MULTIGRADE EFFICIENT CONGRUENCING AND VINOGRADOV'S MEAN VALUE THEOREM

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Abstract. We develop a substantial enhancement of the efficient congruencing method to estimate Vinogradov's integral of degree k for moments of order $2s$, thereby obtaining for the first time near-optimal estimates for $s > \frac{5}{8}k^2$. There are numerous applications. In particular, when k is large, the anticipated asymptotic formula in Waring's problem is established for sums of s kth powers of natural numbers whenever $s \geqslant 1.543k^2$.

1. Introduction

An optimal upper bound in Vinogradov's mean value theorem is now known to hold with a number of variables only twice that conjectured to be best possible (see [17, Theorem 1.1]). Previous to this very recent advance based on "efficient congruencing", available technology required that the number of variables be larger by a factor of order $\log k$, for a system of degree k, a limitation common to all mean value estimates for exponential sums. Hints that the conjectured bounds might be proved in full can be glimpsed in a speculative hypothetical enhancement to efficient congruencing proposed and heuristically analysed in [17, §11]. Our goal in this paper is to realise an approximation to this enhancement, thereby delivering much of what this earlier speculation had promised. In particular, we now come close to establishing an optimal upper bound with a number of variables only twenty-five per cent larger than that conjectured to be best possible. The central role played by Vinogradov's mean value theorem ensures that applications of our new estimates are plentiful.

In order to describe our conclusions, we must introduce some notation. When k and s are natural numbers, denote by $J_{s,k}(X)$ the number of integral solutions of the Diophantine system

$$x_1^j + \ldots + x_s^j = y_1^j + \ldots + y_s^j \quad (1 \leqslant j \leqslant k), \tag{1.1}$$

with $1 \leqslant x_i, y_i \leqslant X$ $(1 \leqslant i \leqslant s)$. The *main conjecture* in Vinogradov's mean value theorem asserts that for each $\varepsilon > 0$, one has

$$J_{s,k}(X) \ll X^{\varepsilon}(X^s + X^{2s-\frac{1}{2}k(k+1)}). \tag{1.2}$$

Here and throughout this paper, the implicit constants associated with Vinogradov's notation $\ll$ and $\gg$ may depend on s, k and ε. This conjecture is

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motivated by the corresponding lower bound

$$J_{s,k}(X) \gg X^s + X^{2s-\frac{1}{2}k(k+1)}, \tag{1.3}$$

that arises by considering the diagonal solutions of (1.1) with $\mathbf{x} = \mathbf{y}$, together with a lower bound for the product of local densities (see [12, equation (7.5)]).

We complete the proof of our new estimate for $J_{s,k}(X)$ in §9.

Theorem 1.1. *Suppose that k, r and s are natural numbers with $k \geqslant 3$,*

$$r \leqslant \min\{k-2, \tfrac{1}{2}(k+1)\} \quad \text{and} \quad s \geqslant k^2 - rk + \tfrac{1}{2}r(r+3) - 1.$$

Then for each $\varepsilon > 0$, one has

$$J_{s,k}(X) \ll X^{2s-\frac{1}{2}k(k+1)+\delta_s+\varepsilon}, \tag{1.4}$$

where $\delta_s = \delta_{s,k,r}$ is defined by

$$\delta_{s,k,r} = \frac{r(r-1)(3k-2r-5)}{6(s-k+1)}.$$

In particular, when $s \geqslant (k-\frac{1}{2}r)^2 + \frac{1}{4}(r+3)^2$, one has $\delta_{s,k,r} < r^2/k$.

We refer the reader to Theorem 9.2 for an alternative bound for $J_{s,k}(X)$ which is in general slightly more precise than that given by the theorem just announced. Theorem 1.1 has the merit of being simpler to state, and also offers slightly sharper bounds in situations where s is close to k^2. The special case $r = 1$ of Theorem 1.1 yields a corollary achieving the upper bound (1.2) asserted by the main conjecture.

Corollary 1.2. *Suppose that s and k are natural numbers with $k \geqslant 3$ and $s \geqslant k^2 - k + 1$. Then for each $\varepsilon > 0$, one has $J_{s,k}(X) \ll X^{2s-\frac{1}{2}k(k+1)+\varepsilon}$.*

This corollary improves on our earlier conclusion [19, Theorem 1.1], in which the same upper bound is achieved subject to the constraint $s \geqslant k^2 - 1$. Prior to the advent of efficient congruencing in [17], meanwhile, estimates of the type supplied by Corollary 1.2 were available only for $s \geqslant (1+o(1))k^2 \log k$ (see [1], [13], [14] and [16]).

The conclusion of Theorem 1.1 improves very significantly on the bounds previously available for $J_{s,k}(X)$ in the range $\frac{5}{8}k^2 \leqslant s < k^2 - 1$. By way of comparison, earlier work of the author joint with Ford [5, Theorem 1.2(i)] shows that the bound (1.4) holds with $\delta_s = m^2$ whenever $2m \leqslant k$ and one has $s \geqslant (k-m)^2 + (k-m)$. Thus, in the situation with $s = \alpha k^2$, in which α is a parameter with $\frac{1}{4} \leqslant \alpha \leqslant 1$, one has a bound of the shape (1.4) with $\delta_s = (1-\sqrt{\alpha})^2 k^2 + O(k)$. Theorem 1.1, on the other hand, shows that when $\frac{5}{8} \leqslant \alpha \leqslant 1$, the estimate (1.4) holds with $\delta_s = C(\alpha)k + O(1)$, where

$$C(\alpha) = \frac{2 - 3\alpha + (2\alpha-1)^{3/2}}{3\alpha}.$$

The superiority of our conclusion in the latter interval is clear, since, for the first time, we demonstrate that the bound (1.4) holds with $\delta_s = O(k)$ throughout the interval $\frac{5}{8}k^2 < s < k^2$. In some sense, therefore, our bounds are near-optimal in the latter range.

We pause at this stage to remark that our methods are by no means restricted to the interval $\frac{5}{8}k^2 < s < k^2$. We have constrained ourselves in this paper to the latter interval in order that the ideas underlying our *multigrade efficient congruencing* method be transparent. At the same time, the new estimates that we make available by imposing this restriction support the bulk of applications stemming from this circle of ideas. In forthcoming work [20], we tackle the considerable technical complications arising from a choice of parameters in which r is permitted to be substantially smaller than $\frac{1}{2}(k+1)$. In this way, when $s = \frac{1}{2}k(k+1)$, we are able to establish an estimate of the shape (1.4) with $\delta_s = \frac{1}{3}k + o(k)$. The transition to exponents s with $s < \frac{1}{2}k(k+1)$ poses further significant challenges. Here, we are able to extend the range $1 \leqslant s \leqslant \frac{1}{4}(k+1)^2$ in which the estimate $J_{s,k}(X) \ll X^{s+\varepsilon}$ is known to hold. The latter, established in [5, Theorem 1.1], substantially extends the classical range $1 \leqslant s \leqslant k+1$ in which the main conjecture (1.2) was previously known to hold in the diagonally dominated regime.

We next explore applications of our methods in the context of Waring's problem. When s and k are natural numbers, let $R_{s,k}(n)$ denote the number of representations of the natural number n as the sum of s kth powers of positive integers. A formal application of the circle method suggests that for $k \geqslant 3$ and $s \geqslant k+1$, one should have

$$R_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}_{s,k}(n)n^{s/k-1} + o(n^{s/k-1}), \tag{1.5}$$

where

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} \Big(q^{-1}\sum_{r=1}^{q} e(ar^k/q)\Big)^s e(-na/q).$$

With suitable congruence conditions imposed on n, one has $1 \ll \mathfrak{S}_{s,k}(n) \ll n^{\varepsilon}$, so that the conjectured relation (1.5) constitutes an honest asymptotic formula. Let $\widetilde{G}(k)$ denote the least integer t with the property that, for all $s \geqslant t$, and all sufficiently large natural numbers n, one has the asymptotic formula (1.5). By combining the conclusion of Theorem 1.1 with our recent work concerning the asymptotic formula in Waring's problem [18], and the enhancement [5, Theorem 8.5] of Ford's work [4], in §10 we are able to derive new upper bounds for $\widetilde{G}(k)$. We defer to §10 a full account of these bounds, contenting ourselves for the present with the enunciation of the most striking consequences.

Theorem 1.3. *Let ξ denote the real root of the polynomial $20\xi^3 + 4\xi^2 - 1$, and put $C = (19 + 75\xi - 12\xi^2)/(8 + 60\xi)$, so that*

$$\xi = 0.312383\ldots \quad \textit{and} \quad C = 1.542749\ldots.$$

Then for large values of k, one has $\widetilde{G}(k) < Ck^2 + O(k)$.

Until recently, the sharpest available estimate for $\widetilde{G}(k)$ for larger k was the bound $\widetilde{G}(k) \leqslant k^2(\log k + \log\log k + O(1))$ due to Ford [4]. This situation was changed with the arrival of efficient congruencing, and the most recent work [5, Corollary 9.4] shows that $\widetilde{G}(k) \leqslant 2k^2 - 2^{2/3}k^{4/3} + O(k)$. Thus the bound supplied by Theorem 1.3 provides the first improvement on that of [17, Theorem 1.4] in which the leading term is reduced by a constant factor. For smaller values of k, one may compute explicitly the upper bounds for $\widetilde{G}(k)$ that stem from our methods.

Theorem 1.4. *With $H(k)$ defined as in Table 1, one has $\widetilde{G}(k) \leqslant H(k)$.*

k	5	6	7	8	9	10	11	12
$H(k)$	28	43	61	83	107	134	165	199

k	13	14	15	16	17	18	19	20
$H(k)$	236	276	320	368	418	473	530	592

Table 1: Upper bounds for $\widetilde{G}(k)$ described in Theorem 1.4.

For comparison, Vaughan [11, Theorem 1] establishes the bound $\widetilde{G}(5) \leqslant 32$, Wooley [19, Corollary 1.7] gives

$$\widetilde{G}(6) \leqslant 52,\ \widetilde{G}(7) \leqslant 75,\ \widetilde{G}(8) \leqslant 103,\ \widetilde{G}(9) \leqslant 135,\ \widetilde{G}(10) \leqslant 171,\ \widetilde{G}(11) \leqslant 211,$$

and Ford and Wooley [5, Corollary 9.3] show that

$$\widetilde{G}(12) \leqslant 253,\ \widetilde{G}(13) \leqslant 299,\ \widetilde{G}(14) \leqslant 349,\ \widetilde{G}(15) \leqslant 403,\ \widetilde{G}(16) \leqslant 460,$$

$$\widetilde{G}(17) \leqslant 521,\ \widetilde{G}(18) \leqslant 587,\ \widetilde{G}(19) \leqslant 656,\ \widetilde{G}(20) \leqslant 729.$$

In particular, we have in Theorem 1.4 the first improvement on the bound of Vaughan, itself closely aligned with that of Hua, for $k = 5$. Methods based on Weyl differencing consequently remain significant only for $k = 3$ and 4. We note that for $k = 4$, in a formal sense our methods show that $\widetilde{G}(4) \leqslant 16.311$, falling somewhat short of the bound $\widetilde{G}(4) \leqslant 16$ established by Vaughan (see [11, Theorem 1]).

We consider further consequences of our new estimates in §§11 and 12. In particular, there are improvements in estimates of Weyl type for exponential sums, in the distribution of polynomials modulo 1, and in Tarry's problem.

We direct the reader to a sketch of the basic efficient congruencing method in [17, §2] for an introduction to such methods. It may be useful, however, to offer some insight concerning the strategy underlying our new multigrade efficient congruencing method. A key step in the efficient congruencing approach to Vinogradov's mean value theorem is that of bounding $J_{s,k}(X)$ in terms of an auxiliary mean value, in which certain variables are related by the congruences

$$\sum_{i=1}^k x_i^j \equiv \sum_{i=1}^k y_i^j \pmod{p^{jb}} \quad (1 \leqslant j \leqslant k). \tag{1.6}$$

Here, for the purpose of illustration, we suppose that $1 \leqslant x_i, y_i \leqslant p^{kb}$, that the x_i are distinct modulo p, and likewise the y_i. The classical approach to Vinogradov's mean value theorem studies the situation here with $b = 1$. In our first work [17] on efficient congruencing, we observe that by lifting solutions modulo p to solutions modulo p^{kb} of the system (1.6), one may suppose without loss that $x_i \equiv y_i \pmod{p^{kb}}$ $(1 \leqslant i \leqslant k)$, provided that one inserts a factor $k!p^{\frac{1}{2}k(k-1)b}$ into the ensuing estimates to reflect the number of solutions modulo p^{kb} for $\mathbf{x}$ given a fixed choice of $\mathbf{y}$. By applying Hölder's inequality, one obtains a new system of the shape (1.6) with b replaced by kb, and a concentration argument establishes the conjectured bound (1.2) whenever $s \geqslant k(k+1)$.

The heuristic argument described in [17, §11] takes as its starting point the conjectural proposition that solutions modulo p of the system (1.6) may be lifted componentwise, in such a manner that one may suppose without loss that $x_i \equiv y_i \pmod{p^{ib}}$ $(1 \leqslant i \leqslant k)$, provided that one inserts a factor $k!$ into the ensuing estimates. The average degree of the congruence concentration is thus essentially halved, greatly improving the efficiency of the method.

Lack of independence amongst the variables in such an approach prevents this idea from being anything other than one of heuristic significance. However, with r a parameter satisfying $1 \leqslant r < k$ to be chosen in due course, one may extract from (1.6) the congruence relation $x_i \equiv y_i \pmod{p^{(k-r)b}}$ $(1 \leqslant i \leqslant k)$, at the cost of inserting a factor $k!p^{\frac{1}{2}(k-r)(k-r-1)b}$ into the ensuing estimates. By applying Hölder's inequality to the associated mean values, one may relate the central mean value to a product of mean values, one in which $k-r$ pairs of variables have been extracted subject to a congruence condition modulo $p^{(k-r)b}$, and another involving the system of congruences

$$\sum_{i=1}^r x_i^j \equiv \sum_{i=1}^r y_i^j \pmod{p^{jb}} \quad (1 \leqslant j \leqslant k). \tag{1.7}$$

One may preserve the condition that the x_i here are distinct modulo p, and likewise the y_i. Thus we may infer that $x_i \equiv y_i \pmod{p^{(k-r+1)b}}$ $(1 \leqslant i \leqslant r)$ at essentially no cost. A further application of Hölder's inequality enables us to relate this mean value to another product of mean values, one in which a further pair of variables have been extracted subject to a congruence condition modulo $p^{(k-r+1)b}$, and another involving a system of the shape (1.7), but now with r replaced by $r-1$. Repeating this procedure, we successively extract pairs of variables, mutually congruent modulo $p^{(k-r+j)b}$ $(1 \leqslant j \leqslant r)$, for use in auxiliary mean values elsewhere in the argument. In this way, one recovers an approximation to the heuristic basis of our analysis in [17, §11]. Needless to say, there are considerable technical complications both in coaxing this approximation to behave like the heuristic approach, and indeed in analysing the consequences only previously discussed in the broadest terms.

A perusal of §§2–9 of this paper will reveal that our multigrade efficient congruencing method is of considerable flexibility. The reader may wonder to what extent the particular arrangement of parameters employed herein is optimal. Thus, the congruence condition modulo $p^{(k-r)b}$ is applied at the outset, and applies to many pairs of variables, with subsequent higher congruences imposed one pair of variables at a time. While this arrangement has been pursued following a great deal of time consuming experimentation, some guidance is possible for readers seeking to become fully immersed in the underlying methods. In this paper we have concentrated on the situation for larger moments, and here as much as possible the full weight of congruence savings must be preserved in order to obtain near-optimal estimates. While the initial step of our procedure realises the full potential of the congruence condition modulo $p^{(k-r)b}$, subsequent steps become possible only following appropriate applications of Hölder's inequality. Each application of the latter slightly diminishes the potential savings associated with these subsequent steps, and it seems that for this reason, it is more profitable to stack the lower congruence levels "up front" rather than spacing out the progress to full level p^{kb} more gradually.

2. Preliminary discussion of infrastructure

We launch our account of the the proof of Theorem 1.1, and the closely allied Theorem 9.2, by assembling the components required for the application of the multigrade efficient congruencing method. Here, where possible, we incorporate the simplifying manœuvres of [5] into the basic infrastructure developed in [17] and [19]. Since we consider the integer k to be fixed, we abbreviate $J_{s,k}(X)$ to $J_s(X)$ without further comment. Let s be an arbitrary natural number, and define the real number λ_s^* by means of the relation

$$\lambda_s^* = \limsup_{X\to\infty} \frac{\log J_s(X)}{\log X}.$$

Thus, for each $\varepsilon > 0$, and any real number X sufficiently large in terms of s, k and ε, one has $J_s(X) \ll X^{\lambda_s^*+\varepsilon}$. In view of the lower bound (1.3), together with a trivial bound for $J_s(X)$, we have

$$\max\{s, 2s - \tfrac{1}{2}k(k+1)\} \leqslant \lambda_s^* \leqslant 2s, \tag{2.1}$$

while the conjectured upper bound (1.2) implies that the first inequality in (2.1) should hold with equality.

We recall some notational conventions from our previous work. The letters s and k denote natural numbers with $k \geqslant 3$, and ε denotes a sufficiently small positive number. The basic parameter occurring in our asymptotic estimates is X, a large real number depending at most on k, s and ε, unless otherwise indicated. Whenever ε appears in a statement, we assert that the statement holds for each $\varepsilon > 0$. As usual, we write $\lfloor \psi \rfloor$ to denote the largest integer no larger than ψ, and $\lceil \psi \rceil$ to denote the least integer no smaller than ψ. We make sweeping and cavalier use of vector notation. Thus, with t implied from the environment at hand, we write $\mathbf{z} \equiv \mathbf{w} \pmod{p}$ to denote that $z_i \equiv w_i \pmod{p}$ $(1 \leqslant i \leqslant t)$, or $\mathbf{z} \equiv \xi \pmod{p}$ to denote that $z_i \equiv \xi \pmod{p}$ $(1 \leqslant i \leqslant t)$, or $[\mathbf{z} \pmod{q}]$ to denote the t-tuple $(\zeta_1, \dots, \zeta_t)$ where for $1 \leqslant i \leqslant t$ one has $1 \leqslant \zeta_i \leqslant q$ and $z_i \equiv \zeta_i \pmod{q}$. Finally, we employ the convention that whenever $G : [0,1)^k \to \mathbb{C}$ is integrable, then

$$\oint G(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} = \int_{[0,1)^k} G(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}.$$

Thus, on writing

$$f_k(\boldsymbol{\alpha};X)=\sum_{1\leqslant x\leqslant X}e(\alpha_1x+\alpha_2x^2+\ldots+\alpha_kx^k),\tag{2.2}$$

where as usual $e(z)$ denotes $e^{2\pi iz}$, it follows from orthogonality that

$$J_{s,k}(X)=\oint|f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}.\tag{2.3}$$

We use the index ι to select choices of parameters appropriate for the proof of our main theorems. We take $\iota=0$ to indicate a choice of parameters appropriate for the proof of Theorem 9.2, and $\iota=1$ for a choice appropriate for the proof of Theorem 1.1. Let r be an integral parameter satisfying

$$1\leqslant r\leqslant\min\{k-2,\tfrac{1}{2}(k+1)\},\tag{2.4}$$

and define

$$\nu_0(r,s)=\sum_{m=1}^{r}\frac{m(k-m-1)}{s-m}\quad\text{and}\quad\nu_1(r,s)=0.\tag{2.5}$$

We take ν to be an integer with $0\leqslant\nu\leqslant\nu_\iota(r,s)$, put

$$s_\iota(\nu)=k^2-(r+1)k+\tfrac{1}{2}r(r+3)-\nu\quad(\iota=0,1),\tag{2.6}$$

and then consider an integer s satisfying the lower bound $s\geqslant s_\iota(\nu)$. For brevity we write $\mathfrak{w}=s+k-1$ and $\lambda=\lambda^*_{\mathfrak{w}}$. Our goal is to establish the upper bound $\lambda\leqslant 2\mathfrak{w}-\frac{1}{2}k(k+1)+\Delta$, where $\Delta=\Delta_\iota(\nu)$ is a carefully chosen target exponent satisfying $0\leqslant\Delta\leqslant\frac{1}{2}k(k+1)$. We define

$$\Delta_0(\nu)=\sum_{m=1}^{r}\frac{(m-1)(k-m-1)}{s-m}-\frac{(\nu_0(r,s)-\nu)(r-1)}{s}\tag{2.7}$$

and

$$\Delta_1(\nu)=s^{-1}\sum_{m=1}^{r}(m-1)(k-m-1).\tag{2.8}$$

Let N be an arbitrary natural number, sufficiently large in terms of s and k, and put

$$\theta=(16(s+k))^{-N-1}\quad\text{and}\quad\delta=(1000N(s+k))^{-N-1}\theta.\tag{2.9}$$

In view of the definition of λ, there exists a sequence of natural numbers $(X_l)_{l=1}^\infty$, tending to infinity, with the property that

$$J_{\mathfrak{w}}(X_l)>X_l^{\lambda-\delta}\quad(l\in\mathbb{N}).\tag{2.10}$$

Also, provided that X_l is sufficiently large, one has the corresponding upper bound

$$J_{\mathfrak{w}}(Y)<Y^{\lambda+\delta}\quad\text{for}\quad Y\geqslant X_l^{1/2}.\tag{2.11}$$

We now consider a fixed element $X=X_l$ of the sequence $(X_l)_{l=1}^\infty$, which we may assume to be sufficiently large in terms of s, k and N. We put $M=X^\theta$, and note from (2.9) that $X^\delta<M^{1/N}$. Throughout, constants implied in the

notation of Landau and Vinogradov may depend on s, k, N, and also on ε in view of our earlier convention, but not on any other variable.

