ADDITIVE REPRESENTATION IN SHORT INTERVALS, II: SUMS OF TWO LIKE POWERS

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Abstract. We establish that, for almost all natural numbers N, there is a sum of two positive integral cubes lying in the interval $[N - N^{7/18+\varepsilon}, N]$. Here, the exponent $7/18$ lies half way between the trivial exponent $4/9$ stemming from the greedy algorithm, and the exponent $1/3$ constrained by the number of integers not exceeding X that can be represented as the sum of two positive integral cubes. We also provide analogous conclusions for sums of two positive integral k-th powers when $k \geqslant 4$.

1. Introduction

The sequence of integers $2 = s_{k,1} < s_{k,2} < \dots$ represented as the sum of two k-th powers of natural numbers is certainly sparse when $k \geqslant 3$, for a simple counting argument confirms that their number, $\nu_k(N)$, not exceeding N is at most $O(N^{2/k})$. Investigations concerning $\nu_k(N)$ date at least as far back as the work of Erdős and Mahler [6, 7], which showed that $\nu_k(N) \gg N^{2/k}$. Hooley [14, 15, 16, 17, 19] has returned to the problem on numerous occasions, and when $h \geqslant 3$ has established the asymptotic formula

$$\nu_h(N) = \frac{\Gamma(1+1/h)^2}{2\Gamma(1+2/h)} N^{2/h} + O(N^{5/(3h)+\varepsilon}). \tag{1.1}$$

This conclusion derives from the paucity of numbers that are represented as the sum of two h-th powers in two essentially distinct ways. Other scholars have augmented and refined Hooley's opera (see Greaves [9, 10], Skinner and Wooley [23], Wooley [29], Heath-Brown [12, 13], Browning [2], Salberger [22]). The distribution of such numbers in short intervals has, thus far, received little attention, although Daniel [5] has considered the corresponding problem for sums of three positive integral cubes. In this memoir we remedy this situation.

Given a large integer n, one may subtract from n the largest integral k-th power not exceeding n, leaving a remainder of size at most $kn^{1-1/k}$. By repeating this greedy algorithm, one finds that for all large N, there is a sum of two positive integral k-th powers between $N - k^2 N^{\phi_k}$ and N, where $\phi_k = (1-1/k)^2$. The main result of this paper shows that the same conclusion

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remains valid, with a smaller exponent in place of ϕ_k, for *almost all* natural numbers N. Denote by $E_k(N,Z)$ the number of natural numbers $N < n \leqslant 2N$ for which the interval $(n, n+Z]$ contains no integer that is the sum of two positive integral k-th powers. When $k \geqslant 3$, we put

$$\sigma_k = \begin{cases} 2^{2-k}, & \text{when } 3 \leqslant k \leqslant 7, \\ (2k^2 - 10k + 12)^{-1}, & \text{when } k \geqslant 8, \end{cases} \tag{1.2}$$

and define

$$\theta_k = 1 - \frac{2}{k} + \frac{1-\sigma_k}{k^2} = \phi_k - \frac{\sigma_k}{k^2}. \tag{1.3}$$

Theorem 1.1. *Suppose that $k \geqslant 3$. Then, whenever $Z \geqslant N^{\theta_k}$, one has*

$$E_k(N,Z) \ll N^{1+\theta_k+\varepsilon} Z^{-1}. \tag{1.4}$$

Whereas the greedy algorithm ensures that $E_k(N, 2k^2 N^{\phi_k}) \ll 1$, the conclusion of Theorem 1.1 yields the bound $E_k(N, N^{\phi_k - \delta}) = o(N)$ whenever $\delta < \sigma_k/k^2$. The spacing of sums of two k-th powers evident in the asymptotic formula (1.1), meanwhile, implies that $E_k(N,Z) \gg N$ whenever $Z \leqslant N^{1-2/k}$. It seems plausible that (1.4) should remain valid provided only that $\theta_k > 1 - 2/k$. Our estimate is particularly strong in the case $k = 3$, where we show that for all $\varepsilon > 0$, and almost all $N \in \mathbb{N}$, there is a sum of two positive integral cubes lying between N and $N + N^{7/18+\varepsilon}$. Here, the exponent $7/18$ lies half way between the trivial exponent $4/9$ stemming from the greedy algorithm, and the exponent $1/3$ constrained by the asymptotic formula (1.1).

The conclusion of Theorem 1.1 also delivers bounds for the size of the gaps between sums of two k-th powers in mean square.

Theorem 1.2. *When $k \geqslant 3$, one has*

$$\sum_{s_{k,n} \leqslant N} (s_{k,n+1} - s_{k,n})^2 \ll N^{1+\theta_k+\varepsilon}.$$

We note in particular that since (1.1) shows that, for almost all $n \in \mathbb{N}$, one has $s_{k,n+1} - s_{k,n} \gg s_{k,n}^{1-2/k}$, then

$$\sum_{N/2 < s_{k,n} \leqslant N} (s_{k,n+1} - s_{k,n})^2 \gg (N^{1-2/k})^2 N^{2/k} = N^{2-2/k}.$$

This lower bound is expected to reflect the asymptotic behaviour of the mean square gap size estimated in Theorem 1.2. Meanwhile, the bound

$$s_{k,n+1} - s_{k,n} \ll s_{k,n}^{\phi_k}, \tag{1.5}$$

immediate from the greedy algorithm, yields the estimate

$$\sum_{s_{k,n} \leqslant N} (s_{k,n+1} - s_{k,n})^2 \ll N^{\phi_k} \sum_{s_{k,n} \leqslant N} (s_{k,n+1} - s_{k,n}) \ll N^{1+\phi_k}.$$

In view of (1.3), one has $2 - 2/k < 1 + \theta_k < 1 + \phi_k$, so that the conclusion of Theorem 1.2 improves on the trivial estimate, but falls short of the aforementioned expectation. In the case $k = 3$, the exponent $1 + \theta_3 = 25/18$ lies half way between the trivial and conjectured bounds.

In the above discussion, we have deliberately restricted attention to the situation in which $k \geqslant 3$. The behaviour of the sequence $(s_{2,n})$, consisting of sums of two squares, is quite different. We refer the reader to Friedlander [8], Harman [11], Hooley [18] and Plaksin [20, 21] for a consideration of the distribution of gaps in this relatively dense sequence.

