1.2.

Theorem 4.3. *Let $k, s \in \mathbb{N}$ and suppose that $s \geq k^2 - k + 1$. Then there exists a positive number $\mathfrak{C}(s, k)$ with $J_{s,k}(X) \sim \mathfrak{C}(s, k)X^{2s-k(k+1)/2}$.*

5. A sketch of the efficient congruencing method

Although complicated in detail, the ideas underlying efficient congruencing are accessible given some simplifying assumptions. In this section, we consider k to be fixed, and drop mention of k from our notation. Let $t = (u + 1)k$, where $u \geq k$ is an integer, and put $s = uk$. We define

$$\lambda_t = \limsup_{X \rightarrow \infty} (\log J_t(X)) / (\log X).$$

Thus, for each $\varepsilon > 0$, one has the bound $J_t(X) \ll X^{\lambda_t + \varepsilon}$. Our goal is to establish that $\lambda_t = 2t - k(k + 1)/2$, as predicted by the Main Conjecture. Define Λ via the relation $\lambda_t = 2t - \frac{1}{2}k(k + 1) + \Lambda$. We suppose that $\Lambda > 0$, and seek a contradiction in order to show that $\Lambda = 0$. Our method rests on an N -fold iteration related to the approach of §2, where N is sufficiently large in terms of u , k and Λ . Let $\theta = (16k)^{-2N}$, put $M = X^\theta$, and consider a prime number p with $M < p \leq 2M$. Also, let $\delta > 0$ be small in terms of all these parameters, so that $8\delta < N(k/u)^N \Lambda \theta$.

Define the mean value

$$K_{a,b}(X) = \max_{\xi, \eta} \oint |\mathfrak{F}_a(\alpha; \xi)^2 \mathfrak{F}_b(\alpha; \eta)^{2u}| d\alpha,$$

and introduce the normalised mean values

$$[[K_{a,b}(X)]] = \frac{K_{a,b}(X)}{(X/M^a)^{2k-k(k+1)/2} (X/M^b)^{2s}} \quad \text{and} \quad [[J_t(X)]] = \frac{J_t(X)}{X^{2t-k(k+1)/2}}.$$

Then whenever X is sufficiently large in terms of the ambient parameters, one has $[[J_t(X)]] > X^{\Lambda - \delta}$ and, when $X^{1/2} \leq Y \leq X$, we have the bound $[[J_t(Y)]] \leq Y^{\Lambda + \delta}$.

We begin by observing that an elaboration of the argument delivering (2.5) can be fashioned to replace (2.6) with the well-conditioned relation

$$J_t(X) \ll M^{2s} \max_{\xi, \eta} \oint |\mathfrak{F}_0(\alpha; \xi)^2 \mathfrak{F}_1(\alpha; \eta)^{2u}| d\alpha = M^{2s} K_{0,1}(X).$$

Here we have exercised considerable expedience in ignoring controllable error terms. Moreover, one may need to replace $K_{0,1}(X)$ by the surrogate $K_{0,1+h}(X)$, for a suitable integer h . An analogue of the argument leading to (2.8) yields the bound

$$K_{0,1}(X) \ll M^{k(k-1)/2} \max_{\eta} \oint \left(\sum_{\nu=1}^{p^k} |\mathfrak{f}_k(\alpha; \nu)|^2 \right)^k |\mathfrak{F}_1(\alpha; \eta)|^{2u} d\alpha.$$

By Hölder's inequality, one finds first that

$$\left(\sum_{\nu=1}^{p^k} |\mathfrak{f}_k(\alpha; \nu)|^2 \right)^k \leq (p^k)^{k-1} \sum_{\nu=1}^{p^k} |\mathfrak{f}_k(\alpha; \nu)|^{2k},$$

and then

$$K_{0,1}(X) \ll M^{k(k-1)/2} (M^k)^k \max_{\eta, \nu} \left(T_1(\eta)^{1-1/u} T_2(\eta, \nu)^{1/u} \right),$$

where

$$T_1(\eta) = \oint |\mathfrak{F}_1(\boldsymbol{\alpha}; \eta)|^{2u+2} d\boldsymbol{\alpha} \quad \text{and} \quad T_2(\eta, \nu) = \oint |\mathfrak{F}_1(\boldsymbol{\alpha}; \eta)^2 \mathfrak{f}_k(\boldsymbol{\alpha}; \nu)^{2s}| d\boldsymbol{\alpha}.$$

On considering the underlying Diophantine systems, one finds that $T_1(\eta)$ may be bounded via (2.3), while $T_2(\eta, \nu)$ may be bounded in terms of $K_{1,k}(X)$. Thus

$$J_t(X) \ll M^{2s+k(k-1)/2} (M^k)^k J_t(X/M)^{1-1/u} K_{1,k}(X)^{1/u}.$$

A modicum of computation therefore confirms that

$$[[J_t(X)]] \ll [[J_t(X/M)]]^{1-1/u} [[K_{1,k}(X)]]^{1/u}. \quad (5.1)$$

The mean value underlying $K_{1,k}(X)$ counts the number of integral solutions of

$$\sum_{i=1}^k (x_i^j - y_i^j) = \sum_{l=1}^s ((p^k u_l + \eta)^j - (p^k v_l + \eta)^j) \quad (1 \leq j \leq k),$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq X$ and $1 \leq p^k \mathbf{u} + \eta, p^k \mathbf{v} + \eta \leq X$ having suitably conditioned coordinates. In particular, one has $\mathbf{x} \equiv \mathbf{y} \equiv \xi \pmod{p}$ but $x_i \not\equiv x_j \pmod{p^2}$ for $i \neq j$, and similarly for $\mathbf{y}$. Translation invariance leads from these equations to

$$\sum_{i=1}^k ((x_i - \eta)^j - (y_i - \eta)^j) = p^{jk} \sum_{l=1}^s (u_l^j - v_l^j) \quad (1 \leq j \leq k),$$

and hence to the congruences

$$(x_1 - \eta)^j + \dots + (x_k - \eta)^j \equiv (y_1 - \eta)^j + \dots + (y_k - \eta)^j \pmod{p^{jk}} \quad (1 \leq j \leq k). \quad (5.2)$$

Since the x_i are distinct modulo p^2 , an application of Hensel's lemma shows that, for each fixed choice of $\mathbf{y}$, there are at most $k!(p^k)^{k(k-1)/2} \cdot p^{k(k-1)/2}$ choices for $\mathbf{x} \pmod{p^{k^2}}$ satisfying (5.2). Here, the factor $p^{k(k-1)/2}$ reflects the fact that, even though $x_i \not\equiv x_j \pmod{p^2}$ for $i \neq j$, one has $x_i \equiv x_j \pmod{p}$ for all i and j . This situation is entirely analogous to that delivering (2.8) above, and thus we obtain

$$K_{1,k}(X) \ll (M^{k+1})^{k(k-1)/2} \max_{\xi, \eta} \oint \left(\sum_{\substack{\nu=1 \\ \nu \equiv \xi \pmod{p}}}^{p^{k^2}} |\mathfrak{f}_{k^2}(\boldsymbol{\alpha}; \nu)|^2 \right)^k |\mathfrak{F}_k(\boldsymbol{\alpha}; \eta)|^{2u} d\boldsymbol{\alpha}.$$

From here, as above, suitable applications of Hölder's inequality show that

$$[[K_{1,k}(X)]] \ll [[J_t(X/M^k)]]^{1-1/u} [[K_{k,k^2}(X)]]^{1/u}. \quad (5.3)$$

By substituting this estimate into (5.1), we obtain the new upper bound

$$[[J_t(X)]] \ll ([[J_t(X/M)]] [[J_t(X/M^k)]]^{1/u})^{1-1/u} [[K_{k,k^2}(X)]]^{1/u^2}.$$

By iterating this process N times, one obtains the relation

$$[[J_t(X)]] \ll \left(\prod_{r=0}^{N-1} [[J_t(X/M^{k^r})]]^{1/u^r} \right)^{1-1/u} [[K_{k^{N-1}, k^N}(X)]]^{1/u^N}. \quad (5.4)$$

While this is a vast oversimplification of what is actually established, it correctly identifies the relationship which underpins the efficient congruencing method.