Let p be a fixed prime number with $M < p \leqslant 2M$ to be chosen in due course. When c and ξ are non-negative integers, and $\boldsymbol{\alpha} \in [0,1)^k$, define

$$\mathfrak{f}_c(\boldsymbol{\alpha};\xi) = \sum_{\substack{1\leqslant x\leqslant X\\ x\equiv \xi \pmod{p^c}}} e(\alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_k x^k). \tag{2.12}$$

As in [17], we must consider well-conditioned tuples of integers belonging to distinct congruence classes modulo a suitable power of p. When $1 \leqslant m \leqslant k-1$, denote by $\Xi_c^m(\xi)$ the set of integral m-tuples $(\xi_1,\ldots,\xi_m)$, with

$$1 \leqslant \boldsymbol{\xi} \leqslant p^{c+1} \quad \text{and} \quad \boldsymbol{\xi} \equiv \xi \pmod{p^c},$$

and satisfying the property that $\xi_i \not\equiv \xi_j \pmod{p^{c+1}}$ for $i \ne j$. We then put

$$\mathfrak{F}_c^m(\boldsymbol{\alpha};\xi) = \sum_{\boldsymbol{\xi}\in \Xi_c^m(\xi)} \prod_{i=1}^m \mathfrak{f}_{c+1}(\boldsymbol{\alpha};\xi_i), \tag{2.13}$$

where the exponential sums $\mathfrak{f}_{c+1}(\boldsymbol{\alpha};\xi_i)$ are defined via (2.12).

As in our previous work on the efficient congruencing method, certain mixed mean values play a critical role within our arguments. When a and b are positive integers, we define

$$I_{a,b}^m(X;\xi,\eta) = \oint |\mathfrak{F}_a^m(\boldsymbol{\alpha};\xi)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2\mathfrak{w}-2m}| \,\mathrm{d}\boldsymbol{\alpha} \tag{2.14}$$

and

$$K_{a,b}^m(X;\xi,\eta) = \oint |\mathfrak{F}_a^m(\boldsymbol{\alpha};\xi)^2 \mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s-2m}| \,\mathrm{d}\boldsymbol{\alpha}. \tag{2.15}$$

We remark that, in order to permit the number of variables subject to the congruencing process to vary, which is tantamount to allowing the parameter m to vary likewise, the definition of the mean value $K_{a,b}^m(X;\xi,\eta)$ is necessarily more complicated than analogues in our previous work on efficient congruencing. This will become apparent in §6.

For future reference, it is useful to note that by orthogonality, the mean value $I_{a,b}^m(X;\xi,\eta)$ counts the number of integral solutions of the system

$$\sum_{i=1}^m (x_i^j - y_i^j) = \sum_{l=1}^{\mathfrak{w}-m} (v_l^j - w_l^j) \quad (1 \leqslant j \leqslant k), \tag{2.16}$$

with

$$1 \leqslant \mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w} \leqslant X, \quad \mathbf{v} \equiv \mathbf{w} \equiv \eta \pmod{p^b},$$
$$[\mathbf{x} \pmod{p^{a+1}}] \in \Xi_a^m(\xi) \quad \text{and} \quad [\mathbf{y} \pmod{p^{a+1}}] \in \Xi_a^m(\xi).$$

Similarly, the mean value $K_{a,b}^m(X;\xi,\eta)$ counts the number of integral solutions of the system

$$\sum_{i=1}^m (x_i^j - y_i^j) = \sum_{l=1}^{k-1} (u_l^j - v_l^j) + \sum_{n=1}^{s-m} (w_n^j - z_n^j) \quad (1 \leqslant j \leqslant k), \tag{2.17}$$

with

$$1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X, \quad [\mathbf{x} \ (\mathrm{mod}\ p^{a+1})] \in \Xi_a^m(\xi), \quad [\mathbf{y} \ (\mathrm{mod}\ p^{a+1})] \in \Xi_a^m(\xi),$$
$$1 \leqslant \mathbf{u}, \mathbf{v} \leqslant X, \quad [\mathbf{u} \ (\mathrm{mod}\ p^{b+1})] \in \Xi_b^{k-1}(\eta), \quad [\mathbf{v} \ (\mathrm{mod}\ p^{b+1})] \in \Xi_b^{k-1}(\eta),$$
$$1 \leqslant \mathbf{w}, \mathbf{z} \leqslant X, \quad \mathbf{w} \equiv \mathbf{z} \equiv \eta \ (\mathrm{mod}\ p^b).$$

Given a solution $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$ of the system (2.17), an application of the Binomial Theorem shows that for $1 \leqslant j \leqslant k$, one has

$$\sum_{i=1}^m ((x_i-\eta)^j - (y_i-\eta)^j) = \sum_{l=1}^{k-1} ((u_l-\eta)^j - (v_l-\eta)^j) + \sum_{n=1}^{s-m} ((w_n-\eta)^j - (z_n-\eta)^j).$$

But in any solution counted by $K_{a,b}^m(X;\xi,\eta)$, one has $\mathbf{u} \equiv \mathbf{v} \equiv \eta \ (\mathrm{mod}\ p^b)$ and $\mathbf{w} \equiv \mathbf{z} \equiv \eta \ (\mathrm{mod}\ p^b)$. We therefore deduce that

$$\sum_{i=1}^m (x_i-\eta)^j \equiv \sum_{i=1}^m (y_i-\eta)^j \ (\mathrm{mod}\ p^{jb}) \quad (1 \leqslant j \leqslant k). \tag{2.18}$$

It is convenient to put

$$I_{a,b}^m(X) = \max_{1 \leqslant \xi \leqslant p^a} \max_{\substack{1 \leqslant \eta \leqslant p^b \\ \eta \not\equiv \xi \ (\mathrm{mod}\ p)}} I_{a,b}^m(X;\xi,\eta) \tag{2.19}$$

and

$$K_{a,b}^m(X) = \max_{1 \leqslant \xi \leqslant p^a} \max_{\substack{1 \leqslant \eta \leqslant p^b \\ \eta \not\equiv \xi \ (\mathrm{mod}\ p)}} K_{a,b}^m(X;\xi,\eta). \tag{2.20}$$

Note here that although these mean values implicitly depend on our choice of the prime p, this choice depends on s, k, r, θ and X_l alone. Since we fix p in the precongruencing step described in §5, following the proof of Lemma 5.1, the particular choice will ultimately be rendered irrelevant.

The precongruencing step requires a definition of $K_{0,b}^m(X)$ aligned with the conditioning idea, and this we now describe. When $\boldsymbol{\zeta}$ is a tuple of integers, we denote by $\Xi^m(\boldsymbol{\zeta})$ the set of m-tuples $(\xi_1,\dots,\xi_m) \in \Xi_0^m(0)$ such that $\xi_i \not\equiv \zeta_j \ (\mathrm{mod}\ p)$ for all i and j. Recalling (2.12), we put

$$\mathfrak{F}^m(\boldsymbol{\alpha};\boldsymbol{\zeta}) = \sum_{\boldsymbol{\xi} \in \Xi^m(\boldsymbol{\zeta})} \prod_{i=1}^m \mathfrak{f}_1(\boldsymbol{\alpha};\xi_i),$$

and then define

$$\widetilde{I}_c^m(X;\eta) = \oint |\mathfrak{F}^m(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_c(\boldsymbol{\alpha};\eta)^{2\mathfrak{w}-2m}| \,\mathrm{d}\boldsymbol{\alpha},$$

$$\widetilde{K}_c^m(X;\eta) = \oint |\mathfrak{F}^m(\boldsymbol{\alpha};\eta)^2 \mathfrak{F}_c^{k-1}(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_c(\boldsymbol{\alpha};\eta)^{2s-2m}| \,\mathrm{d}\boldsymbol{\alpha}, \tag{2.21}$$

$$K_{0,c}^m(X) = \max_{1 \leqslant \eta \leqslant p^c} \widetilde{K}_c^m(X;\eta). \tag{2.22}$$

As in [17], our arguments are simplified by making transparent the relationship between mean values and their anticipated magnitudes, although for present purposes we adopt a more flexible notation than that employed earlier. When $\mathfrak{d}$ and ρ are non-negative numbers, we adopt the convention that

$$[[J_{\mathfrak{w}}(X)]]_{\mathfrak{d}} = \frac{J_{\mathfrak{w}}(X)}{X^{2\mathfrak{w}-\frac{1}{2}k(k+1)+\mathfrak{d}}}, \tag{2.23}$$

$$[[I^m_{a,b}(X)]]_{\mathfrak{d},\rho} = \frac{(M^{ka-b})^\rho I^m_{a,b}(X)}{(X/M^b)^{2\mathfrak{w}-2m}(X/M^a)^{2m-\frac{1}{2}k(k+1)+\mathfrak{d}}} \tag{2.24}$$

and

$$[[K^m_{a,b}(X)]]_{\mathfrak{d},\rho} = \frac{(M^{ka-b})^\rho K^m_{a,b}(X)}{(X/M^b)^{2\mathfrak{w}-2m}(X/M^a)^{2m-\frac{1}{2}k(k+1)+\mathfrak{d}}}. \tag{2.25}$$

Using this notation, the bounds (2.10) and (2.11) may be rewritten as

$$[[J_{\mathfrak{w}}(X)]]_\Delta > X^{\Lambda-\delta} \quad \text{and} \quad [[J_{\mathfrak{w}}(Y)]]_\Delta < Y^{\Lambda+\delta} \quad (Y \geqslant X^{1/2}), \tag{2.26}$$

where $\Lambda = \Lambda(\Delta)$ is defined by

$$\Lambda(\Delta) = \lambda - 2\mathfrak{w} + \tfrac{1}{2}k(k+1) - \Delta. \tag{2.27}$$

We finish this section by recalling two simple estimates that encapsulate the translation-dilation invariance of the Diophantine system (1.1).

Lemma 2.1. *Suppose that c is a non-negative integer with $c\theta \leqslant 1$. Then for each natural number u, one has*

$$\max_{1\leqslant \xi \leqslant p^c} \oint |\mathfrak{f}_c(\boldsymbol{\alpha};\xi)|^{2u}\, \mathrm{d}\boldsymbol{\alpha} \ll_u J_u(X/M^c).$$

Proof. This is [17, Lemma 3.1]. □

Lemma 2.2. *Suppose that c and d are non-negative integers with $c \leqslant \theta^{-1}$ and $d \leqslant \theta^{-1}$. Then whenever $u, v \in \mathbb{N}$ and $\xi, \zeta \in \mathbb{Z}$, one has*

$$\oint |\mathfrak{f}_c(\boldsymbol{\alpha};\xi)^{2u}\mathfrak{f}_d(\boldsymbol{\alpha};\zeta)^{2v}|\, \mathrm{d}\boldsymbol{\alpha} \ll_{u,v} (J_{u+v}(X/M^c))^{u/(u+v)}(J_{u+v}(X/M^d))^{v/(u+v)}.$$

Proof. This is [5, Corollary 2.2]. □

3. Auxiliary systems of congruences

There are two primary regimes of interest so far as auxiliary congruences are concerned. Fortunately, we are able to extract suitable estimates from our previous work [5, 17, 19], though this requires that we recall in detail the notation introduced in the latter papers. When a and b are integers with $1 \leqslant a < b$, we denote by $\mathcal{B}^n_{a,b}(\mathbf{m};\xi,\eta)$ the set of solutions of the system of congruences

$$\sum_{i=1}^n (z_i - \eta)^j \equiv m_j \pmod{p^{jb}} \quad (1 \leqslant j \leqslant k), \tag{3.1}$$

with $1 \leqslant \mathbf{z} \leqslant p^{kb}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \pmod{p^{a+1}}$ for some $\boldsymbol{\xi} \in \Xi^n_a(\xi)$. We define an equivalence relation $\mathcal{R}(\lambda)$ on integral n-tuples by declaring $\mathbf{x}$ and $\mathbf{y}$ to be

$\mathcal{R}(\lambda)$-equivalent when $\mathbf{x} \equiv \mathbf{y} \pmod{p^\lambda}$. We then write $\mathcal{C}_{a,b}^{n,h}(\mathbf{m};\xi,\eta)$ for the set of $\mathcal{R}(hb)$-equivalence classes of $\mathcal{B}_{a,b}^n(\mathbf{m};\xi,\eta)$, and we define $B_{a,b}^{n,h}(p)$ by putting

$$B_{a,b}^{n,h}(p) = \max_{1\leqslant \xi\leqslant p^a} \max_{\substack{1\leqslant \eta\leqslant p^b\\ \eta\not\equiv\xi \pmod p}} \max_{1\leqslant \mathbf{m}\leqslant p^{kb}} \mathrm{card}(\mathcal{C}_{a,b}^{n,h}(\mathbf{m};\xi,\eta)). \tag{3.2}$$

When $a=0$ we modify these definitions, so that $\mathcal{B}_{0,b}^n(\mathbf{m};\xi,\eta)$ denotes the set of solutions of the system of congruences (3.1) with $1 \leqslant \mathbf{z} \leqslant p^{kb}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \pmod p$ for some $\boldsymbol{\xi} \in \Xi_0^n(\xi)$, and for which in addition $\mathbf{z} \not\equiv \eta \pmod p$. As in the situation in which one has $a \geqslant 1$, we write $\mathcal{C}_{0,b}^{n,h}(\mathbf{m};\xi,\eta)$ for the set of $\mathcal{R}(hb)$-equivalence classes of $\mathcal{B}_{0,b}^n(\mathbf{m};\xi,\eta)$, but we define $B_{0,b}^{n,h}(p)$ by putting

$$B_{0,b}^{n,h}(p) = \max_{1\leqslant \eta\leqslant p^b} \max_{1\leqslant \mathbf{m}\leqslant p^{kb}} \mathrm{card}(\mathcal{C}_{0,b}^{n,h}(\mathbf{m};0,\eta)).$$

We note that the choice of ξ in this situation with $a=0$ is irrelevant.

The next lemma records the two estimates for $B_{a,b}^{n,h}(p)$ of use in the regimes of interest to us.

Lemma 3.1. *Let a and b be integers with $0 \leqslant a < b$ and $b \geqslant (r-1)a$. Then*

$$B_{a,b}^{k-1,k-r}(p) \leqslant k!p^{\mu b+\nu a}, \tag{3.3}$$

where

$$\mu = \tfrac{1}{2}(k-r-1)(k-r-2) \quad \textit{and} \quad \nu = \tfrac{1}{2}(k-r-1)(k+r-2). \tag{3.4}$$

In addition, subject to the additional hypothesis $1 \leqslant j \leqslant r$, one has

$$B_{a,b}^{r-j+1,k-r+j}(p) \leqslant k!p^{(r-j)a}. \tag{3.5}$$

Proof. We apply [5, Lemma 3.3]. Thus, provided that k, R and T satisfy

$$k \geqslant 2, \quad \max\{2, \tfrac{1}{2}(k-1)\} \leqslant T \leqslant k, \quad 1 \leqslant R \leqslant k \quad \text{and} \quad R+T \geqslant k, \tag{3.6}$$

and in addition

$$0 \leqslant a < b \quad \text{and} \quad b \geqslant (k-T-1)a, \tag{3.7}$$

one has

$$B_{a,b}^{R,T}(p) \leqslant k!p^{\mu' b+\nu' a},$$

where

$$\mu' = \tfrac{1}{2}(T+R-k)(T+R-k-1) \quad \text{and} \quad \nu' = \tfrac{1}{2}(T+R-k)(k+R-T-1). \tag{3.8}$$

For the first conclusion of the lemma, we take $R=k-1$ and $T=k-r$, noting that the condition (2.4) ensures that $T \geqslant \max\{\frac{1}{2}(k-1),2\}$ and $R+T \geqslant k+1$. Thus, subject to the conditions $0 \leqslant a < b$ and $b \geqslant (r-1)a$ imported from (3.7), one obtains the bound (3.3) by computing the exponents μ' and ν' given by (3.8). For the second conclusion, we take $R=r-j+1$ and $T=k-r+j$. Here, the conditions imposed by (3.6) are easily verified. In this case, the constraints (3.7) are satisfied when $1 \leqslant j \leqslant r$ provided that $0 \leqslant a < b$ and $b \geqslant (r-1)a$, and so the desired conclusion (3.5) again follows by evaluating the exponents delivered by (3.8). $\square$

4. The conditioning process

The mean value $K^m_{a,b}(X;\xi,\eta)$ differs from the analogue previously employed in efficient congruencing methods, and thus we must discuss the conditioning process in some detail. Our goal will now be to replace a factor $\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2k-2}$ occurring in (2.14) by the conditioned factor $\mathfrak{F}^{k-1}_b(\boldsymbol{\alpha};\eta)^2$ in (2.15).