The exceptional set estimate presented in Theorem 1.1 is obtained by applying the Hardy-Littlewood (circle) method to the Diophantine equation

$$x^k + y^k + z = n, \tag{1.6}$$

with z running over a short interval. By applying Bessel's inequality, one is led to consider a mean value estimate implicitly related to the number of integral solutions of the equation

$$x_1^k - x_2^k = y_1^k - y_2^k + z_1 - z_2, \tag{1.7}$$

with x_i and y_i bounded above by $n^{1/k}$, and with z_i in the same short interval. Aficionados of the circle method will recognise the potential for applying arguments based on the use of diminishing ranges, in which the variables y_i are constrained to lie in a slightly shortened interval. Two obstacles prevent a pedestrian treatment of this problem. First, one must apply diminishing ranges in a treatment restricted to minor arcs only. Also, one has the second challenge of handling a problem in which the number of variables is very small. Methods pursued in the first of this series of papers [4] may be adapted to surmount the first of these difficulties (see also [3] and [26] for earlier such treatments). Meanwhile, the second may be overcome by solving a long sequence of pruning exercises, all within range of the accomplished practitioner of such methods.

In this paper, we adopt the convention that whenever ε appears in a statement, either implicitly or explicitly, then the statement holds for each $\varepsilon > 0$. Implicit constants in the notations of Landau and Vinogradov will depend at most on ε and k. Finally, write $\|\theta\| = \min_{y\in\mathbb{Z}} |\theta - y|$ and $e(z)$ for $e^{2\pi i z}$.

Note added 14^{th} *July* 2016*:* Very recent work of Bourgain, Demeter and Guth [1] concerning Vinogradov's mean value theorem allows for some improvement in the exponent σ_k defined in (1.2). Thus, by employing [1, Theorem 1.1] in place of [31, Theorem 1.2] within the argument of the proof of Lemma 4.1, one finds that one may put $\sigma_k = (k^2 - 3k + 2)^{-1}$ for $k \geqslant 7$ without impairing subsequent conclusions.

2. Infrastructure

We begin by introducing the notation and cast of generating functions required to describe our method. We consider a fixed integer k with $k \geqslant 3$, and we define $\sigma = \sigma_k$ and $\theta = \theta_k$ as in (1.2) and (1.3). Let N be a sufficiently large positive number, and define

$$X = (N/3)^{1/k}, \quad Y = X^{1-(1-\sigma)/k}, \quad H = 2^k X^\sigma \quad \text{and} \quad Q = X^{1-\sigma/k}. \tag{2.1}$$

Also, we consider a real number Z with

$$X^{k\theta} \leqslant Z \leqslant 6k^2 X^{k-2+1/k}. \tag{2.2}$$

Let $r(n;Z)$ be the number of integral solutions of the equation (1.6) with $X < x \leqslant 2X$, $Y < y \leqslant 2Y$ and $1 \leqslant z \leqslant Z$. Our goal is an estimate for the quantity

$$\Upsilon(N,Z) = \sum_{N<n\leqslant 2N} \left| r(n;Z) - k^{-1}n^{-1+1/k}YZ \right|^2. \tag{2.3}$$

We bound $\Upsilon(N,Z)$ through the medium of the Hardy-Littlewood method. The exponential sums required in this enterprise are

$$f(\alpha) = \sum_{X<x\leqslant 2X} e(\alpha x^k), \quad g(\alpha) = \sum_{Y<y\leqslant 2Y} e(\alpha y^k), \quad u(\alpha) = \sum_{1\leqslant z\leqslant Z} e(\alpha z). \tag{2.4}$$

It will be expedient on numerous occasions to suppress the argument α from these notations as an aid to exposition and concision. Thus $f(\alpha)$ may be abbreviated to f, for example. By orthogonality, one has

$$r(n;Z) = \int_0^1 f(\alpha)g(\alpha)u(\alpha)e(-n\alpha)\,\mathrm{d}\alpha, \tag{2.5}$$

the relation which provides the starting point for our analysis of $\Upsilon(N,Z)$. With Q defined as in (2.1), we write $\mathfrak{M}$ for the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant QX^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$. Also, we denote by $\mathfrak{M}^\dagger$ the corresponding union of the intervals $\mathfrak{M}(q,a)$ in which $q > 1$. Further, we put $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. When $\mathfrak{B} \subseteq [0,1)$ is measurable, we write

$$r_{\mathfrak{B}}(n;Z) = \int_{\mathfrak{B}} f(\alpha)g(\alpha)u(\alpha)e(-n\alpha)\,\mathrm{d}\alpha.$$

Thus, in view of (2.5), we have

$$r(n;Z) = r_{\mathfrak{M}}(n;Z) + r_{\mathfrak{m}}(n;Z). \tag{2.6}$$

We next introduce the quantities

$$\Upsilon_{\mathfrak{m}} = \sum_{N<n\leqslant 2N} |r_{\mathfrak{m}}(n)|^2 \quad \text{and} \quad \Upsilon_{\mathfrak{M}} = \sum_{N<n\leqslant 2N} |r_{\mathfrak{M}}(n;Z) - k^{-1}n^{-1+1/k}YZ|^2.$$

Substituting (2.6) into (2.3), we thus arrive at the estimate

$$\Upsilon(N,Z) \leqslant 2(\Upsilon_{\mathfrak{m}} + \Upsilon_{\mathfrak{M}}). \tag{2.7}$$

We estimate the contribution of $\Upsilon_{\mathfrak{M}}$ in §3, deferring the consideration of $\Upsilon_{\mathfrak{m}}$ to §§4 and 5.