Since $M^{k^N} < X^{1/3}$, our earlier discussion ensures that

$$[[J_t(X)]] \gg X^{\Lambda-\delta} \quad \text{and} \quad [[J_t(X/M^{k^r})]] \ll (X/M^{k^r})^{\Lambda+\delta} \quad (0 \leq r \leq N).$$

Meanwhile, an application of Hölder's inequality provides the trivial bound

$$[[K_{k^{N-1}, k^N}(X)]] \ll (M^{k(k+1)/2})^{k^N} X^{\Lambda+\delta}.$$

By substituting these estimates into (5.4), we deduce that

$$X^{\Lambda-\delta} \ll \left(X^{1/u^N} \prod_{r=0}^{N-1} (X/M^{k^r})^{(1-1/u)/u^r} \right)^{\Lambda+\delta} (M^{k(k+1)/2})^{(k/u)^N},$$

and hence $X^{\Lambda-\delta} \ll X^{\Lambda+\delta} (M^\Theta)^{(k/u)^N}$, where

$$\Theta = \frac{1}{2}k(k+1) - (1-1/u)(\Lambda+\delta) \sum_{r=1}^N (u/k)^r.$$

But we have $u \geq k$, and so our hypotheses concerning N and δ ensure that

$$\Theta \leq \frac{1}{2}k(k+1) - N(1-1/u)(\Lambda+\delta) < -\frac{1}{2}N\Lambda < -3(u/k)^N \delta/\theta.$$

We therefore conclude that $X^{\Lambda-\delta} \ll X^{\Lambda+\delta} M^{-3\delta/\theta} \ll X^{\Lambda-2\delta}$. This relation yields the contradiction that establishes the desired conclusion $\Lambda = 0$. We may therefore conclude that whenever $t \geq k(k+1)$, one has $J_t(X) \ll X^{2t-k(k+1)/2+\varepsilon}$.

We have sketched the proof of the Main Conjecture for $J_t(X)$ when $t \geq k(k+1)$. Theorem 4.1, which represents the latest state of play in the efficient congruencing method, goes considerably further. Two ideas underpin these advances.

First, one may sacrifice some of the power potentially available from systems of congruences such as (2.7) or (5.2) in order that the efficient congruencing method be applicable when $t < k(k+1)$. Let r be a parameter with $2 \leq r \leq k$, and define the generating function $\mathfrak{F}_c^{(r)}(\alpha; \xi)$ by analogy with $\mathfrak{F}_c(\alpha; \xi)$, though with r (in place of k) underlying exponential sums $f_{c+1}(\alpha; \xi_i)$. One may imitate the basic argument sketched above, with $t = (u+1)r$, to bound the analogue $K_{a,b}^{(r)}(X)$ of the mean value $K_{a,b}(X)$. In place of (5.2) one now obtains the congruences

$$(x_1 - \eta)^j + \dots + (x_r - \eta)^j \equiv (y_1 - \eta)^j + \dots + (y_r - \eta)^j \pmod{p^{jb}} \quad (1 \leq j \leq k). \quad (5.5)$$

For simplicity, suppose that $r \leq (k-1)/2$. Then by considering the r congruence relations of highest degree here, one finds from Hensel's lemma that, for each

fixed choice of $\mathbf{y}$, there are at most $k!$ choices for $\mathbf{x} \pmod{p^{(k-r)b}}$ satisfying (5.5). Although this is a weaker congruence constraint than before on $\mathbf{x}$ and $\mathbf{y}$, the cost in terms of the number of choices is smaller, and so useful estimates may nonetheless be obtained for $J_t(X)$. Ideas along these lines underpin both the work [64] of the author, and in the sharper form sketched above, that of Ford and the author [20].

The second idea conveys us to the threshold of the Main Conjecture. Again we consider the mean values $K_{a,b}^{(r)}(X)$, and for simplicity put $r = k - 1$. The congruences (5.5) yield a constraint on the variables tantamount to $x_i \equiv y_i \pmod{p^{2b}}$ at little cost. Encoding this constraint using exponential sums, and applying Hölder's inequality, one bounds $K_{a,b}^{(k-1)}(X)$ in terms of $K_{a,b}^{(k-2)}(X)$ and $K_{b,2b}^{(k-1)}(X)$. Iterating this process to successively estimate $K_{a,b}^{(k-j)}(X)$ for $j = 1, 2, \dots, k - 1$, we obtain a bound for $K_{a,b}^{(k-1)}(X)$ in terms of $K_{b,jb}^{(k-1)}(X)$ ($2 \leq j \leq k$) and $J_t(X/M^b)$. The heuristic potential of this idea amounts to a relation of the shape

$$[[K_{a,b}^{(k-1)}(X)]] \ll \left(\prod_{j=2}^k [[K_{b,jb}^{(k-1)}(X)]]^{\phi_j} \right) [[J_t(X/M^b)]]^{1-(k-1)/s}, \quad (5.6)$$

where the exponents ϕ_j are approximately equal to $1/s$. Again, this substantially oversimplifies the situation, since non-negligible additional factors occur. However, one discerns a critical advantage over earlier relations such as (5.3). As one iterates (5.6), one bounds $[[K_{a,b}^{(k-1)}(X)]]$ in terms of new expressions $[[K_{b,b'}^{(k-1)}(X)]]$, where the ratio b'/b is on average about $\frac{1}{2}k + 1$, as opposed to the previous ratio k . The relation (5.6) may be converted into a substitute for (5.4) of the shape

$$[[J_t(X)]] \ll \left(\prod_{r=0}^{N-1} [[J_t(X/M^{\rho^r})]]^{1/u^r} \right)^{1-1/u} [[K_{\rho^{N-1}, \rho^N}^{(k-1)}(X)]]^{1/u^N},$$

in which ρ is close to $\frac{1}{2}k + 1$ and $t = (u + 1)(k - 1)$. Thus, when $u \geq \rho$, we find as before that the lower bound $[[J_t(X)]] \gg X^{\Lambda - \delta}$ is tenable only when $\Lambda = 0$, and we have heuristically established the Main Conjecture when t is only slightly larger than $k(k + 1)/2$. Of course, the relation (5.6) represents an idealised situation, and the proof in detail of the results in [65, 66] contains numerous complications requiring the resolution of considerable technical difficulties.

6. Waring's problem

Investigations concerning the validity of the anticipated asymptotic formula in Waring's problem have historically followed one of two paths, associated on the one hand with Weyl, and on the other with Vinogradov. We recall our earlier notation, writing $R_{s,k}(n)$ for the number of representations of the natural number n in the shape $n = x_1^k + \dots + x_s^k$, with $\mathbf{x} \in \mathbb{N}^s$. A heuristic application of the circle method suggests that for $k \geq 3$ and $s \geq k + 1$, one should have

$$R_{s,k}(n) = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}), \quad (6.1)$$

where

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left(q^{-1} \sum_{r=1}^q e(ar^k/q) \right)^s e(-na/q).$$

Under modest congruence conditions, one has $1 \ll \mathfrak{S}_{s,k}(n) \ll n^\varepsilon$, and thus the conjectural relation (6.1) may be seen as an honest asymptotic formula (see [47, §§4.3, 4.5 and 4.6] for details). Let $\tilde{G}(k)$ denote the least integer t with the property that, whenever $s \geq t$, the asymptotic formula (6.1) holds for all large enough n .

