ON SUMS OF POWERS OF ALMOST EQUAL PRIMES

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Abstract. We investigate the Waring-Goldbach problem of representing a positive integer n as the sum of s kth powers of almost equal prime numbers. Define $s_k = 2k(k-1)$ when $k \geqslant 3$, and put $s_2 = 6$. In addition, put $\theta_2 = \frac{19}{24}$, $\theta_3 = \frac{4}{5}$ and $\theta_k = \frac{5}{6}$ $(k \geqslant 4)$. Suppose that n satisfies the necessary congruence conditions, and put $X = (n/s)^{1/k}$. We show that whenever $s > s_k$ and $\varepsilon > 0$, and n is sufficiently large, then n is represented as the sum of s kth powers of prime numbers p with $|p - X| \leqslant X^{\theta_k+\varepsilon}$. This conclusion is based on a new estimate of Weyl-type specific to exponential sums having variables constrained to short intervals.

1. Introduction

A formal application of the circle method suggests that whenever s and k are natural numbers with $s \geqslant k+1$, then all large integers n satisfying appropriate local conditions should be represented as the sum of s kth powers of prime numbers. With this expectation in mind, consider a natural number k and prime p, take $\tau = \tau(k,p)$ to be the integer with $p^\tau | k$ but $p^{\tau+1} \nmid k$, and then define $\gamma = \gamma(k,p)$ by putting $\gamma(k,p) = \tau + 2$, when $p = 2$ and $\tau > 0$, and otherwise $\gamma(k,p) = \tau + 1$. We then define $R = R(k)$ by putting $R(k) = \prod p^\gamma$, where the product is taken over primes p with $(p-1)|k$. In 1938, Hua [8, 9] established that whenever $s > 2^k$, and n is a sufficiently large natural number with $n \equiv s \pmod{R}$, then the equation

$$p_1^k + p_2^k + \ldots + p_s^k = n \tag{1.1}$$

is soluble in prime numbers p_j. The congruence condition here excludes degenerate situations in which variables might otherwise be forced to be prime divisors of k. An intensively studied refinement of Hua's theorem is that in which the variables are constrained to be almost equal. Writing $X = (n/s)^{1/k}$, one seeks an analogue of Hua's theorem in which the variables p_j satisfy $|p_j - X| \leqslant Y$, with Y rather smaller than X. Although limitations in our knowledge concerning the distribution of primes constrain such investigations to intervals with $Y \geqslant X^{7/12+\varepsilon}$ or thereabouts (see [10]), for larger values of k, previous authors have obtained such conclusions only for $Y \geqslant X^{1-\Delta}$, with Δ extremely small. In this paper we decisively improve such conclusions, showing that for large s, one may take Δ large for all k.

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In order to facilitate further discussion, we introduce some additional notation. We say that the exponent $\Delta_{k,s}$ is *admissible* when, provided that Δ is a positive number with $\Delta < \Delta_{k,s}$, then for all sufficiently large positive integers n with $n \equiv s \pmod R$, the equation (1.1) has a solution in prime numbers p_j satisfying $|p_j - X| \leqslant X^{1-\Delta}$ $(1 \leqslant j \leqslant s)$. Old work of Wright [41] on Waring's problem shows that admissible exponents $\Delta_{k,s}$ must always satisfy the condition $0 \leqslant \Delta_{k,s} \leqslant \frac{1}{2}$.

Attention has naturally focused in the first instance on the situation for smaller values of k. We note in this context that, as a consequence of Hua's theorem, all large integers congruent to 5 modulo 24 are the sum of five squares of prime numbers, and all large odd integers are the sum of nine cubes of prime numbers. The first breakthrough was made by Liu and Zhan [19], who in the former setting showed, subject to the truth of the Generalized Riemann Hypothesis (GRH), that the exponent $\Delta_{2,5} = \frac{1}{10}$ is admissible. Subsequently, they introduced an approach to treating enlarged major arcs [20], and this allowed Liu, Lü and Zhan [18, Theorem 1.3] to establish the same conclusion unconditionally. The sharpest unconditional result at present is due to Kumchev and Li [14, Theorem 1], who prove that $\Delta_{2,5} = \frac{1}{9}$ is admissible. Moreover, when more summands are available, the latter authors show [14, Theorem 5] that one has the admissible exponent

$$\Delta_{2,s} = \begin{cases} \frac{9}{40}\left(\frac{s-4}{s-3}\right), & \text{when } 6 \leqslant s \leqslant 16, \\ \frac{5}{24}, & \text{when } s \geqslant 17. \end{cases}$$

We refer the reader to [1, 2, 3, 16, 21, 24, 25, 30] for further results interpolating those already cited.

Turning next to sums of cubes and higher powers, Meng [28] showed that $\Delta_{3,9} = \frac{1}{66}$ is admissible, subject to the truth of GRH, and Lü and Xu [26, Theorem 1] established this conclusion unconditionally. In general, again subject to the truth of GRH, Meng [29] has shown that the exponent

$$\Delta_{k,s} = \frac{1}{(k-1)2^{2k-1} + 2}$$

is admissible whenever $2 \leqslant k \leqslant 10$ and $s > 2^k$, and Sun and Tang [32, Theorem 2] have established this conclusion unconditionally. It is apparent that, in general, these admissible exponents remain small even when s is large.

When $k \geqslant 2$, we define the integer t_k by putting

$$t_k = \begin{cases} 3, & \text{when } k = 2, \\ k(k-1), & \text{when } k \geqslant 3, \end{cases} \tag{1.2}$$

and define the real number θ_k by putting

$$\theta_k = \begin{cases} \frac{19}{24}, & \text{when } k = 2, \\ \frac{4}{5}, & \text{when } k = 3, \\ \frac{5}{6}, & \text{when } k \geqslant 4. \end{cases} \tag{1.3}$$

The main result of this paper shows that there are large admissible exponents $\Delta_{k,s}$ as soon as $s > 2t_k$.

Theorem 1.1. *Let s and k be integers with $k \geqslant 2$ and $s > 2t_k$. Suppose that $\varepsilon > 0$, that n is a sufficiently large natural number satisfying $n \equiv s \pmod{R}$, and write $X = (n/s)^{1/k}$. Then the equation $n = p_1^k + p_2^k + \ldots + p_s^k$ has a solution in prime numbers p_j with $|p_j - X| \leqslant X^{\theta_k + \varepsilon}$ $(1 \leqslant j \leqslant s)$.*

This theorem shows that the exponent $\Delta_{k,s} = \frac{1}{6}$ is admissible whenever $k \geqslant 2$ and $s > 2t_k$. In contrast to the admissible exponents derived in the previous work cited above, this exponent is bounded away from zero as $k \to \infty$. Moreover, only when $k = 2$ and $s \geqslant 17$ does previous work (of Kumchev and Li [14]) match our new conclusions. We remark that since $\frac{19}{24} = \frac{1}{2}\left(1 + \frac{7}{12}\right)$, this exponent is in some sense half way between the trivial exponent 1, and the exponent $\frac{7}{12}$ that, following the work of Huxley [10], represents the effective limit of our knowledge concerning the asymptotic distribution of prime numbers in short intervals.

Aficionados of the circle method will anticipate that similar conclusions may be established in problems with fewer variables if one seeks instead conclusions valid only for *almost all* integers n, so that there are at most $o(N)$ exceptional integers n not exceeding N, as $N \to \infty$. We say that the exponent $\Delta^*_{k,s}$ is *semi-admissible* when, provided that Δ is a positive number with $\Delta < \Delta^*_{k,s}$, then for almost all positive integers n with $n \equiv s \pmod{R}$, the equation (1.1) has a solution in prime numbers p_j satisfying $|p_j - X| \leqslant X^{1-\Delta}$ $(1 \leqslant j \leqslant s)$. In §9 we establish the following conclusion.

Theorem 1.2. *Let s and k be integers with $k \geqslant 2$ and $s > t_k$, and suppose that $\varepsilon > 0$. Then for almost all positive integers n with $n \equiv s \pmod{R}$ (and, in case $k = 3$ and $s = 7$, satisfying also $9 \nmid n$), the equation $n = p_1^k + p_2^k + \ldots + p_s^k$ has a solution in prime numbers p_j with $|p_j - X| \leqslant X^{\theta_k + \varepsilon}$ $(1 \leqslant j \leqslant s)$, where $X = (n/s)^{1/k}$.*

We note that the additional condition $9 \nmid n$ in the case $k = 3$ and $s = 7$ is required to ensure the solubility of (1.1) modulo 9. Previous work on this topic has focused on smaller k. So far as sums of four squares of primes are concerned, Lü and Zhai [27] showed that the exponent $\Delta^*_{2,4} = \frac{4}{25}$ is semi-admissible. Kumchev and Li [14, Theorem 3] improved this conclusion, showing that $\Delta^*_{2,4} = \frac{9}{50}$ is semi-admissible. Theorem 1.2 improves this result further, showing in particular that $\Delta^*_{2,4} = \frac{5}{24}$ is semi-admissible[1]. We direct the reader to Theorem 9.2 for a particularly sharp exceptional set estimate for sums of 6 squares of almost equal primes. Considering next sums of seven or eight cubes of primes (with the additional local solubility condition for seven cubes of primes implied), Liu and Sun [23, Theorem 1] showed that the exponents

[1]A result of Li and Wu [16, Theorem 3], tantamount to the assertion that $\Delta^*_{2,4} = \frac{9}{40}$ is semi-admissible, contains an infelicity discussed in §9 below.

$\Delta^*_{3,7} = \frac{3}{38}$ and $\Delta^*_{3,8} = \frac{1}{10}$ are semi-admissible[2]. We note also that the recent work of Tang and Zhao [34, Theorem 1] shows in particular that the exponent $\Delta^*_{4,13} = \frac{5}{202}$ is semi-admissible. Theorem 1.2, on the other hand, obtains the considerably stronger semi-admissible exponents $\Delta^*_{3,s} = \frac{1}{5}$ for $s \geqslant 7$ and $\Delta^*_{4,s} = \frac{1}{6}$ for $s \geqslant 13$. Indeed, it follows from our new theorem that whenever $k \geqslant 4$ and $s > k(k-1)$, then the exponent $\Delta^*_{k,s} = \frac{1}{6}$ is always semi-admissible.

We outline our proof of Theorem 1.1, which proceeds via the circle method, in §2. By comparison with previous treatments, this argument contains two novel features. The first is an estimate for moments of exponential sums over kth powers in short intervals, of order $2s$, that achieves essentially optimal estimates as soon as $s \geqslant t_k$. This serves as a substitute for the traditional use of Hua's lemma, though for problems involving short intervals is considerably sharper. In §3 we explain how this estimate follows from the analogous work of Daemen [5, 6], based on his use of the so-called *binomial descent method.* The second novel feature is a substitute for a Weyl-type estimate for exponential sums over variables in short intervals that delivers non-trivial estimates on the minor arcs in a Hardy-Littlewood dissection even when the corresponding major arcs are rather narrow. This estimate again makes use of Daemen's estimates via a bilinear form treatment motivated by analogous arguments making use of Vinogradov's mean value theorem. This argument is described in §4. Both the work in §3 and that in §4 makes heavy use of the latest work [40] concerning Vinogradov's mean value theorem. Our sharpest conclusions for $k = 2$ and 3 require a discussion of estimates of Weyl type particular to these exponents, and this we record in §5. The analysis of the major arc estimates required in our application of the circle method is discussed, in stages, in §6, 7 and 8. Finally, we discuss exceptional set estimates in §9, thereby establishing Theorem 1.2.

Throughout this paper, the letter ε will refer to a small positive number. We adopt the convention that whenever ε occurs in a statement, then the statement holds for all sufficiently small $\varepsilon > 0$. Similarly, we write L for $\log X$, and adopt the convention that whenever L^c occurs in a statement, then the statement holds for some $c > 0$. In addition, as usual, we write $e(z)$ for $e^{2\pi i z}$. Finally, we summarize an inequality of the shape $M < m \leqslant 2M$ by writing $m \sim M$.

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2. Outline of the method

Our basic approach to the application of the circle method is straightforward so far as the Waring-Goldbach problem with almost equal summands is

[2]The argument of Zhao [42, Theorem 1.2], underlying the assertion that $\Delta^*_{3,7} = \frac{3}{25}$ and $\Delta^*_{3,8} = \frac{2}{15}$ are semi-admissible, likewise contains an infelicity discussed in §9 below.

concerned. Suppose that k and s are integers with $k \geqslant 2$ and $s \geqslant t_k$, where t_k is defined as in (1.2). Write

$$K = \begin{cases} 36, & \text{when } k = 2, \\ 2t_k(t_k+2), & \text{when } k > 2. \end{cases} \tag{2.1}$$

Let θ be a real number with $\theta_k < \theta < 1$, and let δ be a sufficiently small, but fixed, positive number with $4K\delta < \min\{\theta - \theta_k, 1-\theta\}$. Consider a sufficiently large natural number N, put $X = (N/s)^{1/k}$, and write $Y = X^\theta$. When n is a natural number with $N \leqslant n \leqslant N + X^{k-1}Y$, we denote by $\rho_s(n)$ the weighted number of solutions of the equation (1.1) with $|p_i - X| \leqslant Y$ $(1 \leqslant i \leqslant s)$ given by

$$\rho_s(n) = \sum_{\substack{|p_1 - X| \leqslant Y \\ p_1^k + \ldots + p_s^k = n}} \ldots \sum_{|p_s - X| \leqslant Y} (\log p_1) \ldots (\log p_s).$$

Define

$$f(\alpha) = \sum_{|p-X| \leqslant Y} (\log p) e(p^k \alpha), \tag{2.2}$$

where the summation is over prime numbers p. Then it follows from orthogonality that

$$\rho_s(n) = \int_0^1 f(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha. \tag{2.3}$$

Next we define the Hardy-Littlewood dissection underpinning our application of the circle method. Write

$$P = X^{2K\delta} \quad \text{and} \quad Q = X^{k-2}Y^2P^{-1}. \tag{2.4}$$

We denote by $\mathfrak{M}$ the union of the major arcs

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant Q^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant P$ and $(a,q) = 1$. Finally, we write $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$ for the set of minor arcs complementary to the set of major arcs $\mathfrak{M}$. When $\mathfrak{B}$ is a measurable subset of $[0,1)$, we now define

$$\rho_s(n;\mathfrak{B}) = \int_{\mathfrak{B}} f(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha. \tag{2.5}$$

Thus, since $[0,1)$ is the disjoint union of $\mathfrak{M}$ and $\mathfrak{m}$, one finds from (2.3) that

$$\rho_s(n) = \rho_s(n;\mathfrak{M}) + \rho_s(n;\mathfrak{m}). \tag{2.6}$$

The analysis of the major arc contribution $\rho_s(n;\mathfrak{M})$ is essentially routine, though as is typical with the Waring-Goldbach problem with almost equal summands, the wide major arcs cause some technical difficulties. When $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta \in \mathbb{R}$, define

$$v(\beta) = k^{-1} \sum_{(X-Y)^k \leqslant m \leqslant (X+Y)^k} m^{-1+1/k} e(\beta m)$$

and

$$S(q,a)=\sum_{\substack{r=1\\(r,q)=1}}^{q} e(ar^k/q).$$

We then define the singular integral

$$\mathfrak{J}(n)=\int_0^1 v(\beta)^s e(-\beta n)\,\mathrm{d}\beta$$

and the singular series

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\varphi(q)^{-s}\sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)^s e(-na/q).$$

We temporarily proceed in greater generality than is demanded by our present choice of parameters, so as to permit future reference to our present discussion. The standard theory of major arc contributions in the Waring-Goldbach problem requires little modification to deliver satisfactory estimates for $\mathfrak{J}(n)$ and $\mathfrak{S}(n)$ (see [9, Lemma 8.12] and [22, Lemmata 6.4 and 8.3]). Thus, whenever $s\geqslant 4$, $Y\geqslant X^{1/2+\delta}$ and $|n-sX^k|\leqslant X^{k-1}Y$, there exists a positive number $\mathfrak{C}$ for which $\mathfrak{J}(n)=\mathfrak{C}Y^{s-1}X^{1-k}$. In addition, whenever $s>\max\{3,k(k-1)\}$ and $n\equiv s \pmod{R(k)}$ (and, in the case $k=3$ and $s=7$, one has in addition $9\nmid n$), there is a positive number $\eta=\eta(s,k)$ for which

$$1\ll\mathfrak{S}(n)\ll(\log X)^{\eta}.$$

Indeed, when $k=2$ and $s=4$, one may take $\eta=1$, and when $s\geqslant 5$ one is at liberty to take $\eta=0$. Thus we conclude that, in the circumstances at hand, one has

$$Y^{s-1}X^{1-k}\ll\mathfrak{S}(n)\mathfrak{J}(n)\ll Y^{s-1}X^{1-k}(\log X)^{\eta}. \tag{2.7}$$

We summarise the analysis of the major arc contribution $\rho_s(n;\mathfrak{M})$ in the form of a proposition.