3. The collapse of the major arcs

We set about the task of replacing the generating functions f and u by their natural major arc approximants. We write

$$S(q,a)=\sum_{r=1}^{q}e(ar^k/q)\quad\text{and}\quad V(\beta;P)=\sum_{P^k<x\leqslant(2P)^k}k^{-1}x^{-1+1/k}e(\beta x),\quad(3.1)$$

and put $v(\beta)=V(\beta;X)$ and $w(\beta)=V(\beta;Y)$. Next, we define the function $f^*(\alpha)$ for $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$ by putting

$$f^*(\alpha)=q^{-1}S(q,a)v(\alpha-a/q),$$

and we set $f^*(\alpha)=0$ for $\alpha\in\mathfrak{m}$. Also, we define

$$u^*(\alpha)=\begin{cases}u(\alpha), & \text{when } \|\alpha\|\leqslant QX^{-k},\\ 0, & \text{otherwise.}\end{cases}\qquad(3.2)$$

We record for future reference an estimate of use in replacing $f(\alpha)$ by $f^*(\alpha)$ when $\alpha\in\mathfrak{M}$, with a similar estimate concerning $u(\alpha)$ and $u^*(\alpha)$.

Lemma 3.1. *When $\alpha\in\mathfrak{M}$, one has*

$$f(\alpha)-f^*(\alpha)\ll Q^{1/2+\varepsilon}\quad\textit{and}\quad u(\alpha)-u^*(\alpha)\ll Q.$$

Proof. The claim concerning f is immediate from [27, Theorem 4.1]. Meanwhile, from the relation

$$u(a/q)=\sum_{r=1}^{q}e(ar/q)\left(Z/q+O(1)\right),$$

valid for $a\in\mathbb{Z}$ and $q\in\mathbb{N}$, it follows via partial summation that

$$u(\beta+a/q)=q^{-1}\left(\sum_{r=1}^{q}e(ar/q)\right)u(\beta)+O\left(q(1+Z|\beta|)\right).\qquad(3.3)$$

A similar argument is employed in the proof of [27, Lemma 2.7]. When $q>1$ and $(a,q)=1$, one has

$$\sum_{r=1}^{q}e(ar/q)=0.$$

Thus, when $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$ with $q>1$, one deduces that

$$u(\alpha)\ll q+Z|q\alpha-a|\ll Q+ZQX^{-k}\ll Q.$$

When $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$ with $q=1$, meanwhile, one has $\|\alpha\|\leqslant QX^{-k}$, and hence $u(\alpha)=u^*(\alpha)$. Thus, in any case, we have $u(\alpha)-u^*(\alpha)\ll Q$, and the proof of the lemma is complete. □

We continue with an auxiliary mean value estimate. Write

$$I_1=\int_0^1|g(\alpha)u(\alpha)|^2\,\mathrm{d}\alpha.\qquad(3.4)$$

Lemma 3.2. *One has $I_1\leqslant YZ$.*

Proof. By orthogonality, we see that I_1 counts the number of integral solutions of the equation $y_1^k - y_2^k = z_1 - z_2$, with $Y < y_1, y_2 \leqslant 2Y$ and $1 \leqslant z_1, z_2 \leqslant Z$. When $y_1 \neq y_2$, one has $|y_1^k - y_2^k| \geqslant kY^{k-1} > Z$. The only solutions of this equation counted by I_1 consequently satisfy $y_1 = y_2$, whence $I_1 \leqslant YZ$. This completes the proof of the lemma. □

We are now equipped to pursue the replacement process.

Lemma 3.3. *One has*

$$\int_{\mathfrak{M}} |(f - f^*)gu|^2 \,\mathrm{d}\alpha \ll XYZ \quad \text{and} \quad \int_{\mathfrak{M}^\dagger} |f^*gu|^2 \,\mathrm{d}\alpha \ll X^{1+\varepsilon}YZ. \tag{3.5}$$

Proof. An application of Lemma 3.1 leads from (3.4) via Lemma 3.2 to the estimate

$$\int_{\mathfrak{M}} |(f - f^*)gu|^2 \,\mathrm{d}\alpha \ll Q^{1+\varepsilon} I_1 \ll XYZ,$$

confirming the first bound of (3.5).

For the second bound we must work harder. Note that, from (3.2), one has $u^*(\alpha) = 0$ for $\alpha \in \mathfrak{M}^\dagger$. Hence we deduce from Lemma 3.1 that

$$\int_{\mathfrak{M}^\dagger} |f^*gu|^2 \,\mathrm{d}\alpha \ll Q^2 \int_{\mathfrak{M}} |f^*g|^2 \,\mathrm{d}\alpha.$$

An application of Hölder's inequality shows that

$$\int_{\mathfrak{M}} |f^*g|^2 \,\mathrm{d}\alpha \leqslant \left(\int_{\mathfrak{M}} |f^*|^{k+1} \,\mathrm{d}\alpha \right)^{2/(k+1)} \left(\int_0^1 |g|^4 \,\mathrm{d}\alpha \right)^{1/2}.$$

The first integral on the right hand side here may be estimated through the methods of [27, Chapter 4] (see, in particular, [27, Lemmata 4.9 and 6.2]), and the second integral via Hua's lemma (see [27, Lemma 2.5]). Thus

$$\begin{aligned} \int_{\mathfrak{M}^\dagger} |f^*gu|^2 \,\mathrm{d}\alpha &\ll Q^2 (X^{1+\varepsilon})^{2/(k+1)} (Y^{2+\varepsilon})^{1/2} \\ &\ll X^{1+2\varepsilon} YZ (Q^2 X^{-1+2/(k+1)} Z^{-1}). \end{aligned} \tag{3.6}$$

Since $k \geqslant 3 + 2/(k+1) - (1+\sigma)/k$ when $k \geqslant 3$, it follows that

$$k - 2 + (1-\sigma)/k \geqslant 2/(k+1) - 1 + 2(1 - \sigma/k),$$

so that in view of (1.3), (2.1) and (2.2), the parenthetic factor on the right hand side of (3.6) is at most 1. This confirms the second bound of (3.5) and completes the proof of the lemma. □

We combine the two estimates of Lemma 3.3 in the next lemma.