Lemma 4.1. *Let a and b be integers with $b > a \geqslant 1$. Then one has*

$$I^{k-1}_{a,b}(X) \ll K^{k-1}_{a,b}(X) + M^{2s/3} I^{k-1}_{a,b+1}(X).$$

Proof. Consider fixed integers ξ and η with $\eta \not\equiv \xi \pmod p$. Let T_1 denote the number of integral solutions $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}$, $\mathbf{w}$ of the system (2.16) counted by $I^{k-1}_{a,b}(X;\xi,\eta)$ in which $v_1,\dots,v_s$ together occupy at least $k-1$ distinct residue classes modulo p^{b+1}, and let T_2 denote the corresponding number of solutions in which these integers together occupy at most $k-2$ distinct residue classes modulo p^{b+1}. Then

$$I^{k-1}_{a,b}(X;\xi,\eta) = T_1 + T_2. \tag{4.1}$$

We first estimate T_1. Recall the definitions (2.13), (2.14) and (2.15). Then by orthogonality and an application of Hölder's inequality, one finds that

$$\begin{aligned} T_1 &\leqslant \binom{s}{k-1} \oint |\mathfrak{F}^{k-1}_a(\boldsymbol{\alpha};\xi)|^2 \mathfrak{F}^{k-1}_b(\boldsymbol{\alpha};\eta) \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{s-k+1} \mathfrak{f}_b(-\boldsymbol{\alpha};\eta)^s \,\mathrm{d}\boldsymbol{\alpha} \\ &\ll \left(K^{k-1}_{a,b}(X;\xi,\eta)\right)^{1/2} \left(I^{k-1}_{a,b}(X;\xi,\eta)\right)^{1/2}. \end{aligned} \tag{4.2}$$

Next we estimate T_2. In view of (2.4) and (2.6), one may confirm that the implicit hypothesis $s \geqslant s_\iota(\nu)$ ensures that $s > 2(k-1)$. Consequently, there is an integer $\zeta \equiv \eta \pmod{p^b}$ having the property that three at least of the variables $v_1,\dots,v_s$ are congruent to ζ modulo p^{b+1}. Hence, again recalling the definitions (2.13) and (2.14), one finds by orthogonality in combination with Hölder's inequality that

$$\begin{aligned} T_2 &\leqslant \binom{s}{3} \sum_{\substack{1\leqslant \zeta \leqslant p^{b+1} \\ \zeta \equiv \eta \ (\mathrm{mod}\ p^b)}} \oint |\mathfrak{F}^{k-1}_a(\boldsymbol{\alpha};\xi)|^2 \mathfrak{f}_{b+1}(\boldsymbol{\alpha};\zeta)^3 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{s-3} \mathfrak{f}_b(-\boldsymbol{\alpha};\eta)^s \,\mathrm{d}\boldsymbol{\alpha} \\ &\ll M \max_{\substack{1\leqslant \zeta \leqslant p^{b+1} \\ \zeta \equiv \eta \ (\mathrm{mod}\ p^b)}} (I^{k-1}_{a,b}(X;\xi,\eta))^{1-3/(2s)} (I^{k-1}_{a,b+1}(X;\xi,\zeta))^{3/(2s)}. \end{aligned} \tag{4.3}$$

By substituting (4.2) and (4.3) into (4.1), and recalling (2.19) and (2.20), we therefore conclude that

$$I^{k-1}_{a,b}(X) \ll (K^{k-1}_{a,b}(X))^{1/2} (I^{k-1}_{a,b}(X))^{1/2} + M (I^{k-1}_{a,b}(X))^{1-3/(2s)} (I^{k-1}_{a,b+1}(X))^{3/(2s)},$$

whence

$$I^{k-1}_{a,b}(X) \ll K^{k-1}_{a,b}(X) + M^{2s/3} I^{k-1}_{a,b+1}(X).$$

This completes the proof of the lemma. □

Repeated application of Lemma 4.1, together with a trivial bound for the mean value $K^{k-1}_{a,b+H}(X)$ when H is large enough, yields a relation suitable for our iterative process.

Lemma 4.2. *Let a and b be integers with $1 \leqslant a < b$, and put $H = 15(b-a)$. Suppose that $b + H \leqslant (2\theta)^{-1}$. Then there exists an integer h with $0 \leqslant h < H$ having the property that*

$$I_{a,b}^{k-1}(X) \ll (M^h)^{2s/3} K_{a,b+h}^{k-1}(X) + (M^H)^{-s/4}(X/M^b)^{2s}(X/M^a)^{\lambda-2s}.$$

Proof. By repeated application of Lemma 4.1, we obtain the upper bound

$$I_{a,b}^{k-1}(X) \ll \sum_{h=0}^{H-1} (M^h)^{2s/3} K_{a,b+h}^{k-1}(X) + (M^H)^{2s/3} I_{a,b+H}^{k-1}(X). \tag{4.4}$$

On considering the underlying Diophantine systems, it follows from Lemma 2.2 that, uniformly in ξ and η, one has

$$\begin{aligned} I_{a,b+H}^{k-1}(X;\xi,\eta) &\leqslant \oint |\mathfrak{f}_a(\boldsymbol{\alpha};\xi)^{2k-2} \mathfrak{f}_{b+H}(\boldsymbol{\alpha};\eta)^{2s}| \, \mathrm{d}\boldsymbol{\alpha} \\ &\ll (J_{\mathfrak{w}}(X/M^a))^{(k-1)/\mathfrak{w}} \left(J_{\mathfrak{w}}(X/M^{b+H}) \right)^{1-(k-1)/\mathfrak{w}}. \end{aligned}$$

The argument completing the proof of [5, Lemma 4.2] now applies, delivering the estimate

$$(M^H)^{2s/3} I_{a,b+H}^{k-1}(X) \ll (M^H)^{-s/4}(X/M^b)^{2s}(X/M^a)^{\lambda-2s},$$

and the conclusion of the lemma follows on substituting this bound into (4.4). □

5. The preconguencing step

It is necessary to configure the variables in the initial step of our iteration so that subsequent iterations are not impeded. Here we are able to make use of our earlier work [19, §6] and [5, §6] concerning precongruencing steps so as to abbreviate the discussion, despite our present alteration of the definition of $K_{a,b}^m(X)$ relative to its earlier analogues.

Lemma 5.1. *There exists a prime number p with $M < p \leqslant 2M$, and an integer h with $h \in \{0,1,2,3\}$, for which one has*

$$J_{\mathfrak{w}}(X) \ll M^{2s+2sh/3} K_{0,1+h}^{k-1}(X).$$

Proof. The argument of the proof of [5, Lemma 6.1], leading via equation (6.2) to equation (6.3) of that paper, shows that there is a prime number p with $M < p \leqslant 2M$ for which

$$J_{\mathfrak{w}}(X) \ll p^{2s} \max_{1 \leqslant \eta \leqslant p} \widetilde{I}_1^{k-1}(X;\eta).$$

By modifying the argument of the proof of [5, Lemma 6.1] leading to equation (6.6) of that paper, along the lines easily surmised from our proof of Lemma 4.1 above, one finds that

$$\widetilde{I}_c^{k-1}(X;\eta) \ll \widetilde{K}_c^{k-1}(X;\eta) + M^{2s/3} \max_{1 \leqslant \zeta \leqslant p^{c+1}} \widetilde{I}_{c+1}^{k-1}(X;\zeta). \tag{5.1}$$

Next, we iterate (5.1) in order to bound $\widetilde{I}_1^{k-1}(X;\eta)$, just as in the argument concluding the proof of [5, Lemma 6.1]. In this way, we find either that

$$J_{\mathfrak{w}}(X) \ll M^{2s+2sh/3} \max_{1\leqslant \zeta \leqslant p^{1+h}} \widetilde{K}_{1+h}^{k-1}(X;\zeta) \tag{5.2}$$

for some index $h\in\{0,1,2,3\}$, so that the conclusion of the lemma holds by virtue of the definition (2.22), or else that

$$J_{\mathfrak{w}}(X) \ll X^{\lambda+\delta}M^{-\mathfrak{w}/3}.$$

Thus, on recalling the definition (2.9) of δ, we find that $J_{\mathfrak{w}}(X) \ll X^{\lambda-2\delta}$, contradicting the lower bound (2.10) whenever $X=X_l$ is sufficiently large. We are therefore forced to conclude that the earlier upper bound (5.2) holds, and hence the proof of the lemma is complete. $\square$

It is at this point that we fix the prime number p, once and for all, in accordance with Lemma 5.1.

6. The efficient congruencing step

We extract congruence information from the mean value $K_{a,b}^{k-1}(X)$ in two phases. The reader familiar with earlier efficient congruencing arguments will identify significant complications in each phase associated with our (forced) inhomogeneous definition (2.15) of the mean value $K_{a,b}^{k-1}(X;\xi,\eta)$. Before describing the first phase of the efficient congruencing step, in which we relate $K_{a,b}^{k-1}(X)$ to $I_{b,(k-r)b}^{k-1}(X)$ and $K_{a,b}^{r}(X)$, we introduce some additional notation. We define the generating function

$$\mathfrak{H}_{c,d}^{m}(\boldsymbol{\alpha};\xi) = \sum_{\boldsymbol{\xi}\in\Xi_c^m(\xi)} \sum_{\substack{1\leqslant \boldsymbol{\zeta}\leqslant p^d\\ \boldsymbol{\zeta}\equiv\boldsymbol{\xi}\ (\mathrm{mod}\ p^{c+1})}} \prod_{i=1}^{m} |\mathfrak{f}_d(\boldsymbol{\alpha};\zeta_i)|^2, \tag{6.1}$$

adopting the natural convention that $\mathfrak{H}_{c,d}^{0}(\boldsymbol{\alpha};\xi)=1$. For future reference, we note at this point that successive applications of Hölder's inequality show that when ω is a real number with $m\omega\geqslant 1$, then

$$\begin{aligned}\mathfrak{H}_{c,d}^{m}(\boldsymbol{\alpha};\xi)^{\omega} &\leqslant \Bigg(\sum_{\substack{1\leqslant \zeta\leqslant p^d\\ \zeta\equiv\xi\ (\mathrm{mod}\ p^c)}} |\mathfrak{f}_d(\boldsymbol{\alpha};\zeta)|^2 \Bigg)^{m\omega}\\ &\leqslant (p^{d-c})^{m\omega-1} \sum_{\substack{1\leqslant \zeta\leqslant p^d\\ \zeta\equiv\xi\ (\mathrm{mod}\ p^c)}} |\mathfrak{f}_d(\boldsymbol{\alpha};\zeta)|^{2m\omega}. \end{aligned}\tag{6.2}$$

Finally, we recall the definitions of μ and ν from (3.4).

Lemma 6.1. *Suppose that a and b are integers with $0\leqslant a<b\leqslant \theta^{-1}$ and $b\geqslant (r-1)a$. Then one has*

$$K_{a,b}^{k-1}(X) \ll M^{\mu b+\nu a}\Big((M^{(k-r)b-a})^s I_{b,(k-r)b}^{k-1}(X)\Big)^{\frac{k-r-1}{s-r}} \Big(K_{a,b}^{r}(X)\Big)^{\frac{s-k+1}{s-r}}.$$

Proof. We first consider the situation in which $a \geqslant 1$. The argument associated with the case $a = 0$ is very similar, and so we are able to appeal later to a highly abbreviated argument for this case to complete the proof of the lemma. Consider fixed integers ξ and η with

$$1 \leqslant \xi \leqslant p^a, \quad 1 \leqslant \eta \leqslant p^b \quad \text{and} \quad \eta \not\equiv \xi \pmod{p}. \tag{6.3}$$

The quantity $K_{a,b}^{k-1}(X;\xi,\eta)$ counts the number of integral solutions of the system (2.17) with $m = k-1$ subject to the attendant conditions on $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$. Given such a solution of the system (2.17), the discussion leading to (2.18) shows that

$$\sum_{i=1}^{k-1} (x_i - \eta)^j \equiv \sum_{i=1}^{k-1} (y_i - \eta)^j \pmod{p^{jb}} \quad (1 \leqslant j \leqslant k). \tag{6.4}$$

In the notation introduced in §3, it follows that for some k-tuple of integers $\mathbf{m}$, both $[\mathbf{x} \pmod{p^{kb}}]$ and $[\mathbf{y} \pmod{p^{kb}}]$ lie in $\mathcal{B}_{a,b}^{k-1}(\mathbf{m};\xi,\eta)$. Write

$$\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\mathbf{m}) = \sum_{\boldsymbol{\theta} \in \mathcal{B}_{a,b}^{k-1}(\mathbf{m};\xi,\eta)} \prod_{i=1}^{k-1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i).$$

Then on considering the underlying Diophantine system, we see from (2.17) and (6.4) that

$$K_{a,b}^{k-1}(X;\xi,\eta) = \sum_{m_1=1}^{p^b} \dots \sum_{m_k=1}^{p^{kb}} \oint |\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\mathbf{m})^2 \mathfrak{F}^*(\boldsymbol{\alpha})^2| \,\mathrm{d}\boldsymbol{\alpha},$$

where

$$\mathfrak{F}^*(\boldsymbol{\alpha}) = \mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{s-k+1}. \tag{6.5}$$

We now partition the vectors in each set $\mathcal{B}_{a,b}^{k-1}(\mathbf{m};\xi,\eta)$ into equivalence classes modulo $p^{(k-r)b}$ as in §3. Write $\mathcal{C}(\mathbf{m}) = \mathcal{C}_{a,b}^{k-1,k-r}(\mathbf{m};\xi,\eta)$. By applying Cauchy's inequality and then recalling (3.2), we find by means of Lemma 3.1 that

$$\begin{aligned}
|\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\mathbf{m})|^2 &= \Bigl| \sum_{\mathfrak{C} \in \mathcal{C}(\mathbf{m})} \sum_{\boldsymbol{\theta} \in \mathfrak{C}} \prod_{i=1}^{k-1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i) \Bigr|^2 \\
&\leqslant \operatorname{card}(\mathcal{C}(\mathbf{m})) \sum_{\mathfrak{C} \in \mathcal{C}(\mathbf{m})} \Bigl| \sum_{\boldsymbol{\theta} \in \mathfrak{C}} \prod_{i=1}^{k-1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i) \Bigr|^2 \\
&\ll M^{\mu b + \nu a} \sum_{\mathfrak{C} \in \mathcal{C}(\mathbf{m})} \Bigl| \sum_{\boldsymbol{\theta} \in \mathfrak{C}} \prod_{i=1}^{k-1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i) \Bigr|^2 .
\end{aligned}$$

Hence

$$K_{a,b}^{k-1}(X;\xi,\eta) \ll M^{\mu b + \nu a} \sum_{\mathbf{m}} \sum_{\mathfrak{C} \in \mathcal{C}(\mathbf{m})} \oint \Bigl| \mathfrak{F}^*(\alpha) \sum_{\boldsymbol{\theta} \in \mathfrak{C}} \prod_{i=1}^{k-1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i) \Bigr|^2 \,\mathrm{d}\boldsymbol{\alpha}.$$

For each k-tuple $\mathbf{m}$ and equivalence class $\mathfrak{C}$, the integral above counts solutions of (2.17) with the additional constraint that both $[\mathbf{x} \pmod{p^{kb}}]$ and $[\mathbf{y} \pmod{p^{kb}}]$ lie in $\mathfrak{C}$. In particular, one has $\mathbf{x} \equiv \mathbf{y} \pmod{p^{(k-r)b}}$. Moreover, as the sets $\mathcal{B}^{k-1}_{a,b}(\mathbf{m};\xi,\eta)$ are disjoint for distinct k-tuples $\mathbf{m}$ with $1 \leqslant m_j \leqslant p^{jb}$ $(1 \leqslant j \leqslant k)$, to each pair $(\mathbf{x},\mathbf{y})$ there corresponds at most one pair $(\mathbf{m},\mathfrak{C})$. Thus we deduce that

$$K^{k-1}_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a} H,$$

where H denotes the number of solutions of (2.17) subject to the additional condition $\mathbf{x} \equiv \mathbf{y} \pmod{p^{(k-r)b}}$. Hence, on considering the underlying Diophantine systems and recalling (6.1), we discern that

$$K^{k-1}_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a} \oint \mathfrak{H}^{k-1}_{a,(k-r)b}(\boldsymbol{\alpha};\xi)|\mathfrak{F}^*(\boldsymbol{\alpha})|^2 \,\mathrm{d}\boldsymbol{\alpha}. \tag{6.6}$$

An inspection of the definition of $\Xi^m_a(\xi)$ in the preamble to (2.13) reveals that when $\boldsymbol{\xi} \in \Xi^{k-1}_a(\xi)$, then in particular one has

$$(\xi_1,\dots,\xi_r) \in \Xi^r_a(\xi) \quad \text{and} \quad (\xi_{r+1},\dots,\xi_{k-1}) \in \Xi^{k-r-1}_a(\xi). \tag{6.7}$$

Note here that in the situation with $r = k-1$, the second of these conditions is interpreted as vacuous. In view of (6.7), a further consideration of the underlying Diophantine systems leads from (6.6) via (6.1) to the upper bound

$$K^{k-1}_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a} \oint \mathfrak{H}^{r}_{a,(k-r)b}(\boldsymbol{\alpha};\xi)\mathfrak{H}^{k-r-1}_{a,(k-r)b}(\boldsymbol{\alpha};\xi)|\mathfrak{F}^*(\boldsymbol{\alpha})|^2 \,\mathrm{d}\boldsymbol{\alpha}.$$

By applying Hölder's inequality to the integral on the right hand side of this relation, keeping in mind the definition (6.5), we obtain the bound

$$K^{k-1}_{a,b}(X;\xi,\eta) \ll M^{\mu b+\nu a} U_1^{\omega_1} U_2^{\omega_2} U_3^{\omega_3}, \tag{6.8}$$

where

$$\omega_1 = \frac{s-k+1}{s-r}, \quad \omega_2 = \frac{k-r-1}{s}, \quad \omega_3 = \frac{r(k-r-1)}{s(s-r)}, \tag{6.9}$$

and

$$U_1 = \oint \mathfrak{H}^{r}_{a,(k-r)b}(\boldsymbol{\alpha};\xi)|\mathfrak{F}^{k-1}_b(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s-2r}| \,\mathrm{d}\boldsymbol{\alpha}, \tag{6.10}$$

$$U_2 = \oint |\mathfrak{F}^{k-1}_b(\boldsymbol{\alpha};\eta)|^2 \mathfrak{H}^{k-r-1}_{a,(k-r)b}(\boldsymbol{\alpha};\xi)^{s/(k-r-1)} \,\mathrm{d}\boldsymbol{\alpha}, \tag{6.11}$$

$$U_3 = \oint |\mathfrak{F}^{k-1}_b(\boldsymbol{\alpha};\eta)|^2 \mathfrak{H}^{r}_{a,(k-r)b}(\boldsymbol{\alpha};\xi)^{s/r} \,\mathrm{d}\boldsymbol{\alpha}. \tag{6.12}$$

We note here that the condition (2.4) ensures that $r \leqslant k-2$, so that ω_1, ω_2 and ω_3 are each positive. Thus the argument leading to (6.8) represents a legitimate application of Hölder's inequality.