Proposition 2.1. *Suppose that $k\geqslant 2$ and $s\geqslant\min\{5,k+2\}$. Then, whenever $\frac{19}{24}<\theta<1$, $Y=X^{\theta}$ and n is a natural number with $N\leqslant n\leqslant N+X^{k-1}Y$, one has*

$$\rho_s(n;\mathfrak{M})=\mathfrak{S}(n)\mathfrak{J}(n)+O(Y^{s-1}X^{1-k}(\log X)^{-1}).$$

Proof. The desired conclusion may be established by following the argument of the proof of [14, Proposition 5.1]. The latter argument is superficially restricted to the case $k=2$ and $s=4$, but the generalisation to arbitrary exponents k and $s\geqslant 5$ causes no extra difficulties. We provide details of the necessary argument in §§6–8 below. We note that compared to [14, Proposition 5.1], we have an additional weight $\log p$ within our definition (2.2) of the exponential sum $f(\alpha)$. This again is easily accommodated within the argument of the proof of [14, Proposition 5.1]. The reader may find it illuminating to compare with the argument of the proof of [32, Proposition 2.1], which has the potential to establish the conclusion of our proposition subject to the more restricted hypotheses $s\geqslant 5$ and $Y\geqslant X^{4/5+\varepsilon}$. □

Suppose that $k \geqslant 2$, $s > t_k$ and $Y > X^\theta$. Then by combining (2.7) with the conclusion of Proposition 2.1, it follows that whenever $n \equiv s \pmod{R(k)}$ (and, in case $k = 3$ and $s = 7$, one has also $9 \nmid n$), then

$$\rho_s(n; \mathfrak{M}) \gg Y^{s-1} X^{1-k}. \tag{2.8}$$

In order to estimate the minor arc contribution $\rho_s(n; \mathfrak{m})$, in §3 we prepare an analogue of Hua's lemma. We recall the definition of t_k from (1.2).

Proposition 2.2. *Suppose that Y is a real number with $Y \geqslant X^{1/2}$. Then whenever $s \geqslant 2t_k$ and $\varepsilon > 0$, one has*

$$\int_0^1 |f(\alpha)|^s \, \mathrm{d}\alpha \ll Y^{s-1} X^{1-k+\varepsilon}.$$

Next, in §4, we establish an estimate of Weyl-type that delivers non-trivial estimates throughout the set of minor arcs $\mathfrak{m}$. Recall here the definition of K given in (2.1).

Proposition 2.3. *Let θ be a real number with $\frac{5}{6} < \theta < 1$, and suppose that X and Y are real numbers with $X^\theta \leqslant Y \leqslant X$. Then, whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha - a/q| \leqslant q^{-2}$, one has*

$$f(\alpha) \ll X^\varepsilon Y \left(\Xi^{-1} + X^{-1/2} + \Xi Y^{-2} X^{2-k} \right)^{1/K},$$

where $\Xi = q + Y^2 X^{k-2} |q\alpha - a|$.

We are now equipped to dispose of the minor arc contribution. Suppose that $\alpha \in \mathfrak{m}$. By Dirichlet's theorem on Diophantine approximation, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $q \leqslant Q$, $(a, q) = 1$ and $|q\alpha - a| \leqslant Q^{-1}$. The definition of $\mathfrak{m}$ ensures that $q > P$, and thus when $k \geqslant 4$ and $Y = X^\theta$, Proposition 2.3 combines with the definition (2.4) to deliver the bound

$$f(\alpha) \ll X^\varepsilon Y (P^{-1} + X^{-1/2} + Q Y^{-2} X^{2-k})^{1/K} \ll X^{\varepsilon - 2\delta} Y.$$

Meanwhile, when $k = 2$ or 3, we deduce from the conclusion of §5 that one has likewise the bound

$$f(\alpha) \ll X^{\varepsilon - 2\delta} Y. \tag{2.9}$$

We therefore conclude from Proposition 2.2 that whenever $s > 2t_k$, then for each $\varepsilon > 0$, one has

$$\begin{aligned} \rho_s(n; \mathfrak{m}) &\leqslant \Big(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \Big)^{s - 2t_k} \int_{\mathfrak{m}} |f(\alpha)|^{2t_k} \, \mathrm{d}\alpha \\ &\ll (X^{\varepsilon - 2\delta} Y)^{s - 2t_k} Y^{2t_k - 1} X^{1-k+\varepsilon} \ll Y^{s-1} X^{1-k-\delta}. \end{aligned}$$

Combining this estimate with (2.6) and (2.8), we see that under the hypotheses of Theorem 1.1, one has the lower bound $\rho_s(n) \gg Y^{s-1} X^{1-k}$, and thus the proof of Theorem 1.1 is complete.

3. Mean value estimates for primes in short intervals

Recall the definition (2.2) of the exponential sum $f(\alpha)$. A heuristic application of the circle method suggests that for all positive integers t, and for all real numbers Y with $X^{1/2} \leqslant Y \leqslant X$, one should have the bound

$$\int_0^1 |f(\alpha)|^{2t}\,\mathrm{d}\alpha \ll Y^t(\log X)^t + Y^{2t-1}X^{1-k}. \tag{3.1}$$

It may be shown using Hua's lemma, meanwhile, that when $2t \geqslant 2^k$, then

$$\int_0^1 |f(\alpha)|^{2t}\,\mathrm{d}\alpha \ll X^\varepsilon Y^{2t-k}. \tag{3.2}$$

It is apparent that when $2t \geqslant 2^k$ and $Y^{t-1} \geqslant X^{k-1}$, the bound (3.1) is sharper than (3.2) by a factor $(Y/X)^{k-1}$. It transpires that the methods of Daemen [5, 6] permit the proof of a serviceable substitute for (3.1) when $t > t_k$, and it is this that explains in part the relative success of our approach over that of previous authors.

In order to proceed further, we must introduce some additional notation. Let

$$F(\alpha) = \sum_{|m-X|\leqslant Y} e(m^k\alpha). \tag{3.3}$$

Also, denote by $J_{s,k}(X)$ the number of integral solutions of the Diophantine system

$$\sum_{i=1}^s (x_i^j - y_i^j) = 0 \quad (1 \leqslant j \leqslant k),$$

with $1 \leqslant x_i, y_i \leqslant X$ $(1 \leqslant j \leqslant k)$.

Lemma 3.1. *Suppose that k and t are natural numbers with $k \geqslant 2$. Then whenever $X^{1/3} \leqslant Y \leqslant X$, one has*

$$\int_0^1 |F(\alpha)|^{2t}\,\mathrm{d}\alpha \ll (1 + Y^2/X)X^{2-k}Y^{\frac{1}{2}k(k+1)-3}J_{t,k}(Y).$$

Proof. This is essentially [5, Theorem 3]. The slightly smaller lower bound on Y is accommodated by employing precisely the same proof as described in the latter source. □

The latest developments on Vinogradov's mean value theorem (for example, see [37, 38, 40]) supply more powerful estimates for $J_{s,k}(X)$ than were available to Daemen.

Lemma 3.2. *When $k \geqslant 2$ and $t \geqslant t_k$, one has $J_{t,k}(X) \ll X^{2t-\frac{1}{2}k(k+1)+\varepsilon}$.*

Proof. The estimate $J_{3,2}(X) \ll X^{3+\varepsilon}$ is immediate from [9, Theorem 7]. Meanwhile, when $k \geqslant 3$ and $t \geqslant k(k-1)$, the bound $J_{t,k}(X) \ll X^{2t-\frac{1}{2}k(k+1)+\varepsilon}$ is supplied by [40, Theorem 1.2]. □

We now establish Proposition 2.2. Suppose that Y is a real number with $X^{1/2} \leqslant Y \leqslant X$, and that $s \geqslant 2t_k$. Then by applying the trivial estimate $|f(\alpha)| = O(Y)$, we find that

$$\int_0^1 |f(\alpha)|^s \,\mathrm{d}\alpha \ll Y^{s-2t_k} I(t_k), \tag{3.4}$$

where

$$I(t) = \int_0^1 |f(\alpha)|^{2t} \,\mathrm{d}\alpha.$$

On recalling (2.2), one finds by orthogonality that $I(t)$ counts the number of solutions of the equation

$$\sum_{i=1}^{t} (p_i^k - p_{i+t}^k) = 0,$$

with p_i prime and $|p_i - X| \leqslant Y$ $(1 \leqslant i \leqslant t)$, in which each solution is counted with weight

$$\prod_{i=1}^{2t} (\log p_i) \ll (\log X)^{2t}.$$

On considering the number of solutions of the underlying Diophantine equation, we therefore see from (3.3) that

$$I(t) \ll (\log X)^{2t} \int_0^1 |F(\alpha)|^{2t} \,\mathrm{d}\alpha.$$

Notice that primality has now made an exit from the discussion. We thus deduce from Lemma 3.1 that

$$I(t) \ll (1 + Y^2/X) X^{2-k} Y^{\frac{1}{2}k(k+1)-3} J_{t,k}(Y) (\log X)^{2t},$$

whence by Lemma 3.2 and the hypothesis $Y \geqslant X^{1/2}$, one obtains the bound

$$I(t_k) \ll X^{1-k} Y^{\frac{1}{2}k(k+1)-1} (\log X)^{2t} Y^{2t_k - \frac{1}{2}k(k+1)+\varepsilon} \ll X^{1-k+2\varepsilon} Y^{2t_k - 1}.$$

The conclusion of Proposition 2.2 is confirmed by substituting this estimate into (3.4).

4. Estimates of Weyl type, I: $k \geqslant 4$

Our goal in this section is the proof of Proposition 2.3. We begin with an auxiliary lemma concerning exponential sums having a bilinear structure. Let $a(m)$ and $b(n)$ be arithmetic functions satisfying the property that for all natural numbers m and n, one has

$$a(m) \ll m^{\varepsilon} \quad \text{and} \quad b(n) \ll n^{\varepsilon}. \tag{4.1}$$

Let M and N be positive parameters, and define the exponential sum $T(\alpha) = T(\alpha; M, N)$ by

$$T(\alpha; M, N) = \max_{M' \leqslant 2M} \max_{N' \leqslant 2N} \Bigg| \sum_{M < m \leqslant M'} a(m) \sum_{\substack{N < n \leqslant N' \\ x < mn \leqslant x+y}} b(n) e((mn)^k \alpha) \Bigg|. \tag{4.2}$$

Lemma 4.1. *Let $c > 0$ be fixed, and let x and y be positive numbers with $x^{3/4} \leqslant y \leqslant x$. Suppose that M and N are positive numbers with $MN \asymp x$ and $cx^2/y^2 \leqslant N \leqslant c^{-1}y^2/x$. Suppose in addition that $t \geqslant t_k$ and $2w \geqslant t_k + 2$. Then whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$ and $|\alpha - a/q| \leqslant q^{-2}$, one has*

$$T(\alpha; M, N) \ll_c x^\varepsilon y \left(\frac{1}{q} + \frac{x}{yN^k} + \frac{N^k}{x^{k-1}y} + \frac{q}{x^{k-2}y^2} \right)^{1/(4tw)}.$$

Proof. By means of a double application of Hölder's inequality, the bilinear structure of $T(\alpha; M, N)$ permits a familiar linearisation argument to be executed. The difficulty to be faced is that of handling the short interval constraint on the implicit product of variables. Throughout this proof, implicit constants may depend on c. Let M' and N' be real numbers corresponding to the maxima in (4.2). Given a $2w$-tuple $\mathbf{n}$, write

$$\widetilde{b}(\mathbf{n}) = b(n_1) \dots b(n_w) \overline{b(n_{w+1})} \dots \overline{b(n_{2w})}$$

and

$$\sigma(\mathbf{n}) = \sum_{i=1}^{w} (n_i^k - n_{w+i}^k).$$

Then in view of the hypothesis (4.1), an application of Hölder's inequality leads from (4.2) to the bound

$$|T(\alpha)|^{2w} \ll x^\varepsilon y^{2w} T_1(\alpha), \tag{4.3}$$

where

$$T_1(\alpha) = (M/y)^{2w} M^{-1} \sum_{m \sim M} \sum_{\mathbf{n}} \widetilde{b}(\mathbf{n}) e(m^k \sigma(\mathbf{n}) \alpha), \tag{4.4}$$

in which the summation is over $2w$-tuples $\mathbf{n}$ satisfying

$$N < n_i \leqslant N' \quad \text{and} \quad x/m < n_i \leqslant (x+y)/m \quad (1 \leqslant i \leqslant 2w).$$

By interchanging the order of summation in (4.4), we obtain the bound

$$T_1(\alpha) \ll (M/y)^{2w} M^{-1} \sum_{n_1 \sim N} \dots \sum_{n_{2w} \sim N} |\widetilde{b}(\mathbf{n})| \left| \sum_{m \sim M} e(m^k \sigma(\mathbf{n}) \alpha) \right|, \tag{4.5}$$

where the summation over m is subject to the constraint

$$x/n_i < m \leqslant (x+y)/n_i \quad (1 \leqslant i \leqslant 2w). \tag{4.6}$$

Next, invoking symmetry, one discerns that the bound (4.5) remains valid if we impose the additional condition $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_{2w}$. The condition (4.6) then becomes $x/n_1 < m \leqslant (x+y)/n_{2w}$, a constraint that implies the relation $n_{2w} < n_1(1 + y/x)$. We therefore see that, for the $2w$-tuples $\mathbf{n}$ at hand, one has

$$|n_{i+1}^k - n_{w+i}^k| \leqslant (n_1(1+y/x))^k - n_1^k \leqslant k(n_1 y/x) \left(n_1(1+y/x)\right)^{k-1},$$

whence

$$\left| \sum_{i=1}^{w-1} (n_{i+1}^k - n_{w+i}^k) \right| \leqslant wk(2n_1)^k y/x.$$

Denote by $\kappa_{\mu,\nu}(u)$ the number of integral solutions of the equation

$$\sum_{i=1}^{w-1}(n_{i+1}^k - n_{w+i}^k) = u,$$

with

$$\mu \leqslant n_2 \leqslant \ldots \leqslant n_{2w-1} \leqslant \nu.$$

Then, again applying (4.1), we arrive at the upper bound

$$T_1(\alpha) \ll x^{\varepsilon}(M/y)^{2w}M^{-1} \sum_{\substack{N<\mu\leqslant\nu\leqslant 2N \\ \nu<\mu(1+y/x)}} \sum_u \kappa_{\mu,\nu}(u)|T_2(\alpha;\mu,\nu;u)|, \tag{4.7}$$

where the summation over u is subject to the condition

$$|u| \leqslant wk(2\mu)^k y/x, \tag{4.8}$$

and

$$T_2(\alpha;\mu,\nu;u) = \sum_{\substack{m\sim M \\ x/\mu<m\leqslant(x+y)/\nu}} e(m^k(\mu^k-\nu^k-u)\alpha). \tag{4.9}$$

Notice that the variables μ and ν assist in encoding the short interval constraints implict on the variables. As we shall see, there is some loss associated with isolating these two variables, though this is largely recovered through diligent accountancy.