Lemma 3.4. *One has*

$$\int_{\mathfrak{M}^\dagger} |fgu|^2 \,\mathrm{d}\alpha \ll X^{1+\varepsilon}YZ.$$

Proof. The elementary inequality $|f|^2 \ll |f-f^*|^2+|f^*|^2$ implies that

$$\int_{\mathfrak{M}^\dagger} |fgu|^2\,\mathrm{d}\alpha \ll \int_{\mathfrak{M}} |(f-f^*)gu|^2\,\mathrm{d}\alpha + \int_{\mathfrak{M}^\dagger} |f^*gu|^2\,\mathrm{d}\alpha,$$

and the desired conclusion is now immediate from Lemma 3.3. □

We define the central interval $\mathfrak{C}=[-QX^{-k},QX^{-k}]$, and note that

$$r_{\mathfrak{M}}(n;Z)=r_{\mathfrak{M}^\dagger}(n;Z)+r_{\mathfrak{C}}(n;Z).$$

It is useful to observe that when $\alpha\in\mathfrak{C}$, one has $f^*(\alpha)=v(\alpha)$. Next, put

$$\rho_1(n;Z)=\int_{\mathfrak{C}} v(\alpha)g(\alpha)u(\alpha)e(-n\alpha)\,\mathrm{d}\alpha.$$

Since $\mathfrak{C}\subseteq\mathfrak{M}+\mathbb{Z}$, an application of Bessel's inequality leads us via Lemma 3.3 to the bound

$$\sum_{N<n\leqslant 2N} |r_{\mathfrak{C}}(n;Z)-\rho_1(n;Z)|^2 \leqslant \int_{\mathfrak{C}} |(f-f^*)gu|^2\,\mathrm{d}\alpha \ll XYZ. \tag{3.7}$$

Likewise, we deduce via Lemma 3.4 that

$$\sum_{N<n\leqslant 2N} |r_{\mathfrak{M}^\dagger}(n;Z)|^2 \leqslant \int_{\mathfrak{M}^\dagger} |fgu|^2\,\mathrm{d}\alpha \ll X^{1+\varepsilon}YZ. \tag{3.8}$$

The singular integral is

$$\rho_2(n;Z)=\int_{-1/2}^{1/2} v(\alpha)g(\alpha)u(\alpha)e(-n\alpha)\,\mathrm{d}\alpha, \tag{3.9}$$

and we next compare this expression to $\rho_1(n;Z)$.

Lemma 3.5. *One has*

$$\sum_{N<n\leqslant 2N} |\rho_1(n;Z)-\rho_2(n;Z)|^2 \ll XYZ.$$

Proof. An application of Bessel's inequality conveys us from (3.9) via [27, Lemma 6.2] to the bound

$$\begin{aligned}\sum_{N<n\leqslant 2N} |\rho_1(n;Z)-\rho_2(n;Z)|^2 &\ll \int_{QX^{-k}}^{1/2} |v(\alpha)g(\alpha)u(\alpha)|^2\,\mathrm{d}\alpha \\ &\ll (XYZ)^2\int_{QX^{-k}}^{1/2}(1+X^k\alpha)^{-2}\,\mathrm{d}\alpha.\end{aligned}$$

Thus we conclude that

$$\sum_{N<n\leqslant 2N} |\rho_1(n;Z)-\rho_2(n;Z)|^2 \ll XYZ(X^{1-k}YZQ^{-1}).$$

Since $Q\geqslant Y$ and $Z\leqslant X^{k-1}$, the parenthetic factor on the right hand side here does not exceed 1, and so the proof of the lemma is complete. □

The singular integral may be evaluated with an error acceptable in mean square.

Lemma 3.6. *One has*
$$\sum_{N<n\leqslant 2N} \left|\rho_2(n;Z)-k^{-1}YZn^{-1+1/k}\right|^2 \ll XYZ.$$

Proof. By orthogonality, it follows from (3.9) that
$$\rho_2(n;Z)=k^{-1}\sum_{Y<y\leqslant 2Y}\sum_{1\leqslant z\leqslant Z}\sum_{\substack{X^k<m\leqslant (2X)^k\\ m+y^k+z=n}} m^{-1+1/k}.$$
Observe that when $n>N$, $y\leqslant 2Y$ and $z\leqslant Z$, one has
$$m=n-y^k-z=n\left(1+O(HX^{-1}+X^{-2+1/k})\right).$$
Hence
$$m^{-1+1/k}=n^{-1+1/k}(1+O(HX^{-1})),$$
and so it follows that
$$\rho_2(n;Z)=k^{-1}YZn^{-1+1/k}(1+O(HX^{-1})).$$
We thus deduce that
$$\begin{aligned}\sum_{N<n\leqslant 2N} \left|\rho_2(n;Z)-k^{-1}YZn^{-1+1/k}\right|^2 &\ll (YZ)^2N^{-1+2/k}H^2X^{-2}\\ &\ll XYZ(X^{-k-1}YZH^2).\end{aligned}$$
The parenthetic factor on the right hand side is at most $X^{-2}H^{2+1/k}\ll 1$. This completes the proof of the lemma. □

Write
$$S_1=r_{\mathfrak{M}^\dagger}(n;Z),\quad S_2=r_{\mathfrak{C}}(n;Z)-\rho_1(n;Z),$$
and
$$S_3=\rho_1(n;Z)-\rho_2(n;Z),\quad S_4=\rho_2(n;Z)-k^{-1}n^{-1+1/k}YZ.$$
Then since
$$r_{\mathfrak{M}}(n;Z)-k^{-1}n^{-1+1/k}YZ=S_1+\ldots+S_4,$$
an application of the elementary inequality $|S_1+\ldots+S_4|^2\leqslant |S_1|^2+\ldots+|S_4|^2$ combines with (3.7), (3.8), and Lemmata 3.5 and 3.6 to give
$$\Upsilon_{\mathfrak{M}}\ll X^{1+\varepsilon}YZ. \tag{3.10}$$

4. Minor arcs with a difference

We now estimate $\Upsilon_{\mathfrak{m}}$, noting that by Bessel's inequality, one has
$$\Upsilon_{\mathfrak{m}}\leqslant \int_{\mathfrak{m}}|fgu|^2\,\mathrm{d}\alpha = T-\int_{\mathfrak{M}}|fgu|^2\,\mathrm{d}\alpha, \tag{4.1}$$
in which
$$T=\int_0^1 |fgu|^2\,\mathrm{d}\alpha.$$
By orthogonality, the mean value T counts the number of integral solutions of the equation (1.7) with $X<x_i\leqslant 2X$, $Y<y_i\leqslant 2Y$ and $1\leqslant z_i\leqslant Z$ for