Our next task is to relate the mean values U_i to the more familiar ones introduced in §2. Observe first that a consideration of the underlying Diophantine system leads from (6.10) via (6.1) and (2.13) to the upper bound

$$U_1 \leqslant \oint |\mathfrak{F}^r_a(\boldsymbol{\alpha};\xi)^2 \mathfrak{F}^{k-1}_b(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s-2r}| \,\mathrm{d}\boldsymbol{\alpha}. \tag{6.13}$$

Indeed, the Diophantine system underlying the mean value U_1 is subject to additional diagonal structure that we have discarded in the mean value on the right hand side of (6.13). It is worth noting here that this manœuvre, though superficially inefficient, loses nothing in the ensuing argument, since the additional diagonal constraint is recovered without cost in the next stage of our argument, in Lemma 6.2. On recalling (2.15) and (2.20), we thus deduce from (6.13) that

$$U_1 \leqslant K_{a,b}^r(X). \tag{6.14}$$

Next, by employing (6.2) within (6.11) and (6.12), we find that

$$U_2+U_3 \ll (M^{(k-r)b-a})^s \max_{\substack{1\leqslant \zeta\leqslant p^{(k-r)b}\\ \zeta\equiv \xi \ (\mathrm{mod}\ p^a)}} \oint |\mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)^2\mathfrak{f}_{(k-r)b}(\boldsymbol{\alpha};\zeta)^{2s}|\,\mathrm{d}\boldsymbol{\alpha}.$$

Notice here that since the condition (6.3) implies that $\eta\not\equiv \xi \pmod p$, and we have $\zeta\equiv \xi \pmod{p^a}$ with $a\geqslant 1$, then $\zeta\not\equiv \eta \pmod p$. In this way we deduce from (2.14) and (2.19) that

$$U_2+U_3 \ll (M^{(k-r)b-a})^s I_{b,(k-r)b}^{k-1}(X). \tag{6.15}$$

By substituting (6.14) and (6.15) into the relation

$$K_{a,b}^{k-1}(X;\xi,\eta) \ll M^{\mu b+\nu a}U_1^{\omega_1}(U_2+U_3)^{1-\omega_1},$$

that is immediate from (6.8), and then recalling (6.9) and (2.20), the conclusion of the lemma follows when $a\geqslant 1$. When $a=0$, we must modify this argument slightly. In this case, from (2.21) and (2.22) we find that

$$K_{0,b}^{k-1}(X) = \max_{1\leqslant \eta\leqslant p^b} \oint |\mathfrak{F}^{k-1}(\boldsymbol{\alpha};\eta)^2\mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)^2\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s-2k+2}|\,\mathrm{d}\boldsymbol{\alpha}.$$

The desired conclusion follows in this instance by pursuing the proof given above in the case $a\geqslant 1$, noting that the definition of $\mathfrak{F}^{k-1}(\boldsymbol{\alpha};\eta)$ ensures that the variables resulting from the congruencing argument avoid the congruence class η modulo p. This completes the proof of the lemma. □

We now establish the machinery for an iteration by relating the mean value $K_{a,b}^{r-j+1}(X)$ to $I_{b,(k-r+j)b}^{k-1}(X)$ and $K_{a,b}^{r-j}(X)$ for $j=1,\dots,r$. Each step of this iteration effectively extracts (approximately) two variables mutually congruent modulo $p^{(k-r+j)b}$ for use in the next stage of the efficient congruencing argument, leaving a mean value of similar type to the original one on which a stronger congruencing process is applicable. It is at this point that we make effective the argument sketched at the end of the introduction. It is useful here and later to write

$$k_j=k-r+j \quad \text{and} \quad s_j=s-r+j. \tag{6.16}$$

Lemma 6.2. *Suppose that a and b are integers with $0\leqslant a<b\leqslant \theta^{-1}$ and $b\geqslant (r-1)a$. Then for $1\leqslant j\leqslant r$, one has*

$$K_{a,b}^{r-j+1}(X) \ll M^{(r-j)a}\left((M^{k_jb-a})^s I_{b,k_jb}^{k-1}(X)\right)^{1/s_j}\left(K_{a,b}^{r-j}(X)\right)^{s_{j-1}/s_j}.$$

Proof. We follow closely the argument of the proof of previous lemma. We again suppose in the first instance that $a \geqslant 1$. Consider fixed integers ξ and η satisfying the conditions imposed by (6.3). The quantity $K_{a,b}^{r-j+1}(X;\xi,\eta)$ counts the number of integral solutions of the system (2.17) with $m=r-j+1$ subject to the attendant conditions on $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$. Given such a solution of the system (2.17), the argument leading to (2.18) shows that

$$\sum_{i=1}^{r-j+1} (x_i-\eta)^l \equiv \sum_{i=1}^{r-j+1} (y_i-\eta)^l \pmod{p^{lb}} \quad (1\leqslant l\leqslant k). \tag{6.17}$$

In the notation introduced in §3, it follows that for some k-tuple of integers $\mathbf{m}$, both $[\mathbf{x} \pmod{p^{kb}}]$ and $[\mathbf{y} \pmod{p^{kb}}]$ lie in $\mathcal{B}_{a,b}^{r-j+1}(\mathbf{m};\xi,\eta)$. Write

$$\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\mathbf{m}) = \sum_{\boldsymbol{\theta}\in \mathcal{B}_{a,b}^{r-j+1}(\mathbf{m};\xi,\eta)} \prod_{i=1}^{r-j+1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i).$$

Then on considering the underlying Diophantine system, we see from (2.17) and (6.17) that

$$K_{a,b}^{r-j+1}(X;\xi,\eta) = \sum_{m_1=1}^{p^b} \dots \sum_{m_k=1}^{p^{kb}} \oint |\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\mathbf{m})^2 \mathfrak{F}_j^*(\boldsymbol{\alpha})^2|\, \mathrm{d}\boldsymbol{\alpha},$$

where we write

$$\mathfrak{F}_j^*(\boldsymbol{\alpha}) = \mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{s_{j-1}}. \tag{6.18}$$

We now partition the vectors in each set $\mathcal{B}_{a,b}^{r-j+1}(\mathbf{m};\xi,\eta)$ into equivalence classes modulo $p^{k_j b}$ as in §3. Write $\mathcal{C}(\mathbf{m}) = \mathcal{C}_{a,b}^{r-j+1,k_j}(\mathbf{m};\xi,\eta)$. By applying Cauchy's inequality and then recalling (3.2), we find by means of Lemma 3.1 that

$$\begin{aligned}
|\mathfrak{G}_{a,b}(\boldsymbol{\alpha};\mathbf{m})|^2 &= \Big| \sum_{\mathfrak{C}\in\mathcal{C}(\mathbf{m})} \sum_{\boldsymbol{\theta}\in\mathfrak{C}} \prod_{i=1}^{r-j+1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2 \\
&\leqslant \mathrm{card}(\mathcal{C}(\mathbf{m})) \sum_{\mathfrak{C}\in\mathcal{C}(\mathbf{m})} \Big| \sum_{\boldsymbol{\theta}\in\mathfrak{C}} \prod_{i=1}^{r-j+1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2 \\
&\ll M^{(r-j)a} \sum_{\mathfrak{C}\in\mathcal{C}(\mathbf{m})} \Big| \sum_{\boldsymbol{\theta}\in\mathfrak{C}} \prod_{i=1}^{r-j+1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2 .
\end{aligned}$$

Hence

$$K_{a,b}^{r-j+1}(X;\xi,\eta) \ll M^{(r-j)a} \sum_{\mathbf{m}} \sum_{\mathfrak{C}\in\mathcal{C}(\mathbf{m})} \oint \Big| \mathfrak{F}_j^*(\alpha) \sum_{\boldsymbol{\theta}\in\mathfrak{C}} \prod_{i=1}^{r-j+1} \mathfrak{f}_{kb}(\boldsymbol{\alpha};\theta_i)\Big|^2 \mathrm{d}\boldsymbol{\alpha}.$$

For each k-tuple $\mathbf{m}$ and equivalence class $\mathfrak{C}$, the integral above counts solutions of (2.17) with the additional constraint that both $[\mathbf{x} \pmod{p^{kb}}]$ and $[\mathbf{y} \pmod{p^{kb}}]$ lie in $\mathfrak{C}$. In particular, one has $\mathbf{x}\equiv\mathbf{y} \pmod{p^{k_j b}}$. Moreover, as the sets $\mathcal{B}_{a,b}^{r-j+1}(\mathbf{m};\xi,\eta)$ are disjoint for distinct k-tuples $\mathbf{m}$ with $1\leqslant m_j\leqslant p^{jb}$

($1\leqslant j\leqslant k$), to each pair $(\mathbf{x},\mathbf{y})$ there corresponds at most one pair $(\mathbf{m},\mathfrak{C})$. Thus we deduce that

$$K_{a,b}^{r-j+1}(X;\xi,\eta)\ll M^{(r-j)a}H,$$

where H denotes the number of solutions of (2.17) subject to the additional condition $\mathbf{x}\equiv\mathbf{y}\pmod{p^{k_jb}}$. Hence, on considering the underlying Diophantine systems and recalling (6.1), we discern that

$$K_{a,b}^{r-j+1}(X;\xi,\eta)\ll M^{(r-j)a}\oint \mathfrak{H}_{a,k_jb}^{r-j+1}(\boldsymbol{\alpha};\xi)|\mathfrak{F}_j^*(\boldsymbol{\alpha})|^2\,\mathrm{d}\boldsymbol{\alpha}. \tag{6.19}$$

An inspection of the definition of $\Xi_a^m(\xi)$, given in the preamble to (2.13), reveals on this occasion that when $\boldsymbol{\xi}\in\Xi_a^{r-j+1}(\xi)$, then

$$(\xi_1,\dots,\xi_{r-j})\in\Xi_a^{r-j}(\xi)\quad\text{and}\quad(\xi_{r-j+1})\in\Xi_a^1(\xi).$$

Then a further consideration of the underlying Diophantine systems leads from (6.19) via (6.1) to the upper bound

$$K_{a,b}^{r-j+1}(X;\xi,\eta)\ll M^{(r-j)a}\oint \mathfrak{H}_{a,k_jb}^{r-j}(\boldsymbol{\alpha};\xi)\mathfrak{H}_{a,k_jb}^{1}(\boldsymbol{\alpha};\xi)|\mathfrak{F}_j^*(\boldsymbol{\alpha})|^2\,\mathrm{d}\boldsymbol{\alpha}.$$

By applying Hölder's inequality to the integral on the right hand side of this relation, bearing in mind the definitions (6.16) and (6.18), we obtain the bound

$$K_{a,b}^{r-j+1}(X;\xi,\eta)\ll M^{(r-j)a}U_1^{\omega_1}U_2^{\omega_2}U_3^{\omega_3}, \tag{6.20}$$

where

$$\omega_1=s_{j-1}/s_j,\quad \omega_2=1/s,\quad \omega_3=(r-j)/(ss_j), \tag{6.21}$$

and

$$U_1=\oint \mathfrak{H}_{a,k_jb}^{r-j}(\boldsymbol{\alpha};\xi)|\mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)^2\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s_j}|\,\mathrm{d}\boldsymbol{\alpha}, \tag{6.22}$$

$$U_2=\oint |\mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)|^2\mathfrak{H}_{a,k_jb}^{1}(\boldsymbol{\alpha};\xi)^s\,\mathrm{d}\boldsymbol{\alpha}, \tag{6.23}$$

$$U_3=\oint |\mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)|^2\mathfrak{H}_{a,k_jb}^{r-j}(\boldsymbol{\alpha};\xi)^{s/(r-j)}\,\mathrm{d}\boldsymbol{\alpha}. \tag{6.24}$$

We again relate the mean values U_i to those introduced in §2. Observe first that a consideration of the underlying Diophantine system leads from (6.22) via (6.1) and (2.13) to the upper bound

$$U_1\leqslant\oint |\mathfrak{F}_a^{r-j}(\boldsymbol{\alpha};\xi)^2\mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)^2\mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s_j}|\,\mathrm{d}\boldsymbol{\alpha}.$$

On recalling (2.15) and (2.20), we thus deduce that

$$U_1\leqslant K_{a,b}^{r-j}(X). \tag{6.25}$$

Next, by employing (6.2) within (6.23) and (6.24), we find that

$$U_2+U_3\ll (M^{k_jb-a})^s\max_{\substack{1\leqslant\zeta\leqslant p^{k_jb}\\ \zeta\equiv\xi\ (\mathrm{mod}\ p^a)}}\oint |\mathfrak{F}_b^{k-1}(\boldsymbol{\alpha};\eta)^2\mathfrak{f}_{k_jb}(\boldsymbol{\alpha};\zeta)^{2s}|\,\mathrm{d}\boldsymbol{\alpha}.$$

Notice here that since the condition (6.3) implies that $\eta \not\equiv \xi \pmod{p}$, and we have $\zeta \equiv \xi \pmod{p^a}$ with $a \geqslant 1$, then once more one has $\zeta \not\equiv \eta \pmod{p}$. In this way we deduce from (2.14) and (2.19) that

$$U_2 + U_3 \ll (M^{k_j b - a})^s I^{k-1}_{b,k_j b}(X). \tag{6.26}$$

By substituting (6.25) and (6.26) into the relation

$$K^{r-j+1}_{a,b}(X;\xi,\eta) \ll M^{(r-j)a} U_1^{\omega_1}(U_2+U_3)^{1-\omega_1},$$

that is immediate from (6.20), and then recalling (6.21) and (2.20), the conclusion of the lemma follows when $a \geqslant 1$. When $a = 0$, we must modify this argument slightly. In this case, from (2.21) and (2.22) we find that

$$K^{r-j+1}_{0,b}(X) = \max_{1 \leqslant \eta \leqslant p^b} \oint |\mathfrak{F}^{r-j+1}(\boldsymbol{\alpha};\eta)^2 \mathfrak{F}^{k-1}_b(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{2s_{j-1}}| \,\mathrm{d}\boldsymbol{\alpha}.$$

The desired conclusion follows in this instance by pursuing the proof given above in the case $a \geqslant 1$, noting that the definition of $\mathfrak{F}^{r-j+1}(\boldsymbol{\alpha};\eta)$ ensures that the variables resulting from the congruencing argument avoid the congruence class η modulo p. This completes the proof of the lemma. □

There are, of course, similarities between the arguments applied to establish Lemmata 6.1 and 6.2. Some economy of space would be afforded by the proof of a common lemma, of which these respective lemmata would be special cases. However, the considerable complications associated with such a unified approach would, on the one hand, obscure the strategy underlying the proof of these lemmata, and on the other hand consume not inconsiderable space to accommodate these complications. Thus, we have deliberately opted for clarity over concision in offering two separate treatments.

7. The multigrade combination

We next combine the estimates supplied by Lemmata 6.1 and 6.2 so as to bound $K^{k-1}_{a,b}(X)$ in terms of the mean values $I^{k-1}_{b,k_j b}(X)$ $(0 \leqslant j \leqslant r)$. We achieve this goal by initiating this process with Lemma 6.1, and then iterate the application of Lemma 6.2. Before announcing our basic asymptotic estimate, we recall the definition (6.16) and then define the exponents

$$\phi_j = (s-k+1)/(s_{j-1}s_j) \quad (1 \leqslant j \leqslant r). \tag{7.1}$$

In addition, we write

$$\phi_0 = (k-r-1)/s_0 \quad \text{and} \quad \phi^* = (s-k+1)/s, \tag{7.2}$$

so that

$$\begin{aligned} \phi^* + \sum_{j=0}^r \phi_j &= \frac{s-k+1}{s} + \frac{k-r-1}{s-r} + (s-k+1)\sum_{j=1}^r (s_{j-1}^{-1} - s_j^{-1}) \\ &= \frac{k-r-1}{s-r} + \frac{s-k+1}{s-r} = 1. \end{aligned} \tag{7.3}$$

Notice here that ϕ_j is roughly equal to $1/s$ for $1 \leqslant j \leqslant r$. With this in mind, the reader will find that the conclusion of our next lemma is an approximate analogue of the formula presented in the display preceding [17, equation (11.3)], a key element in the heuristic argument that inspired our present work.

Lemma 7.1. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant \theta^{-1}$ and $b \geqslant (r-1)a$. Then one has*

$$K_{a,b}^{k-1}(X) \ll M^{\mu' b+\nu' a} \left(J_{\mathfrak{w}}(X/M^b)\right)^{\phi^*} \prod_{j=0}^{r} \left(I_{b,k_jb}^{k-1}(X)\right)^{\phi_j},$$

where

$$\mu' = \mu + sk_0(k_0-1)/s_0 + s(s-k+1)\sum_{j=1}^{r} k_j/(s_{j-1}s_j) \tag{7.4}$$

and

$$\nu' = \nu - s(k_0-1)/s_0 + (s-k+1)\sum_{j=1}^{r} s_{j-1}^{-1}\left(r-j-s/s_j\right). \tag{7.5}$$

Proof. We prove by induction that for $0 \leqslant l \leqslant r$, one has

$$K_{a,b}^{k-1}(X) \ll M^{\mu_l b+\nu_l a} \left(K_{a,b}^{r-l}(X)\right)^{\phi_l^*} \prod_{j=0}^{l} \left(I_{b,k_jb}^{k-1}(X)\right)^{\phi_j}, \tag{7.6}$$

where

$$\phi_l^* = (s-k+1)/s_l,$$

$$\mu_l = \mu + sk_0(k_0-1)/s_0 + s(s-k+1)\sum_{j=1}^{l} k_j/(s_{j-1}s_j)$$

and

$$\nu_l = \nu - s(k_0-1)/s_0 + (s-k+1)\sum_{j=1}^{l} s_{j-1}^{-1}(r-j-s/s_j).$$

The conclusion of the lemma follows from the case $l=r$ of (7.6), on noting that Lemma 2.1 delivers the estimate $K_{a,b}^0(X) \ll J_{\mathfrak{w}}(X/M^b)$.

We observe first that the inductive hypothesis (7.6) holds when $l=0$, as a consequence of Lemma 6.1, definition (6.16), and the familiar convention that an empty sum is zero. Suppose then that J is a positive integer not exceeding r, and that the inductive hypothesis (7.6) holds for $0 \leqslant l < J$. An application of Lemma 6.2 supplies the estimate

$$K_{a,b}^{r-J+1}(X) \ll M^{(r-J)a} \left((M^{k_Jb-a})^s I_{b,k_Jb}^{k-1}(X)\right)^{1/s_J} \left(K_{a,b}^{r-J}(X)\right)^{s_{J-1}/s_J}.$$

On substituting this bound into the estimate (7.6) with $l=J-1$, one obtains the new upper bound

$$K_{a,b}^{k-1}(X) \ll M^{\Omega} \left(K_{a,b}^{r-J}(X)\right)^{\phi_J^*} \prod_{j=0}^{J} \left(I_{b,k_jb}^{k-1}(X)\right)^{\phi_j},$$

where

$$\Omega = \mu_{J-1}b + \nu_{J-1}a + ((r-J)a + s(k_Jb-a)/s_J)\,\phi^*_{J-1}.$$

Since

$$\mu_J = \mu_{J-1} + s(s-k+1)k_J/(s_{J-1}s_J),$$

and

$$\nu_J = \nu_{J-1} + (s-k+1)s_{J-1}^{-1}\left(r-J-s/s_J\right),$$

we find that the estimate (7.6) holds with $l=J$, completing the proof of the inductive step. In view of our earlier remarks, the conclusion of the lemma now follows. □

We next recall the anticipated magnitude operator $[[\,\cdot\,]]_{\mathfrak{d},\rho}$ defined in equations (2.23) to (2.25), and convert Lemma 7.1 into a more portable form. Before announcing our conclusions, we recall the definition (2.27) of $\Lambda(\Delta)$.

Lemma 7.2. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant \theta^{-1}$ and $b \geqslant (r-1)a$. Then one has*

$$[[K_{a,b}^{k-1}(X)]]_{\Delta^\dagger,1} \ll \left((X/M^b)^{\Lambda(\Delta^\dagger)+\delta}\right)^{\phi^*} \prod_{j=0}^r [[I_{b,k_jb}^{k-1}(X)]]_{\Delta^\dagger,1}^{\phi_j},$$

where

$$\Delta^\dagger = \sum_{m=1}^r \frac{(m-1)(k-m-1)}{s-m}.$$

Proof. In order to promote greater transparency, we begin by utilising the operator $[[\,\cdot\,]]_{\Delta^\dagger,0}$, only later translating our statements into analogues for the operator $[[\,\cdot\,]]_{\Delta^\dagger,1}$. Write $\Lambda^\dagger = \Lambda(\Delta^\dagger)$,

$$\kappa = 2k - 2 - \tfrac{1}{2}k(k+1) + \Delta^\dagger, \tag{7.7}$$

and define μ' and ν' as in (7.4) and (7.5). Then we find from Lemma 7.1 in combination with (2.11) that

$$[[K_{a,b}^{k-1}(X)]]_{\Delta^\dagger,0} \ll M^{\mu^*b+\nu^*a}\left((X/M^b)^{\Lambda^\dagger+\delta}\right)^{\phi^*} \prod_{j=0}^r [[I_{b,k_jb}^{k-1}(X)]]_{\Delta^\dagger,0}^{\phi_j}, \tag{7.8}$$

where

$$\mu^* = \mu' + 2s - \phi^*(\kappa+2s) - \sum_{j=0}^r \phi_j(\kappa + 2sk_j) \quad \text{and} \quad \nu^* = \nu' + \kappa.$$

On recalling (7.1) to (7.5), we find that

$$\mu^* = \mu + 2k - 2 - \kappa - sk_0(k_0-1)s_0^{-1} - s(s-k+1)\sum_{j=1}^r k_j/(s_{j-1}s_j).$$

But in view of (6.16), one has

$$\sum_{j=1}^r \frac{(s-k+1)k_j}{s_{j-1}s_j} = \sum_{j=1}^r \left(\frac{k_{j-1}k_j}{s_j} - \frac{k_{j-2}k_{j-1}}{s_{j-1}} - \frac{k_{j-2}}{s_{j-1}} \right)$$
$$= \frac{k_{r-1}k_r}{s_r} - \frac{k_0(k_0-1)}{s_0} - \sum_{j=1}^r \frac{k_{j-2}}{s_{j-1}}. \tag{7.9}$$

Thus, again recalling (6.16) and making the change of variable $m = r - j + 1$, we discern that

$$\mu^* = \mu + \tfrac{1}{2}k(k+1) - \Delta^\dagger - k(k-1) + \sum_{m=1}^r \frac{s(k-m-1)}{s-m}.$$

Consequently, by reference now to (3.4) and the definition of $\Delta^\dagger$, we obtain

$$\mu^* = \tfrac{1}{2}(k-r-1)(k-r-2) - \tfrac{1}{2}k(k-3) + \sum_{m=1}^r \left(k-m-1 + \frac{k-m-1}{s-m} \right),$$

whence

$$\mu^* = 1 + \sum_{m=1}^r \frac{k-m-1}{s-m}. \tag{7.10}$$

Observe next that

$$\sum_{j=1}^r \frac{s-k+1}{s_{j-1}s_j} = \sum_{j=1}^r \left(\frac{k_{j-1}}{s_j} - \frac{k_{j-2}}{s_{j-1}} \right) = \frac{k_{r-1}}{s_r} - \frac{k_0 - 1}{s_0}.$$