Suppose next that $\nu < \mu(1+y/x)$, consistent with the summation condition in (4.7). Then another application of Hölder's inequality conveys us from (4.9) to the bound

$$\sum_u \kappa_{\mu,\nu}(u)|T_2(\alpha;\mu,\nu;u)| \leqslant T_3(\mu,\nu)^{1-1/t}T_4(\mu,\nu)^{1/(2t)}T_5(\alpha;\mu,\nu)^{1/(2t)}, \tag{4.10}$$

where

$$T_3(\mu,\nu) = \sum_u \kappa_{\mu,\nu}(u), \quad T_4(\mu,\nu) = \sum_u \kappa_{\mu,\nu}(u)^2, \tag{4.11}$$

$$T_5(\alpha;\mu,\nu) = \sum_u |T_2(\alpha;\mu,\nu;u)|^{2t}, \tag{4.12}$$

and the summations over u are again subject to (4.8).

We estimate the sums T_3, T_4 and T_5 in turn. Note that our hypothesis $cx^2/y^2 \leqslant N$ ensures that, in the situation at hand, one has

$$(\mu y/x)^2\mu^{-1} \geqslant (y^2/x^2)N \geqslant c,$$

whence $\mu y/x \geqslant (c\mu)^{1/2}$. Then the definition of $\kappa_{\mu,\nu}(u)$ takes us from (4.11) to the bound

$$T_3(\mu,\nu) \leqslant (\nu-\mu+1)^{2w-2} \ll (\mu y/x)^{2w-2}. \tag{4.13}$$

Next, when U and V are non-negative numbers, define the Weyl sum

$$W(\alpha;U,V) = \sum_{U\leqslant m\leqslant U+V} e(m^k\alpha). \tag{4.14}$$

Since $\mu y/x \geqslant (c\mu)^{1/2}$, we are led from (4.11) via orthogonality and Lemma 3.1 to the bound

$$\begin{aligned} T_4(\mu,\nu) &\leqslant \int_0^1 |W(\alpha;\mu,\nu-\mu)|^{4w-4}\,\mathrm{d}\alpha \\ &\ll (\mu y/x)^{\frac{1}{2}k(k+1)-1}\mu^{1-k}J_{2w-2,k}(\mu y/x). \end{aligned}$$

Provided that $N<\mu\leqslant\nu\leqslant 2N$ and $2w-2\geqslant t_k$, therefore, we deduce from Lemma 3.2 that

$$T_4(\mu,\nu) \ll x^\varepsilon \mu^{1-k}(\mu y/x)^{4w-5}. \tag{4.15}$$

Recall that $MN \asymp x$, and that the ambient summation conditions ensure that $N\leqslant\mu\leqslant 2N$. Then by substituting (4.13) and (4.15) into (4.10), we deduce from (4.7) that

$$\begin{aligned} |T_1(\alpha)|^{2t} &\ll x^\varepsilon (M/y)^{4tw}M^{-2t}(N^2y/x)^{2t}(Ny/x)^{(2w-2)(2t-2)} \\ &\quad\times N^{1-k}(Ny/x)^{4w-5}\max_{\mu,\nu} T_5(\alpha;\mu,\nu) \\ &\ll x^\varepsilon (N^ky/x)^{-1}(y/N)^{-2t}\max_{\mu,\nu} T_5(\alpha;\mu,\nu), \end{aligned} \tag{4.16}$$

where the maximum over μ and ν is subject to the conditions

$$N<\mu\leqslant\nu\leqslant 2N \quad\text{and}\quad \nu<\mu(1+y/x). \tag{4.17}$$

We next turn to the estimation of $T_5(\alpha;\mu,\nu)$. Denote by $\omega_{\mu,\nu}(v)$ the number of integral solutions of the equation

$$\sum_{i=1}^t (m_i^k - m_{t+i}^k) = v,$$

with

$$x/\mu < m_i \leqslant (x+y)/\nu \quad\text{and}\quad m_i\sim M \quad (1\leqslant i\leqslant 2t). \tag{4.18}$$

Note that our hypothesis $N\leqslant c^{-1}y^2/x$ combines with (4.17) to ensure that, in the situation at hand, one has

$$(y/\nu)^2(x/\nu)^{-1} \geqslant (y^2/x)(2N)^{-1} \geqslant \tfrac{1}{2}c,$$

whence $y/\nu \geqslant (\frac{1}{2}c)^{1/2}(x/\nu)^{1/2}$. Observe also that since $\mu\leqslant\nu$, one has $\omega_{\mu,\nu}(v)\leqslant\omega_{\nu,\nu}(v)$. Then on recalling the definition (4.14), we are led via orthogonality and Lemma 3.1 to the bound

$$\begin{aligned} \omega_{\mu,\nu}(v) &\leqslant \int_0^1 |W(\alpha;x/\nu,y/\nu)|^{2t}\,\mathrm{d}\alpha \\ &\ll (y/\nu)^{\frac{1}{2}k(k+1)-1}(x/\nu)^{1-k}J_{t,k}(y/\nu). \end{aligned}$$

Provided that $N<\mu\leqslant\nu\leqslant 2N$ and $t\geqslant t_k$, therefore, we deduce from Lemma 3.2 that

$$\omega_{\mu,\nu}(v) \ll (x/\nu)^{1-k}(y/\nu)^{2t-1+\varepsilon}. \tag{4.19}$$

Observe next that the conditions (4.18) ensure that

$$\left|\sum_{i=1}^{t}(m_i^k - m_{t+i}^k)\right| \leqslant tk(y/\nu)\left((x+y)/\nu\right)^{k-1}.$$

Then by expanding (4.12) and interchanging the order of summation, we find that

$$T_5(\alpha;\mu,\nu) \leqslant \sum_v \omega_{\nu,\nu}(v)\left|\sum_u e(v(\mu^k - \nu^k - u)\alpha)\right|,$$

where the summations over u and v are subject to the conditions (4.8) and

$$|v| \leqslant tk(y/\nu)(2x/\nu)^{k-1}.$$

We therefore deduce from (4.19) that

$$\begin{aligned} T_5(\alpha;\mu,\nu) &\ll \sum_v \omega_{\nu,\nu}(v)\min\{N^k y/x, \|v\alpha\|^{-1}\} \\ &\ll (x/\nu)^{1-k}(y/\nu)^{2t-1+\varepsilon}\sum_v \min\{x^{k-2}y^2/v, \|v\alpha\|^{-1}\}. \end{aligned}$$

Thus, as a consequence of [36, Lemma 2.2], we conclude that

$$(N^k y/x)^{-1}(y/N)^{-2t}\max_{\mu,\nu} T_5(\alpha;\mu,\nu) \ll (qx)^{\varepsilon}\Theta, \tag{4.20}$$

where, in view of the hypotheses of the lemma concerning α, one has

$$\Theta \ll \frac{1}{q} + \frac{x}{yN^k} + \frac{N^k}{x^{k-1}y} + \frac{q}{x^{k-2}y^2}.$$

The conclusion of the lemma now follows by substituting (4.20) into (4.16), and thence into (4.3). □

In advance of the next lemma, we introduce the positive real number

$$\sigma_k = (8t_k)^{-1} \quad (k \geqslant 2). \tag{4.21}$$

The following result is an analogue for short intervals of [12, Lemma 3.2]. We proceed in greater generality than is necessary for the application within this paper, since this conclusion is likely to find application elsewhere and the additional generality comes at little cost.

Lemma 4.2. *Let x and y be positive numbers with $1 \leqslant y \leqslant x$, and suppose that σ is a real number with $0 < \sigma \leqslant \sigma_k$. Suppose also that M and N are positive numbers with $(MN)^{2/3-\sigma} \leqslant x^{-1}y^{5/3-\sigma}$,*

$$M^{1-2\sigma}N^{2-2\sigma} \leqslant x^{1-2/k}y^{2/k-2\sigma} \quad \text{and} \quad M^{2-2k\sigma}N^{-2k\sigma} \leqslant x^{1-2/k}y^{2/k-2k\sigma}. \tag{4.22}$$

Suppose that (a_m), (b_n) and (c_n) are sequences of complex numbers satisfying $|a_m| \leqslant 1 + \log m$ and $|b_n| \leqslant 1$ for each m and n, and with $c_l = 1$ for all l, or $c_l = \log l$ for all l. Suppose further that α is a real number, and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying

$$(a,q) = 1, \quad 1 \leqslant q \leqslant (x^{k-2}y^2)^{1/2} \quad \text{and} \quad |q\alpha - a| \leqslant (x^{k-2}y^2)^{-1/2}. \tag{4.23}$$

Then one has

$$\sum_{1\leqslant m\leqslant M} a_m \sum_{1\leqslant n\leqslant N} b_n \sum_{|x-lmn|\leqslant y} c_l e\left((lmn)^k\alpha\right)$$
$$\ll y^{1-\sigma+\varepsilon}(MN)^\sigma + \frac{y^{1+\varepsilon}}{(q+x^{k-1}y|q\alpha-a|)^{1/k^2}}.$$

Proof. Write

$$\Psi(\theta)=\sum_{|l-x/(mn)|\leqslant y/(mn)} c_l e(l^k\theta).$$

The reader may find it useful to note that this function will shortly be applied with $\theta=(mn)^k\alpha$. Put $\mathcal{Q}=(y/(mn))^{1/3}$, and define $\mathfrak{W}$ to be the union of the arcs

$$\mathfrak{W}(q,a)=\{\alpha\in[0,1):|\theta-a/q|\leqslant \mathcal{Q}(x/(mn))^{2-k}(y/(mn))^{-2}\},$$

with $0\leqslant a\leqslant q\leqslant \mathcal{Q}$ and $(a,q)=1$. Also, put $\mathfrak{w}=[0,1)\setminus\mathfrak{W}$. Then on noting that (4.21) implies that $(2t_k)^{-1}=4\sigma_k$, it follows from the argument underlying the proof of [5, equation (3.5)] that

$$\sup_{\theta\in\mathfrak{w}}|\Psi(\theta)|\ll (y/(mn))\mathcal{Q}^{\varepsilon-1/(2t_k)}\ll (y/(mn))^{1-\sigma_k}. \tag{4.24}$$

Meanwhile, from [5, equations (5.1)-(5.5) and §6], we see that when $\theta\in\mathfrak{W}(q,a)\subseteq\mathfrak{W}$, one has

$$\Psi(\theta)\ll \frac{y/(mn)}{(q+(x/(mn))^{k-1}(y/(mn))|q\theta-a|)^{1/k}}+\Delta, \tag{4.25}$$

where

$$\Delta\ll \mathcal{Q}^{1/2+\varepsilon}\left(1+\frac{\mathcal{Q}(x/(mn))^k}{(x/(mn))^{k-2}(y/(mn))^2}\right)^{1/2}\ll \mathcal{Q}^{1+\varepsilon}x/y.$$

Provided that $(mn)^{2/3-\sigma_k}\leqslant x^{-1}y^{5/3-\sigma_k}$, one discerns that

$$\Delta\ll (y/(mn))^{1/3+\varepsilon}x/y\leqslant (y/(mn))^{1-\sigma_k+\varepsilon}.$$

In combination with (4.25), therefore, we conclude that whenever $0<\sigma\leqslant\sigma_k$, then

$$\Psi(\theta)\ll (y/(mn))^{1-\sigma+\varepsilon}+\frac{y(mn)^{-1}}{(q+(x/(mn))^{k-1}(y/(mn))|q\theta-a|)^{1/k}}. \tag{4.26}$$

For each integer m with $1\leqslant m\leqslant M$, denote by $\mathcal{N}$ the set of natural numbers n with $1\leqslant n\leqslant N$ for which there exist integers b and r with $(b,r)=1$,

$$1\leqslant r\leqslant \tfrac{1}{3}\left(\frac{y}{mn}\right)^{k\sigma},\quad |r(mn)^k\alpha-b|\leqslant \tfrac{1}{2}\left(\frac{x}{mn}\right)^{2-k}\left(\frac{y}{mn}\right)^{k\sigma-2}. \tag{4.27}$$

Now consider the situation with $\theta=(mn)^k\alpha$. By Dirichlet's theorem on Diophantine approximation, there exist integers b and r with $(b,r)=1$ and

$$1\leqslant r\leqslant \mathcal{Q}^{-1}\left(\frac{x}{mn}\right)^{k-2}\left(\frac{y}{mn}\right)^2$$

such that

$$|(mn)^k\alpha - b/r| \leqslant r^{-1}\mathcal{Q}\left(\frac{x}{mn}\right)^{2-k}\left(\frac{y}{mn}\right)^{-2}.$$

We distinguish two cases, case (A) in which $1 \leqslant r \leqslant \mathcal{Q}$, and case (B) in which $r > \mathcal{Q}$. First, in case (A), one has $1 \leqslant r \leqslant \mathcal{Q}$, and thus it follows that $(mn)^k\alpha \in \mathfrak{W}$, whence from (4.26) one sees that

$$\Psi((mn)^k\alpha) \ll \left(\frac{y}{mn}\right)^{1-\sigma+\varepsilon} + \frac{y(mn)^{-1}}{(r+(x/(mn))^{k-1}(y/(mn))|(mn)^k r\alpha - b|)^{1/k}}.$$