$i = 1, 2$. Put $h = x_1 - x_2$, and for concision write $x = x_2$. Then the equation (1.7) becomes

$$h\Psi(x,h) = y_1^k - y_2^k + z_1 - z_2, \tag{4.2}$$

where

$$\Psi(x,h) = \sum_{j=1}^{k} \binom{k}{j} x^{k-j} h^{j-1}.$$

For any solution of (4.2) counted by T, we have

$$|h| \leqslant X^{1-k}((2^k - 1)Y^k + Z) \leqslant H.$$

Thus, on putting

$$F(\alpha) = \sum_{|h| \leqslant H} \sum_{\substack{X < x \leqslant 2X \\ X < x+h \leqslant 2X}} e(h\Psi(x,h)\alpha), \tag{4.3}$$

we infer via orthogonality that

$$T = \int_0^1 F(\alpha)|g(\alpha)u(\alpha)|^2 \, \mathrm{d}\alpha.$$

In view of (4.1), therefore, we obtain the relation

$$\Upsilon_{\mathfrak{m}} \leqslant \int_0^1 F|g^2u^2| \, \mathrm{d}\alpha - \int_{\mathfrak{M}} |fgu|^2 \, \mathrm{d}\alpha. \tag{4.4}$$

We require a modified Hardy-Littlewood dissection for the discussion of the mean value T. Put $C = k^{-3k}$, and let $\mathfrak{N}$ denote the union of the intervals

$$\mathfrak{N}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant CXY^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant X$ and $(a,q) = 1$. Also, we denote by $\mathfrak{N}^\dagger$ the corresponding union of the intervals $\mathfrak{N}(q,a)$ in which $q > 1$. Further, we put $\mathfrak{n} = [0,1) \setminus \mathfrak{N}$.

Lemma 4.1. *One has*

$$\int_{\mathfrak{n}} |Fg^2u^2| \, \mathrm{d}\alpha \ll X^{1+\varepsilon}YZ.$$

Proof. Suppose that $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$ and $|\alpha - a/q| \leqslant q^{-2}$. Then it follows from a pedestrian generalisation of the proof of [25, Lemma 1] with $\nu = \sigma$ that, when $4 \leqslant k \leqslant 7$, one has

$$F(\alpha) \ll X^{1+\sigma+\varepsilon}(q^{-1} + X^{-1} + qX^{1-k-\sigma})^{2^{2-k}}. \tag{4.5}$$

Here, we have observed that the term with $h = 0$ in (4.3) contributes $O(X)$ to $|F(\alpha)|$, this being majorised by the term X^{-1} in the parenthetic expression on the right hand side of (4.5), since $\sigma = 2^{2-k}$ for $4 \leqslant k \leqslant 7$. The same conclusion follows from the proof of the lemma of [24] in the case $k = 3$.

Let $\alpha \in \mathfrak{n}$. An application of Dirichlet's theorem on Diophantine approximation shows that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, with $0 \leqslant a \leqslant q \leqslant (CX)^{-1}Y^k$ and $(a,q) = 1$, for which $|q\alpha - a| \leqslant CXY^{-k}$. In such circumstances, the definition of $\mathfrak{N}$ shows that $q > X$, and hence (4.5) yields the bound

$$F(\alpha) \ll X^{1+\sigma+\varepsilon}(X^{-1} + Y^kX^{-k-\sigma})^\sigma \ll X^{1+\varepsilon}.$$

When $k \geqslant 8$, meanwhile, we apply the method of proof of [28, Lemma 10.3] in which we formally take $M = \frac{1}{2}$ and $R = 2$. By substituting the conclusion of [30, Theorem 1.5], in the enhanced form made available via [31, Theorem 1.2], for [28, Lemma 10.2], one finds that the bound

$$\sup_{\alpha \in \mathfrak{n}} |F(\alpha)| \ll X^{1-\sigma+\varepsilon} H$$

holds with $\sigma = (2(k-2)(k-3))^{-1}$. Hence, when $\alpha \in \mathfrak{n}$, one has $F(\alpha) \ll X^{1+\varepsilon}$ in all cases. We note that both here, in considering the exponents $k \geqslant 8$, and in our earlier treatment for $3 \leqslant k \leqslant 7$, the exponential sum $F(\alpha)$ differs from the analogues occurring in the cited sources only by the presence of the additional summation condition $X < x + h \leqslant 2X$ in (4.3). However, the latter is easily accommodated in the respective proofs of the desired conclusions.

On recalling (3.4) and Lemma 3.2, we now see that

$$\int_{\mathfrak{n}} |Fg^2u^2| \, \mathrm{d}\alpha \ll X^{1+\varepsilon} \int_0^1 |gu|^2 \, \mathrm{d}\alpha \ll X^{1+\varepsilon} YZ.$$

This completes the proof of the lemma. □

It is convenient to isolate the diagonal contribution within $F(\alpha)$. Write

$$F_1(\alpha) = \sum_{1 \leqslant h \leqslant H} \sum_{\substack{X < x \leqslant 2X \\ X < x+h \leqslant 2X}} e(h\Psi(x,h)\alpha), \tag{4.6}$$

and observe that, in view of (4.3), one then has

$$F(\alpha) = 2\,\mathrm{Re}\, F_1(\alpha) + O(X). \tag{4.7}$$

Lemma 4.2. *One has*

$$\int_{\mathfrak{N}^\dagger} |Fg^2u^2| \, \mathrm{d}\alpha \ll X^{1+\varepsilon} YZ.$$

Proof. On recalling (3.4) and the estimate supplied by Lemma 3.2, one finds that (4.7) yields the relation

$$\int_{\mathfrak{N}^\dagger} |Fg^2u^2| \, \mathrm{d}\alpha \ll XI_1 + \int_{\mathfrak{N}^\dagger} |F_1g^2u^2| \, \mathrm{d}\alpha. \tag{4.8}$$