Leaving aside the smallest exponents $k = 1$ and 2 accessible to classical methods, the first to obtain a bound for $\tilde{G}(k)$ were Hardy and Littlewood [21], who devised a method based on Weyl differencing to show that $\tilde{G}(k) \leq (k-2)2^{k-1} + 5$. In 1938, Hua [24] obtained a refinement based on the estimate

$$\int_0^1 |g_k(\alpha; X)|^{2^k} d\alpha \ll X^{2^k - k + \varepsilon}, \quad (6.2)$$

in which $g_k(\alpha; X)$ is defined via (1.5), showing that $\tilde{G}(k) \leq 2^k + 1$. For small values of k , this estimate remained the strongest known for nearly half a century. Finally, Vaughan [45, 46] succeeded in wielding Hooley's Δ -functions to deduce that $\tilde{G}(k) \leq 2^k$ for $k \geq 3$. For slightly larger exponents $k \geq 6$, this bound was improved by Heath-Brown [22] by combining Weyl differencing with a novel cubic mean value estimate. His bound $\tilde{G}(k) \leq \frac{7}{8}2^k + 1$ was, in turn, refined by Boklan [8], who exploited Hooley's Δ -functions in this new setting to deduce that $\tilde{G}(k) \leq \frac{7}{8}2^k$ for $k \geq 6$.

Turning now to large values of k , the story begins with Vinogradov [50], who showed that $\tilde{G}(k) \leq 183k^9(\log k + 1)^2$, reducing estimates previously exponential in k to polynomial bounds. As Vinogradov's mean value theorem progressed to the state essentially captured by Theorem 1.1, bounds were rapidly refined to the form $\tilde{G}(k) \leq (C + o(1))k^2 \log k$, culminating in 1949 with Hua's bound [27] of this shape with $C = 4$. The connection with Vinogradov's mean value theorem is simple to explain, for on considering the underlying Diophantine systems, one finds that

$$\int_0^1 |g_k(\alpha; X)|^{2s} d\alpha = \sum_{\mathbf{h}} \oint |f_k(\boldsymbol{\alpha}; X)|^{2s} e(-h_1\alpha_1 - \dots - h_{k-1}\alpha_{k-1}) d\boldsymbol{\alpha},$$

where the summation is over $|h_j| \leq sX^j$ ($1 \leq j \leq k-1$). The bound (1.4) therefore leads via the triangle inequality and (1.2) to the estimate

$$\int_0^1 |g_k(\alpha; X)|^{2s} d\alpha \ll X^{k(k-1)/2} J_{s,k}(X) \ll X^{2s-k+\Delta_{s,k}}, \quad (6.3)$$

which serves as a surrogate for (6.2). In 1992, the author reduced the permissible value of C from 4 to 2 by applying the repeated efficient differencing method [56]. A more efficient means of utilising Vinogradov's mean value theorem to bound $\tilde{G}(k)$ was found by Ford [18] (see also [44]), showing that $C = 1$ is permissible.

Refinements for smaller values of k show that this circle of ideas surpasses the above-cited bound $\tilde{G}(k) \leq \frac{7}{8}2^k$ when $k \geq 9$ (see Boklan and Wooley [9]).

We summarise the classical state of affairs in the following theorem.

Theorem 6.1 (Classical status of $\tilde{G}(k)$). *One has:*

- (i) $\tilde{G}(k) \leq 2^k$ ($k = 3, 4, 5$) and $\tilde{G}(k) \leq \frac{7}{8}2^k$ ($k = 6, 7, 8$);
- (ii) $\tilde{G}(9) \leq 365$, $\tilde{G}(10) \leq 497$, $\tilde{G}(11) \leq 627$, $\tilde{G}(12) \leq 771$, ...;
- (iii) $\tilde{G}(k) \leq (1 + o(1))k^2 \log k$ (k large).

The most immediate impact of the new efficient congruencing method in Vinogradov's mean value theorem [62] was the bound $\tilde{G}(k) \leq 2k^2 + 2k - 3$, valid for $k \geq 2$. This already supersedes the previous work presented in Theorem 6.1 when $k \geq 7$. In particular, the obstinate factor of $\log k$ is definitively removed for large values of k . Subsequent refinements [20, 63, 64, 65, 66] have delivered further progress, especially for smaller values of k , which we summarise as follows.

Theorem 6.2 (Status of $\tilde{G}(k)$ after efficient congruencing). *One has:*

- (i) $\tilde{G}(k) \leq 2^k$ ($k = 3, 4$);
- (ii) $\tilde{G}(5) \leq 28$, $\tilde{G}(6) \leq 43$, $\tilde{G}(7) \leq 61$, $\tilde{G}(8) \leq 83$, $\tilde{G}(9) \leq 107$, $\tilde{G}(10) \leq 134$, $\tilde{G}(11) \leq 165$, $\tilde{G}(12) \leq 199$, ...;
- (iii) $\tilde{G}(k) \leq (C + o(1))k^2$ (k large), where $C = 1.54079$ is an approximation to the number $(5 + 6\xi - 3\xi^2)/(2 + 6\xi)$, in which ξ is the real root of $6\xi^3 + 3\xi^2 - 1$.

A comparison of Theorems 6.1 and 6.2 reveals that the classical Weyl-based bounds have now been superseded for $k \geq 5$. The latest developments [65, 67] hint, indeed, at further progress even when $k = 4$. These advances for smaller values of k stem in part, of course, from the substantial progress in our new bounds for $J_{s,k}(X)$, as outlined in Theorems 4.1 and 4.2. However, an important role is also played by a novel mean value estimate for moments of $g_k(\alpha; X)$. Define the minor arcs $\mathfrak{m} = \mathfrak{m}_k$ to be the set of real numbers $\alpha \in [0, 1)$ satisfying the property that, whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|q\alpha - a| \leq X^{1-k}$, then $q > X$. The argument of the proof of [63, Theorem 2.1] yields the bound

$$\int_{\mathfrak{m}} |g_k(\alpha; X)|^{2s} d\alpha \ll X^{\frac{1}{2}k(k-1)-1} (\log X)^{2s+1} J_{s,k}(X). \quad (6.4)$$

We thus infer from Theorem 4.1(iii) that whenever $k \geq 3$ and $s \geq k(k-1)$, then

$$\int_{\mathfrak{m}} |g_k(\alpha; X)|^{2s} d\alpha \ll X^{2s-k-1+\varepsilon}.$$

As compared to the classical approach embodied in (6.3), an additional factor X has been saved in these estimates at no cost in terms of the number of variables, and for smaller values of k this is a very substantial gain.

For large values of k , the enhancement of Ford [18] given by Ford and Wooley [20, Theorem 8.5] remains of value. When $k, s \in \mathbb{N}$, denote by $\eta(s, k)$ the least number η with the property that, whenever X is sufficiently large in terms of s and k , one has $J_{s,k}(X) \ll_{\varepsilon} X^{2s-k(k+1)/2+\eta+\varepsilon}$. Let $r \in \mathbb{N}$ satisfy $1 \leq r \leq k-1$. Then [20, Theorem 8.5] shows that whenever $s \geq r(r-1)/2$, one has

$$\int_0^1 |g_k(\alpha; X)|^{2s} d\alpha \ll X^{2s-k+\varepsilon} (X^{\eta_r^*(s,k)-1/r} + X^{\eta_r^*(s,k-1)}),$$

where $\eta_r^*(s, w) = r^{-1}\eta(s - r(r-1)/2, w)$ for $w = k-1, k$.

Finally, we note that familiar conjectures concerning mean values of the exponential sum $g_k(\alpha; X)$ imply that one should have $\tilde{G}(k) \leq 2k+1$ for each $k \geq 3$, and indeed it may even be the case that $\tilde{G}(k) = k+1$.