We divide the analysis here into two subcases, that in which $n \notin \mathcal{N}$ that we presently discuss, and another case with $n \in \mathcal{N}$ that we defer for future attention. Observe that in the first of these cases, where $n \notin \mathcal{N}$, it follows that either $r > \frac{1}{3}(y/(mn))^{k\sigma}$, or else that

$$|r(mn)^k\alpha - b| > \tfrac{1}{2}\left(\frac{x}{mn}\right)^{2-k}\left(\frac{y}{mn}\right)^{k\sigma-2},$$

whence

$$\Psi((mn)^k\alpha) \ll \left(\frac{y}{mn}\right)^{1-\sigma+\varepsilon}.$$

Meanwhile, in case (B) one has $r > \mathcal{Q}$, and here one discerns that $(mn)^k\alpha \in \mathfrak{w}$. Thus we deduce from (4.24) that

$$\Psi((mn)^k\alpha) \ll (y/(mn))^{1-\sigma+\varepsilon}.$$

The discussion of the previous paragraph supplies the estimate

$$\sum_{1\leqslant m\leqslant M} a_m \sum_{1\leqslant n\leqslant N} b_n \sum_{|x-lmn|\leqslant y} c_l e\left((lmn)^k\alpha\right) \ll E_0 + E_1, \tag{4.28}$$

where

$$E_0 = \sum_{1\leqslant m\leqslant M}\sum_{1\leqslant n\leqslant N} |a_m b_n| \left(\frac{y}{mn}\right)^{1-\sigma+\varepsilon}$$

and

$$E_1 = \sum_{1\leqslant m\leqslant M}\sum_{n\in\mathcal{N}} |a_m b_n| \frac{y(mn)^{-1}\log x}{(r+(x/(mn))^{k-1}(y/(mn))|(mn)^k r\alpha - b|)^{1/k}}.$$

Notice that E_1 contains the contribution of the terms arising in the second subcase of case (A). We note also that in case $c_l = \log l$ for all l, then the desired conclusion follows in like manner from the aforementioned work of Daemen by partial summation. In view of our hypotheses concerning (a_m) and (b_n), one has

$$E_0 \ll y^{1-\sigma+\varepsilon}(MN)^\sigma. \tag{4.29}$$

Also, it is evident from an application of Hölder's inequality that

$$E_1 \ll (\log x) E_2^{1-1/k}(yE_3)^{1/k}, \tag{4.30}$$

where

$$E_2 = \sum_{1\leqslant m\leqslant M}\sum_{n\in\mathcal{N}} |a_m b_n|^{k/(k-1)} y/(mn) \ll y^{1+\varepsilon} \tag{4.31}$$

and

$$E_3 = \sum_{1\leqslant m\leqslant M} \sum_{n\in\mathcal{N}} \frac{(mn)^{-1}}{r + (x/(mn))^{k-1}(y/(mn))|(mn)^k r\alpha - b|}. \tag{4.32}$$

For each integer m with $1 \leqslant m \leqslant M$, we apply Dirichlet's approximation theorem to deduce the existence of $c \in \mathbb{Z}$ and $s \in \mathbb{N}$ with $(c,s)=1$,

$$1 \leqslant s \leqslant \left(\frac{x}{mN}\right)^{k-2}\left(\frac{y}{mN}\right)^{2-k\sigma} \quad \text{and} \quad |sm^k\alpha - c| \leqslant \left(\frac{x}{mN}\right)^{2-k}\left(\frac{y}{mN}\right)^{k\sigma-2}. \tag{4.33}$$

By combining (4.27) and (4.33), we obtain the bound

$$\begin{aligned}|rn^k c - sb| &\leqslant rn^k\left(\frac{x}{mN}\right)^{2-k}\left(\frac{y}{mN}\right)^{k\sigma-2} + \tfrac{1}{2}s\left(\frac{x}{mn}\right)^{2-k}\left(\frac{y}{mn}\right)^{k\sigma-2}\\ &\leqslant \tfrac{1}{2} + \tfrac{1}{3}\left(\frac{x}{m}\right)^{2-k}\left(\frac{y}{m}\right)^{2k\sigma-2}(nN)^{k-k\sigma}.\end{aligned}$$

Then it follows from (4.22) that $|rn^k c - sb| < 1$, whence

$$\frac{b}{rn^k} = \frac{c}{s} \quad \text{and} \quad r = \frac{s}{(s,n^k)}.$$

We therefore deduce from (4.32) that

$$E_3 = \sum_{1\leqslant m\leqslant M} \frac{m^{-1}}{s + (x/m)^{k-1}(y/m)|m^k s\alpha - c|} \sum_{n\in\mathcal{N}} \frac{(s,n^k)}{n}.$$

Observe next that

$$\sum_{1\leqslant n\leqslant N} \frac{(s,n^k)}{n} \leqslant \sum_{d|s} \sum_{\substack{1\leqslant n\leqslant N\\ n^k\equiv 0 \pmod{d}}} \frac{d}{n} \ll \sum_{d|s} dv_k(d)^{-1}\log N,$$

in which $v_k(w)$ is the multiplicative function of w defined by taking

$$v_k(p^{uk+v}) = p^{u+1},$$

for prime numbers p, when $u \geqslant 0$ and $1 \leqslant v \leqslant k$. Hence we deduce that

$$\sum_{1\leqslant n\leqslant N} \frac{(s,n^k)}{n} \ll (\log N)\sum_{d|s} d^{1-1/k} \ll s^{1-1/k+\varepsilon}\log N,$$

whence

$$E_3 \ll \sum_{1\leqslant m\leqslant M} \frac{s^{\varepsilon-1/k}\log N}{m\,(1 + (x/m)^{k-1}(y/m)|m^k\alpha - c/s|)}. \tag{4.34}$$

We next define $\mathcal{M}$ to be the set of natural numbers m with $1 \leqslant m \leqslant M$ such that the integers c and s defined in (4.33) satisfy

$$1 \leqslant s \leqslant \tfrac{1}{3}\left(\frac{y}{MN}\right)^{k^2\sigma}, \quad |sm^k\alpha - c| \leqslant \tfrac{1}{3}\left(\frac{y}{MN}\right)^{k^2\sigma}\left(\frac{x}{m}\right)^{1-k}\left(\frac{y}{m}\right)^{-1}. \tag{4.35}$$

In view of (4.34), we find that

$$E_3 \ll \sum_{m\in\mathcal{M}} \frac{s^{\varepsilon-1/k}\log N}{m\,(1 + (x/m)^{k-1}(y/m)|m^k\alpha - c/s|)} + y^{\varepsilon-k\sigma}(MN)^{k\sigma}. \tag{4.36}$$

When a and q satisfy (4.23) and $m \in \mathcal{M}$, it follows from (4.35) that

$$\begin{aligned}|sm^k a - qc| &\leqslant sm^k(x^{k-2}y^2)^{-1/2} + \tfrac{1}{3}q\left(\frac{y}{MN}\right)^{k^2\sigma}\left(\frac{x}{m}\right)^{1-k}\left(\frac{y}{m}\right)^{-1}\\ &\leqslant \tfrac{1}{3}\left(x^{1-k/2}y^{k^2\sigma-1} + x^{-k/2}y^{k^2\sigma}\right)M^{k-k^2\sigma}N^{-k^2\sigma}.\end{aligned}$$

The second condition of (4.22) therefore reveals that

$$|sm^k a - qc| < 1,$$

whence

$$\frac{c}{sm^k} = \frac{a}{q} \quad \text{and} \quad s = \frac{q}{(q, m^k)}.$$

We therefore deduce that

$$\begin{aligned}&\sum_{m\in\mathcal{M}} \frac{s^{\varepsilon-1/k}\log N}{m\,(1+(x/m)^{k-1}(y/m)|m^k\alpha - c/s|)}\\ &\qquad \leqslant \frac{q^\varepsilon \log N}{1+x^{k-1}y|\alpha - a/q|}\sum_{1\leqslant m\leqslant M}\frac{(q/(q,m^k))^{-1/k}}{m}.\end{aligned}$$

In this case, we observe as above that

$$\begin{aligned}\sum_{1\leqslant m\leqslant M}\frac{(q/(q,m^k))^{-1/k}}{m} &\leqslant \sum_{d|q}\sum_{\substack{1\leqslant m\leqslant M\\ m^k\equiv 0 \ (\mathrm{mod}\ d)}}\frac{(q/d)^{-1/k}}{m}\\ &\ll \sum_{d|q}(q/d)^{-1/k}v_k(d)^{-1}\log M\\ &\ll (\log M)q^{-1/k}\sum_{d|q}1 \ll q^{\varepsilon-1/k}\log M.\end{aligned}$$

We therefore infer that

$$\sum_{m\in\mathcal{M}} \frac{s^{\varepsilon-1/k}\log N}{m\,(1+(x/m)^{k-1}(y/m)|m^k\alpha - c/s|)} \ll \frac{q^{\varepsilon-1/k}(\log x)^2}{1+x^{k-1}y|\alpha - a/q|},$$

whence by (4.36) we find that

$$E_3 \ll y^{\varepsilon-k\sigma}(MN)^{k\sigma} + \frac{(\log x)^2}{(q+x^{k-1}y|q\alpha - a|)^{1/k}}.$$

On substituting this last estimate together with (4.31) into (4.30), we deduce that

$$E_1 \ll y^{1-\sigma+\varepsilon}(MN)^\sigma + \frac{y^{1+\varepsilon}}{(q+x^{k-1}y|q\alpha - a|)^{1/k^2}}.$$

The conclusion of the lemma now follows by substituting this estimate along with (4.29) into (4.28). □

We now turn to the problem of estimating the exponential sum $f(\alpha)$ defined in (2.2). We suppose throughout that $\frac{5}{6} < \theta \leqslant 1$, $\sigma = \sigma_k$ and $Y = X^\theta$, and further that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$ and $|\alpha - a/q| \leqslant q^{-2}$. Let $\Lambda(n)$ denote the von Mangoldt function, defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{when } n = p^l \text{ for some prime } p \text{ and natural number } l, \\ 0, & \text{otherwise,} \end{cases}$$

and let $\mu(n)$ denote the Möbius function. Suppose that p is a prime number and $l \geqslant 2$. Then whenever $|p^l - X| \leqslant Y$, one has $|p - X^{1/l}| \leqslant YX^{-1+1/l}$, and hence

$$f(\alpha) = \sum_{|n-X|\leqslant Y} \Lambda(n) e(n^k\alpha) + O(YX^{-1/2}). \tag{4.37}$$

We put $U = V = 4X^{2-2\theta}$, and apply Vaughan's identity (see [35]) in the shape

$$\Lambda(n) = \sum_{\substack{md=n\\ 1\leqslant d\leqslant V}} \mu(d)\log m - \sum_{\substack{lmd=n\\ 1\leqslant d\leqslant V\\ 1\leqslant m\leqslant U}} \mu(d)\Lambda(m) - \sum_{\substack{lmd=n\\ 1\leqslant d\leqslant V\\ m>U\\ ld>V}} \mu(d)\Lambda(m).$$

Thus we deduce from (4.37) that

$$f(\alpha) = S_1 - S_2 - S_3 + O(YX^{-1/2}),$$

where

$$S_1 = \sum_{1\leqslant d\leqslant V} \mu(d) \sum_{|m-X/d|\leqslant Y/d} (\log m) e\left((md)^k\alpha\right),$$

$$S_2 = \sum_{1\leqslant v\leqslant UV} \lambda_0(v) \sum_{|l-X/v|\leqslant Y/v} e\left((lv)^k\alpha\right),$$

$$S_3 = \sum_{V<u\leqslant (X+Y)/U} \lambda_1(u) \sum_{\substack{|m-X/u|\leqslant Y/u\\ m>U}} \Lambda(m) e\left((mu)^k\alpha\right),$$

in which

$$\lambda_0(v) = \sum_{\substack{md=v\\ 1\leqslant d\leqslant V\\ 1\leqslant m\leqslant U}} \mu(d)\Lambda(m) \quad \text{and} \quad \lambda_1(u) = \sum_{\substack{d|u\\ 1\leqslant d\leqslant V}} \mu(d).$$

We begin by estimating the sum S_3. Note here that when $\theta > \frac{5}{6}$ and $Y = X^\theta$, then our choices for U and V ensure that

$$X^2/Y^2 < V < (X+Y)/U < Y^2/X.$$

We cover the interval $|n-X| \leqslant Y$ by intervals of the shape $[x, x+y]$ with $|x-X| \leqslant Y$ and $x^\theta \leqslant y \leqslant x$. On noting that $|\lambda_1(u)| \leqslant \tau(u)$, we find that we may apply Lemma 4.1 with $t = t_k$ and $w = \lceil \frac{1}{2}(t_k+2) \rceil$. First, on interchanging the order of summation, we see that

$$S_3 = \sum_{U<m\leqslant (X+Y)/V} \Lambda(m) \sum_{\substack{|u-X/m|\leqslant Y/m\\ u>V}} \lambda_1(u) e((mu)^k\alpha).$$

We next divide the summations over u and m into dyadic intervals

$$2^iV < u \leqslant \min\{2^{i+1}V, (X+Y)/U\}, \quad 2^jU < m \leqslant \min\{2^{j+1}U, (X+Y)/V\},$$

and thereby deduce by applying Lemma 4.1 with $M = 2^jU$ and $N = 2^iV$, and summing over i and j, that

$$\begin{aligned} S_3 &\ll (\log X) \max_{V \leqslant N \leqslant (X+Y)/U} X^\varepsilon Y \left(\frac{1}{q} + \frac{X}{YN^k} + \frac{N^k}{X^{k-1}Y} + \frac{q}{X^{k-2}Y^2} \right)^{1/K} \\ &\ll X^\varepsilon Y \left(q^{-1} + X^{-1/2} + qX^{2-k}Y^{-2} \right)^{1/K}, \end{aligned} \tag{4.38}$$

where K is defined as in (2.2).

Next we estimate S_2. Write

$$S_4(Z, W) = \sum_{Z < v \leqslant W} \lambda_0(v) \sum_{|l - X/v| \leqslant Y/v} e((lv)^k \alpha).$$

Then we find that

$$S_2 = S_4(0, V) + S_4(V, UV). \tag{4.39}$$

Note that

$$X^2/Y^2 < V < UV < 16Y^2/X.$$

In view of the bound $|\lambda_0(v)| \leqslant \log v$, we may again divide the summation over v into dyadic intervals to deduce from Lemma 4.1 that

$$S_4(V, UV) \ll X^\varepsilon Y (q^{-1} + X^{-1/2} + qX^{2-k}Y^{-2})^{1/K}. \tag{4.40}$$

In order to estimate $S_4(0, V)$, we begin by applying Dirichlet's theorem on Diophantine approximation to obtain integers b and r with $(b, r) = 1$,

$$1 \leqslant r \leqslant (X^{k-2}Y^2)^{1/2} \quad \text{and} \quad |r\alpha - b| \leqslant (X^{k-2}Y^2)^{-1/2}.$$

Next, noting that

$$V^{2-2k\sigma} \leqslant X^{1-2/k}Y^{2/k-2k\sigma} \quad \text{and} \quad V^{1-2\sigma} < X^{1-2/k}Y^{2/k-2\sigma},$$

and further that

$$V^{2/3-\sigma} \leqslant X^{-1}Y^{5/3-\sigma},$$

an application of Lemma 4.2 with $N = 1$, $M = V$ and $b_1 = 1$ yields

$$S_4(0, V) \ll Y^{1-\sigma+\varepsilon}V^\sigma + Y^{1+\varepsilon}(r + X^{k-1}Y|r\alpha - b|)^{-1/k^2}. \tag{4.41}$$

Note here that if one were to have both $r \leqslant \frac{1}{2}q$ and $|r\alpha - b| \leqslant q^{-1}Y/X$, then it follows from the triangle inequality that

$$qr \left| \frac{a}{q} - \frac{b}{r} \right| \leqslant rq^{-1} + Y/X < 1,$$

so that $a/q = b/r$, and indeed $q = r$ and $a = b$. In such circumstances, therefore, one has

$$\begin{aligned} S_4(0, V) &\ll Y^{1-\sigma+\varepsilon}V^\sigma + Y^{1+\varepsilon} \left(q + X^{k-1}Y|q\alpha - a| \right)^{-1/k^2} \\ &\ll Y^{1+\varepsilon} \left(q^{-1} + X^{-1/2} + qX^{2-k}Y^{-2} \right)^{1/K}. \end{aligned} \tag{4.42}$$

Meanwhile, when one has either $r > \frac{1}{2}q$ or $|r\alpha - b| > q^{-1}Y/X$, the same conclusion follows directly from (4.41). Thus, by combining (4.40) and (4.42), we deduce from (4.39) that

$$S_2 \ll X^\varepsilon Y(q^{-1} + X^{-1/2} + qX^{2-k}Y^{-2})^{1/K}. \tag{4.43}$$

Finally, in order to estimate S_1, we apply Lemma 4.2 directly, proceeding as in the treatment of $S_4(0, V)$. Thus we again obtain the bound

$$S_1 \ll X^\varepsilon Y(q^{-1} + X^{-1/2} + qX^{2-k}Y^{-2})^{1/K}. \tag{4.44}$$

Finally, by combining (4.38), (4.43) and (4.44), we obtain the following conclusion.