By reference to the argument leading to (3.3), we find that when $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta + a/q \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}^\dagger$, one has

$$u(\beta + a/q) \ll q + ZXY^{-k} \ll X. \tag{4.9}$$

Suppose first that $k \geqslant 4$. Then an application of Schwarz's inequality in combination with Lemma 3.2 reveals that

$$\int_{\mathfrak{N}^\dagger} |Fg^2u^2| \, \mathrm{d}\alpha \ll XYZ + X^2 I_2^{1/2} I_3^{1/2},$$

where

$$I_2 = \int_0^1 |F_1(\alpha)|^2 \, \mathrm{d}\alpha \quad \text{and} \quad I_3 = \int_0^1 |g(\alpha)|^4 \, \mathrm{d}\alpha. \tag{4.10}$$

By orthogonality, the integral I_2 counts the number of integral solutions of the equation $h_1\Psi(x_1,h_1) = h_2\Psi(x_2,h_2)$, with $X < x_i \leqslant 2X$ and $1 \leqslant h_i \leqslant H$ for $i = 1, 2$. A divisor function estimate confirms that, for each fixed choice of x_2 and h_2, there are $O((XH)^\varepsilon)$ possible choices for x_1 and h_1, whence $I_2 \ll (XH)^{1+\varepsilon}$. Meanwhile, the bound $I_3 \ll Y^{2+\varepsilon}$ follows from Hua's lemma (see [27, Lemma 2.5]). Hence

$$\int_{\mathfrak{N}^\dagger} |Fg^2u^2|\,\mathrm{d}\alpha \ll XYZ + X^{2+\varepsilon}(XH)^{1/2}Y \ll X^{1+\varepsilon}YZ(1 + X^{3/2}Z^{-1}H^{1/2}).$$

Since $X^{-k+7/2-1/k}H^{1/2+1/k} \leqslant 1$, the conclusion of the lemma follows for $k \geqslant 4$.

We turn next to the situation in which $k = 3$. Put $A = Z^{-1/2}X^{-1/8}$, and divide the set $\mathfrak{N}^\dagger$ into the two subsets

$$\mathfrak{N}_0^\dagger = \{\alpha \in \mathfrak{N}^\dagger : \|\alpha\| \leqslant A\} \quad \text{and} \quad \mathfrak{N}_1^\dagger = \{\alpha \in \mathfrak{N}^\dagger : \|\alpha\| > A\}.$$

Making use of the familiar estimate $u(\alpha) \ll \|\alpha\|^{-1}$, we find that

$$\int_{\mathfrak{N}_1^\dagger} |F_1g^2u^2|\,\mathrm{d}\alpha \ll (Z^{1/2}X^{1/8})^2 \int_{\mathfrak{N}_1^\dagger} |F_1g^2|\,\mathrm{d}\alpha.$$

An application of Schwarz's inequality yields the bound

$$\int_{\mathfrak{N}_1^\dagger} |F_1g^2|\,\mathrm{d}\alpha \ll I_2^{1/2}I_3^{1/2},$$

where I_2 and I_3 are defined as in (4.10). We observe that our earlier bounds for I_2 and I_3 remain valid also when $k = 3$. Thus, we conclude that

$$\int_{\mathfrak{N}_1^\dagger} |F_1g^2u^2|\,\mathrm{d}\alpha \ll ZX^{1/4+\varepsilon}(XH)^{1/2}(Y^2)^{1/2} \ll X^{1+\varepsilon}YZ(X^{-1/4}H^{1/2}). \tag{4.11}$$

For the treatment of $\mathfrak{N}_0^\dagger$, we require a sharp upper bound for

$$I_4 = \int_{\mathfrak{N}_0^\dagger} |g(\alpha)|^4\,\mathrm{d}\alpha.$$

Recall (3.1), and define

$$g^*(\alpha) = q^{-1}S(q,a)w(\alpha - a/q),$$

when $\alpha \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}$, and otherwise set $g^*(\alpha) = 0$. Then we find from [27, Theorem 4.1] that whenever $\alpha \in \mathfrak{N}$, one has $g(\alpha) - g^*(\alpha) \ll X^{1/2+\varepsilon}$. Hence

$$I_4 \ll \int_{\mathfrak{N}} |g^*(\alpha)|^4\,\mathrm{d}\alpha + X^{2+\varepsilon}\mathrm{mes}(\mathfrak{N}_0^\dagger). \tag{4.12}$$

From [27, Lemmata 4.9 and 6.2], one readily infers the bound

$$\int_{\mathfrak{N}} |g^*(\alpha)|^4\,\mathrm{d}\alpha \ll Y^{1+\varepsilon}.$$

Meanwhile

$$\begin{aligned}\operatorname{mes}(\mathfrak{N}_0^\dagger) &\leqslant \sum_{1\leqslant q\leqslant X} \sum_{\substack{1\leqslant a\leqslant q\\ \|a/q\|\leqslant 2Z^{-1/2}X^{-1/8}}} \operatorname{mes}(\mathfrak{N}(q,a))\\ &\ll \sum_{1\leqslant q\leqslant X} \left(qZ^{-1/2}X^{-1/8}\right)(q^{-1}XY^{-3}) \ll X^{15/8}Y^{-3}Z^{-1/2}.\end{aligned}$$

On substituting these estimates into (4.12), we discern that

$$I_4 \ll Y^{1+\varepsilon} + X^{31/8+\varepsilon}Y^{-3}Z^{-1/2} \ll Y^{1+\varepsilon}, \tag{4.13}$$

since $\frac{31}{8} - 3\left(\frac{5}{6}\right) - \frac{1}{2}\left(\frac{7}{6}\right) = \frac{19}{24} < \frac{5}{6}$.