7. Estimates of Weyl-type, and distribution mod 1

Pointwise estimates for exponential sums appear already in the work of Weyl [54] in 1916. By applying $k-1$ Weyl-differencing steps, one bounds the exponential sum $f_k(\alpha; X)$ in terms of a new exponential sum over a linear polynomial, and this may be estimated by summing what is, after all, a geometric progression. In this way, one obtains the classical version of Weyl's inequality (see [47, Lemma 2.4]).

Theorem 7.1 (Weyl's inequality). *Let $\alpha \in \mathbb{R}^k$, and suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha_k - a/q| \leq q^{-2}$. Then one has*

$$|f_k(\alpha; X)| \ll X^{1+\varepsilon} (q^{-1} + X^{-1} + qX^{-k})^{2^{1-k}}. \quad (7.1)$$

This provides a non-trivial estimate for $f_k(\alpha; X)$ when the leading coefficient α_k is not well-approximated by rational numbers. Consider, for example, the set $\mathfrak{m} = \mathfrak{m}_k$ defined in the preamble to (6.4). When $\alpha_k \in \mathfrak{m}$, an application of Dirichlet's theorem on Diophantine approximation shows that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$ such that $q \leq X^{k-1}$ and $|q\alpha - a| \leq X^{1-k}$. The definition of $\mathfrak{m}$ then implies that $q > X$, and so Theorem 7.1 delivers the bound

$$\sup_{\alpha_k \in \mathfrak{m}} |f_k(\alpha; X)| \ll X^{1-\sigma(k)+\varepsilon}, \quad (7.2)$$

in which $\sigma(k) = 2^{1-k}$. Heath-Brown's variant [22, Theorem 1] of Weyl's inequality applies mean value estimates for certain cubic exponential sums that, for $k \geq 6$, give bounds superior to (7.1) when q lies in the range $X^{5/2} < q < X^{k-5/2}$. By making use of the cubic case of the Main Conjecture in Vinogradov's mean value theorem [67], the author [68] has extended this range to $X^2 < q < X^{k-2}$.

Theorem 7.2. *Let $k \geq 6$, and suppose that $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha - a/q| \leq q^{-2}$. Then one has*

$$|g_k(\alpha; X)| \ll X^{1+\varepsilon} \Theta^{2^{-k}} + X^{1+\varepsilon} (\Theta/X)^{\frac{2}{3}2^{-k}},$$

where $\Theta = q^{-1} + X^{-3} + qX^{-k}$.

For comparison, we note that Heath-Brown [22, Theorem 1] obtains the bound $|g_k(\alpha; X)| \ll X^{1+\varepsilon}(X\Theta)^{\frac{4}{3}2^{-k}}$. Robert and Sargos [40, Théorème 4 et Lemme 7] extend these ideas when $k \geq 8$ to show that $|g_k(\alpha; X)| \ll X^{1+\varepsilon}(X^{17/8}\Theta')^{\frac{8}{5}2^{-k}}$, in which $\Theta' = q^{-1} + X^{-4} + qX^{-k}$. See Parsell [37] for a refinement when $k = 8$.

The above methods yield exponents exponentially small in k . By substituting estimates for $J_{s,k}(X)$ into the conclusion of Theorem 1.3, one obtains analogous bounds polynomial in k . Classical versions of Vinogradov's mean value theorem yield estimates of the shape (7.2) with $\sigma(k)^{-1} = (C + o(1))k^2 \log k$. Thus, Linnik [33] obtained the permissible value $C = 22\,400$, and Hua [27] obtained $C = 4$ in 1949. This was improved via efficient differencing [56] in 1992 to $C = 2$, and subsequently the author [59] obtained $C = 3/2$ by incorporating some ideas of Bombieri [11]. The latest developments in efficient congruencing yield the new exponent $\sigma(k)^{-1} = 2(k-1)(k-2)$ for $k \geq 3$, a conclusion that removes the factor $\log k$ from earlier estimates, and improves on Weyl's inequality for $k \geq 7$.

Theorem 7.3. *Let k be an integer with $k \geq 3$, and let $\alpha \in \mathbb{R}^k$. Suppose that there exists a natural number j with $2 \leq j \leq k$ such that, for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, one has $|\alpha_j - a/q| \leq q^{-2}$. Then one has*

$$|f_k(\alpha; X)| \ll X^{1+\varepsilon}(q^{-1} + X^{-1} + qX^{-j})^{\sigma(k)},$$

where $\sigma(k)^{-1} = 2(k-1)(k-2)$.

This conclusion makes use of Theorem 4.1(iii), and improves slightly on [65, Theorem 11.1]. When $k \geq 6$ and α lies on a suitable subset of $\mathbb{R}$, the above-cited work of Heath-Brown may deliver estimates for $|g_k(\alpha; X)|$ superior to those stemming from Weyl's inequality, and similar comments apply to the work of Robert and Sargos, and of Parsell, when $k \geq 8$. However, Theorem 7.3 proves superior in all circumstances to the estimates of Heath-Brown when $k > 7$, and to the estimates of Robert and Sargos for all exponents k .

In many applications, it is desirable to have available estimates for $|f_k(\alpha; X)|$ that depend on simultaneous approximations to $\alpha_1, \dots, \alpha_k$ of a given height. This is a subject to which R. C. Baker and W. M. Schmidt have made significant contributions. By exploiting such methods in combination with our new estimates for $J_{s,k}(X)$, one obtains the following conclusion (compare [65, Theorem 11.2]).

Theorem 7.4. *Let k be an integer with $k \geq 3$, and let τ and δ be real numbers with $\tau^{-1} > 4(k-1)(k-2)$ and $\delta > k\tau$. Suppose that X is sufficiently large in terms of k , δ and τ , and further that $|f_k(\alpha; X)| > X^{1-\tau}$. Then there exist integers $q, a_1, \dots, a_k$ such that $1 \leq q \leq X^\delta$ and $|q\alpha_j - a_j| \leq X^{\delta-j}$ ($1 \leq j \leq k$).*

Here, the constraint $\tau^{-1} > 4(k-1)(k-2)$ may be compared with the corresponding hypothesis $\tau^{-1} > (8 + o(1))k^2 \log k$ to be found in [5, Theorem 4.5], and the Weyl-based bound $\tau^{-1} > 2^{k-1}$ obtained in [5, Theorem 5.2] (see also [6]). The conclusion of Theorem 7.4 is superior to the latter for $k > 8$.

Bounds for exponential sums of Weyl-type may be converted into equidistribution results for polynomials modulo 1 by applying estimates of Erdős-Turán type.

Write $\|\theta\|$ for the least value of $|\theta - n|$ for $n \in \mathbb{Z}$, and consider a sequence $(x_n)_{n=1}^\infty$ of real numbers. Then it follows from [5, Theorem 2.2], for example, that whenever $\|x_n\| \geq M^{-1}$ for $1 \leq n \leq N$, then

$$\sum_{1 \leq m \leq M} \left| \sum_{n=1}^N e(mx_n) \right| > \frac{1}{6}N.$$

By carefully exploiting this result using the methods of Baker [5], one deduces from Theorem 7.4 the following conclusion (compare [65, Theorem 11.3]).

Theorem 7.5. *When $k \geq 3$, put $\tau(k) = 1/(4(k-1)(k-2))$. Then whenever $\alpha \in \mathbb{R}^k$ and N is sufficiently large in terms of k and ε , one has*

$$\min_{1 \leq n \leq N} \|\alpha_1 n + \alpha_2 n^2 + \dots + \alpha_k n^k\| < N^{\varepsilon - \tau(k)}.$$

8. Further applications

Vinogradov's mean value theorem finds application in numerous number-theoretic problems, besides those discussed in the previous two sections. We take the opportunity now to outline several applications, emphasising recent developments.