Lemma 4.3. *Let θ be a real number with $\frac{5}{6} < \theta < 1$, and suppose that X and Y are real numbers with $X^\theta \leqslant Y \leqslant X$. Then whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha - a/q| \leqslant q^{-2}$, one has*

$$\sum_{|p-X|\leqslant Y} (\log p)e(p^k\alpha) \ll X^\varepsilon Y \left(q^{-1} + X^{-1/2} + qY^{-2}X^{2-k}\right)^{1/K},$$

where K is defined as in (2.1).

Proposition 2.3 follows from this lemma by applying a standard transference principle (see [39, Lemma 14.1]) to the conclusion of Lemma 4.3. We note that Kumchev [13, Theorem 1.2] has stronger conclusions than those available via Lemma 4.3 in circumstances wherein $k \geqslant 3$ and

$$X^{(2k+2)/(2k+3)} < Y \leqslant X.$$

A key feature of Lemma 4.3, however, is the validity of its estimates for values of Y almost as small as $X^{5/6}$.

5. Estimates of Weyl type, II: $k = 2$ and 3

Lemma 4.3 would suffice to obtain viable minor arc estimates for $f(\alpha)$, but only in circumstances wherein $Y > X^{5/6}$. By making use of the earlier literature on the subject, we are able to obtain an estimate of the shape (2.9) in the cases $k = 2$ and 3 of use when Y is a somewhat smaller power of X. This we accomplish by dividing the minor arcs $\mathfrak{m}$ into two parts, and treating each part in turn. Let

$$Q_0 = X^{1-k}Y^{2k-1}P^{-1}, \tag{5.1}$$

and note that $Q_0 < Q$. Denote by $\mathfrak{m}_1$ the union of the arcs

$$\{\alpha \in [0, 1) : |q\alpha - a| \leqslant Q_0^{-1}\},$$

with $(a, q) = 1$, $0 \leqslant a \leqslant q$ and $P < q \leqslant Q_0$, and by $\mathfrak{m}_2$ the union of the arcs

$$\{\alpha \in [0, 1) : Q^{-1} < |q\alpha - a| \leqslant Q_0^{-1}\},$$

with $(a, q) = 1$ and $0 \leqslant a \leqslant q \leqslant P$. By Dirichlet's theorem on Diophantine approximation, each real number $\alpha \in \mathfrak{m}$ can be written in the shape $\alpha =$

$\lambda + a/q$, with $(a,q) = 1$, $0 \leqslant a \leqslant q \leqslant Q_0$ and $|q\lambda| < Q_0^{-1}$. In view of the definition of the minor arcs $\mathfrak{m}$, we have either

$$P < q \leqslant Q_0 \quad \text{and} \quad |q\lambda| \leqslant Q_0^{-1}, \tag{5.2}$$

or else

$$1 \leqslant q \leqslant P \quad \text{and} \quad Q^{-1} < |q\lambda| \leqslant Q_0^{-1}. \tag{5.3}$$

Thus $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2$.

We obtain a bound for $f(\alpha)$ when $\alpha \in \mathfrak{m}_1$ by means of the following lemma due to Tang [33].

Lemma 5.1. *Let k be an integer with $k \geqslant 2$, and put $\omega = 2^{k-1}$. Suppose that $y > x^{1/2}$, and that $\alpha \in \mathbb{R}$ satisfies the property that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$1 \leqslant q \leqslant x^{\frac{k-1}{\omega-1}} y^{\frac{k(\omega-2)+1}{\omega-1}} \quad \text{and} \quad |\alpha - a/q| \leqslant q^{-2}.$$

Then for any $\varepsilon > 0$, one has

$$\sum_{x \leqslant n \leqslant x+y} \Lambda(n) e(n^k \alpha) \ll y^{1+\varepsilon} \left(\frac{1}{q} + \frac{x^{1/2}}{y} + \frac{x^{\frac{\omega^2}{\omega+1}}}{y^\omega} + \frac{x^{\frac{(k-1)(\omega+1)-1}{\omega+1}}}{y^{2k-2}} + + \frac{qx^{k-1}}{y^{2k-1}} \right)^{1/\omega^2}.$$

Proof. This is [33, Lemma 2]. □

Recall that we may suppose that $Y = X^\theta$ with $\theta_k + 4K\delta < \theta < 1$. Then the definition (5.1) implies that when $k = 2$, we have

$$Q_0 = X^{-1} Y^3 P^{-1} < XY = X^{\frac{k-1}{\omega-1}} Y^{\frac{k(\omega-2)+1}{\omega-1}}.$$

Meanwhile, when $k = 3$, then instead

$$Q_0 = X^{-2} Y^5 P^{-1} < X^{2/3} Y^{7/3} = X^{\frac{k-1}{\omega-1}} Y^{\frac{k(\omega-2)+1}{\omega-1}}.$$

Then on recalling (2.4), (5.1) and (5.2), we may apply Lemma 5.1 to show that whenever $\alpha \in \mathfrak{m}_1$, then

$$\sum_{|n-X| \leqslant Y} \Lambda(n) e(n^k \alpha) \ll X^\varepsilon Y P^{-1/\omega^2} \ll Y X^{\varepsilon - 2\delta}. \tag{5.4}$$

When $\alpha \in \mathfrak{m}_2$, meanwhile, we bound $f(\alpha)$ via the following estimate of Liu, Lü and Zhan [18].

Lemma 5.2. *Let k be an integer with $k \geqslant 1$, and suppose that $2 \leqslant y \leqslant x$. Let $\alpha \in \mathbb{R}$, suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$, and define*

$$\Xi = x^k |\alpha - a/q| + x^2 y^{-2}.$$

Then for any $\varepsilon > 0$, one has

$$\sum_{x < n \leqslant x+y} \Lambda(n) e(n^k \alpha) \ll (qx)^\varepsilon \left(\frac{y(q\Xi)^{1/2}}{x^{1/2}} + (qx)^{1/2} \Xi^{1/6} + y^{1/2} x^{3/10} + \frac{x^{4/5}}{\Xi^{1/6}} + \frac{x}{(q\Xi)^{1/2}} \right).$$

Proof. This is [18, Theorem 1.1]. □

We again recall that $Y = X^\theta$ with $\theta_k + 4K\delta < \theta < 1$. Consequently, under the hypotheses of Lemma 5.2, when $\alpha \in \mathfrak{m}_2$, we have $\Xi > X^2Y^{-2}$ and

$$X^kQ^{-1} < q\Xi = |q\alpha - a|X^k + qX^2Y^{-2} \leqslant X^kQ_0^{-1} + PX^2Y^{-2} < X^kQ_0^{-1}.$$

Thus on recalling (2.4), (5.1) and (5.2), we find via Lemma 5.2 that whenever $\alpha \in \mathfrak{m}_2$, then

$$\sum_{|n-X|\leqslant Y} \Lambda(n)e(n^k\alpha) \ll X^\varepsilon \Big(X^{(k-1)/2}YQ_0^{-1/2} + P^{1/3}X^{(k+3)/6}Q_0^{-1/6} + Y^{1/2}X^{3/10} + Y^{1/3}X^{7/15} + Q^{1/2}X^{1-k/2}\Big),$$

whence the estimate (5.4) follows also when $\alpha \in \mathfrak{m}_2$. Since $\mathfrak{m} = \mathfrak{m}_1 \cup \mathfrak{m}_2$, it follows from (5.4) that when $k = 2$ and 3, and $\alpha \in \mathfrak{m}$, one has

$$\sum_{|n-X|\leqslant Y} \Lambda(n)e(n^k\alpha) \ll YX^{\varepsilon-2\delta}.$$

This combined with (4.37) delivers the bound (2.9) for $k = 2, 3$.

6. The major arc analysis: preliminaries

We analyse the major arc contribution by applying the iterative idea of [17]. Recall the definition (2.4) of P and Q, and write

$$N_1 = X - Y \quad \text{and} \quad N_2 = X + Y.$$

Suppose that $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ satisfy $(a, q) = 1$ and $|\lambda| \leqslant (qQ)^{-1}$, and consider the value of $f(\alpha)$ when $\alpha = \lambda + a/q$.

Write χ for a typical Dirichlet character modulo q, and denote the principal character by χ_0. Also, let δ_χ be 1 or 0 according to whether χ is principal or not. Observe that when $1 \leqslant q \leqslant P$ and p is a prime number with $N_1 \leqslant p \leqslant N_2$, then $(q, p) = 1$. Then we may rewrite $f(\lambda + a/q)$ in the form

$$\begin{aligned} f(\lambda + a/q) &= \sum_{\substack{N_1\leqslant p\leqslant N_2\\ (p,q)=1}} (\log p)e\left(p^k(\lambda + a/q)\right) \\ &= \varphi(q)^{-1}C(q, a)V(\lambda) + \varphi(q)^{-1} \sum_{\chi \bmod q} C(\chi, a)W(\chi, \lambda), \end{aligned}$$

where

$$\begin{aligned} C(\chi, a) &= \sum_{h=1}^{q} \overline{\chi}(h)e(h^ka/q), \quad C(q, a) = C(\chi_0, a), \\ V(\lambda) &= \sum_{N_1\leqslant m\leqslant N_2} e(m^k\lambda), \\ W(\chi, \lambda) &= \sum_{N_1\leqslant p\leqslant N_2} (\log p)\chi(p)e(p^k\lambda) - \delta_\chi V(\lambda). \end{aligned}$$

Thus we discern that

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\,\mathrm{d}\alpha = \sum_{j=0}^{s} \binom{s}{j} I_j, \tag{6.1}$$

where

$$I_j = \sum_{1\leqslant q\leqslant P} \sum_{\substack{a=1\\(a,q)=1}}^{q} \varphi(q)^{-s} C(q,a)^{s-j} e(-na/q)\mathcal{I}_j(q,a), \tag{6.2}$$

in which

$$\mathcal{I}_j(q,a) = \int_{-1/(qQ)}^{1/(qQ)} V(\lambda)^{s-j} \Bigg(\sum_{\chi \bmod q} C(\chi,a)W(\chi,\lambda)\Bigg)^j e(-n\lambda)\,\mathrm{d}\lambda. \tag{6.3}$$

We shall find that I_0 provides the main contribution on the right hand side of (6.1), while $I_1, I_2, \ldots, I_s$ contribute to an error term. We begin by computing the main term I_0. Define

$$B(n,q;\chi_1,\ldots,\chi_s) = \sum_{\substack{a=1\\(a,q)=1}}^{q} C(\chi_1,a)\ldots C(\chi_s,a)e(-na/q),$$

$$B(n,q) = B(n,q;\chi_0,\ldots,\chi_0),$$

and

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \varphi(q)^{-s} B(n,q).$$

We observe that $\mathfrak{S}(n)$ is the usual singular series in the Waring-Goldbach problem defined in §2. Thus, whenever $s > \max\{3, k(k-1)\}$ and $n \equiv s \pmod{R(k)}$ (and, in the case $k=3$ and $s=7$, one has in addition $9 \nmid n$), there is a positive number $\eta = \eta(s,k)$ for which $1 \ll \mathfrak{S}(n) \ll (\log X)^{\eta}$. An auxiliary estimate facilitates our transition from truncated singular series to the completed singular series $\mathfrak{S}(n)$.

Lemma 6.1. *Let q and $r_1,\ldots,r_s$ be natural numbers, and denote by r_0 the least common multiple $[r_1,\ldots,r_s]$. Let $\chi_j \bmod r_j$ be primitive characters for $1 \leqslant j \leqslant s$, and write $\chi_0 \bmod q$ for the principal character. Then there is a positive number c, independent of q and $r_1,\ldots,r_s$, such that*

$$\sum_{\substack{1\leqslant q\leqslant x\\ r_0|q}} \varphi(q)^{-s} |B(n,q;\chi_1\chi_0,\ldots,\chi_s\chi_0)| \ll r_0^{1+\varepsilon-s/2} L^c.$$

Proof. The desired conclusion follows by means of the argument of the proof of [15, Lemma 6.7(a)]. □

We now examine the expression

$$I_0 = \sum_{1\leqslant q\leqslant P} \varphi(q)^{-s} B(n,q) \int_{-1/(qQ)}^{1/(qQ)} V(\lambda)^s e(-n\lambda)\,\mathrm{d}\lambda. \tag{6.4}$$

Write

$$V^*(\lambda)=k^{-1}\sum_{N_1^k\leqslant m\leqslant N_2^k} m^{-1+1/k}e(\lambda m)$$

and

$$\mathcal{J}(n)=\int_{-1/2}^{1/2} V^*(\lambda)^s e(-n\lambda)\,\mathrm{d}\lambda.$$

We remark that, by orthogonality, one finds that $\mathcal{J}(n)$ counts the number of solutions of the equation

$$m_1+\ldots+m_s=n,$$

with $N_1^k\leqslant m_i\leqslant N_2^k$ $(1\leqslant i\leqslant s)$, and with each solution $\mathbf{m}$ counted with weight $k^{-s}(m_1\ldots m_s)^{-1+1/k}$. Thus we have

$$\mathcal{J}(n)\asymp Y^{s-1}X^{1-k}. \tag{6.5}$$

Our first step in the analysis of the relation (6.4) is to replace the integral on the right hand side by $\mathcal{J}(n)$, with an acceptable error term.