Next, by (4.9) and the inequalities of Cauchy and Schwarz, one has

$$\int_{\mathfrak{N}_0^\dagger} |F_1 g^2 u^2|\,\mathrm{d}\alpha \ll XI_4^{1/2}(HI_5)^{1/2}, \tag{4.14}$$

where

$$I_5 = \int_0^1 F_2(\alpha)|u(\alpha)|^2\,\mathrm{d}\alpha,$$

in which we write

$$F_2(\alpha) = \sum_{1\leqslant h\leqslant H} \Bigg| \sum_{\substack{X<x\leqslant 2X\\ X<x+h\leqslant 2X}} e(h\Psi(x,h)\alpha)\Bigg|^2.$$

The integral I_5 does not exceed the number of integral solutions of the equation

$$h(\Psi(x_1,h) - \Psi(x_2,h)) = z_1 - z_2,$$

with $1 \leqslant h \leqslant H$, $X < x_1, x_2 \leqslant 2X$ and $1 \leqslant z_1, z_2 \leqslant Z$. Since $x_1 - x_2$ divides the polynomial $\Psi(x_1,h) - \Psi(x_2,h)$, it follows via an elementary divisor function estimate that, whenever z_1 and z_2 are fixed with $z_1 \neq z_2$, then there are $O(Z^\varepsilon)$ possible choices for h, x_1 and x_2. Hence we deduce that

$$I_5 \ll HXZ + Z^{2+\varepsilon} \ll HXZ.$$

On substituting this bound together with (4.13) into (4.14), we see that

$$\int_{\mathfrak{N}_0^\dagger} |F_1 g^2 u^2|\,\mathrm{d}\alpha \ll XY^{1/2}(H^2XZ)^{1/2}.$$

This, in combination with Lemma 3.2 and equations (4.8) and (4.11), gives

$$\int_{\mathfrak{N}^\dagger} |Fg^2u^2|\,\mathrm{d}\alpha \ll X^{1+\varepsilon}YZ(1 + X^{-1/4}H^{1/2} + X^{1/2}Y^{-1/2}Z^{-1/2}H).$$

Since $\frac{1}{2} - \frac{5}{12} - \frac{7}{12} + \frac{1}{2} = 0$, the conclusion of the lemma follows for $k=3$. □

The treatment of the minor arcs is now coming to an end. Define

$$\mathfrak{D} = \{\alpha \in [0,1) : \|\alpha\| \leqslant CXY^{-k}\}.$$

Note that $\mathfrak{M} = \mathfrak{M}^\dagger \cup \mathfrak{C}$ and $\mathfrak{N} = \mathfrak{N}^\dagger \cup \mathfrak{D}$. Since $[0, 1) = \mathfrak{D} \cup \mathfrak{N}^\dagger \cup \mathfrak{n}$, it follows by combining Lemmata 4.1 and 4.2 that

$$\int_0^1 F|gu|^2 \,\mathrm{d}\alpha = \int_{\mathfrak{D}} F|gu|^2 \,\mathrm{d}\alpha + O(X^{1+\varepsilon}YZ).$$

Likewise, we obtain from Lemma 3.4 the relation

$$\int_{\mathfrak{M}} |fgu|^2 \,\mathrm{d}\alpha = \int_{\mathfrak{C}} |fgu|^2 \,\mathrm{d}\alpha + O(X^{1+\varepsilon}YZ).$$

Hence, we conclude from (4.4) that

$$\Upsilon_{\mathfrak{m}} \leqslant \int_{\mathfrak{D}} F|gu|^2 \,\mathrm{d}\alpha - \int_{\mathfrak{C}} |fgu|^2 \,\mathrm{d}\alpha + O(X^{1+\varepsilon}YZ). \tag{4.15}$$

5. The annihilation of the central intervals

In this penultimate section, we complete the estimation of $\Upsilon_{\mathfrak{m}}$ by exploiting cancellations between the two integrals on the right hand side of equation (4.15). With this in view, we put $\mathfrak{c} = \mathfrak{D} \setminus \mathfrak{C}$ and recast the relation (4.15) as

$$\Upsilon_{\mathfrak{m}} \leqslant \int_{\mathfrak{C}} (F - |f|^2)|gu|^2 \,\mathrm{d}\alpha + \int_{\mathfrak{c}} F|gu|^2 \,\mathrm{d}\alpha + O(X^{1+\varepsilon}YZ). \tag{5.1}$$

We first show that the integral over $\mathfrak{c}$ can be absorbed into the error term. The argument will depend on the following simple estimate.

Lemma 5.1. *Let Δ be a positive number. Then*

$$\int_{-\Delta}^{\Delta} |g(\alpha)|^2 \,\mathrm{d}\alpha \ll \Delta Y + Y^{2-k+\varepsilon}.$$

Proof. By (2.4), one has

$$\int_{-\Delta}^{\Delta} |g(\alpha)|^2 \,\mathrm{d}\alpha = \sum_{Y < y_1, y_2 \leqslant 2Y} \int_{-\Delta}^{\Delta} e(\alpha(y_1^k - y_2^k)) \,\mathrm{d}\alpha.$$

The terms with $y_1 = y_2$ contribute $2\Delta Y$. The remaining terms contribute an amount not exceeding

$$\sum_{\substack{Y < y_1, y_2 \leqslant 2Y \\ y_1 \neq y_2}} \frac{2}{|y_1^k - y_2^k|}.$$

Here, we write $l = y_1 - y_2$, and observe that by symmetry, it suffices to estimate the part of the sum where $l > 0$. But then $y_1^k - y_2^k \gg lY^{k-1}$, and the sum in the preceding display is therefore bounded by

$$\sum_{1 \leqslant l \leqslant Y} \sum_{Y < y_2 \leqslant 2Y} \frac{1}{lY^{k-1}} \ll Y^{2-k+\varepsilon}.$$

The desired conclusion now follows. □

Lemma 5.2. *One has*

$$\int_{\mathfrak{c}} F(\alpha)|g(\alpha)u(\alpha)|^2 \,\mathrm{d}\alpha \ll X^{1+\varepsilon}YZ.$$

Proof. We note that when $\alpha \in \mathfrak{D}$ one has

$$HX^{k-2}\|\alpha\| \leqslant 2^k C X^{\sigma+k-2} XY^{-k} \leqslant 2^k C.$$

Hence, temporarily assuming that $k \geqslant 4$ and estimating the sum $F_1(\alpha)$ defined in (4.6) via [25, Lemma 2], we first deduce that

$$F_1(\alpha) \ll HX(1 + HX^{k-1}\|\alpha\|)^{-1} + H,$$

and then infer from (4.7) the bound

$$F(\alpha) \ll HX(1 + HX^{k-1}\|\alpha\|)^{-1} + X. \tag{5.2}$$

The proof of [25, Lemma 2] remains valid when $k = 3$ and $q = 1$ (in the notation of this reference). Hence (5.2) holds for all $k \geqslant 3$, and consequently,

$$\int_{\mathfrak{c}} F|gu|^2 \, \mathrm{d}\alpha \ll XI_1 + \Gamma, \tag{5.3}$$

where I_1 is given by (3.4), and

$$\Gamma = HX \int_{\mathfrak{c}} \frac{|g(\alpha)u(\alpha)|^2}{1 + HX^{k-1}\|\alpha\|} \, \mathrm{d}\alpha.$$