Thus, in view of (6.16), it follows from (7.5) that

$$\nu^* = \nu + \kappa - (k-1) + (s-k+1) \sum_{j=1}^r \frac{r-j}{s-r+j-1}.$$

On making the change of variable $m = r-j+1$ once again, we therefore obtain

$$\nu^* = \nu + \kappa - (k-1) + \sum_{m=1}^r \left(m - 1 - \frac{(m-1)(k-m-1)}{s-m} \right).$$

In this way, now invoking (3.4), (7.7) and the definition of $\Delta^\dagger$, we arrive at the relation

$$\nu^* = \tfrac{1}{2}(k-r-1)(k+r-2) + k - 1 - \tfrac{1}{2}k(k+1) + \tfrac{1}{2}r(r-1) = -k. \tag{7.11}$$

At this point we note from (2.24) that

$$[[I_{b,k_jb}^{k-1}(X)]]_{\Delta^\dagger,1} = M^{(k-k_j)b}[[I_{b,k_jb}^{k-1}(X)]]_{\Delta^\dagger,0}.$$

Furthermore, in view of (6.16), (7.1) and (7.2), one has the relation

$$\sum_{j=1}^r (k-k_j)\phi_j = (s-k+1)\sum_{j=1}^r \frac{r-j}{s_{j-1}s_j}$$
$$= \sum_{j=1}^r \left(\frac{(r-j)k_{j-1}}{s_j} - \frac{(r-j+1)k_{j-2}}{s_{j-1}} + \frac{k_{j-2}}{s_{j-1}} \right),$$

whence

$$\sum_{j=0}^r (k-k_j)\phi_j = \frac{r(k_0-1)}{s_0} + \sum_{j=1}^r (k-k_j)\phi_j = \sum_{m=1}^r \frac{k-m-1}{s-m}.$$

Thus, collecting together our formulae (7.10) and (7.11) for μ^* and ν^* within (7.8), we obtain the upper bound

$$[[K_{a,b}^{k-1}(X)]]_{\Delta^\dagger,0} \ll M^{b-ka}\left((X/M^b)^{\Lambda^\dagger+\delta} \right)^{\phi^*} \prod_{j=0}^r [[I_{b,k_jb}^{k-1}(X)]]_{\Delta^\dagger,1}^{\phi_j}.$$

The conclusion of the lemma now follows from (2.25), since the latter implies the relation

$$M^{ka-b}[[K_{a,b}^{k-1}(X)]]_{\Delta^\dagger,0} = [[K_{a,b}^{k-1}(X)]]_{\Delta^\dagger,1}.$$

□

Our penultimate result in this section is a reconfiguration of Lemma 7.2 that facilitates an alternative bound equipped with equally weighted exponents. In this context, we recall the definitions (2.8) and (2.27) of $\Delta_1(\nu)$ and $\Lambda(\Delta)$.

Lemma 7.3. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant \theta^{-1}$ and $b \geqslant (r-1)a$. Then one has*

$$[[K_{a,b}^{k-1}(X)]]_{\Delta,1} \ll \left((X/M^b)^{\Lambda+\delta} \right)^{(s-k+1)/s} [[I_{b,k_0b}^{k-1}(X)]]_{\Delta,1}^{(k-r-1)/s} \prod_{j=1}^r [[I_{b,k_jb}^{k-1}(X)]]_{\Delta,1}^{1/s},$$

where $\Delta = \Delta_1(\nu)$ and $\Lambda = \Lambda(\Delta_1(\nu))$.

Proof. Write

$$\Delta_\rho = \sum_{m=1}^\rho \frac{(m-1)(k-m-1)}{s-m}. \tag{7.12}$$

Then on recalling (2.24) and (2.25), we find that

$$[[I_{b,(k-\rho+j)b}^{k-1}(X)]]_{0,1} = (X/M^b)^{\Delta_\rho}[[I_{b,(k-\rho+j)b}^{k-1}(X)]]_{\Delta_\rho,1} \quad (0 \leqslant j \leqslant \rho) \tag{7.13}$$

and

$$[[K_{a,b}^{k-1}(X)]]_{0,1} = (X/M^a)^{\Delta_\rho}[[K_{a,b}^{k-1}(X)]]_{\Delta_\rho,1}. \tag{7.14}$$

By reference to (6.16) and (7.1) to (7.3), it therefore follows from Lemma 7.2 that for $0 \leqslant \rho \leqslant r$, one has

$$[[K_{a,b}^{k-1}(X)]]_{0,1} \ll (M^{b-a})^{\Delta_\rho} \left((X/M^b)^{\Lambda(0)+\delta}\right)^{\frac{s-k+1}{s}} [[I_{b,(k-\rho)b}^{k-1}(X)]]_{0,1}^{\frac{k-\rho-1}{s-\rho}} \times \prod_{j=1}^{\rho} [[I_{b,(k-\rho+j)b}^{k-1}(X)]]_{0,1}^{\frac{s-k+1}{(s-\rho+j-1)(s-\rho+j)}}. \tag{7.15}$$

By applying this bound successively for $\rho = r, r-1, \dots, 0$, we deduce that

$$\begin{aligned}[[K_{a,b}^{k-1}(X)]]_{0,1} &= [[K_{a,b}^{k-1}(X)]]_{0,1}^{(s-r)/s} \prod_{\rho=0}^{r-1} [[K_{a,b}^{k-1}(X)]]_{0,1}^{1/s} \\ &\ll (M^{b-a})^{\Delta^*/s} \left((X/M^b)^{\Lambda(0)+\delta}\right)^{\frac{s-k+1}{s}} [[I_{b,(k-r)b}^{k-1}(X)]]_{0,1}^{(k-r-1)/s} \\ &\quad \times \prod_{j=1}^{r} [[I_{b,(k-r+j)b}^{k-1}(X)]]_{0,1}^{\omega_j/s},\end{aligned}$$

where

$$\Delta^* = (s-r)\Delta_r + \sum_{\rho=0}^{r-1} \Delta_\rho$$

and

$$\omega_j = \frac{(s-r)(s-k+1)}{(s-r+j-1)(s-r+j)} + \frac{k-r+j-1}{s-r+j} + \sum_{\rho=r-j+1}^{r-1} \frac{s-k+1}{(s-r+j-1)(s-r+j)}.$$

We observe first that, as a consequence of (7.12), one has

$$\begin{aligned}\Delta^* &= (s-r)\sum_{m=1}^{r} \frac{(m-1)(k-m-1)}{s-m} + \sum_{\rho=0}^{r-1}\sum_{m=1}^{\rho} \frac{(m-1)(k-m-1)}{s-m} \\ &= (s-r)\sum_{m=1}^{r} \frac{(m-1)(k-m-1)}{s-m} + \sum_{m=1}^{r} \frac{(m-1)(k-m-1)}{s-m} \sum_{\rho=m}^{r-1} 1 \\ &= \sum_{m=1}^{r} (m-1)(k-m-1).\end{aligned}$$

Recalling the definition (2.8) of $\Delta_1(\nu)$, therefore, we conclude that $\Delta^* = s\Delta$. Also, one sees that

$$\omega_j = \frac{s-k+1}{s-r+j} + \frac{k-r+j-1}{s-r+j} = 1.$$

Thus we infer that

$$M^{\Delta a}[[K_{a,b}^{k-1}(X)]]_{0,1} \ll M^{\Delta b}\left((X/M^b)^{\Lambda(0)+\delta}\right)^{\frac{s-k+1}{s}}[[I_{b,(k-r)b}^{k-1}(X)]]_{0,1}^{(k-r-1)/s} \times \prod_{j=1}^{r}[[I_{b,(k-r+j)b}^{k-1}(X)]]_{0,1}^{1/s},$$

and the conclusion of the lemma follows on recalling the relations (7.13) and (7.14), though now with Δ_ρ replaced by Δ. $\square$

Finally, we extract from Lemma 7.2 a conclusion related to that of Lemma 7.3, but one that makes available estimates utilising the full power underlying our methods. In this context, it is useful to observe that our analysis of the iteration process in §9 requires that we work with an integral number of variables. The interpolation residing in the next lemma addresses this requirement. Before announcing this refinement, we recall the definitions (2.5) and (2.7) of $\nu_0(r,s)$ and $\Delta_0(\nu)$, and also the definition (7.1) of ϕ_j.

Lemma 7.4. *Let r and ν be integers with $r \geqslant 2$ and $0 \leqslant \nu \leqslant \nu_0(r,s)$, and put*

$$\sigma = \frac{(\nu_0(r,s)-\nu)(s-r)}{(k-r-1)s}.$$

Suppose that a and b are integers with $0 \leqslant a < b \leqslant \theta^{-1}$ and $b \geqslant (r-1)a$. Then one has

$$[[K_{a,b}^{k-1}(X)]]_{\Delta,1} \ll \left((X/M^b)^{\Lambda+\delta}\right)^{(s-k+1)/s}\prod_{j=0}^{r}[[I_{b,k_jb}^{k-1}(X)]]_{\Delta,1}^{\phi_j(\sigma)},$$

where $\Delta = \Delta_0(\nu)$, $\Lambda = \Lambda(\Delta_0(\nu))$, and

$$\phi_0(\sigma) = \frac{(k-r-1)(1-\sigma)}{s-r}, \tag{7.16}$$

$$\phi_1(\sigma) = \frac{(s-k+1)(1-\sigma)}{(s-r)(s-r+1)} + \frac{(k-r)\sigma}{s-r+1}, \tag{7.17}$$

$$\phi_j(\sigma) = \phi_j \quad (2 \leqslant j \leqslant r). \tag{7.18}$$

Proof. We adapt the argument of the proof of Lemma 7.3, making use of the notation (7.12) and the relations (7.13) to (7.15). By reference to (2.4) and (2.5), one finds that when $k \geqslant 4$, one has

$$\sigma \leqslant \frac{s-r}{(k-r-1)s}\sum_{m=1}^{r}\frac{m(k-m-1)}{s-m} \leqslant \sum_{1\leqslant m\leqslant (k-1)/2}\frac{k-m-1}{s} + \frac{(k+1)/2}{s}.$$

In this way, it is easily verified that when $k \geqslant 3$, the parameter σ satisfies $0 \leqslant \sigma < 1$, and hence, by applying (7.15) for $\rho = r$ and $r-1$, we deduce that

$$\begin{aligned}[][[K_{a,b}^{k-1}(X)]]_{0,1} &= [[K_{a,b}^{k-1}(X)]]_{0,1}^{1-\sigma}[[K_{a,b}^{k-1}(X)]]_{0,1}^{\sigma}\\ &\ll (M^{b-a})^{\Delta^*}\left((X/M^b)^{\Lambda(0)+\delta}\right)^{\frac{s-k+1}{s}}\prod_{j=0}^{r}[[I_{b,(k-r+j)b}^{k-1}(X)]]_{0,1}^{\phi_j(\sigma)},\end{aligned}$$

where

$$\Delta^* = (1-\sigma)\Delta_r + \sigma\Delta_{r-1}.$$

By making use of the definitions (2.5) and (2.7), we find that

$$\Delta^* = \Delta_r - \left(\frac{(\nu_0(r,s)-\nu)(s-r)}{(k-r-1)s}\right)\left(\frac{(r-1)(k-r-1)}{s-r}\right) = \Delta_0(\nu).$$

The conclusion of the lemma follows on recalling the relations (7.13) and (7.14), though in this instance we replace Δ_ρ by the value of Δ given in the statement of the lemma. □

8. The latent monograde process

The estimates supplied by Lemmata 7.2, 7.3 and 7.4 could, in principle, be applied in an iterative manner so as to bound $K_{a,b}^{k-1}(X)$ in terms of the $r+1$ mean values $K_{b,k_jb}^{k-1}(X)$ $(0 \leqslant j \leqslant r)$, each of which could be bounded in terms of $r+1$ new mean values of the shape $K_{b',k_jb'}^{k-1}(X)$ $(0 \leqslant j \leqslant r)$, and so on. This, indeed, is the strategy proposed in the speculative heuristic argument described in [17, §11]. After N iterations, one then has a bound for $K_{a,b}^{k-1}(X)$ in terms of $(r+1)^N$ new mean values, and one is left with the task of analysing the consequences of this iteration. Since we have yet to take account of the need to condition the mean values $I_{b,k_jb}^{k-1}(X)$ $(0 \leqslant j \leqslant r)$ occurring as intermediate steps in this process, the complexity of this analysis would be formidable indeed. Fortunately, we are able to make use of a simplified analysis by focusing attention on just one of the $r+1$ mean values at each stage, this being achieved by applying a weighted version of an estimate related to Hölder's inequality (see Lemma 8.1 below). In this way, the complicated product of mean values produced by Lemma 7.4 is bounded in terms of a sum of mean values, and one may then focus on the single summand which is maximal. Thus, it transpires that one may convert the multigrade iteration into a monograde process that loses none of the potential of a full-blown analysis.

Lemma 8.1. *Suppose that $z_0,\dots,z_r \in \mathbb{C}$, and that β_i and γ_i are positive real numbers for $0 \leqslant i \leqslant r$. Put $\Omega = \beta_0\gamma_0 + \ldots + \beta_r\gamma_r$. Then one has*

$$|z_0^{\beta_0}\dots z_r^{\beta_r}| \leqslant \sum_{i=0}^{r}|z_i|^{\Omega/\gamma_i}.$$

Proof. We apply the elementary inequality

$$|Z_0^{\theta_0}\dots Z_r^{\theta_r}|\leqslant \sum_{i=0}^r |Z_i|^{\theta_0+\dots+\theta_r}.$$

Thus, on taking $Z_i=z_i^{1/\gamma_i}$ and $\theta_i=\beta_i\gamma_i$ for $0\leqslant i\leqslant r$, we obtain the bound

$$\prod_{i=0}^r |z_i^{1/\gamma_i}|^{\beta_i\gamma_i}\leqslant \sum_{i=0}^r (|z_i|^{1/\gamma_i})^\Omega.$$

This completes the proof of the lemma. □

Before announcing the lemma that encodes the latent monograde iteration process, we recall the definitions (2.6) to (2.8), put $s_\iota^*=s_\iota(\nu)$, and define $\rho_j=\rho_{j,\iota}(k,r,s)$ by

$$\rho_j=k_js/s_\iota^*\quad (\iota=0,1). \tag{8.1}$$

Note also that, as in all of the work of §§2–9, we assume throughout that the parameter r satisfies the condition (2.4), and also that $s\geqslant s_\iota^*$.

Lemma 8.2. *Let ι be either 0 or 1, and put $\Delta=\Delta_\iota(\nu)$ and $\Lambda=\Lambda(\Delta_\iota(\nu))$. Suppose that $\Lambda\geqslant 0$, and let a and b be integers with*

$$0\leqslant a<b\leqslant (32k\theta)^{-1}\quad \textit{and}\quad b\geqslant (r-1)a.$$

Suppose in addition that there are real numbers ψ, c and γ, with

$$0\leqslant c\leqslant (2\delta)^{-1}\theta,\quad \gamma\geqslant -b\quad \textit{and}\quad \psi\geqslant 0,$$

such that

$$X^\Lambda M^{\Lambda\psi}\ll X^{c\delta}M^{-\gamma}[[K_{a,b}^{k-1}(X)]]_{\Delta,1}. \tag{8.2}$$

Then, for some integers j and h with $0\leqslant j\leqslant r$ and $0\leqslant h\leqslant 15(k_j-1)b$, one has the upper bound

$$X^\Lambda M^{\Lambda\psi'}\ll X^{c'\delta}M^{-\gamma'}[[K_{b,k_jb+h}^{k-1}(X)]]_{\Delta,1},$$

where

$$\psi'=\rho_j\left(\psi+(1-(k-1)/s)\,b\right),\quad c'=\rho_j(c+1),\quad \gamma'=\rho_j\gamma+(\tfrac{4}{3}s-1)h.$$

Proof. We begin by establishing the lemma when $\iota=0$, the corresponding argument for $\iota=1$ being analogous, though simpler. By hypothesis, we have $X^{c\delta}<M^{1/2}$. We therefore deduce from the postulated bound (8.2) and Lemma 7.4 that

$$X^\Lambda M^{\Lambda\psi}\ll X^{(c+1)\delta}M^{-\gamma}(X/M^b)^{\Lambda\phi^*}\prod_{j=0}^r [[I_{b,k_jb}^{k-1}(X)]]_{\Delta,1}^{\phi_j(\sigma)},$$

where the exponents ϕ^* and $\phi_j(\sigma)$ are defined by means of (7.2) and (7.16) to (7.18). Thus, on verifying that $\phi_0(\sigma)+\phi_1(\sigma)=\phi_0+\phi_1$, and then making use of (7.3), we deduce that

$$\prod_{j=0}^r \Bigl(X^{-\Lambda}[[I_{b,k_jb}^{k-1}(X)]]_{\Delta,1}\Bigr)^{\phi_j(\sigma)}\gg X^{-(c+1)\delta}M^{\Lambda(\psi+\phi^*b)+\gamma}. \tag{8.3}$$

We next prepare for our application of Lemma 8.1, but we first examine a related situation. Put $\beta_j = \phi_j$ and $\gamma_j = k_j$ for $0 \leqslant j \leqslant r$. We write $\Omega_r = \beta_0\gamma_0 + \ldots + \beta_r\gamma_r$, and recall the relation (7.9). Then we find from (6.16), (7.1) and (7.2) that

$$\Omega_r = \frac{k_0(k_0-1)}{s_0} + \sum_{j=1}^{r} \frac{(s-k+1)k_j}{s_{j-1}s_j} = \frac{k_{r-1}k_r}{s_r} - \sum_{j=1}^{r} \frac{k_{j-2}}{s_{j-1}}.$$

Thus, on making the change of variable $m = r-j+1$, and referring once more to (6.16), we find that

$$\begin{aligned} s\Omega_r &= k(k-1) - \sum_{m=1}^{r}(k-m-1) - \sum_{m=1}^{r} \frac{m(k-m-1)}{s-m} \\ &= k^2 - (r+1)k + \tfrac{1}{2}r(r+3) - \sum_{m=1}^{r} \frac{m(k-m-1)}{s-m}. \end{aligned}$$

On writing Ω_{r-1} for the analogue of Ω_r in which r is replaced by $r-1$, therefore, we find that

$$s\left((1-\sigma)\Omega_r + \sigma\Omega_{r-1}\right) = k^2 - (r+1)k + \tfrac{1}{2}r(r+3) - \nu^+,$$

where

$$\nu^+ = \sum_{m=1}^{r} \frac{m(k-m-1)}{s-m} - \sigma\left(k-r-1+\frac{r(k-r-1)}{s-r}\right).$$

By reference to (2.5) and the definition of σ from Lemma 7.4, we find that

$$\nu^+ = \nu_0(r,s) - \frac{(\nu_0(r,s)-\nu)(s-r)}{s}\left(1+\frac{r}{s-r}\right) = \nu,$$

and hence it follows from (2.6) that

$$s\left((1-\sigma)\Omega_r + \sigma\Omega_{r-1}\right) = s_0(\nu).$$

With the last equation in hand, we now write $\beta_j = \phi_j(\sigma)$ and $\gamma_j = k_j$ for $0 \leqslant j \leqslant r$, and put $\Omega = \beta_0\gamma_0 + \ldots + \beta_r\gamma_r$. Then we find from (7.1), (7.2) and (7.16) to (7.18) that

$$\Omega = (1-\sigma)\Omega_r + \sigma\Omega_{r-1} = s_0(\nu)/s.$$

A comparison with (8.1) therefore reveals that

$$\Omega/k_j = s_0^*/(sk_j) = 1/\rho_j.$$

By wielding Lemma 8.1 against (8.3), we thus deduce that

$$\sum_{j=0}^{r} \left(X^{-\Lambda}[[I_{b,k_jb}^{k-1}(X)]]_{\Delta,1}\right)^{1/\rho_j} \gg X^{-(c+1)\delta} M^{\Lambda(\psi+\phi^*b)+\gamma}.$$