(i) *Tarry's problem.* When h, k and s are positive integers with $h \geq 2$, consider the Diophantine system

$$\sum_{i=1}^s x_{i1}^j = \sum_{i=1}^s x_{i2}^j = \dots = \sum_{i=1}^s x_{ih}^j \quad (1 \leq j \leq k). \quad (8.1)$$

Let $W(k, h)$ denote the least natural number s having the property that the simultaneous equations (8.1) possess an integral solution $\mathbf{x}$ with

$$\sum_{i=1}^s x_{iu}^{k+1} \neq \sum_{i=1}^s x_{iv}^{k+1} \quad (1 \leq u < v \leq h).$$

The problem of estimating $W(k, h)$ was intensely investigated by E. M. Wright and L.-K. Hua (see [25, 28, 69]), the latter obtaining $W(k, h) \leq k^2(\log k + O(1))$ for $h \geq 2$. The argument of the proof of [62, Theorem 1.3] shows that $W(k, h) \leq s$ whenever one can establish the estimate $J_{s, k+1}(X) = o(X^{2s-k(k+1)/2})$. By using this criterion together with the estimates for $J_{s, k+1}(X)$ obtained via the latest efficient congruencing methods, one obtains substantial improvements in these earlier conclusions (see [65, Theorem 12.1] and [66, Theorem 12.1]).

Theorem 8.1. *When h and k are natural numbers with $h \geq 2$ and $k \geq 3$, one has $W(k, h) \leq \frac{5}{8}(k+1)^2$. Moreover, when k is large, one has $W(k, h) \leq \frac{1}{2}k(k+1) + 1$.*

Although the last of these conclusions achieves the limit of current analytic approaches to bounding $W(k, h)$, explicit numerical examples are available⁴ which may be applied to show that $W(k, 2) = k + 1$ for $1 \leq k \leq 9$ and $k = 11$.

⁴See the website <http://euler.free.fr/eslp/eslp.htm>.

(ii) *Sum-product theorems.* When A is a finite set of real numbers, define the sets $A + A = \{x + y : x, y \in A\}$ and $A \cdot A = \{xy : x, y \in A\}$, and more generally

$$hA = \{x_1 + \dots + x_h : \mathbf{x} \in A^h\} \quad \text{and} \quad A^{(h)} = \{x_1 \dots x_h : \mathbf{x} \in A^h\}.$$

A conjecture of Erdős and Szemerédi [17] asserts that for any finite set of integers A , one has $|A + A| + |A \cdot A| \gg_\varepsilon |A|^{2-\varepsilon}$. It is also conjectured that whenever A is a finite set of *real* numbers, then for each $h \in \mathbb{N}$, one should have $|hA| + |A^{(h)}| \gg_{\varepsilon, h} |A|^{h-\varepsilon}$. Chang [13] has made progress towards this conjecture by showing that when A is a finite set of integers, and $|A \cdot A| < |A|^{1+\varepsilon}$, then $|hA| \gg_{\varepsilon, h} |A|^{h-\delta}$, where $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$. Subsequently, Bourgain and Chang [12] showed that for any $b \geq 1$, there exists $h \geq 1$ with the property that $|hA| + |A^{(h)}| \gg |A|^b$. By exploiting bounds for $W(k, h)$ of the type given by Theorem 8.1, Croot and Hart [16] have made progress toward an analogue of such conclusions for sets of real numbers.

Theorem 8.2. *Suppose that $\varepsilon > 0$ and $|A \cdot A| \leq |A|^{1+\varepsilon}$. Then there exists a number $\lambda > 0$ such that, when h is large enough in terms of ε , one has $|h(A \cdot A)| > |A|^{\lambda h^{1/3}}$.*

This conclusion (see [64, Theorem 11.5]) improves on [16, Theorem 2], where a similar result is obtained with $(h/\log h)^{1/3}$ in place of the exponent $h^{1/3}$.

(iii) *The Hilbert-Kamke problem and its brethren.* Hilbert [23] considered an extension of Waring's problem related to Vinogradov's mean value theorem. When $n_1, \dots, n_k \in \mathbb{N}$, let $R_{s,k}(\mathbf{n})$ denote the number of solutions of the system

$$x_1^j + \dots + x_s^j = n_j \quad (1 \leq j \leq k), \quad (8.2)$$

with $\mathbf{x} \in \mathbb{N}^s$. Put $X = \max_{1 \leq j \leq k} n_j^{1/j}$, and then write

$$\mathcal{J}_{s,k}(\mathbf{n}) = \int_{\mathbb{R}^k} \left(\int_0^1 e(\beta_1 \gamma + \dots + \beta_k \gamma^k) d\gamma \right)^s e(-\beta_1 n_1/X - \dots - \beta_k n_k/X^k) d\boldsymbol{\beta}$$

and

$$\mathcal{S}_{s,k}(\mathbf{n}) = \sum_{q=1}^{\infty} \sum_{\substack{1 \leq a_1, \dots, a_k \leq q \\ (q, a_1, \dots, a_k)=1}} (q^{-1} f_k(\mathbf{a}/q; q))^s e(-(a_1 n_1 + \dots + a_k n_k)/q).$$

See [1, 34, 35] for an account of the analysis of this problem, and in particular for a discussion of the conditions under which real and p -adic solutions exist for the system (8.2). While the conditions $n_k^{j/k} \leq n_j \leq s^{1-j/k} n_k^{j/k}$ ($1 \leq j \leq k$) are plainly necessary, one finds that p -adic solubility is not assured when $s < 2^k$. This classical technology gives an asymptotic formula for $R_{s,k}(\mathbf{n})$ provided that $s \geq (4+o(1))k^2 \log k$. Efficient congruencing methods lead to considerable progress. The following result improves on [62, Theorem 9.2] using Theorem 4.1(iii).

Theorem 8.3. *Let $s, k \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^k$. Suppose that $X = \max_j n_j^{1/j}$ is sufficiently large in terms of s and k , and that the system (8.2) has non-singular real and p -adic solutions. Then whenever $k \geq 3$ and $s \geq 2k^2 - 2k + 1$, one has*

$$R_{s,k}(\mathbf{n}) = \mathcal{J}_{s,k}(\mathbf{n}) \mathcal{S}_{s,k}(\mathbf{n}) X^{s-k(k+1)/2} + o(X^{s-k(k+1)/2}).$$

Similar arguments apply to more general Diophantine systems. Let $k_1, \dots, k_t$ be distinct positive integers. Suppose that $s, k \in \mathbb{N}$, and that $a_{ij} \in \mathbb{Z}$ for $1 \leq i \leq t$ and $1 \leq j \leq s$. Write

$$\phi_i(\mathbf{x}) = a_{i1}x_1^{k_i} + \dots + a_{is}x_s^{k_i} \quad (1 \leq i \leq t),$$

and consider the Diophantine system $\phi_i(\mathbf{x}) = 0$ ($1 \leq i \leq t$). We write $N(B; \phi)$ for the number of integral solutions of this system with $|\mathbf{x}| \leq B$. When $L > 0$, define

$$\sigma_\infty = \lim_{L \rightarrow \infty} \int_{|\xi| \leq 1} \prod_{i=1}^t \max\{0, L(1 - L|\phi_i(\xi)|)\} d\xi.$$

Also, for each prime number p , put

$$\sigma_p = \lim_{H \rightarrow \infty} p^{H(t-s)} \text{card}\{\mathbf{x} \in (\mathbb{Z}/p^H\mathbb{Z})^s : \phi_i(\mathbf{x}) \equiv 0 \pmod{p^H} \ (1 \leq i \leq t)\}.$$

By applying the Hardy-Littlewood method, a fairly routine application of Theorem 4.1(iii) delivers the following conclusion (compare [62, Theorem 9.1]).