First, by partial summation and a change of variable (compare the discussion following the proof of [36, Lemma 2.7]), one finds that

$$\begin{aligned} V(\lambda)&=\int_{N_1}^{N_2} e(\lambda u^k)\,\mathrm{d}u+O(1+X^{k-1}Y|\lambda|)\\ &=V^*(\lambda)+O(1+X^{k-1}Y|\lambda|). \end{aligned}\tag{6.6}$$

By partial summation, one obtains the estimate

$$V^*(\lambda)\ll Y(1+X^{k-1}Y\|\lambda\|)^{-1}. \tag{6.7}$$

Here, of course, when $|\lambda|\leqslant\frac{1}{2}$, one has $\|\lambda\|=|\lambda|$. Recall from (2.4) that $Q=X^{k-2}Y^2P^{-1}$, and recall also that our hypotheses concerning Y ensure that $Y\geqslant X^{3/4}$. Thus $X^{k-1}Y/Q=XP/Y\ll Y^{1/2}$, and we see from (6.6) that whenever $|\lambda|\leqslant 1/(qQ)$, then $V(\lambda)=V^*(\lambda)+O(Y^{1/2})$. These estimates lead us, via (6.7), to the relation

$$\begin{aligned} &\int_{-1/(qQ)}^{1/(qQ)} V(\lambda)^s e(-n\lambda)\,\mathrm{d}\lambda-\int_{-1/(qQ)}^{1/(qQ)} V^*(\lambda)^s e(-n\lambda)\,\mathrm{d}\lambda\\ &\qquad\ll Y^{1/2}\int_{-1/(qQ)}^{1/(qQ)} Y^{s-1}(1+X^{k-1}Y|\lambda|)^{1-s}\,\mathrm{d}\lambda+\int_{-1/(qQ)}^{1/(qQ)} Y^{s/2}\,\mathrm{d}\lambda\\ &\qquad\ll Y^{s-3/2}X^{1-k}+PY^{-2+s/2}X^{2-k}. \end{aligned}$$

The reader may find it useful to note that the last line follows from the previous estimate on noting that $Q=X^{k-2}Y^2P^{-1}$. When $s\geqslant 3$, the hypothesis $Y=X^\theta>X^{19/24}$ implies that one has

$$\frac{PY^{-2+s/2}X^{2-k}}{Y^{s-3/2}X^{1-k}}=PXY^{-(s+1)/2}\ll X^{-1/2}.$$

Thus, when $s \geqslant 3$, we see from (6.7) that when $1 \leqslant q \leqslant P$, one has

$$\begin{aligned}\int_{-1/(qQ)}^{1/(qQ)} V(\lambda)^s e(-n\lambda)\,\mathrm{d}\lambda - \mathcal{J}(n) &\ll \int_{1/(qQ)}^{1/2} \frac{Y^s}{(1+X^{k-1}Y\lambda)^s}\,\mathrm{d}\lambda + Y^{s-3/2}X^{1-k}\\ &\ll Y^{s-1}X^{1-k}\left(\frac{X^{k-1}Y}{PQ}\right)^{-1} + Y^{s-3/2}X^{1-k}\\ &\ll Y^sX^{-k} + Y^{s-3/2}X^{1-k}.\end{aligned}$$

Hence we deduce that

$$\int_{-1/(qQ)}^{1/(qQ)} V(\lambda)^s e(-n\lambda)\,\mathrm{d}\lambda = \mathcal{J}(n) + O(Y^{s-1}X^{1-k-2\delta}). \tag{6.8}$$

Recall next from [9, Lemma 8.5] that when $(a,q)=1$, one has

$$C(\chi_0,a) = \sum_{\substack{h=1\\(h,q)=1}}^{q} e(ah^k/q) \ll q^{1/2+\varepsilon}.$$

Thus we deduce that whenever $s \geqslant 5$, one has

$$\sum_{q>P} \varphi(q)^{-s}B(n,q) \ll \sum_{q>P} \varphi(q)^{1-s}(q^{1/2+\varepsilon})^s \ll P^{\varepsilon-1/2}$$

and

$$\sum_{1\leqslant q\leqslant P} \varphi(q)^{-s}|B(n,q)| \ll 1.$$

By substituting (6.8) into (6.4), we therefore deduce that

$$\begin{aligned}I_0 - \mathcal{J}(n)\sum_{1\leqslant q\leqslant P}\varphi(q)^{-s}B(n,q) &\ll Y^{s-1}X^{1-k-2\delta}\sum_{1\leqslant q\leqslant P}\varphi(q)^{-s}|B(n,q)|\\ &\ll Y^{s-1}X^{1-k-\delta},\end{aligned}$$

whence, in view of (6.5), one sees that

$$I_0 - \mathfrak{S}(n)\mathcal{J}(n) \ll P^{\varepsilon-1/2}Y^{s-1}X^{1-k} + Y^{s-1}X^{1-k-\delta}.$$

We thus conclude that

$$I_0 = \mathfrak{S}(n)\mathcal{J}(n) + O(Y^{s-1}X^{1-k-\delta}). \tag{6.9}$$

In particular, provided that $s > \max\{4, k(k-1)\}$ and $n \equiv s \pmod{R(k)}$ (and, in the case $k=3$ and $s=7$, one has in addition $9 \nmid n$), then

$$I_0 \gg Y^{s-1}X^{1-k}. \tag{6.10}$$

It remains to handle the contribution of the remaining terms I_j within (6.1). The discussion of these terms requires us to analyse the auxiliary expressions

$$J_\nu(g) = \sum_{1\leqslant r\leqslant P} [g,r]^{1+\nu-s/2} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\leqslant 1/(rQ)} |W(\chi,\lambda)|$$

and

$$K_\nu(g)=\sum_{1\leqslant r\leqslant P}[g,r]^{1+\nu-s/2}\sideset{}{^*}\sum_{\chi \bmod r}\biggl(\int_{-1/(rQ)}^{1/(rQ)}|W(\chi,\lambda)|^2\,\mathrm{d}\lambda\biggr)^{1/2},$$

in which the asterisk decorating the summations over characters χ is used to denote that the sums are restricted to primitive characters modulo r. Throughout, we define P and Q as in (2.4). In §7 we establish the following estimate for $K_\nu(g)$.

Lemma 6.2. *Let ν be a sufficiently small positive number. Then there is a positive number c with the property that*

$$K_\nu(g)\ll g^{1+2\nu-s/2}(YX^{1-k})^{1/2}L^c.$$

In §8 we turn our attention toward the expression $J_\nu(g)$, providing the following estimates.

Lemma 6.3. *Let ν be a sufficiently small positive number. Then there is a positive number c with the property that*

$$J_\nu(g)\ll g^{1+2\nu-s/2}YL^c.$$

Also, for any positive number B, one has

$$J_\nu(1)\ll YL^{-B}.$$

Granted the validity of Lemmata 6.2 and 6.3, we are now equipped to establish that for each positive number A, one has

$$I_j\ll Y^{s-1}X^{1-k}L^{-A}. \tag{6.11}$$

By substituting this estimate together with (6.9) and (6.10) into (6.1), we conclude that

$$\int_{\mathfrak{M}}f(\alpha)^se(-n\alpha)\,\mathrm{d}\alpha=\mathfrak{S}(n)\mathfrak{J}(n)+O(Y^{s-1}X^{1-k}L^{-1})\gg Y^{s-1}X^{1-k},$$

thereby completing the proof of Proposition 2.1.

We begin by confirming the estimate (6.11) in the case $j=s$. Recall that $r_0=[r_1,\dots,r_s]$. By reference to (6.2) and (6.3), we may reduce the characters occurring in the summation into primitive characters, thereby obtaining the upper bound

$$|I_s|\leqslant\sum_{\mathbf{r}}\sideset{}{^*}\sum_{\boldsymbol{\chi}\bmod \mathbf{r}}\sum_{\substack{1\leqslant q\leqslant P\\ r_0|q}}\varphi(q)^{-s}|B(n,q;\chi_1\chi_0,\dots,\chi_s\chi_0)|\mathcal{I}(r_0,\boldsymbol{\chi}),$$

where the first summation denotes a sum over $1\leqslant r_j\leqslant P$ $(1\leqslant j\leqslant s)$, the second denotes one over $\chi_j \bmod r_j$ $(1\leqslant j\leqslant s)$, and

$$\mathcal{I}(r_0,\boldsymbol{\chi})=\int_{-1/(r_0Q)}^{1/(r_0Q)}|W(\chi_1,\lambda)\dots W(\chi_s,\lambda)|\,\mathrm{d}\lambda.$$

Notice that the various occurrences of χ_0 here correspond to different moduli, as appropriate, and that the interval of integration in the definition of $\mathcal{I}(r_0,\boldsymbol{\chi})$

expands whenever we pass to smaller moduli r_0. An application of Lemma 6.1 yields the bound

$$I_s \ll L^c \sum_{\mathbf{r}} r_0^{1+\varepsilon-s/2} \sum_{\boldsymbol{\chi} \bmod \mathbf{r}}^{*} \mathcal{I}(r_0, \boldsymbol{\chi}).$$

By Cauchy's inequality, therefore, we discern that

$$I_s \ll L^c \sum_{\mathbf{r}'} \sum_{\boldsymbol{\chi}' \bmod \mathbf{r}'}^{*} \Bigg(\prod_{j=1}^{s-2} \max_{|\lambda| \leqslant 1/(r_j Q)} |W(\chi_j, \lambda)| \Bigg) \mathfrak{T}(r_1, \ldots, r_{s-2}), \tag{6.12}$$

where the first summation denotes a sum over $1 \leqslant r_j \leqslant P$ $(1 \leqslant j \leqslant s-2)$, the second denotes one over χ_j mod r_j $(1 \leqslant j \leqslant s-2)$, and

$$\mathfrak{T}(r_1, \ldots, r_{s-2}) = \sum_{1 \leqslant r_{s-1} \leqslant P} \; \sum_{\chi_{s-1} \bmod r_{s-1}}^{*} \mathfrak{T}_1(r_{s-1}, \chi_{s-1}) \mathfrak{T}_2(r_1, \ldots, r_{s-1}), \tag{6.13}$$

in which

$$\mathfrak{T}_1(r_{s-1}, \chi_{s-1}) = \Bigg(\int_{-1/(r_{s-1}Q)}^{1/(r_{s-1}Q)} |W(\chi_{s-1}, \lambda)|^2 \, \mathrm{d}\lambda \Bigg)^{1/2}$$

and

$$\mathfrak{T}_2(r_1, \ldots, r_{s-1}) = \sum_{1 \leqslant r_s \leqslant P} r_0^{1+\varepsilon-s/2} \sum_{\chi_s \bmod r_s}^{*} \Bigg(\int_{-1/(r_s Q)}^{1/(r_s Q)} |W(\chi_s, \lambda)|^2 \, \mathrm{d}\lambda \Bigg)^{1/2}.$$

Now we follow the iterative procedure of [17] in order to bound the above sums over $r_s, r_{s-1}, \ldots, r_1$ in turn. First, on noting that

$$r_0 = [r_1, \ldots, r_s] = [[r_1, \ldots, r_{s-1}], r_s],$$

we deduce from Lemma 6.2 that

$$\mathfrak{T}_2(r_1, \ldots, r_{s-1}) = K_\varepsilon([r_1, \ldots, r_{s-1}]) \ll [r_1, \ldots, r_{s-1}]^{1+2\varepsilon-s/2} Y^{1/2} X^{(1-k)/2} L^c.$$

Substituting this estimate into (6.13), and applying Lemma 6.2 once again, we deduce that

$$\begin{aligned} \mathfrak{T}(r_1, \ldots, r_{s-2}) &\ll Y^{1/2} X^{(1-k)/2} L^c K_{2\varepsilon}([r_1, \ldots, r_{s-2}]) \\ &\ll [r_1, \ldots, r_{s-2}]^{1+4\varepsilon-s/2} Y X^{1-k} L^{2c}. \end{aligned}$$

Next substituting this estimate into (6.12), we may apply Lemma 6.3 to bound the sum over r_{s-2} and χ_{s-2} on the right hand side to obtain the bound

$$\begin{aligned} \sum_{r_{s-2}} \sum_{\chi_{s-2}} \max_{|\lambda| \leqslant 1/(r_{s-2}Q)} &|W(\chi_{s-2}, \lambda)| \mathfrak{T}(r_1, \ldots, r_{s-2}) \\ &\ll Y X^{1-k} L^{2c} J_{4\varepsilon}([r_1, \ldots, r_{s-3}]) \\ &\ll [r_1, \ldots, r_{s-3}]^{1+8\varepsilon-s/2} Y^2 X^{1-k} L^{3c}. \end{aligned}$$

Proceeding in a similar manner to bound successively the summations over $r_{s-3}, \ldots, r_1$, we arrive in the final step at the bound

$$I_s \ll Y^{s-2} X^{1-k} L^{(s-1)c} \sum_{1 \leqslant r_1 \leqslant P} r_1^{1+2^{s-1}\varepsilon-s/2} \sum_{\chi_1 \bmod r_1}^{*} \max_{|\lambda| \leqslant 1/(r_1 Q)} |W(\chi_1, \lambda)|.$$

We therefore conclude from an application of Lemma 6.3, with B taken to be sufficiently large in terms of c, that for each positive number A one has

$$I_s \ll Y^{s-2}X^{1-k}L^{(s-1)c}J_{2^{s-1}\varepsilon}(1) \ll Y^{s-1}X^{1-k}L^{-A}.$$

This confirms the estimate (6.11) in the case $j = s$.

The other terms I_j with $1 \leqslant j \leqslant s-1$ may be handled in a manner similar to, though simpler than, that applied in the case $j = s$. Recall (2.4) and the hypothesis $Y \geqslant X^{3/4}$. Then one observes that, as a consequence of (6.6) and (6.7), when $|\lambda| \leqslant 1/Q$, one has

$$V(\lambda) = k^{-1} \sum_{N_1^k \leqslant m \leqslant N_2^k} m^{-1+1/k} e(\lambda m) + O(Y^{1/2}), \qquad \max_{|\lambda| \leqslant Q^{-1}} |V(\lambda)| \ll Y,$$

and

$$\begin{aligned}\int_{-1/Q}^{1/Q} |V(\lambda)|^2 \,\mathrm{d}\lambda &\ll Y^2(X^{k-1}Y)^{-1} + YQ^{-1} \\ &\ll YX^{1-k} + PY^{-1}X^{2-k} \ll YX^{1-k}.\end{aligned}$$

This completes our account of the proof of the estimates (6.11), and hence also of the proof of Proposition 2.1, subject to the account in the following two sections of the proof of Lemmata 6.2 and 6.3.

7. The estimation of $K_\nu(g)$

The approach that we adopt in the proofs of Lemmata 6.2 and 6.3 is similar to that of [16, §2], which combines the standard approach to such problems (as in [26] and [32]) and the mean value theorem of Choi and Kumchev [4]. We begin by introducing a special case of an immediate generalisation of [16, Lemma 2.2].