Consequently, for some index j with $0 \leqslant j \leqslant r$, one has

$$X^{-\Lambda}[[I_{b,k_jb}^{k-1}(X)]]_{\Delta,1} \gg X^{-\rho_j(c+1)\delta} M^{\Lambda\rho_j(\psi+\phi^*b)+\rho_j\gamma},$$

whence, by reference to (7.2), we conclude that

$$X^\Lambda M^{\Lambda\psi'} \ll X^{c'\delta} M^{-\rho_j\gamma} [[I^{k-1}_{b,k_jb}(X)]]_{\Delta,1}. \tag{8.4}$$

We fix the integer j so that the upper bound (8.4) holds, and write $H = 15(k_j-1)b$. The estimate (8.4) comes close to achieving the bound claimed in the conclusion of the lemma, though it remains to condition the mean value $I^{k-1}_{b,k_jb}(X)$ so as to replace it with a suitable mean value of the form $K^{k-1}_{b,c}(X)$. As a consequence of Lemma 4.2, there exists an integer h with $0 \leqslant h < H$ such that

$$I^{k-1}_{b,k_jb}(X) \ll (M^h)^{2s/3} K^{k-1}_{b,k_jb+h}(X) + (M^H)^{-s/4}(X/M^{k_jb})^{2s}(X/M^b)^{\lambda-2s}.$$

Then in view of (6.16), we infer from (2.24) and (2.25) that

$$[[I^{k-1}_{b,k_jb}(X)]]_{\Delta,1} \ll M^\omega [[K^{k-1}_{b,k_jb+h}(X)]]_{\Delta,1} + (M^H)^{-s/4} M^{(r-j)b} X^\Lambda,$$

where

$$\omega = h + 2sh/3 - 2sh = (1-4s/3)h.$$

Substituting this estimate into (8.4), therefore, we deduce that

$$X^\Lambda M^{\Lambda\psi'} \ll X^{c'\delta} M^{-\gamma'} [[K^{k-1}_{b,k_jb+h}(X)]]_{\Delta,1} + \Psi, \tag{8.5}$$

where

$$\Psi = (M^H)^{-s/4} M^{(r-j)b-\rho_j\gamma} X^{\Lambda+c'\delta}. \tag{8.6}$$

We now set about analysing the term Ψ, and seek to show that it makes a negligible contribution in (8.5). Observe that from (2.4) and (2.5), one has

$$\nu_0(r,s) \leqslant \frac{r(r+1)(k-2)/2}{k(k+1)/2} < \tfrac{1}{2}r,$$

so that (2.6) delivers the lower bound

$$s_0^* \geqslant k^2 - (r+1)k + \tfrac{1}{2}r(r+2) > (k - \tfrac{1}{2}(r+1))^2 - 1 \geqslant k.$$

Our ambient hypotheses ensure also that $k_j \geqslant 2$, and thus $H = 15(k_j-1)b \geqslant 15b$. We therefore obtain the lower bound

$$\tfrac{1}{4}Hs \geqslant 3bs \geqslant 3kbs/s_0^*.$$

Meanwhile, from (6.16) and (8.1), one has

$$\begin{aligned}\rho_j + (r-j)b - \rho_j\gamma &\leqslant (2k_j + (r-j)s_0^*/s)\, bs/s_0^* \\ &\leqslant 2kbs/s_0^* < \tfrac{1}{4}Hs.\end{aligned}$$

Since our hypotheses ensure also that

$$c'\delta \leqslant \rho_j(c+1)\delta \leqslant \rho_j\theta - \delta,$$

we conclude from (8.6) that

$$\Psi \ll (M^H)^{-s/4} M^{(r-j)b-\rho_j\gamma+\rho_j} X^{\Lambda-\delta} \ll X^{-\delta}\left(X^\Lambda M^{\Lambda\psi'}\right).$$

The conclusion of the lemma for $\iota = 0$ follows by substituting this estimate into (8.5).

We now turn to the situation with $\iota = 1$. We proceed as before, but now deduce from Lemma 7.3 that the lower bound (8.3) holds with the exponents $\phi_j(\sigma)$ in this instance modified so that

$$\phi_0(\sigma) = (k-r-1)/s \quad \text{and} \quad \phi_j(\sigma) = 1/s \quad (1 \leqslant j \leqslant r).$$

With this modification in hand, and β_j and γ_j defined in the same manner as before, we find by reference to (2.5) and (2.6) that

$$s\Omega = (k-r-1)(k-r) + \sum_{j=1}^{r}(k-r+j) = s_1^*.$$

Thus, with the modified definitions automatically implied by our shift from $\iota = 0$ to $\iota = 1$, one finds that (8.4) holds also in this case. From here we may follow precisely the same argument as in the case $\iota = 0$, delivering again the conclusion of the lemma in this second case. □

9. The iterative process

In common with our previous efficient congruencing methods, the conclusion of Lemma 8.2 provides the basis for a concentration argument. Thus, if the mean value $K_{a,b}^{k-1}(X)$ is significantly larger than its "expected" magnitude, then for some index j and a suitable non-negative integer h, the related mean value $K_{b,k_jb+h}^{k-1}(X)$ exceeds its "expected" magnitude by an even larger margin. By iterating this process, we amplify this excess to the point that we obtain a contradiction. Since Lemma 5.1 bounds $J_{\mathfrak{w}}(X)$ in terms of $K_{0,1+h}^{k-1}(X)$, for some $h \in \{0,1,2,3\}$, we are able to infer that $J_{\mathfrak{w}}(X)$ is very close to its "expected" magnitude. The main difficulty we face in this paper, as opposed to previous work [5, 17, 19], is that the modulus amplification factor k_j varies from one iteration to the next. Fortunately, with care, our previous analyses may be adapted to accommodate this complication. We begin by recalling a crude upper bound for $K_{a,b}^{k-1}(X)$.

Lemma 9.1. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant (2\theta)^{-1}$, and let $\Lambda = \Lambda(\Delta)$. Then provided that $\Lambda \geqslant 0$, one has*

$$[[K_{a,b}^{k-1}(X)]]_{\Delta,1} \ll X^{\Lambda+\delta}(M^{b-a})^s M^{ka-b}.$$

Proof. The desired conclusion follows by the argument applied in the proof of [5, Lemma 5.3], on noting that $[[K_{a,b}^{k-1}(X)]]_{\Delta,1} = M^{ka-b}[[K_{a,b}^{k-1}(X)]]_{\Delta,0}$. □

We can now announce a mean value estimate for $J_{\mathfrak{w}}(X)$ that in many circumstances is a little sharper than that recorded in Theorem 1.1. Since this estimate is likely to be of use in future applications, as indeed is the case in §§10–12 of this paper, we deliberately opt for a relatively transparent form.

Theorem 9.2. *Suppose that k, r and s are natural numbers with $k \geqslant 3$,*

$$r \leqslant \min\{k-2, \tfrac{1}{2}(k+1)\} \quad \textit{and} \quad s \geqslant s_0,$$

where

$$s_0 = k^2 - (r+1)k + \tfrac{1}{2}r(r+3) - \nu_0(r,s)$$

and

$$\nu_0(r,s)=\sum_{m=1}^r \frac{m(k-m-1)}{s-m}.$$

Put

$$\nu=\max\{k^2-(r+1)k+\tfrac{1}{2}r(r+3)-s,0\}.$$

Then for each $\varepsilon>0$*, one has*

$$J_{s+k-1}(X)\ll X^{2s+2k-2-\frac{1}{2}k(k+1)+\Delta+\varepsilon},$$

where

$$\Delta=\sum_{m=1}^r \frac{(m-1)(k-m-1)}{s-m}-\frac{(\nu_0(r,s)-\nu)(r-1)}{s}.$$

Proof. Write $\Lambda=\Lambda(\Delta)$, and note that $\Delta=\Delta_0(\nu)$. We prove that $\Lambda\leqslant 0$, for the conclusion of the lemma then follows at once from (2.26). Recall the definition (2.6). Then we may suppose also that $s=s_0(\nu)$, for if $t>\mathfrak{w}$, then a trivial estimate delivers the estimate $J_t(X)\ll X^{2(t-\mathfrak{w})}J_{\mathfrak{w}}(X)$, and thus the desired conclusion follows from the upper bound provided by the theorem in the case $s=s_0(\nu)$. Assume then that $\Lambda\geqslant 0$, for otherwise there is nothing to prove. We begin by noting that as a consequence of Lemma 5.1, one finds from (2.23) and (2.25) that there exists an integer $h_{-1}\in\{0,1,2,3\}$ such that

$$\begin{aligned}[][[J_{\mathfrak{w}}(X)]]_\Delta &\ll (M^{h_{-1}})^{-4s/3}[[K^{k-1}_{0,1+h_{-1}}(X)]]_{\Delta,0}\\ &\ll M^{1-(4s/3-1)h_{-1}}[[K^{k-1}_{0,1+h_{-1}}(X)]]_{\Delta,1}.\end{aligned}$$

We therefore deduce from (2.26) that

$$X^\Lambda\ll X^\delta[[J_{\mathfrak{w}}(X)]]_\Delta\ll X^\delta M(M^{h_{-1}})^{-(4s/3-1)}[[K^{k-1}_{0,1+h_{-1}}(X)]]_{\Delta,1}. \tag{9.1}$$

Next we define sequences (κ_n), (h_n), (a_n), (b_n), (c_n), (ψ_n) and (γ_n), for $0\leqslant n\leqslant N$, in such a way that

$$k-r\leqslant \kappa_{n-1}\leqslant k,\quad 0\leqslant h_{n-1}\leqslant 15(\kappa_{n-1}-1)b_{n-1}\quad (n\geqslant 1), \tag{9.2}$$

and

$$X^\Lambda M^{\Lambda\psi_n}\ll X^{c_n\delta}M^{-\gamma_n}[[K^{k-1}_{a_n,b_n}(X)]]_{\Delta,1}. \tag{9.3}$$

Given a fixed choice for the sequences (κ_n) and (h_n), the remaining sequences are defined by means of the relations

$$a_{n+1}=b_n,\quad b_{n+1}=\kappa_nb_n+h_n, \tag{9.4}$$

$$c_{n+1}=\kappa_n(c_n+1), \tag{9.5}$$

$$\psi_{n+1}=\kappa_n\psi_n+\kappa_n(1-(k-1)/s)b_n, \tag{9.6}$$

$$\gamma_{n+1}=\kappa_n\gamma_n+(4s/3-1)h_n. \tag{9.7}$$

We put

$$\kappa_{-1}=k,\quad a_0=0,\quad b_{-1}=1,\quad b_0=1+h_{-1},$$

$$\psi_0=0,\quad c_0=1,\quad \gamma_0=(4s/3-1)h_{-1}-1,$$

so that both (9.2) and (9.3) hold with $n = 0$ as a consequence of our initial choice of h_{-1} together with (9.1). We prove by induction that for each non-negative integer n with $n < N$, the sequences $(\kappa_m)_{m=0}^n$ and $(h_m)_{m=-1}^n$ may be chosen in such a way that

$$0 \leqslant a_n < b_n \leqslant (32\kappa_n\theta)^{-1}, \quad b_n \geqslant (r-1)a_n, \tag{9.8}$$

$$\psi_n \geqslant 0, \quad \gamma_n \geqslant -b_n, \quad 0 \leqslant c_n \leqslant (2\delta)^{-1}\theta, \tag{9.9}$$

and so that (9.2) and (9.3) both hold with n replaced by $n+1$.

Let $0 \leqslant n < N$, and suppose also that (9.2) and (9.3) both hold for the index n. We have already shown such to be the case for $n = 0$. We observe first that the relation (9.4) demonstrates that $b_n > a_n$ for all n. Moreover, since our hypotheses on r ensure that $\kappa_n \geqslant k - r \geqslant r - 1$, it follows from (9.4) that one has $b_n \geqslant (r-1)a_n$. Also, from (9.2) and (9.4), we find that

$$b_n \leqslant 4 \cdot 16^n \kappa_0 \dots \kappa_{n-1} \leqslant 4(16k)^n,$$

whence, by invoking (2.9), we see that for $0 \leqslant n < N$ one has

$$b_n \leqslant (32k\theta)^{-1} \leqslant (32\kappa_n\theta)^{-1}.$$

It is apparent from (9.5) and (9.6) that c_n and ψ_n are non-negative for all n. Observe also that since $\kappa_m \leqslant k$, then by iterating (9.5) we obtain the bound

$$c_n \leqslant k^n + k\Big(\frac{k^n - 1}{k-1}\Big) \leqslant 3k^n \quad (n \geqslant 0), \tag{9.10}$$

and by reference to (2.9) we see that $c_n \leqslant (2\delta)^{-1}\theta$ for $0 \leqslant n < N$.

In order to bound γ_n, we begin by noting from (9.4) that for $m \geqslant 1$, one has

$$h_m = b_{m+1} - \kappa_m b_m \quad \text{and} \quad a_m = b_{m-1}.$$

Then it follows from (9.7) that for $m \geqslant 1$ one has

$$\gamma_{m+1} - (4s/3 - 1)b_{m+1} = \kappa_m \left(\gamma_m - (4s/3-1)b_m\right).$$

By iterating this relation, we deduce that for $m \geqslant 1$, one has

$$\gamma_m = (4s/3-1)b_m + \kappa_0 \dots \kappa_{m-1}(\gamma_0 - (4s/3-1)b_0).$$

Recall next that $b_0 = 1 + h_{-1}$ and $\gamma_0 = (4s/3-1)h_{-1} - 1$. Then we discern that

$$\gamma_m = (4s/3-1)b_m - (4s/3)\kappa_0 \dots \kappa_{m-1} \quad (m \geqslant 1). \tag{9.11}$$

Finally, we find from (9.4) that for $m \geqslant 0$ one has $b_{m+1} \geqslant \kappa_m b_m$, so that an inductive argument yields the lower bound $b_m \geqslant \kappa_0 \dots \kappa_{m-1}$ for $m \geqslant 0$. Hence we deduce that

$$\gamma_m \geqslant \tfrac{4}{3}s(b_m - \kappa_0 \dots \kappa_{m-1}) - b_m \geqslant -b_m.$$

Assembling this conclusion together with those of the previous paragraph, we have shown that both (9.8) and (9.9) hold for $0 \leqslant n \leqslant N$.

At this point in the argument, we may suppose that (9.3), (9.8) and (9.9) hold for the index n. An application of Lemma 8.2 therefore reveals that there exist integers κ_n and h_n satisfying the constraints implied by (9.2) with n replaced by $n+1$, for which the upper bound (9.3) holds also with n replaced by $n+1$. This completes the inductive step, so that in particular the upper bound (9.3) holds for $0 \leqslant n \leqslant N$.

We now exploit the bound just established. Since we have the upper bound $b_N \leqslant 4(16k)^N \leqslant (2\theta)^{-1}$, it is a consequence of Lemma 9.1 that

$$[[K^{k-1}_{a_N,b_N}(X)]]_{\Delta,1} \ll X^{\Lambda+\delta}(M^{b_N-b_{N-1}})^s M^{kb_{N-1}-b_N}. \tag{9.12}$$

By combining (9.3) with (9.11) and (9.12), we obtain the bound

$$\begin{aligned} X^\Lambda M^{\Lambda\psi_N} &\ll X^{\Lambda+(c_N+1)\delta} M^{kb_{N-1}-b_N+s(b_N-b_{N-1})-\gamma_N} \\ &\ll X^{\Lambda+(c_N+1)\delta} M^{-\frac{1}{3}sb_N-(s-k)b_{N-1}+\frac{4}{3}s\kappa_0\dots\kappa_{N-1}}. \end{aligned} \tag{9.13}$$

Meanwhile, an application of (9.10) in combination with (2.9) shows that $X^{(c_N+1)\delta} < M$. We therefore deduce from (9.13) and our previous lower bound $b_N \geqslant \kappa_0 \dots \kappa_{N-1}$ that

$$\Lambda\psi_N \leqslant \tfrac{4}{3}s\kappa_0\dots\kappa_{N-1} - \tfrac{1}{3}b_N s \leqslant s\kappa_0\dots\kappa_{N-1}.$$

Temporarily, we write $\chi = (k-1)/s$. Then a further application of the lower bound $b_n \geqslant \kappa_0 \dots \kappa_{n-1}$ leads from (9.6) to the relation

$$\psi_{n+1} = \kappa_n\psi_n + \kappa_n(1-\chi)b_n \geqslant \kappa_n\psi_n + \kappa_n(1-\chi)\kappa_0\dots\kappa_{n-1},$$

whence, by an inductive argument, one finds that

$$\psi_N \geqslant N(1-\chi)\kappa_0\dots\kappa_{N-1}.$$

Thus we deduce that

$$\Lambda \leqslant \frac{s\kappa_0\dots\kappa_{N-1}}{N(1-1/\chi)\kappa_0\dots\kappa_{N-1}} = \frac{s(1-1/\chi)^{-1}}{N}.$$

Since we are at liberty to take N as large as we please in terms of s and k, we are forced to conclude that $\Lambda \leqslant 0$. In view of our opening discussion, this completes the proof of the theorem. □

The proof of Theorem 1.1 follows by precisely the same argument as that employed to establish Theorem 9.2. We have merely to adjust the choice of parameters so that $s_0(\nu)$ and Δ are replaced by

$$s_1(0) = k^2 - (r+1)k + \tfrac{1}{2}r(r+3) \quad \text{and} \quad \Delta = \frac{1}{s}\sum_{m=1}^{r}(m-1)(k-m-1).$$

The argument of the proof then applies just as before, and when $s \geqslant s_1(0)$ one obtains the bound

$$J_{s+k-1}(X) \ll X^{2s+2k-2-\frac{1}{2}k(k+1)+\Delta+\varepsilon},$$

where, following a modest computation, one finds that

$$\Delta = \frac{r(r-1)(3k-2r-5)}{6s}.$$

The conclusion of Theorem 1.1 follows on replacing s in the bound just described by $s-k+1$ in the statement of the theorem.

10. The asymptotic formula in Waring's problem

Our first applications of the improved mean value estimates supplied by Theorems 1.1 and 9.2 concern the asymptotic formula in Waring's problem. In this section we establish Theorems 1.3 and 1.4, as well as a number of auxiliary estimates of use in related topics. In this context, we define the exponential sum $g(\alpha) = g_k(\alpha; X)$ by

$$g_k(\alpha; X) = \sum_{1 \leqslant x \leqslant X} e(\alpha x^k).$$

Also, we define the set of minor arcs $\mathfrak{m} = \mathfrak{m}_k$ to be the set of real numbers $\alpha \in [0, 1)$ satisfying the property that, whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|q\alpha - a| \leqslant (2k)^{-1} X^{1-k}$, then $q > (2k)^{-1} X$. We begin by applying the methods of [18] to derive a mean value estimate restricted to minor arcs.

The introduction of some additional notation eases our exposition. We define exponents $\nu_\iota^*(r, s)$ and $\Delta_\iota^*(r, s)$ consistent with the definitions (2.5), (2.7) and (2.8), save that we shift the parameter s by $k - 1$. Thus, we put

$$\nu_0^*(r, s) = \sum_{m=1}^{r} \frac{m(k - m - 1)}{s - k - m + 1} \quad \text{and} \quad \nu_1^*(r, s) = 0.$$

We then take ν to be an integer with $0 \leqslant \nu \leqslant \nu_\iota^*(r, s)$, and put

$$\Delta_0^*(r, s; \nu) = \sum_{m=1}^{r} \frac{(m - 1)(k - m - 1)}{s - k - m + 1} - \frac{(\nu_0^*(r, s) - \nu)(r - 1)}{s - k + 1}, \tag{10.1}$$

$$\Delta_1^*(r, s; \nu) = \sum_{m=1}^{r} \frac{(m - 1)(k - m - 1)}{s - k + 1}. \tag{10.2}$$

Our first result of this section provides a mean value estimate restricted to minor arcs of use in many applications.