Theorem 8.4. *Let s and k be natural numbers with $k \geq 3$ and $s \geq 2k^2 - 2k + 1$. Suppose that $\max k_i \leq k$, and that a_{ij} ($1 \leq i \leq t$, $1 \leq j \leq s$) are non-zero integers. Suppose in addition that the system of equations $\phi_i(\mathbf{x}) = 0$ ($1 \leq i \leq t$) has non-singular real and p -adic solutions, for each prime number p . Then*

$$N(B; \phi) \sim \sigma_\infty \left(\prod_p \sigma_p \right) B^{s-k_1-\dots-k_t}.$$

(iv) *Solutions of polynomial congruences in short intervals.* There has been much activity in recent years concerning the solubility of polynomial congruences in short intervals, some of which makes use of estimates associated with Vinogradov's mean value theorem. Let $f \in \mathbb{F}_p[X]$ have degree $m \geq 3$, and let M be a positive integer with $M < p$. Denote by $I_f(M; R, S)$ the number of solutions of the congruence $y^2 \equiv f(x) \pmod{p}$, with $(x, y) \in [R+1, R+M] \times [S+1, S+M]$. Weil's bounds for exponential sums yield the estimate $I_f(M; R, S) = M^2 p^{-1} + O(p^{1/2}(\log p)^2)$, one that is worse than trivial for $M \leq p^{1/2}(\log p)^2$. The work of Chang et al. [14] gives estimates that remain non-trivial for significantly smaller values of M .

Theorem 8.5. *Let $f \in \mathbb{F}_p[X]$ be any polynomial of degree $m \geq 4$. Then whenever M is a positive integer with $1 \leq M < p$, we have*

$$I_f(M; R, S) \ll M^{1+\varepsilon} (M^3 p^{-1} + M^{3-m})^{1/(2m(m-1))}.$$

This follows from [14, Theorem 4] on applying Theorem 4.1(iii). In particular, for any $\varepsilon > 0$, one finds that there exists a $\delta > 0$, depending only on ε and $\deg(f)$, such that whenever $M < p^{1/3-\varepsilon}$ and $\deg(f) \geq 4$, then $I_f(M; R, S) \ll M^{1-\delta}$.

(v) *The zero-free region for the Riemann zeta function.* We would be remiss not to mention the role of Vinogradov's mean value theorem in the proof of the widest

available zero-free region for the Riemann zeta function. The sharpest estimates date from work of Vinogradov [52] and Korobov [31] in 1958 (see also [53]). Thus, there is a positive constant c_1 with the property that $\zeta(s) \neq 0$ when $s = \sigma + it$, with $\sigma, t \in \mathbb{R}$, whenever $|t| \geq 3$ and $\sigma \geq 1 - c_1(\log |t|)^{-2/3}(\log \log |t|)^{-1/3}$. More recently, Ford [19] has shown that one may take $c_1 = 1/57.54$. This, in turn leads to an effective version of the prime number theorem of the shape

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x \exp(-c_2(\log x)^{3/5}(\log \log x)^{-1/5})\right),$$

where $c_2 = 0.2098$. Using currently available methods, the nature of the constant $C(k, r)$ in estimates of the shape (1.4) is significant for estimates of this type, while the precise nature of the defect in the exponent $\Delta_{s,k}$ less so. Thus, although the new estimates for $J_{s,k}(X)$ stemming from efficient congruencing have the potential to impact the numerical constants c_1 and c_2 , the dependence on t and x , respectively, in the above estimates has not been affected.

9. Generalisations

Thus far, we have focused on estimates for $J_{s,k}(X)$, the number of solutions (over the ring $\mathbb{Z}$) of the translation-dilation invariant system (1.3) with $1 \leq \mathbf{x}, \mathbf{y} \leq X$. Previous authors have considered generalisations in which either the ring, or else the translation-dilation invariant system, is varied.

(i) *Algebraic number fields.* The arguments underlying the proof of Theorem 4.1 change little when the setting is shifted from $\mathbb{Z}$ to the ring of integers of a number field. When $s \geq k(k-1)$, the ensuing estimates are at most a factor X^ε away from the upper bound predicted by a heuristic application of the circle method. In common with Birch's use of Hua's lemma in number fields [7], our estimates are therefore robust to variation in the degree of the field extension, since Weyl-type estimates for exponential sums no longer play a significant role in applications. In forthcoming work we apply such ideas to establish the following result.

Theorem 9.1. *Let $L/\mathbb{Q}$ be a field extension of finite degree. Suppose that $d \geq 3$, $s > 2d(d-1)$ and $\mathbf{a} \in (L^\times)^s$. Then the hypersurface defined by $a_1x_1^d + \dots + a_sx_s^d = 0$ satisfies weak approximation and the Hasse principle over L .*

For comparison, Birch [7, Theorem 3] gives such a conclusion only for $s > 2^d$, while the work of Körner [30] yields analogous conclusions in which the number of variables is larger, and depends also on the degree of the field extension $L/\mathbb{Q}$.

(ii) *Function fields.* Consider a finite field $\mathbb{F}_q$ of characteristic p . Let $B \in \mathbb{N}$ be large enough in terms of q , k and s , and denote by $J_{s,k}(B; q)$ the number of solutions of (1.3) with $x_i, y_i \in \mathbb{F}_q[t]$ ($1 \leq i \leq s$) having degree at most B . When $p < k$, one can reduce (1.3) to a minimal translation-invariant system in which certain equations are omitted. We write K for the sum of the degrees of this minimal system, so that $K = k(k+1)/2$ when $p > k$, and $K < k(k+1)/2$ when $p < k$. Then, when

$s \geq k(k+1)$, the efficient congruencing method adapts to give the upper bound $J_{s,k}(B; q) \ll (q^B)^{2s-K+\varepsilon}$. This and much more is contained in forthcoming work of the author joint with Y.-R. Liu, generalisations of which are described in [32].

(iii) *Multidimensional analogues.* Vinogradov's methods have been generalised to multidimensional settings by Arkhipov, Chubarikov and Karatsuba [2, 4], Parsell [36] and Prendiville [39]. Variants of the efficient congruencing method deliver much sharper conclusions in far greater generality. Let $r, s, d \in \mathbb{N}$, and consider a linearly independent system of homogeneous polynomials $\mathbf{F} = (F_1, \dots, F_r)$, where $F_j(\mathbf{z}) \in \mathbb{Z}[z_1, \dots, z_d]$. Suppose that for $1 \leq j \leq r$ and $1 \leq l \leq d$, the polynomial $\partial F_j / \partial z_l$ lies in $\text{span}(1, F_1, \dots, F_r)$. Such a *reduced translation-dilation invariant* system is said to have *rank* r , *dimension* d , *degree* $k = \max \deg F_j$, and *weight* $K = \sum_1^r \deg F_j$. Denote by $J_s(X; \mathbf{F})$ the number of integral solutions of the system of equations

$$\sum_{i=1}^s (F_j(\mathbf{x}_i) - F_j(\mathbf{y}_i)) = 0 \quad (1 \leq j \leq r),$$

with $1 \leq \mathbf{x}_i, \mathbf{y}_i \leq X$ ($1 \leq i \leq s$). The work of Parsell, Prendiville and the author [38, Theorem 2.1] provides a general estimate for $J_s(X; \mathbf{F})$ matching the predictions of the appropriate analogue of the Main Conjecture.

Theorem 9.2. *Let $\mathbf{F}$ be a reduced translation-dilation invariant system of rank r , dimension d , degree k and weight K . Then $J_s(X; \mathbf{F}) \ll X^{2sd-K+\varepsilon}$ for $s \geq r(k+1)$.*

Reduced translation-dilation invariant systems are easy to generate by taking successive partial derivatives and reducing to a linearly independent spanning set. Thus, for example, the initial seed $x^5 + 3x^2y^3$ gives rise to just such a system

$$\mathbf{F} = \{x^5 + 3x^2y^3, 5x^4 + 6xy^3, x^2y^2, 10x^3 + 3y^3, xy^2, x^2y, x^2, xy, y^2, x, y\},$$

with $d = 2$, $r = 11$, $k = 5$, $K = 30$. We therefore see from Theorem 9.2 that $J_s(X; \mathbf{F}) \ll X^{4s-30+\varepsilon}$ for $s \geq 66$. Theorem 9.2 should be susceptible to improvement by using the ideas underlying multigrade efficient congruencing [65, 66, 67].