Lemma 7.1. *Let χ be a Dirichlet character modulo r. In addition, let $\mathcal{Q}$, $\mathcal{X}$ and $\mathcal{Y}$ be real numbers with $\mathcal{Q} \geqslant r$, $2 \leqslant \mathcal{X} < \mathcal{Y} \leqslant 2\mathcal{X}$ and $\|\mathcal{X}\| \asymp \|\mathcal{Y}\| \asymp 1$, and put*

$$\mathcal{T}_0 = (\log(\mathcal{Y}/\mathcal{X}))^{-1}, \quad \mathcal{T}_1 = \min\left\{(\log(\mathcal{Y}/\mathcal{X}))^{-2}, 4k\pi\mathcal{X}^k/(r\mathcal{Q})\right\},$$

$$\mathcal{T}_2 = 4k\pi\mathcal{X}^k/(r\mathcal{Q}), \quad \mathcal{T}_3 = \mathcal{X}^{2k} \quad \text{and} \quad \kappa = (\log \mathcal{X})^{-1}.$$

Define $F(s) = F(s,\chi)$ by

$$F(s,\chi) = \sum_{\mathcal{X} < n \leqslant 2\mathcal{X}} \Lambda(n)\chi(n)n^{-s}$$

and $W(\beta) = W_\chi(\beta;\mathcal{X},\mathcal{Y})$ by

$$W_\chi(\beta;\mathcal{X},\mathcal{Y}) = \Bigl|\sum_{\mathcal{X} \leqslant n \leqslant \mathcal{Y}} \Lambda(n)\chi(n)e(\beta n^k)\Bigr|.$$

Then we have

$$\max_{|\beta|\leqslant 1/(r\mathcal{Q})} W(\beta) \ll \log(\mathcal{X}/\mathcal{Y}) \int\limits_{|\tau|\leqslant \mathcal{T}_1} |F(\kappa+i\tau)|\,\mathrm{d}\tau + \int\limits_{\mathcal{T}_1<|\tau|\leqslant \mathcal{T}_2} \frac{|F(\kappa+i\tau)|}{|\tau|^{1/2}}\,\mathrm{d}\tau + \int\limits_{\mathcal{T}_2<|\tau|\leqslant \mathcal{T}_3} \frac{|F(\kappa+i\tau)|}{|\tau|}\,\mathrm{d}\tau + 1$$

and

$$W(0) \ll \log(\mathcal{X}/\mathcal{Y}) \int\limits_{|\tau|\leqslant \mathcal{T}_0} |F(\kappa+i\tau)|\,\mathrm{d}\tau + \int\limits_{\mathcal{T}_0<|\tau|\leqslant \mathcal{T}_3} \frac{|F(\kappa+i\tau)|}{|\tau|}\,\mathrm{d}\tau + 1.$$

We next record the mean value theorem of Choi and Kumchev in the modified form presented by Li and Wu (see the formulation of [4, Theorem 1.1] given in [16, Lemma 2.1]).

Lemma 7.2. *Let $l \in \mathbb{N}$, and let R, T and $\mathcal{X}$ be real numbers with $R \geqslant 1$, $T \geqslant 1$ and $\mathcal{X} \geqslant 1$. Finally, put $\kappa = (\log \mathcal{X})^{-1}$. Then there is an absolute constant $c > 0$ for which*

$$\sum_{\substack{r\sim R\\ l|r}} \sideset{}{^*}\sum_{\chi \bmod r} \int_{-T}^{T} \Bigg| \sum_{\mathcal{X}\leqslant n\leqslant 2\mathcal{X}} \frac{\Lambda(n)\chi(n)}{n^{\kappa+i\tau}} \Bigg|\,\mathrm{d}\tau \ll (l^{-1}R^2T\mathcal{X}^{11/20}+\mathcal{X})\,(\log(RT\mathcal{X}))^c\,.$$

We are now equipped to estimate $K_\nu(g)$, thus completing the proof of Lemma 6.2. Let ν be a sufficiently small positive number. Recall that δ_χ is 1 or 0 according to whether χ is principal or not. Write

$$\widehat{W}(\chi,\lambda) = \sum_{N_1\leqslant m\leqslant N_2} (\Lambda(m)\chi(m)-\delta_\chi)\,e(m^k\lambda),$$

and then put

$$\Upsilon = \sum_{1\leqslant r\leqslant P} [g,r]^{1+\nu-s/2} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |W(\chi,\lambda)-\widehat{W}(\chi,\lambda)|^2\,\mathrm{d}\lambda \right)^{1/2}.$$

Next, observe that when X is large, the condition $|p^j - X| \leqslant Y$ implies that $|p - X^{1/j}| \leqslant YX^{-1+1/j}$, and hence for all $\lambda \in \mathbb{R}$, one has

$$\widehat{W}(\chi,\lambda) - W(\chi,\lambda) = \sum_{j\geqslant 2} \sum_{N_1\leqslant p^j\leqslant N_2} (\log p)\chi(p^j)e(p^{jk}\lambda) \ll YX^{-1/2}.$$

Then on making use of the relation $[g,r](g,r) = gr$, we deduce that

$$\begin{aligned}\Upsilon &\ll YX^{-1/2} \sum_{1\leqslant r\leqslant P} [g,r]^{1+\nu-s/2}(r/Q)^{1/2}\\ &\ll YX^{-1/2}g^{1+\nu-s/2}Q^{-1/2} \sum_{1\leqslant r\leqslant P} \left(\frac{r}{(g,r)}\right)^{1+\nu-s/2} r^{1/2}.\end{aligned}$$

Consequently, recalling the definition (2.4) of P and Q, and noting that we are permitted to assume that $s \geqslant 5$, we see that

$$\begin{aligned}\Upsilon &\ll YX^{-1/2}g^{1+\nu-s/2}Q^{-1/2}P^{1/2}\sum_{\substack{1\leqslant d\leqslant P\\ d|g}}\sum_{1\leqslant\rho\leqslant P/d}\rho^{1+\nu-s/2}\\ &\ll YX^{-1/2}g^{1+2\nu-s/2}(X^{k-2}Y^2P^{-1})^{-1/2}P^{1/2+\varepsilon}.\end{aligned}$$

Thus we conclude that $\Upsilon \ll g^{1+2\nu-s/2}(YX^{1-k})^{1/2}$, so that in order to establish Lemma 6.2, it suffices to confirm that whenever $\frac{1}{2} \leqslant R \leqslant P$, one has

$$\sum_{r\sim R}[g,r]^{1+\nu-s/2}\sideset{}{^*}\sum_{\chi \bmod r}\biggl(\int_{-1/(rQ)}^{1/(rQ)}|\widehat{W}(\chi,\lambda)|^2\,\mathrm{d}\lambda\biggr)^{1/2} \ll g^{1+2\nu-s/2}(YX^{1-k})^{1/2}L^c. \tag{7.1}$$

We observe next that an application of Gallagher's lemma (see [7, Lemma 1]) shows that whenever $r \sim R$, then

$$\begin{aligned}\int_{-1/(rQ)}^{1/(rQ)}|\widehat{W}(\chi,\lambda)|^2\,\mathrm{d}\lambda &\ll (RQ)^{-2}\int_{-\infty}^{\infty}\biggl|\sum_{\substack{N_1^k\leqslant m^k\leqslant N_2^k\\ |m^k-v|\leqslant RQ/3}}(\Lambda(m)\chi(m)-\delta_\chi)\biggr|^2\,\mathrm{d}v\\ &\ll (RQ)^{-2}\int_{N_1^k-RQ/3}^{N_2^k+RQ/3}\biggl|\sum_{U\leqslant m\leqslant V}(\Lambda(m)\chi(m)-\delta_\chi)\biggr|^2\,\mathrm{d}v,\end{aligned} \tag{7.2}$$

where we write

$$U=\max\{N_1,(v-RQ/3)^{1/k}\}\quad\text{and}\quad V=\min\{N_2,(v+RQ/3)^{1/k}\}.$$

We begin by examining the situation with $R=\frac{1}{2}$ and $r=1$. Here we have

$$\begin{aligned}\biggl|\sum_{U\leqslant m\leqslant V}(\Lambda(m)\chi(m)-\delta_\chi)\biggr| &= \biggl|\sum_{U\leqslant m\leqslant V}(\Lambda(m)-1)\biggr|\\ &\ll \bigl((v+Q/3)^{1/k}-(v-Q/3)^{1/k}\bigr)\,L\\ &\ll X^{1-k}QL.\end{aligned}$$

On substituting this conclusion into (7.2), we deduce that

$$\int_{-1/Q}^{1/Q}|\widehat{W}(\chi_0,\lambda)|^2\,\mathrm{d}\lambda \ll Q^{-2}(X^{k-1}Y+Q)(X^{1-k}QL)^2,$$

whence

$$g^{1+\nu-s/2}\biggl(\int_{-1/Q}^{1/Q}|\widehat{W}(\chi_0,\lambda)|^2\,\mathrm{d}\lambda\biggr)^{1/2} \ll g^{1+\nu-s/2}(YX^{1-k})^{1/2}L.$$

This confirms (7.1) in the case $R=\frac{1}{2}$.

Suppose next that $R\geqslant 1$ and $r\sim R$. In these circumstances, we have $\delta_\chi=0$. We apply Lemma 7.1 to estimate the integral on the right hand side of (7.2), taking $\mathcal{X} = U$ and $\mathcal{Y} = V$, and making use of the notation of the statement of that lemma. We observe in this context that

$$(N_2^k + RQ/3) - (N_1^k - RQ/3) \ll X^{k-1}Y.$$

Define $T_i = T_i(\chi, r)$ for $i = 1$ and 2 by putting

$$T_1(\chi, r) = \mathcal{T}_0^{-1} \int_{|\tau| \leqslant \mathcal{T}_0} |F(\kappa + i\tau)| \, \mathrm{d}\tau \quad \text{and} \quad T_2(\chi, r) = \int_{\mathcal{T}_0 < |\tau| \leqslant \mathcal{T}_3} \frac{|F(\kappa + i\tau)|}{|\tau|} \, \mathrm{d}\tau.$$

Then we infer from Lemma 7.1 that

$$\left(\int_{N_1^k - RQ/3}^{N_2^k + RQ/3} \Big| \sum_{U \leqslant m \leqslant V} \Lambda(m) \chi(m) \Big|^2 \, \mathrm{d}v \right)^{1/2} \ll (X^{k-1}Y)^{1/2} (T_1 + T_2 + 1).$$

By substituting this conclusion first into (7.2), and then into (7.1), we conclude thus far that

$$\sum_{r \sim R} [g, r]^{1+\nu-s/2} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |\widehat{W}(\chi, \lambda)|^2 \, \mathrm{d}\lambda \right)^{1/2} \ll \frac{(X^{k-1}Y)^{1/2}}{QR} \sum_{i=1}^{3} \mathfrak{T}_i, \quad (7.3)$$

where

$$\mathfrak{T}_1 = \mathcal{T}_0^{-1} \sum_{r \sim R} [g, r]^{1+\nu-s/2} \sideset{}{^*}\sum_{\chi \bmod r} \int_{|\tau| \leqslant \mathcal{T}_0} |F(\kappa + i\tau)| \, \mathrm{d}\tau,$$

$$\mathfrak{T}_2 = \sum_{r \sim R} [g, r]^{1+\nu-s/2} \sideset{}{^*}\sum_{\chi \bmod r} \int_{\mathcal{T}_0 < |\tau| \leqslant \mathcal{T}_3} \frac{|F(\kappa + i\tau)|}{|\tau|} \, \mathrm{d}\tau$$

$$\mathfrak{T}_3 = \sum_{r \sim R} [g, r]^{1+\nu-s/2} \sideset{}{^*}\sum_{\chi \bmod r} 1.$$

We estimate the terms $\mathfrak{T}_i$ in turn. Observe first that

$$\mathcal{T}_0^{-1} = \log(\mathcal{Y}/\mathcal{X}) \ll RQv^{-1} \ll RQX^{-k}. \quad (7.4)$$

Recall once again our assumption that $s \geqslant 5$. Then an application of Lemma 7.2 yields the bound

$$\begin{aligned} \mathfrak{T}_1 &\ll g^{1+\nu-s/2} \mathcal{T}_0^{-1} \sum_{\substack{1 \leqslant l \leqslant 2R \\ l | g}} (R/l)^{1+\nu-s/2} (l^{-1} R^2 \mathcal{T}_0 X^{11/20} + X) L^c \\ &\ll g^{1+\nu-s/2+\varepsilon} L^c \left(RX^{11/20} + RQX^{1-k} \right). \end{aligned}$$

On recalling the definition (2.4) of P and Q, and noting that by hypothesis, one has $Y > X^{19/24} > PX^{31/40}$, we therefore deduce that

$$\begin{aligned} \mathfrak{T}_1 &\ll g^{1+2\nu-s/2} RQX^{1-k} L^c (1 + PX^{31/20} Y^{-2}) \\ &\ll g^{1+2\nu-s/2} RQX^{1-k} L^c. \end{aligned} \quad (7.5)$$

In order to estimate $\mathfrak{T}_2$, we introduce the auxiliary function

$$M(l,R,\Theta,X)=\sum_{\substack{r\sim R\\ l|r}}\sum_{\chi \bmod r}^{*}\int_{\Theta}^{2\Theta}|F(\kappa+i\tau,\chi)\,\mathrm{d}\tau.$$

Then by dividing the range of integration into dyadic intervals, one finds in a similar manner to the treatment of $\mathfrak{T}_1$ that

$$\begin{aligned}\mathfrak{T}_2&\ll g^{1+\nu-s/2}L^c\sum_{\substack{1\leqslant l\leqslant 2R\\ l|g}}(R/l)^{1+\nu-s/2}\max_{\mathcal{T}_0\leqslant\Theta\leqslant\mathcal{T}_3}\Theta^{-1}M(l,R,\Theta,X)\\&\ll g^{1+\nu-s/2}\sum_{\substack{1\leqslant l\leqslant 2R\\ l|g}}(R/l)^{1+\nu-s/2}(l^{-1}R^2X^{11/20}+\mathcal{T}_0^{-1}X)L^{2c},\end{aligned}$$

whence, by reference to (7.4), one obtains the bound

$$\mathfrak{T}_2\ll g^{1+2\nu-s/2}RQX^{1-k}L^c. \tag{7.6}$$

Finally, one has

$$\begin{aligned}\mathfrak{T}_3&\ll\sum_{r\sim R}[g,r]^{1+\nu-s/2}r\ll g^{1+\nu-s/2}\sum_{\substack{1\leqslant l\leqslant 2R\\ l|g}}(R/l)^{1+\nu-s/2}R\\&\ll g^{1+2\nu-s/2}RQX^{1-k}(PX/Y^2)\ll g^{1+2\nu-s/2}RQX^{1-k}.\end{aligned} \tag{7.7}$$

By substituting the estimates (7.5), (7.6) and (7.7) into (7.3), we conclude that the estimate (7.1) does indeed hold, and thus the conclusion of Lemma 6.2 is finally confirmed.

8. The estimation of $J_\nu(g)$

The argument of the proof of Lemma 6.2 adapts with only modest complications to establish Lemma 6.3, though making use of the first estimate of Lemma 7.1 in place of the second, and we suppress details of the necessary adjustments in the interests of concision. We refer the reader to [26, §6] and [32, §6] for the necessary details. One issue deserves additional attention, this being associated with the proof of the second estimate $J_\nu(1)\ll YL^{-B}$ of Lemma 6.3. When $g=1$, we divide the summation over r in the definition of $J_\nu(1)$ into dyadic intervals $r\sim R$, and then consider separately the situations with $R\leqslant L^B$ and $L^B<R\leqslant P$, in which B is a positive number depending only on A. Since $g=1$, in the latter situation we may extract a factor $R^{-1/4}$ from the estimates involving summations over r without damaging convergence, and hence replace the bound $g^{1+2\nu-s/2}YL^c$ by $g^{1+2\nu-s/2}YL^cR^{-1/4}\ll YL^{-A}$ in the estimates that result. Thus it suffices to restrict attention to the situation with $1\leqslant R\leqslant L^B$.