Theorem 10.1. *Let ι be either 0 or 1. Suppose that r, s and k are integers with $k \geqslant 3$,*

$$1 \leqslant r \leqslant \min\{k - 2, \tfrac{1}{2}(k + 1)\}, \tag{10.3}$$

and

$$s \geqslant k^2 - rk + \tfrac{1}{2} r(r + 3) - 1 - \nu_\iota^*(r, s). \tag{10.4}$$

Put

$$\nu = \max\{k^2 - rk + \tfrac{1}{2} r(r + 3) - 1 - s, 0\}.$$

Then for each $\varepsilon > 0$, one has

$$\int_{\mathfrak{m}} |g_k(\alpha; X)|^{2s} \, d\alpha \ll X^{2s - k - 1 + \Delta + \varepsilon},$$

where $\Delta = \Delta_\iota^*(r, s; \nu)$.

Proof. According to [18, Theorem 2.1], one has

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^{2s}\,\mathrm{d}\alpha \ll X^{\frac{1}{2}k(k-1)-1}(\log X)^{2s+1}J_{s,k}(2X).$$

By combining Theorems 1.1 and 9.2, it follows that when s satisfies the lower bound (10.4), one has

$$J_{s,k}(2X) \ll X^{2s-\frac{1}{2}k(k+1)+\Delta+\varepsilon},$$

and the conclusion of the theorem now follows. □

The dependence on s of $\nu_0^*(r,s)$ suggests that the lower bound (10.4) may be difficult to interpret. However, since s exceeds $\frac{1}{2}k(k+1)$ and $r \leqslant \frac{1}{2}(k+1)$, a crude computation confirms that $\nu_0^*(r,s) \leqslant k$. Put

$$s_0 = k^2 - rk + \tfrac{1}{2}r(r+3) - 1.$$

In practice one may check successively for the largest integral value of ν with $0 \leqslant \nu \leqslant k$ for which $\nu_0^*(r, s_0 - \nu) \geqslant \nu$. This isolates the largest integer ν for which (10.4) holds with $s = s_0 - \nu$. As we have noted, this maximal value of ν is no larger than k, and so this is not particularly expensive computationally.

The special case of Theorem 10.1 with $r = 1$ merits particular attention.

Corollary 10.2. *Suppose that $s \geqslant k^2 - k + 1$. Then for each $\varepsilon > 0$, one has*

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^{2s}\,\mathrm{d}\alpha \ll X^{2s-k-1+\varepsilon}.$$

The mean value over major arcs $\mathfrak{M} = [0,1) \setminus \mathfrak{m}$ corresponding to that bounded in this corollary has order of magnitude X^{2s-k}. Thus, as is clear already in [18], estimates of the type provided by Corollary 10.2 may be employed in applications as powerful substitutes for estimates of Weyl type.

We apply these bounds so as to handle the minor arc contribution in Waring's problem, beginning with a sketch of the arguments required for smaller values of k. For each natural number k, one begins by computing permissible exponents $\Delta_{s,k}$ having the property that

$$J_{s,k}(X) \ll X^{2s-\frac{1}{2}k(k+1)+\Delta_{s,k}+\varepsilon}. \tag{10.5}$$

It is apparent from Corollary 1.2 that one may take $\Delta_{s,k} = 0$ for $s \geqslant k^2 - k + 1$, and thus we may concentrate on the interval $1 \leqslant s \leqslant k(k-1)$. Consider each integer s in this interval in turn. For each integer r satisfying (10.3), one may check (for $\iota \in \{0,1\}$) whether the lower bound (10.4) is satisfied or not. If this lower bound is satisfied, then the exponent $\Delta_\iota^*(r,s;\nu)$ given by (10.1) or (10.2) is permissible. As a preliminary value, one takes $\Delta_{s,k}^*$ to be the least of these permissible exponents $\Delta_\iota^*(r,s;\nu)$ as one runs through the available choices for r and ι. Next, by applying Hölder's inequality to (2.3), it is apparent that whenever s_1 and s_2 are integers with

$$1 \leqslant s_1 \leqslant s \leqslant s_2 \leqslant k^2 - k + 1,$$

then the upper bound (10.5) holds with

$$\Delta_{s,k}=\frac{(s-s_1)\Delta^*_{s_2,k}+(s_2-s)\Delta^*_{s_1,k}}{s_2-s_1}.$$

For each integer s with $1\leqslant s\leqslant k^2-k+1$, therefore, one may linearly interpolate in this manner amongst all possible choices of s_1 and s_2 so as to obtain the smallest available value of $\Delta_{s,k}$. It is this exponent $\Delta_{s,k}$ that we now fix, and use computationally in what follows. We note that if one is content to make use of potentially non-optimal conclusions, then one has the alternative option of applying Theorems 1.1 and 9.2 for the specific values of s given by integral choices of r with $1\leqslant r\leqslant \frac{1}{2}(k+1)$.

We are now equipped to negotiate the details of our analysis of the asymptotic formula in Waring's problem. We employ two strategies, the first of which interpolates between the minor arc estimate supplied by Theorem 10.1, and that offered by Hua's lemma (see [12, Lemma 2.5]). Given natural numbers k and t with $k\geqslant 3$ and $1\leqslant t\leqslant k^2-k+1$, define the positive number $s_0(j)=s_0(k,t,j)$ by means of the relation

$$s_0(k,t,j)=2t-\frac{(1-\Delta_{t,k})(2t-2^{j+1})}{k-j-\Delta_{t,k}},$$

and then put

$$s_1(k)=\min_{\substack{1\leqslant t\leqslant k^2-k+1\\ \Delta_{t,k}<1}}\ \min_{\substack{0\leqslant j\leqslant k-2\\ 2^j<t}} s_0(k,t,j). \tag{10.6}$$

Lemma 10.3. *Suppose that k is a natural number with $k\geqslant 3$. Then*

$$\int_0^1 |g_k(\alpha;X)|^{s_1(k)}\,\mathrm{d}\alpha \ll X^{s_1(k)-k+\varepsilon}.$$

Moreover, when s is a real number with $s>s_1(k)$, there exists a positive number $\delta=\delta(k,s)$ with the property that

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^{s}\,\mathrm{d}\alpha \ll X^{s-k-\delta}.$$

Proof. We first establish the second conclusion of the lemma. Let the parameters j and t correspond to the minimum implict in (10.6). Then the second estimate claimed in the lemma is immediate from [18, Theorem 2.1] when $s\geqslant 2t$, since we have $\Delta_{t,k}<1$. Here, if necessary, we make use of the trivial estimate $|g_k(\alpha;X)|\leqslant X$. Indeed, the latter theorem shows that

$$\int_{\mathfrak{m}} |g(\alpha)|^{2t}\,\mathrm{d}\alpha \ll X^{\frac{1}{2}k(k-1)-1+\varepsilon}J_{t,k}(2X)\ll X^{2t-k+\Delta_{t,k}-1+2\varepsilon}. \tag{10.7}$$

We suppose therefore that $s_1(k)<s\leqslant 2t$, and we put $\tau=s-s_1(k)$. Then by Hölder's inequality, one has

$$\int_{\mathfrak{m}} |g(\alpha)|^{s}\,\mathrm{d}\alpha \leqslant \Bigl(\int_{\mathfrak{m}} |g(\alpha)|^{2t}\,\mathrm{d}\alpha\Bigr)^a\Bigl(\int_0^1 |g(\alpha)|^{2^{j+1}}\,\mathrm{d}\alpha\Bigr)^b,$$

where

$$a=\frac{s-2^{j+1}}{2t-2^{j+1}}\quad\text{and}\quad b=\frac{2t-s}{2t-2^{j+1}}.$$

An application of Theorem 10.1, in the guise of the estimate (10.7), in combination with Hua's lemma (see [12, Lemma 2.5]) therefore yields the bound

$$\int_{\mathfrak{m}}|g(\alpha)|^s\,\mathrm{d}\alpha\ll X^\varepsilon(X^{2t-k-1+\Delta_{t,k}})^a(X^{2^{j+1}-j-1})^b\ll X^{s-k-\omega+\varepsilon},$$

where $\omega=a(1-\Delta_{t,k})-(k-j-1)b$. A modicum of computation reveals that

$$\begin{aligned}\omega&=\frac{(k-j-\Delta_{t,k})(s-2t)+(1-\Delta_{t,k})(2t-2^{j+1})}{2t-2^{j+1}}\\&=\frac{(k-j-\Delta_{t,k})(s-s_1(k))}{2t-2^{j+1}}\geqslant\tau/(2k^2),\end{aligned}$$

and consequently the second conclusion of the lemma follows with $\delta=\tau/(4k^2)$.

When $s=s_1(k)$, the above discussion shows that

$$\int_{\mathfrak{m}}|g(\alpha)|^s\,\mathrm{d}\alpha\ll X^{s-k+\varepsilon}.$$

But on writing $\mathfrak{M}=[0,1)\setminus\mathfrak{m}$, the methods of [12, Chapter 4] confirm that whenever $s\geqslant k+2$, one has

$$\int_{\mathfrak{M}}|g(\alpha)|^s\,\mathrm{d}\alpha\ll X^{s-k}.$$

The first conclusion of the lemma follows by combining these two estimates. $\square$

The argument following the proof of [18, Lemma 3.1] may now be adapted to deliver the upper bound contained in the following lemma.

Lemma 10.4. *When $k\geqslant 3$, define $s_1(k)$ as in equation* (10.6). *Then one has $\widetilde{G}(k)\leqslant\lfloor s_1(k)\rfloor+1$.*

This upper bound may of course be made explicit for smaller values of k. By using a naïve computer program to optimise the choice of parameters, one obtains the values of $s_1(k)$ reported in Table 2 below. Here, we have rounded up in the final decimal place reported. The conclusion of Theorem 1.4 now follows by inserting the bounds for $s_1(k)$ supplied by Table 2 into Lemma 10.4. When $k=3$ and 4, the bounds for $s_1(k)$ supplied by Table 2 may be compared with the bounds for $\widetilde{G}(k)$ available from Vaughan's refinements [10, 11] of Hua's work. The latter work supplies bounds in a sense tantamount to $s_1(3)\leqslant 8$ and $s_1(4)\leqslant 16$. Thus our present work, while coming close to these bounds, nonetheless fails the cigar test.

k	3	4	5	6	7	8
$s_1(k)$	9.000	16.311	27.413	42.710	60.799	82.023

k	9	10	11	12	13	14
$s_1(k)$	106.492	133.724	164.453	198.448	235.389	275.661

k	15	16	17	18	19	20
$s_1(k)$	319.462	367.221	417.870	472.973	529.938	591.528

Table 2: Upper bounds for $s_1(k)$ described in equation (10.6).

For concreteness, we note that reasonable bounds may be computed by hand with relative ease. Thus a good approximation to the bound for $s_1(5)$ recorded in Table 2 derives from the permissible exponent $\Delta_{18,5} = \frac{2}{7}$ that stems from Theorem 1.1 with $r = 3$ and $k = 5$, and then the exponent

$$s_0(5, 18, 3) = 36 - \frac{(1 - \frac{2}{7})(36 - 16)}{5 - 3 - \frac{2}{7}} = 27\tfrac{2}{3}$$

that determines $s_1(5)$ by means of (10.6). Similarly, one finds that the permissible exponent $\Delta_{26,6} = \frac{1}{3}$ is made available by Theorem 1.1 with $r = 3$ and $k = 6$, and then the exponent

$$s_0(6, 26, 3) = 52 - \frac{(1 - \frac{1}{3})(52 - 16)}{6 - 3 - \frac{1}{3}} = 43$$

determines an approximation to $s_1(6)$ by means of (10.6).

For a clean, easy to state upper bound for $\widetilde{G}(k)$ valid for $k \geqslant 5$, one may proceed as follows. First, apply Corollary 1.2 to obtain the permissible exponent $\Delta_{t,k} = 0$ with $t = k^2 - k + 1$. One then finds from (10.6) that

$$\begin{aligned} s_1(k) \leqslant s_0(k, k^2 - k + 1, 4) &= 2(k^2 - k + 1) - \frac{2(k^2 - k + 1) - 32}{k - 4} \\ &= 2k^2 - 4k - 4 + \frac{6}{k - 4}, \end{aligned}$$

so that $s_1(k) < 2k^2 - 4k - 2$ whenever $k \geqslant 8$. Consequently, by reference to Lemma 10.4, one obtains the following upper bound on $\widetilde{G}(k)$.

Corollary 10.5. *Whenever $k \geqslant 5$, one has $\widetilde{G}(k) \leqslant 2k^2 - 4k - 2$.*

For larger values of k, one may employ the methods of [5, §8] in order to improve on the bound given in Corollary 10.5. The statement of our most general conclusion requires a little preparation. Let $\Delta_{s,k}$ $(s \in \mathbb{N})$ be the exponents defined in the discussion following (10.5). For each $v \in \mathbb{N}$, we define

$$\Delta^+_{v,k} = \min\{\Delta_{v,k} - 1, \Delta_{v,k-1}\}.$$

When $1 \leqslant t \leqslant k^2 - k + 1$, we now define the positive number $u_0(k, t, v, w)$ by means of the relation

$$u_0(k, t, v, w) = 2t - \frac{(1 - \Delta_{t,k})(2t - 2v - w(w - 1))}{1 - \Delta_{t,k} + \Delta^+_{v,k}/w}.$$

We then put

$$u_1(k) = \min_{\substack{1 \leqslant t \leqslant k^2 - k + 1 \\ \Delta_{t,k} < 1}} \min_{\substack{1 \leqslant w \leqslant k-1 \\ 2v + w(w-1) < 2t}} \min_{v \geqslant 1} u_0(k, t, v, w). \tag{10.8}$$

We begin by announcing an analogue of Lemma 10.3 useful for intermediate and larger values of k.

Lemma 10.6. *Let k be a natural number with $k \geqslant 3$, and suppose that s is a real number with $s > u_1(k)$. Then there exists a positive number $\delta = \delta(k,s)$ with the property that*

$$\int_{\mathfrak{m}} |g_k(\alpha;X)|^s \, \mathrm{d}\alpha \ll X^{s-k-\delta}.$$

Proof. Let the parameters v, w and t correspond to the minimum implicit in (10.8). Then in view of the implicit hypothesis $\Delta_{t,k} < 1$, we find just as in the proof of Lemma 10.3 that the desired conclusion is an immediate consequence of [18, Theorem 2.1] when $s \geqslant 2t$, on making use of the trivial estimate $|g_k(\alpha;X)| \leqslant X$. We suppose therefore that $u_1(k) < s \leqslant 2t$, and we put $\tau = s - u_1(k)$. Then by Hölder's inequality, one has

$$\int_{\mathfrak{m}} |g(\alpha)|^s \, \mathrm{d}\alpha \leqslant \Bigl(\int_{\mathfrak{m}} |g(\alpha)|^{2t} \, \mathrm{d}\alpha \Bigr)^a \Bigl(\int_0^1 |g(\alpha)|^{2v+w(w-1)} \, \mathrm{d}\alpha \Bigr)^{1-a},$$

where

$$a = \frac{s - 2v - w(w-1)}{2t - 2v - w(w-1)}.$$

We next apply Theorem 10.1, as embodied in (10.7), and then wield the estimate supplied by [5, Theorem 8.5], thus obtaining the bound

$$\int_{\mathfrak{m}} |g(\alpha)|^s \, \mathrm{d}\alpha \ll X^\varepsilon \left(X^{2t-k-1+\Delta_{t,k}} \right)^a \left(X^{2v+w(w-1)-k+\Delta^+_{v,k}/w} \right)^{1-a}. \tag{10.9}$$

Since we may suppose that

$$s > u_0(k,t,v,w) = \frac{2t\Delta^+_{v,k} + w(1-\Delta_{t,k})(2v + w(w-1))}{w(1-\Delta_{t,k}) + \Delta^+_{v,k}},$$

we see that

$$a(1-\Delta_{t,k}) > (1-a)\Delta^+_{v,k}/w,$$

and thus the conclusion of the lemma follows at once from (10.9). □

The argument following the proof of [18, Lemma 3.1] may be adapted on this occasion to give the following upper bound for $\widetilde{G}(k)$.

Lemma 10.7. *When $k \geqslant 3$, define $u_1(k)$ as in equation* (10.8). *Then one has $\widetilde{G}(k) \leqslant \lfloor u_1(k) \rfloor + 1$.*

It would appear that Lemma 10.7 yields superior bounds for $\widetilde{G}(k)$ as compared to Lemma 10.4 only for k exceeding 25 or thereabouts. However, for large values of k one obtains substantial quantitative improvements on previous bounds for $\widetilde{G}(k)$, these being reported in Theorem 1.3. Suppose that k is a large natural number, and let β be a positive parameter to be determined in due course. We take

$$r = \lfloor \tfrac{1}{2}(k+1) \rfloor, \quad w = \lfloor \beta k \rfloor, \quad \text{and} \quad v = k^2 - rk + \tfrac{1}{2} r(r+3).$$

Then one has

$$\tfrac{5}{8}k^2 \leqslant v \leqslant \tfrac{5}{8}k^2 + k,$$

so by Theorem 1.1 one finds that the exponent $\Delta_{v,k}$ is permissible, where

$$\Delta_{v,k} \leqslant \frac{r(r-1)(2k-5)}{6(\frac{5}{8}k^2 - k + 1)} = \tfrac{2}{15}k + O(1).$$

Also, by taking $t = k^2 - k + 1$, one sees from Corollary 1.2 that $\Delta_{t,k} = 0$. Then we deduce from (10.8) that

$$u_1(k) \leqslant 2(k^2 - k + 1) - \frac{2k^2 - 2(\frac{5}{8}k^2) - (\beta k)^2 + O(k)}{1 + \frac{2}{15}k/(\beta k) + O(1/k)}.$$

It follows that

$$\begin{aligned} u_1(k)/(2k^2) &\leqslant 1 - \frac{\beta(\frac{3}{8} - \frac{1}{2}\beta^2)}{\beta + \frac{2}{15}} + O(1/k) \\ &= \frac{60\beta^3 + 75\beta + 16}{120\beta + 16} + O(1/k). \end{aligned}$$

A modest computation confirms that the optimal choice for the parameter β is ξ, where ξ is the real root of the polynomial equation $20\xi^3 = 1 - 4\xi^2$. With this choice for β, one finds that

$$u_1(k) \leqslant \left(\frac{19 + 75\xi - 12\xi^2}{8 + 60\xi} \right) k^2 + O(k).$$

The conclusion of Theorem 1.3 is now immediate from Lemma 10.7.

We finish by noting that the proof of [18, Theorem 4.2] may be adapted in the obvious manner so as to establish that when $s > \min\{s_1(k), u_1(k)\}$, then the anticipated asymptotic formula holds for the number of integral solutions of the diagonal equation

$$a_1 x_1^k + \ldots + a_s x_s^k = 0,$$

with $|\mathbf{x}| \leqslant B$. Here, the coefficients a_i $(1 \leqslant i \leqslant s)$ are fixed integers. Similar improvements may be wrought in upper bounds for $\widetilde{G}^+(k)$, the least number of variables required to establish that the anticipated asymptotic formula in Waring's problem holds for almost all natural numbers n. Thus, one may adapt the methods of [18, §5] to show that

$$\widetilde{G}^+(k) \leqslant 1 + \min\{\lfloor \tfrac{1}{2}s_1(k) \rfloor, \lfloor \tfrac{1}{2}u_1(k) \rfloor\}.$$

In this way, one finds that for large values of k, one has $\widetilde{G}^+(k) \leqslant 0.772k^2$, and further that for $5 \leqslant k \leqslant 20$, one has $\widetilde{G}^+(k) \leqslant H^+(k)$, where $H^+(k)$ is given in Table 3 below.

k	5	6	7	8	9	10	11	12
$H^+(k)$	14	22	31	42	54	67	83	100

k	13	14	15	16	17	18	19	20
$H^+(k)$	118	138	160	184	209	237	265	296

Table 3: Upper bounds for $H^+(k)$.

11. Estimates of Weyl type

In this section we briefly discuss some applications of the mean value estimates supplied by Theorems 1.1 and 9.2 to analogues of Weyl's inequality. Our first conclusion has the merit of being simple to state, and improves on [19, Theorem 11.1] for $k \geqslant 4$. We recall the definition of $f_k(\boldsymbol{\alpha}; X)$ from (2.2).