10. Challenges

The remarkable success of the efficient congruencing method encourages ambitious speculation concerning other potential applications, a topic we briefly explore.

(i) *The Main Conjecture for larger s .* In Theorem 4.1, one sees that the upper bound $J_{s,k}(X) \ll X^{s+\varepsilon}$ predicted by the Main Conjecture is now known to hold for $1 \leq s \leq \frac{1}{2}k(k+1) - t_k$, where $t_k = \frac{1}{3}k + O(k^{2/3})$. In striking contrast, on the other side of the critical value $s = \frac{1}{2}k(k+1)$, the upper bound $J_{s,k}(X) \ll X^{2s-k(k+1)/2+\varepsilon}$ is known to hold only when $s \geq \frac{1}{2}k(k+1) + u_k$, where $u_k = \frac{1}{2}k(k-3)$. Plainly, the value of u_k is substantially larger than t_k , and an intriguing possibility is that a hitherto unseen refinement of the method might reduce u_k to a size more similar to that of t_k . This would have great significance in numerous applications.

(ii) *Paucity.* When $k \geq 3$ and $1 \leq s < \frac{1}{2}k(k+1)$, we have precise asymptotics for $J_{s,k}(X)$ only when $s \leq k+1$. Since the formula $J_{s,k}(X) = T_s(X) \sim s!X^s$ is trivial for $1 \leq s \leq k$, the case $s = k+1$ is the only one with content. It is tempting to speculate that a suitable adaptation of efficient congruencing might confirm that $J_{s,k}(X) = T_s(X) + O(X^{s-\delta})$, for some $\delta > 0$, for some exponent $s \geq k+2$.

(iii) *Minor arc bounds.* When $q \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^k$, denote by $\mathfrak{M}(q, \mathbf{a})$ the set of points $\alpha \in [0, 1)^k$ such that $|q\alpha_j - a_j| \leq X^{1-j}$ ($1 \leq j \leq k$). Write $\mathfrak{M}$ for the union of the boxes $\mathfrak{M}(q, \mathbf{a})$ with $0 \leq a_j \leq q \leq X$ ($1 \leq j \leq k$) and $(q, a_1, \dots, a_k) = 1$, and put $\mathfrak{m} = [0, 1)^k \setminus \mathfrak{M}$. The methods of §7 provide estimates of the shape $|f_k(\alpha; X)| \ll X^{1-\sigma_k+\varepsilon}$ for $\alpha \in \mathfrak{m}$. However, when $s = k(k-1) + t$ and $t \geq 1$, our most efficient means of estimating moments of $f_k(\alpha; X)$ of order $2s$, restricted to minor arcs, proceeds by applying Theorem 4.1(iii) via the trivial bound

$$\begin{aligned} \int_{\mathfrak{m}} |f_k(\alpha; X)|^{2s} d\alpha &\ll \left(\sup_{\alpha \in \mathfrak{m}} |f_k(\alpha; X)| \right)^{2t} \oint |f_k(\alpha; X)|^{2k(k-1)} d\alpha \\ &\ll X^{2s - \frac{1}{2}k(k+1) - 2t\sigma_k + \varepsilon}. \end{aligned}$$

This bound is relatively weak, even when t is large. Efficient congruencing provides a possible means of deriving estimates directly for such moments, and might even lead to improvements in our lower bounds for permissible exponents σ_k .

(iv) *Non-translation invariant systems.* The system (1.3) is translation-dilation invariant. A major desideratum is to apply a variant of efficient congruencing to systems of equations that are *not* translation invariant. The author has forthcoming work applicable to systems that are only approximately translation invariant.

References

- [1] Arkhipov, G. I., On the Hilbert-Kamke problem, *Izv. Akad. Nauk SSSR Ser. Mat.* **48** (1984), no. 1, 3–52.
- [2] Arkhipov, G. I., Chubarikov, V. N. and Karatsuba, A. A., *Trigonometric sums in number theory and analysis*, Walter de Gruyter, Berlin, 2004.
- [3] Arkhipov, G. I. and Karatsuba, A. A., A new estimate of an integral of I. M. Vinogradov, *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978), no. 4, 751–762.
- [4] Arkhipov, G. I., Karatsuba, A. A., and Chubarikov, V. N., Multiple Trigonometric Sums, *Trudy Mat. Inst. Steklov* **151** (1980), 1–126.
- [5] Baker, R. C., *Diophantine inequalities*, London Mathematical Society Monographs, vol. **1**, Oxford University Press, Oxford, 1986.
- [6] Baker, R. C., Correction to: “Weyl sums and Diophantine approximation” [J. London Math. Soc. (2) 25 (1982), no. 1, 25–34], J. London Math. Soc. (2) **46** (1992), no. 2, 202–204.
- [7] Birch, B. J., Waring’s problem in algebraic number fields, *Proc. Cambridge Philos. Soc.* **57** (1961), 449–459.
- [8] Boklan, K. D., The asymptotic formula in Waring’s problem, *Mathematika* **41** (1994), no. 2, 329–347.

- [9] Boklan, K. D. and Wooley, T. D., On Weyl sums for smaller exponents, *Funct. Approx. Comment. Math.* **46** (2012), no. 1, 91–107.
- [10] Blomer, V. and Brüdern, J., The number of integer points on Vinogradov's quadric, *Monatsh. Math.* **160** (2010), no. 3, 243–256.
- [11] Bombieri, E., On Vinogradov's mean value theorem and Weyl sums, *Automorphic forms and analytic number theory (Montreal, PQ, 1989)*, pp. 7–24, Univ. Montréal, Montreal, QC, 1990.
- [12] Bourgain, J. and Chang, M.-C., On the size of k -fold sum and product sets of integers, *J. Amer. Math. Soc.* **17** (2004), no. 2, 473–497.
- [13] Chang, M.-C., The Erdős-Szemerédi problem on sum set and product set, *Ann. of Math.* (2) **157** (2003), no. 3, 939–957.
- [14] Chang, M.-C., Cilleruelo, J., Garaev, M. Z., Hernández, J., Shparlinski, I. E. and Zumulacárregui, A., Points on curves in small boxes and applications, *Michigan Math. J.*, to appear; available as arXiv:1111.1543.
- [15] van der Corput, J. G., Verschärfung der Abschätzungen beim Teilerproblem, *Math. Ann.* **87** (1922), 39–65.
- [16] Croot, E. and Hart, D., h -fold sums from a set with few products, *SIAM J. Discrete Math.* **24** (2010), no. 2, 505–519.
- [17] Erdős, P. and Szemerédi, E., On sums and products of integers, *Studies in Pure Mathematics*, Birkhäuser, Basel, 1983, pp. 213–218.
- [18] Ford, K. B., New estimates for mean values of Weyl sums, *Internat. Math. Res. Notices* (1995), no. 3, 155–171.
- [19] Ford, K. B., Vinogradov's integral and bounds for the Riemann zeta function, *Proc. London Math. Soc.* (3) **85** (2002), no. 3, 565–633.
- [20] Ford, K. B. and Wooley, T. D., On Vinogradov's mean value theorem: strongly diagonal behaviour via efficient congruencing, submitted, arXiv:1304.6917.
- [21] Hardy, G. H. and Littlewood, J. E., Some problems of 'Partitio Numerorum': IV. The singular series in Waring's Problem and the value of the number $G(k)$, *Math. Zeit.* **12** (1922), 161–188.
- [22] Heath-Brown, D. R., Weyl's inequality, Hua's inequality, and Waring's problem, *J. London Math. Soc.* (2) **38** (1988), no. 2, 216–230.
- [23] Hilbert, D., Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n^{ter} Potenzen (Waringsches Problem), *Math. Ann.* **67** (1909), no. 3, 281–300.
- [24] Hua, L.-K., On Waring's problem, *Quart. J. Math. Oxford* **9** (1938), 199–202.
- [25] Hua, L.-K., On Tarry's problem, *Quart. J. Math. Oxford* **9** (1938), 315–320.
- [26] Hua, L. K., *The additive prime number theory*, Trav. Inst. Math. Stekloff, **22**, Acad. Sci. USSR, Moscow-Leningrad, 1947.
- [27] Hua, L.-K., An improvement of Vinogradov's mean value theorem and several applications, *Quart. J. Math. Oxford* **20** (1949), 48–61.
- [28] Hua, L.-K., Improvement of a result of Wright, *J. London Math. Soc.* **24** (1949), 157–159.
- [29] Karatsuba, A. A., The mean value of the modulus of a trigonometric sum, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 1203–1227.