When $1 \leqslant u \leqslant n$, $r \leqslant 2L^B$ and χ is a Dirichlet character modulo r, we may make use of the explicit formula

$$\sum_{1\leqslant m\leqslant u} \Lambda(m)\chi(m) = \delta_\chi u - \sum_{|\gamma|\leqslant T} \frac{u^\rho}{\rho} + O\left(\left(\frac{u}{T}+1\right)\log^2(ruT)\right), \tag{8.1}$$

where we denote by $\rho = \beta + i\gamma$ a typical non-trivial zero of the Dirichlet L-function $L(s,\chi)$, and T is a parameter with $2 \leqslant T \leqslant u$. Taking $T = X^{5/12-\nu}$ and substituting (8.1) into the formula for $\widehat{W}(\chi,\lambda)$, we deduce that whenever $|\lambda| \leqslant (rQ)^{-1}$, one has

$$\begin{aligned}\widehat{W}(\chi,\lambda) &= \int_{N_1}^{N_2} e(u^k\lambda) \sum_{|\gamma|\leqslant T} u^{\rho-1}\,\mathrm{d}u + O\left((1+X^{k-1}Y|\lambda|)XT^{-1}L^2\right)\\ &\ll Y \sum_{|\gamma|\leqslant T} X^{\beta-1} + O(X^kYQ^{-1}T^{-1}L^2).\end{aligned}$$

Recalling (2.4) once again, and taking $\nu > 0$ and $\delta > 0$ sufficiently small in terms of θ, one finds that

$$X^kQ^{-1}T^{-1} \ll X^2Y^{-2}PT^{-1} \ll X^{\nu+2K\delta-2(\theta-19/24)} \ll X^{-2\nu/3},$$

whence

$$\widehat{W}(\chi,\lambda) \ll Y \sum_{|\gamma|\leqslant T} X^{\beta-1} + O(X^{-\nu/2}Y).$$

Now put $\eta(T) = c(\log T)^{-4/5}$, with c a suitably small positive number. Then by Page's Lemma (see [31, Corollary 11.10]), one sees that $\prod_{\chi \bmod r} L(s,\chi)$ has no zeros in the region $\sigma \geqslant 1-\eta(T)$ and $|t| \leqslant T$, except potentially for a single Siegel zero. But as a consequence of Siegel's theorem (see [31, Corollary 11.15]), no Siegel zero plays any role when $r \sim R \leqslant L^B$. Thus, on making use of the zero-density estimates for Dirichlet L-functions of large sieve type given in [11, equation (1.1)], we deduce that

$$\begin{aligned}\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma|\leqslant T} X^{\beta-1} &\ll L^c \int_0^{1-\eta(T)} T^{12(1-\alpha)/5} X^{\alpha-1}\,\mathrm{d}\alpha,\\ &\ll L^c \int_0^{1-\eta(T)} X^{12(\alpha-1)\nu/5}\,\mathrm{d}\alpha,\end{aligned}$$

whence

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \sum_{|\gamma|\leqslant T} X^{\beta-1} \ll L^c X^{-12\eta(T)\nu/5} \ll \exp(-c'L^{1/5}),$$

for a suitable positive number c'. Assembling these estimates together, we conclude that

$$\sum_{r\leqslant P} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\leqslant 1/(rQ)} |\widehat{W}(\chi,\lambda)| \ll YL^{-A},$$

where $A > 0$ is arbitrary. This completes the proof of the second estimate of Lemma 6.3.

9. Exceptional sets

The basic exceptional set conclusions embodied in Theorem 1.2 follow by the standard arguments employing Bessel's inequality. Let $\mathcal{Z}$ denote the set of integers n with $N \leqslant n \leqslant N + X^{k-1}Y$ and $n \equiv s \pmod{R}$ (and, in case $k = 3$ and $s = 7$, satisfying also $9 \nmid n$), for which the equation

$$n = p_1^k + p_2^k + \ldots + p_s^k \tag{9.1}$$

has no solution in prime numbers p_j with $|p_j - X| < Y$ $(1 \leqslant j \leqslant s)$. In addition, put $Z = \mathrm{card}(\mathcal{Z})$. Then we find from Bessel's inequality and (2.5) that

$$\sum_{n\in\mathcal{Z}} |\rho_s(n;\mathfrak{m})|^2 \leqslant \sum_{N\leqslant n\leqslant N+X^{k-1}Y} |\rho_s(n;\mathfrak{m})|^2 \leqslant \int_{\mathfrak{m}} |f(\alpha)|^{2s}\,\mathrm{d}\alpha.$$

We therefore conclude from Propositions 2.2 and 2.3, together with (2.9), that whenever $s > t_k$, then

$$\begin{aligned}\sum_{n\in\mathcal{Z}} |\rho_s(n;\mathfrak{m})|^2 &\leqslant \Big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\Big)^{2s-2t_k} \int_{\mathfrak{m}} |f(\alpha)|^{2t_k}\,\mathrm{d}\alpha\\ &\ll (X^{\varepsilon-2\delta}Y)^{2s-2t_k} Y^{2t_k-1}X^{1-k+\delta} \ll Y^{2s-1}X^{1-k-\delta}.\end{aligned} \tag{9.2}$$

Meanwhile, it follows from Proposition 2.1 and the lower bound in (2.7) that whenever $s \geqslant \min\{5, k+2\}$, then

$$\sum_{n\in\mathcal{Z}} |\rho_s(n;\mathfrak{M})|^2 \gg Z(Y^{s-1}X^{1-k})^2. \tag{9.3}$$

Since for $n \in \mathcal{Z}$, one necessarily has

$$\rho_s(n;\mathfrak{M}) + \rho_s(n;\mathfrak{m}) = \rho_s(n) = 0,$$

it follows from (9.2) and (9.3) that

$$Z(Y^{s-1}X^{1-k})^2 \ll Y^{2s-1}X^{1-k-\delta},$$

whence $Z \ll X^{k-1-\delta}Y$.

Since the number of integers n satisfying $N \leqslant n \leqslant N + X^{k-1}Y$, together with other associated conditions described in the opening paragraph, is of order $X^{k-1}Y$, it follows that for almost all of these integers one has $\rho_s(n) \neq 0$. We observe also that when $|p_j - X| < Y$ and $N \leqslant n \leqslant N + X^{k-1}Y$, one has also $|p_j - (n/s)^{1/k}| \ll Y$, and thus the conclusion of Theorem 1.2 is confirmed. This completes our proof of Theorem 1.2.

It may be worth observing at this point that, since one has $|p_j - X| < Y$ in the representation (9.1), then necessarily one has $|n - sX^k| \ll X^{k-1}Y$, or equivalently $|n - N| \ll X^{k-1}Y$. We therefore see that, in order for an exceptional set estimate to constitute a non-trivial assertion to the effect that almost all eligible integers are represented in the form (9.1), one must establish an estimate of the shape $Z = o(X^{k-1}Y)$. Assertions in [16] and [42] are only slightly stronger than $Z = o(X^k)$, and consequently yield non-trivial exceptional sets only when Y is very nearly as large as X. This explains the origin of the infelicities noted in the discussion following the statement of Theorem 1.2 above.

We finish this section by noting that the ideas underlying the proof of Proposition 2.2 may be used to good effect in sharpening estimates for exceptional sets. In order to illustrate such ideas, we begin by recording a lemma of use in estimating exceptional sets for sums of squares of almost equal primes. Here, we adopt the notation of §§2 and 3.

Lemma 9.1. *Let* $\mathcal{Z} \subseteq [N, N + XY] \cap \mathbb{Z}$, *and put*

$$K(\alpha) = \sum_{n\in\mathcal{Z}} e(n\alpha).$$

Then one has

$$\int_0^1 |f(\alpha)^4 K(\alpha)^2|\,\mathrm{d}\alpha \ll X^\varepsilon(Y^2Z + Y^3X^{-1}Z^2).$$

Proof. By adjusting the value of Y by an amount at most $O(1)$, we may suppose that X is an integer, and consequently we are free to suppose the latter in the proof of this lemma. Next, by orthogonality, the mean value in question is bounded above by the number of integral solutions of the equation

$$(X + y_1)^2 + (X + y_2)^2 - (X + y_3)^2 - (X + y_4)^2 = n_1 - n_2, \tag{9.4}$$

with $|y_i| \leqslant Y$ $(1 \leqslant i \leqslant 4)$ and $n_1, n_2 \in \mathcal{Z}$, and with each solution counted with weight

$$\prod_{i=1}^{4} \log(X + y_i).$$

Let T_1 denote the number of such solutions in which $n_1 = n_2$, and let T_2 denote the corresponding number of solutions with $n_1 \neq n_2$. Then we have

$$\int_0^1 |f(\alpha)^4 K(\alpha)^2|\,\mathrm{d}\alpha \ll X^\varepsilon(T_1 + T_2). \tag{9.5}$$

By orthogonality and an application of Hua's lemma, we find from (9.4) that

$$T_1 \leqslant \mathrm{card}(\mathcal{Z}) \int_0^1 |f(\alpha)|^4\,\mathrm{d}\alpha \ll ZY^{2+\varepsilon}. \tag{9.6}$$

Consider then a solution $\mathbf{y}, \mathbf{n}$ counted by T_2. Write $m = m(n_1, n_2)$ for the integer closest to $(n_1 - n_2)/(2X)$. Then since it follows from (9.4) that

$$2X(y_1 + y_2 - y_3 - y_4) + (y_1^2 + y_2^2 - y_3^2 - y_4^2) = n_1 - n_2,$$

we find that

$$|y_1 + y_2 - y_3 - y_4 - m| \leqslant 1 + Y^2/X \leqslant 2Y^2/X.$$

Thus we discern that for each fixed choice of n_1 and n_2 with $n_1, n_2 \in \mathcal{Z}$ and $n_1 \neq n_2$, there exists an integer h with $|h - m| \leqslant 2Y^2/X$ and

$$\left.\begin{aligned} y_1^2 + y_2^2 - y_3^2 - y_4^2 &= n_1 - n_2 - 2Xh \\ y_1 + y_2 - y_3 - y_4 &= h \end{aligned}\right\}. \tag{9.7}$$

We divide the solutions $\mathbf{y}, \mathbf{n}$ counted by T_2 into two types. Let T_3 denote the number of the solutions counted by T_2 in which

$$n_1 - n_2 = h(2X + 2y_3 + h), \tag{9.8}$$

and let T_4 denote the corresponding number of solutions in which the latter equation does not hold.

Given a fixed choice of $n_1, n_2 \in \mathcal{Z}$ with $n_1 \neq n_2$, a divisor function estimate shows that the number of possible choices for h and y_3 satisfying (9.8) is $O(X^\varepsilon)$. Fix any one such choice, and eliminate y_4 between the equations (9.7). We deduce that

$$(y_1 + y_2 - y_3 - h)^2 = (y_1^2 + y_2^2 - y_3^2) - (n_1 - n_2 - 2Xh),$$

whence

$$2(y_1 - y_3 - h)(y_2 - y_3 - h) = h(2X + 2y_3 + h) - (n_1 - n_2). \tag{9.9}$$

Thus, for solutions counted by T_3 in which the right hand side vanishes, one has $y_1 = y_3 + h$ or $y_2 = y_3 + h$. In the former case, the value of y_1 is fixed, and it follows from the linear equation of (9.7) that $y_2 = y_4$. A symmetrical argument yields a symmetrical conclusion in the latter case, and thus we deduce that

$$T_3 \ll X^\varepsilon Y Z^2 \ll X^\varepsilon (Y^3/X) Z^2. \tag{9.10}$$

Meanwhile, for solutions counted by T_4 in which the right hand side of (9.9) is non-zero, we find by a divisor estimate that for each of the $O(Y^2/X)$ possible choices for h, and each of the $O(Y)$ possible choices for y_3, there are $O(X^\varepsilon)$ possible choices for $y_1 - y_3 - h$ and $y_2 - y_3 - h$. Given any fixed such choices, we find that y_1 and y_2 are fixed, and then y_4 is also fixed by virtue of the linear equation in (9.7). Thus we conclude that

$$T_4 \ll X^\varepsilon Y (Y^2/X) Z^2. \tag{9.11}$$

By combining (9.6), (9.10) and (9.11), and substituting into (9.5), the conclusion of the lemma now follows. ☐

Equipped with this lemma, we may establish a powerful estimate for the exceptional set associated with six almost equal squares of prime numbers. When $Y \geqslant 1$, denote by $E_6(N; Y)$ the number of positive integers n, with $n \equiv 6 \pmod{24}$ and $|n - N| \leqslant XY$, for which the equation $n = p_1^2 + p_2^2 + \ldots + p_6^2$ has no solution in prime numbers p_j with $|p_j - (n/6)^{1/2}| < Y$ $(1 \leqslant j \leqslant 6)$.

Theorem 9.2. *Suppose that $Y \geqslant X^{19/24+\varepsilon}$, for some positive number ε. Then there is a positive number δ for which $E_6(N; Y) \ll Y^{-1} X^{1-\delta}$.*

Proof. Let $\mathcal{Z}$ denote the set of integers counted by $E_6(N; Y)$. Then for each $n \in \mathcal{Z}$ one has

$$\int_{\mathfrak{m}} f(\alpha)^6 e(-n\alpha)\, \mathrm{d}\alpha = -\int_{\mathfrak{M}} f(\alpha)^6 e(-n\alpha)\, \mathrm{d}\alpha,$$

whence Proposition 2.1 and the associated discussion yields the lower bound

$$\left| \int_{\mathfrak{m}} f(\alpha)^6 K(-\alpha)\, \mathrm{d}\alpha \right| \gg \sum_{n\in\mathcal{Z}} Y^5 X^{-1} = ZY^5 X^{-1}.$$

By Schwarz's inequality, we therefore deduce that

$$ZY^5X^{-1} \ll \left(\int_0^1 |f(\alpha)^4 K(\alpha)^2|\, \mathrm{d}\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} |f(\alpha)|^8\, \mathrm{d}\alpha \right)^{1/2}.$$

An application of Proposition 2.2 together with (2.9) shows that

$$\begin{aligned}\int_{\mathfrak{m}} |f(\alpha)|^8\, \mathrm{d}\alpha &\leqslant \Big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)| \Big)^2 \int_0^1 |f(\alpha)|^6\, \mathrm{d}\alpha \\ &\ll (YX^{-2\delta})^2 Y^5 X^{\varepsilon-1},\end{aligned}$$

and so we deduce from Lemma 9.1 that

$$\begin{aligned} ZY^5X^{-1} &\ll X^{\varepsilon}(Y^2Z + Y^3X^{-1}Z^2)^{1/2}(Y^7X^{-1-2\delta})^{1/2} \\ &\ll Y^{9/2}X^{-(1+\delta)/2}Z^{1/2} + Y^5X^{-1-\delta/2}Z. \end{aligned}$$

Consequently, one sees that $Z \ll X^{1-\delta}Y^{-1}$, and the proof of the theorem is complete. □

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