Theorem 11.1. *Let k be an integer with $k \geqslant 4$, and let $\boldsymbol{\alpha} \in \mathbb{R}^k$. Suppose that there exists a natural number j with $2 \leqslant j \leqslant k$ such that, for some $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, one has $|\alpha_j - a/q| \leqslant q^{-2}$ and $q \leqslant X^j$. Then one has*

$$f_k(\boldsymbol{\alpha}; X) \ll X^{1+\varepsilon}(q^{-1} + X^{-1} + qX^{-j})^{\sigma(k)},$$

where $\sigma(k)^{-1} = 2(k^2 - 3k + 3)$.

Proof. Under the hypotheses of the statement of the theorem, we find that [12, Theorem 5.2] shows that for $s \in \mathbb{N}$, one has

$$f_k(\boldsymbol{\alpha}; X) \ll (J_{s,k-1}(2X)X^{\frac{1}{2}k(k-1)}(q^{-1} + X^{-1} + qX^{-j}))^{1/(2s)} \log(2X).$$

The conclusion of the theorem therefore follows on taking

$$s = (k-1)^2 - (k-1) + 1 = k^2 - 3k + 3,$$

for in such circumstances Corollary 1.2 delivers the bound

$$J_{s,k-1}(2X) \ll X^{2s-\frac{1}{2}k(k-1)+\varepsilon}.$$

□

The proof of [17, Theorem 1.6] may be easily adapted to deliver estimates depending on common Diophantine approximations.

Theorem 11.2. *Let k be an integer with $k \geqslant 4$, and let τ and δ be real numbers with $\tau^{-1} > 4(k^2 - 3k + 3)$ and $\delta > k\tau$. Suppose that X is sufficiently large in terms of k, δ and τ, and further that $|f_k(\boldsymbol{\alpha}; X)| > X^{1-\tau}$. Then there exist integers $q, a_1, \ldots, a_k$ such that $1 \leqslant q \leqslant X^\delta$ and $|q\alpha_j - a_j| \leqslant X^{\delta-j}$ $(1 \leqslant j \leqslant k)$.*

The proof of [17, Theorem 1.7] likewise delivers the following result concerning the distribution modulo 1 of polynomial sequences. Here, we write $\|\theta\|$ for $\min_{y\in\mathbb{Z}}|\theta - y|$.

Theorem 11.3. *Let k be an integer with $k \geqslant 4$, and define $\tau(k)$ by $\tau(k)^{-1} = 4(k^2 - 3k + 3)$. Then whenever $\boldsymbol{\alpha} \in \mathbb{R}^k$ and N is sufficiently large in terms of k and ε, one has*

$$\min_{1\leqslant n\leqslant N} \|\alpha_1 n + \alpha_2 n^2 + \ldots + \alpha_k n^k\| < N^{\varepsilon-\tau(k)}.$$

In each of Theorems 11.2 and 11.3, the exponent $4(k^2 - 3k + 3)$ represents an improvement on the exponent $4k(k-2)$ made available in [19, Theorems 11.2 and 11.3]. In [19, Theorem 11.1], meanwhile, we established a conclusion similar to that of Theorem 11.1, though with a weaker exponent $\sigma(k)$ satisfying $\sigma(k)^{-1} = 2k(k-2)$. Our estimates supersede the Weyl exponent $\sigma(k) = 2^{1-k}$ when $k \geqslant 7$ (see [12, Lemma 2.4] and [2, Theorem 5.1]).

If one restricts to the situation where all coefficients save α_k are zero, then further modest improvements may be obtained. When $\theta \in (0, k)$, let $\mathfrak{m}_\theta$ denote the set of real numbers α having the property that, whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|q\alpha - a| \leqslant P^{\theta - k}$, then one has $q > P^\theta$. The simplest improvements in earlier Weyl exponents stem from the following result of Boklan and Wooley [3, Theorem 1.1].

Lemma 11.4. *Let $k \in \mathbb{N}$ with $k \geqslant 4$, and suppose that the exponent $\Delta_{s,k-1}$ is permissible for $s \geqslant k$. Then for each $\varepsilon > 0$, one has*

$$\sup_{\alpha \in \mathfrak{m}_1} |g_k(\alpha; X)| \ll X^{1-\sigma(k)+\varepsilon},$$

where

$$\sigma(k) = \max_{s \geqslant k} \left(\frac{3 - \Delta_{s,k-1}}{6s + 2} \right).$$

By making use of the permissible exponents $\Delta_{s,k}$ stemming from the discussion following (10.5) one obtains the following conclusion by means of a naïve computer program.

Theorem 11.5. *Suppose that $4 \leqslant k \leqslant 20$ and that the positive numbers $\Sigma_1(k)$ are defined as in Table 4. Then for each $\varepsilon > 0$, one has*

$$\sup_{\alpha \in \mathfrak{m}_1} |g_k(\alpha; X)| \ll X^{1-\sigma(k)+\varepsilon},$$

where $\sigma(k) = 1/\Sigma_1(k)$.

k	6	7	8	9	10
$\Sigma_1(k)$	39.023	58.093	80.867	107.396	137.763

k	11	12	13	14	15
$\Sigma_1(k)$	172.027	210.222	252.370	298.487	348.580

k	16	17	18	19	20
$\Sigma_1(k)$	402.655	460.718	522.771	588.815	658.854

Table 4: Upper bounds for $\Sigma_1(k)$ used in Theorem 11.5.

As we have noted, our estimates supersede the Weyl exponent $\sigma(k) = 2^{1-k}$ when $k \geqslant 7$. Heath-Brown [6] obtains the estimate

$$\sup_{\alpha \in \mathfrak{m}_{3-\varepsilon}} |g_k(\alpha; X)| \ll X^{1-\sigma(k)+\varepsilon},$$

with $\sigma(k)^{-1} = 3 \cdot 2^{k-3}$, an estimate that is superseded by Theorems 11.1 and 11.5 for $k \geqslant 8$. The work of Robert and Sargos [9] and of Parsell [8] yields sharper results subject to more restrictive Diophantine approximation hypotheses. These are also superseded by our conclusions for $k \geqslant 9$, though Parsell's work [8, Theorem 1.2] shows that

$$\sup_{\alpha \in \mathfrak{m}_{4-\varepsilon}} |g_8(\alpha; X)| \ll X^{1-\sigma+\varepsilon},$$

with $\sigma^{-1} = 80$, a conclusion slightly sharper than that implied by Theorem 11.5, though under more restrictive hypotheses.

An asymptotic analysis of the argument establishing Theorem 11.5 shows that its conclusion holds in general with $\Sigma_1(k) = 2k^2 - 8k + O(1)$. However, for large values of k one may derive a sharper bound by applying our earlier work [15], which we now recall.

Lemma 11.6. *Let R be an integer with $1 \leqslant R \leqslant \frac{1}{2}k$, and write $\lambda = 1 - R/k$. Suppose that s and t are positive integers with $s \geqslant \frac{1}{2}k(k-1)$, and suppose further that the exponents $\Delta_{s,k-1}$ and $\Delta_{t,k}$ are permissible. Then we have*

$$\sup_{\alpha_k \in \mathfrak{m}_\lambda} |f_k(\boldsymbol{\alpha}; X)| \ll P^\varepsilon (P^{1-\mu(k)} + P^{1-\nu(k)}),$$

where

$$\mu(k) = \frac{R - \Delta_{s,k-1}}{2Rs} \quad \textit{and} \quad \nu(k) = \frac{k - R(1+\Delta_{t,k})}{2tk}.$$

Proof. This is [15, Theorem 2]. □

Theorem 11.7. *Suppose that k is a large positive integer. Then for each $\varepsilon > 0$, one has*

$$\sup_{\alpha \in \mathfrak{m}_1} |g_k(\alpha; X)| \ll X^{1-\sigma(k)+\varepsilon},$$

where

$$\sigma(k)^{-1} = 2k^2 - \frac{1}{\sqrt{3}} k^{3/2} + O(k).$$

Proof. We apply Lemma 11.6 with

$$s = (k-1)^2 - r(k-1) + \tfrac{1}{2}r(r+3) - 1$$

and

$$t = k^2 - uk + \tfrac{1}{2}u(u+3) - 1,$$

with the permissible exponents $\Delta_{s,k-1}$ and $\Delta_{t,k}$ determined via Theorem 1.1. With a little experimentation, one finds that the optimal choices of the parameters r, u and R in the application of Lemma 11.6 are all of order $\sqrt{k}$. We therefore put $R = \lfloor \theta\sqrt{k} \rfloor$, $r = \lfloor \phi\sqrt{k} \rfloor$ and $u = \lfloor \psi\sqrt{k} \rfloor$, with θ, ϕ and ψ positive parameters to be chosen in due course. One finds from Theorem 1.1 that one has permissible exponents $\Delta_{s,k-1}$ and $\Delta_{t,k}$, with

$$\Delta_{s,k-1} = \tfrac{1}{2}\phi^2 + O(k^{-1/2}) \quad \text{and} \quad \Delta_{t,k} = \tfrac{1}{2}\psi^2 + O(k^{-1/2}).$$

In addition, one has

$$s = k^2(1 - \phi k^{-1/2} + O(1/k)) \quad \text{and} \quad t = k^2(1 - \psi k^{-1/2} + O(1/k)).$$

Thus, in our application of Lemma 11.6, we obtain

$$\begin{aligned}2k^2\mu(k) &= (1-\tfrac{1}{2}\phi^2\theta^{-1}k^{-1/2})(1+\phi k^{-1/2})+O(1/k)\\ &= 1+\tfrac{1}{2}(2\phi-\phi^2\theta^{-1})k^{-1/2}+O(1/k),\end{aligned}$$

and

$$\begin{aligned}2k^2\nu(k) &= (1-(1+\tfrac{1}{2}\psi^2)\theta k^{-1/2})(1+\psi k^{-1/2})+O(1/k)\\ &= 1+\tfrac{1}{2}(2\psi-(2+\psi^2)\theta)k^{-1/2}+O(1/k).\end{aligned}$$

A rapid optimisation reveals that we should take $\phi=\theta$ and $\psi=\theta^{-1}$ in order to optimise these two expressions, and then the optimal choice for θ is determined by the equation $\theta=\theta^{-1}-2\theta$. Thus we deduce that one should take $\theta=1/\sqrt{3}$, $\phi=1/\sqrt{3}$ and $\psi=\sqrt{3}$, delivering the exponents

$$\mu(k)^{-1}=2k^2(1-1/(2\sqrt{3k})+O(1/k))$$

and

$$\nu(k)^{-1}=2k^2(1-1/(2\sqrt{3k})+O(1/k)).$$

The conclusion of the theorem now follows at once from Lemma 11.6. □

It would appear that the exponents provided by means of Lemma 11.6 do not supersede those provided by Theorem 11.5, by reference to Table 4, in the range $6\leqslant k\leqslant 20$.

12. Further applications

We turn next to Tarry's problem. When h, k and s are positive integers with $h\geqslant 2$, consider the Diophantine system

$$\sum_{i=1}^s x_{i1}^j=\sum_{i=1}^s x_{i2}^j=\ldots=\sum_{i=1}^s x_{ih}^j\quad (1\leqslant j\leqslant k). \tag{12.1}$$

Let $W(k,h)$ denote the least natural number s having the property that the simultaneous equations (12.1) possess an integral solution $\mathbf{x}$ with

$$\sum_{i=1}^s x_{iu}^{k+1}\neq\sum_{i=1}^s x_{iv}^{k+1}\quad (1\leqslant u<v\leqslant h).$$

Theorem 12.1. *When h and k are natural numbers with $h\geqslant 2$ and $k\geqslant 3$, one has $W(k,h)\leqslant \frac{5}{8}(k+1)^2$.*

Proof. The argument of the proof of [17, Theorem 1.3] shows that $W(k,h)\leqslant s$ whenever one can establish the existence of a permissible exponent $\Delta_{s,k+1}$ with $\Delta_{s,k+1}<k+1$. But by taking $r=\lfloor\frac{1}{2}(k+1)\rfloor$ in Theorem 1.1, one finds that whenever $s\geqslant\frac{5}{8}(k+1)^2$, one has $\Delta_{s,k+1}<\frac{2}{15}k$. The conclusion of the theorem follows immediately. □

In [19, Theorem 11.4], we obtained the weaker bound

$$W(k,h) \leqslant k^2 - \sqrt{2}k^{3/2} + 4k.$$

It is plain that there is plenty of room to spare in the above proof of Theorem 12.1. This is a topic we intend to pursue elsewhere.

We note also that on writing

$$\mathfrak{S}(s,k) = \sum_{q=1}^{\infty} \sum_{\substack{a_1=1 \\ (a_1,\ldots,a_k,q)=1}}^{q} \cdots \sum_{a_k=1}^{q} \Bigl| q^{-1} \sum_{r=1}^{q} e((a_1 r + \ldots + a_k r^k)/q) \Bigr|^{2s}$$

and

$$\mathcal{J}(s,k) = \int_{\mathbb{R}^k} \Bigl| \int_0^1 e(\beta_1 \gamma + \ldots + \beta_k \gamma^k)\, \mathrm{d}\gamma \Bigr|^{2s} \mathrm{d}\boldsymbol{\beta},$$

the method of proof of [17, Theorem 1.2] may be modified in the light of Corollary 1.2 to obtain the following conclusion.

Theorem 12.2. *Suppose that $k \geqslant 3$ and $s \geqslant k^2 - k + 2$. Then one has the asymptotic formula*

$$J_{s,k}(X) \sim \mathfrak{S}(s,k)\mathcal{J}(s,k) X^{2s - \frac{1}{2}k(k+1)}.$$

In [19, §11], such a conclusion was obtained for $s \geqslant k^2$. A similar improvement holds also for work on the asymptotic formula in the Hilbert-Kamke problem.

Finally, write

$$F_k(\boldsymbol{\beta}; X) = \sum_{1 \leqslant x \leqslant X} e(\beta_k x^k + \beta_{k-2} x^{k-2} + \ldots + \beta_1 x).$$

L.-K. Hua [7] investigated the problem of bounding the least integer C_k such that, whenever $s \geqslant C_k$, one has

$$\oint |f_k(\boldsymbol{\alpha}; X)|^s \, \mathrm{d}\boldsymbol{\alpha} \ll X^{s - \frac{1}{2}k(k+1) + \varepsilon},$$

and likewise the least integer S_k such that, whenever $s \geqslant S_k$, one has

$$\oint |F_k(\boldsymbol{\beta}; X)|^s \, \mathrm{d}\boldsymbol{\beta} \ll X^{s - \frac{1}{2}(k^2 - k + 2) + \varepsilon}. \tag{12.2}$$

We are able to reduce the upper bounds for C_k and S_k provided by Hua [7], and also the subsequent improved bounds given in our earlier work [17, 19].

Theorem 12.3. *When $k \geqslant 3$, one has $C_k \leqslant 2k^2 - 2k + 2$. Meanwhile, one has*

$$S_4 \leqslant 22, \quad S_5 \leqslant 34, \quad S_6 \leqslant 52, \quad S_7 \leqslant 66, \quad S_8 \leqslant 88$$

and $S_k \leqslant 2k^2 - 6k + 6$ for $k \geqslant 9$.

Proof. The bound on C_k is immediate from Corollary 1.2 via (2.3) and orthogonality. In order to establish the bound on S_k, we begin by observing that [17, equation (10.10)] supplies the estimate

$$\oint |F_k(\boldsymbol{\beta}; X)|^{2t}\,\mathrm{d}\boldsymbol{\beta} \ll X^{k-2+\varepsilon} J_{t,k}(2X) + X^{\varepsilon-1} J_{t,k-1}(2X). \tag{12.3}$$

It follows from Corollary 1.2 that when $t \geqslant k^2 - 3k + 3$, one has

$$J_{t,k-1}(2X) \ll X^{2t-\frac{1}{2}k(k-1)+\varepsilon}. \tag{12.4}$$

In addition, explicit computations of the type described following (10.5) show that the exponent $\Delta_{t,k} = 1$ is permissible whenever $t \geqslant t^*(k)$ and $4 \leqslant k \leqslant 8$, where

$$t^*(4) = 11, \quad t^*(5) = 17, \quad t^*(6) = 26, \quad t^*(7) = 33, \quad t^*(8) = 44.$$

For these exponents, therefore, when $t \geqslant t^*(k)$, one has the upper bound

$$J_{t,k}(2X) \ll X^{2t-\frac{1}{2}k(k+1)+1+\varepsilon}. \tag{12.5}$$

By substituting (12.4) and (12.5) into (12.3), we obtain the desired conclusion (12.2) with $s = 2t$ for $4 \leqslant k \leqslant 8$.

When $k \geqslant 9$, we instead apply Theorem 1.1 with $r = 5$ to show that whenever $t \geqslant k^2 - 5k + 19$, the permissible exponent $\Delta_{t,k}$ is permissible, where

$$\Delta_{t,k} = \frac{10(k-5)}{k^2 - 6k + 20} < 1.$$

When $k \geqslant 9$ and $t \geqslant k^2 - 3k + 3$, therefore, we again have the estimates (12.4) and (12.5), and the estimate (12.2) with $s = 2t$ follows just as before. This completes the proof of the theorem. ☐

For comparison, in [19, Theorem 11.6] we derived the somewhat weaker bounds $C_k \leqslant 2k^2 - 2$ and $S_k \leqslant 2k^2 - 2k$.

References

[1] G. I. Arkhipov, V. N. Chubarikov and A. A. Karatsuba, *Trigonometric sums in number theory and analysis*, de Gruyter Expositions in Mathematics, **39**, Walter de Gruyter, Berlin, 2004.

[2] R. C. Baker, *Diophantine inequalities*, London Mathematical Society Monographs, vol. **1**, Oxford University Press, 1986.

[3] K. D. Boklan and T. D. Wooley, *On Weyl sums for smaller exponents*, Funct. Approx. Comment. Math. **46** (2012), no. 1, 91–107.

[4] K. B. Ford, *New estimates for mean values of Weyl sums*, Internat. Math. Res. Notices (1995), no. 3, 155–171.

[5] K. B. Ford and T. D. Wooley, *On Vinogradov's mean value theorem: strongly diagonal behaviour via efficient congruencing*, Acta Math. **213** (2014), no. 2, 199–236.

[6] D. R. Heath-Brown, *Weyl's inequality, Hua's inequality, and Waring's problem*, J. London Math. Soc. (2) **38** (1988), no. 2, 216–230.

[7] L.-K. Hua, *Additive theory of prime numbers*, American Math. Soc., Providence, RI, 1965.

[8] S. T. Parsell, *A note on Weyl's inequality for eighth powers*, Rocky Mountain J. Math. **44** (2014), no. 1, 259–268.

[9] O. Robert and P. Sargos, *Un théorème de moyenne pour les sommes d'exponentielles. Application á l'inégalité de Weyl*, Publ. Inst. Math. (Beograd) (N.S.) **67** (2000), 14–30.
[10] R. C. Vaughan, *On Waring's problem for cubes*, J. Reine Angew. Math. **365** (1986), 122–170.
[11] R. C. Vaughan, *On Waring's problem for smaller exponents. II*, Mathematika **33** (1986), no. 1, 6–22.
[12] R. C. Vaughan, *The Hardy-Littlewood method*, Cambridge University Press, Cambridge, 1997.
[13] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Trav. Inst. Math. Stekloff **23** (1947), 109pp.
[14] T. D. Wooley, *On Vinogradov's mean value theorem*, Mathematika **39** (1992), no. 2, 379–399.
[15] T. D. Wooley, *New estimates for Weyl sums*, Quart. J. Math. Oxford (2) **46** (1995), no. 1, 119-127
[16] T. D. Wooley, *Some remarks on Vinogradov's mean value theorem and Tarry's problem*, Monatsh. Math. **122** (1996), no. 3, 265–273.
[17] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing*, Annals of Math. (2) **175** (2012), no. 3, 1575–1627.
[18] T. D. Wooley, *The asymptotic formula in Waring's problem*, Internat. Math. Res. Notices (2012), no. 7, 1485–1504.
[19] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing, II*, Duke Math. J. **162** (2013), no. 4, 673–730.
[20] T. D. Wooley, *Approximating the main conjecture in Vinogradov's mean value theorem*, submitted; arXiv:1401.2932.

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