- [30] Körner, O., Über Mittelwerte trigonometrischer Summen und ihre Anwendung in algebraischen Zahlkörpern, *Math. Ann.* **147** (1962), 205–239.
- [31] Korobov, N. M., Estimates of trigonometric sums and their applications, *Uspehi Mat. Nauk* **13** (1958), no. 4 (82), 185–192.
- [32] Kuo, W., Liu, Y.-R. and Zhao, X., Multidimensional Vinogradov-type estimates in function fields, *Canad. J. Math.*, in press.
- [33] Linnik, Yu. V., On Weyl's sums, *Mat. Sbornik (Rec. Math.) N. S.* **12** (1943), 28–39.
- [34] Mit'kin, D. A., Estimate for the number of summands in the Hilbert-Kamke problem, *Mat. Sbornik (N.S.)* **129** (1986), no. 4, 549–577.
- [35] Mit'kin, D. A., Estimate for the number of summands in the Hilbert-Kamke problem, II, *Mat. Sbornik (N.S.)* **132** (1987), no. 3, 345–351.
- [36] Parsell, S. T., A generalization of Vinogradov's mean value theorem, *Proc. London Math. Soc.* (3) **91** (2005), no. 1, 1–32.
- [37] Parsell, S. T., A note on Weyl's inequality for eighth powers, *Rocky Mountain J. Math.*, in press.
- [38] Parsell, S. T., Prendiville, S. M. and Wooley, T. D., Near-optimal mean value estimates for multidimensional Weyl sums, *Geom. Funct. Anal.* **23** (2013), no. 6, 1962–2024.
- [39] Prendiville, S. M., Solution-free sets for sums of binary forms, *Proc. London Math. Soc.* (3) **107** (2013), no. 2, 267–302.
- [40] Robert, O. and Sargos, P., Un théorème de moyenne pour les sommes d'exponentielles. Application à l'inégalité de Weyl, *Publ. Inst. Math. (Beograd) (N.S.)* **67** (2000), 14–30.
- [41] Rogovskaya, N. N., An asymptotic formula for the number of solutions of a system of equations, *Diophantine Approximations*, Part II, Moskov. Gos. Univ., Moscow, 1986, pp. 78–84.
- [42] Stechkin, S. B., On mean values of the modulus of a trigonometric sum, *Trudy Mat. Inst. Steklov* **134** (1975), 283–309.
- [43] Tyrina, O. V., A new estimate for a trigonometric integral of I. M. Vinogradov, *Izv. Akad. Nauk SSSR Ser. Mat.* **51** (1987), no. 2, 363–378.
- [44] Ustinov, A. V., On the number of summands in the asymptotic formula for the number of solutions of the Waring equation, *Mat. Zametki* **64** (1998), no. 2, 285–296.
- [45] Vaughan, R. C., On Waring's problem for cubes, *J. Reine Angew. Math.* **365** (1986), 122–170.
- [46] Vaughan, R. C., On Waring's problem for smaller exponents, II, *Mathematika* **33** (1986), no. 1, 6–22.
- [47] Vaughan, R. C., *The Hardy-Littlewood method*, 2nd edition, Cambridge University Press, Cambridge, 1997.
- [48] Vaughan, R. C. and Wooley, T. D., On a certain nonary cubic form and related equations, *Duke Math. J.* **80** (1995), no. 3, 669–735.
- [49] Vaughan, R. C. and Wooley, T. D., A special case of Vinogradov's mean value theorem, *Acta Arith.* **79** (1997), no. 3, 193–204.

- [50] Vinogradov, I. M., New estimates for Weyl sums, *Dokl. Akad. Nauk SSSR* **8** (1935), 195–198.
- [51] Vinogradov, I. M., The method of trigonometrical sums in the theory of numbers, *Trav. Inst. Math. Stekloff* **23** (1947), 109pp.
- [52] Vinogradov, I. M., A new estimate of the function $\zeta(1+it)$, *Izv. Akad. Nauk SSSR. Ser. Mat.* **22** (1958), 161–164.
- [53] Walfisz, A. Z., *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Deutscher Verlag der Wissenschaften, Berlin, 1963.
- [54] Weyl, H., Über die Gleichverteilung von Zahlen mod Eins, *Math. Ann.* **77** (1916), 313–352.
- [55] Wooley, T. D., Large improvements in Waring's problem, *Ann. of Math.* (2) **135** (1992), no. 1, 131–164.
- [56] Wooley, T. D., On Vinogradov's mean value theorem, *Mathematika* **39** (1992), no. 2, 379–399.
- [57] Wooley, T. D., On Vinogradov's mean value theorem, II, *Michigan Math. J.* **40** (1993), no. 1, 175–180.
- [58] Wooley, T. D., Quasi-diagonal behaviour in certain mean value theorems of additive number theory, *J. Amer. Math. Soc.* **7** (1994), no. 1, 221–245.
- [59] Wooley, T. D., New estimates for Weyl sums, *Quart. J. Math. Oxford* (2) **46** (1995), no. 1, 119–127.
- [60] Wooley, T. D., Some remarks on Vinogradov's mean value theorem and Tarry's problem, *Monatsh. Math.* **122** (1996), no. 3, 265–273.
- [61] Wooley, T. D., Diophantine methods for exponential sums, and exponential sums for Diophantine problems, *Proceedings of the International Congress of Mathematicians, August 20–28, 2002, Beijing*, Volume II, Higher Education Press, 2002, pp. 207–217.
- [62] Wooley, T. D., Vinogradov's mean value theorem via efficient congruencing, *Ann. of Math.* (2) **175** (2012), no. 3, 1575–1627.
- [63] Wooley, T. D., The asymptotic formula in Waring's problem, *Internat. Math. Res. Notices* (2012), no. 7, 1485–1504.
- [64] Wooley, T. D., Vinogradov's mean value theorem via efficient congruencing, II, *Duke Math. J.* **162** (2013), no. 4, 673–730.
- [65] Wooley, T. D., Multigrade efficient congruencing and Vinogradov's mean value theorem, submitted, arXiv:1310.8447.
- [66] Wooley, T. D., Approximating the main conjecture in Vinogradov's mean value theorem, submitted, arXiv:1401.2932.
- [67] Wooley, T. D., The cubic case of the main conjecture in Vinogradov's mean value theorem, submitted, arXiv:1401.3150.
- [68] Wooley, T. D., Mean value estimates for odd cubic Weyl sums, submitted, arXiv:1401.7152.
- [69] Wright, E. M., The Prouhet-Lehmer problem, *J. London Math. Soc.* **23** (1948), 279–285.

School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, United Kingdom
 E-mail: matdw@bristol.ac.uk