Note that $\mathfrak{c}$ is the union of two intervals, one of which being $[QX^{-k}, XY^{-k}]$. By symmetry, and since the integrand has period 1, it suffices to estimate the contribution from this interval. This we cover by $O(\log X)$ disjoint intervals $[AY^{-k}, 2AY^{-k}]$, with $QX^{\sigma-1} \leqslant A \leqslant X$. By Lemma 5.1, making use of the trivial bound $|u(\alpha)| \leqslant Z$, we find that

$$\begin{aligned}\int_{AY^{-k}}^{2AY^{-k}} \frac{|g(\alpha)u(\alpha)|^2}{1 + HX^{k-1}\alpha} \, \mathrm{d}\alpha &\ll Z^2 H^{-1} X^{1-k} A^{-1} Y^k \int_{AY^{-k}}^{2AY^{-k}} |g(\alpha)|^2 \, \mathrm{d}\alpha \\ &\ll Z^2 A^{-1}(AY^{1-k} + Y^{2-k+\varepsilon}) \\ &\ll Z^2 Y^{1-k} + Z^2 Q^{-1} X^{1-\sigma} Y^{2-k+\varepsilon}.\end{aligned}$$

Here the second term on the right hand side dominates, and we infer the bound

$$\Gamma \ll HX^{2-\sigma} Y^{2-k+\varepsilon} Z^2 Q^{-1} \ll X^{1+\varepsilon} YZ(XY^{1-k} Q^{-1} Z).$$

Since $Y/Q = X^{(2\sigma-1)/k}$ and $XY^{-k}Z \ll X^{-\sigma+1/k}$, it follows that $\Gamma \ll X^{1+\varepsilon}YZ$. The lemma now follows from (5.3) and Lemma 3.2. □

Lemma 5.3. *One has*

$$\int_{\mathfrak{C}} (F - |f|^2)|gu|^2 \, \mathrm{d}\alpha = \int_{\mathfrak{C}} (F - |f|^2)|wu|^2 \, \mathrm{d}\alpha + O(XYZ).$$

Proof. When $\alpha \in \mathfrak{C}$, we find from [27, Theorem 4.1] that $g(\alpha) = w(\alpha) + O(1)$, and hence $|g(\alpha)|^2 = |w(\alpha)|^2 + O(|w(\alpha)|)$. On multiplying this relation with $(F - |f|^2)|u|^2$, one finds that the lemma will follow from the estimate

$$\int_{\mathfrak{C}} |(F - |f|^2) w u^2| \, \mathrm{d}\alpha \ll XYZ, \tag{5.4}$$

that we now establish in two steps.

First we observe that [27, Lemma 6.2] delivers the bound

$$\int_{\mathfrak{C}} |w(\alpha)|\,\mathrm{d}\alpha \ll Y \int_{-1/2}^{1/2} (1+Y^k|\alpha|)^{-1}\,\mathrm{d}\alpha \ll Y^{1-k+\varepsilon}.$$

Hence, the trivial bounds $F(\alpha) \ll HX$ and $u(\alpha) \ll Z$ suffice to conclude that

$$\int_{\mathfrak{C}} |Fwu^2|\,\mathrm{d}\alpha \ll HXY^{1-k+\varepsilon}Z^2 = XYZ(HY^{\varepsilon-k}Z), \tag{5.5}$$

and we note that $HY^{\varepsilon-k}Z \ll 1$.

Another appeal to [27, Theorem 4.1] shows that whenever $\alpha \in \mathfrak{C}$, one has $f(\alpha) = v(\alpha) + O(1)$, and [27, Lemma 6.2] then delivers the estimate

$$f(\alpha) \ll X(1 + X^k\|\alpha\|)^{-1}.$$

Using trivial bounds for $w(\alpha)$ and $u(\alpha)$, we now infer that

$$\int_{\mathfrak{C}} |f^2wu^2|\,\mathrm{d}\alpha \ll YZ^2X^2 \int_{-1/2}^{1/2} (1+X^k|\alpha|)^{-2}\,\mathrm{d}\alpha \ll YZ^2X^{2-k} \ll XYZ.$$

On combining this bound with (5.5), we arrive at (5.4). This completes the proof of the lemma. □

Lemma 5.4. *Let* $\mathfrak{K} = [0,1] \setminus \mathfrak{C}$. *Then*

$$\int_{\mathfrak{K}} (F(\alpha) - |f(\alpha)|^2)|w(\alpha)u(\alpha)|^2\,\mathrm{d}\alpha \ll XYZ.$$

Proof. The argument is similar to the one used to demonstrate the previous lemma. We again use [27, Lemma 6.2], this time providing the bound

$$\begin{aligned}\int_{\mathfrak{K}} |w(\alpha)|^2\,\mathrm{d}\alpha &\ll Y^2 \int_{Q/X^k}^{1/2} (1+Y^k\alpha)^{-2}\,\mathrm{d}\alpha \\ &\ll Y^{2-k}Q^{-1}X^kY^{-k} \ll Y^{2-k}H^{-1+1/k}.\end{aligned} \tag{5.6}$$

The trivial bound for $F(\alpha)|u(\alpha)|^2$ now implies that

$$\int_{\mathfrak{K}} F|wu|^2\,\mathrm{d}\alpha \ll Y^{2-k}H^{-1+1/k}HXZ^2 \ll XYZ, \tag{5.7}$$

because one has $Y^{1-k}H^{1/k}Z \ll H^{-1+2/k} \ll 1$.

More care is required for the term involving $|f(\alpha)|^2$. Here, we split $\mathfrak{K}$ into its subsets $\mathfrak{c}$ and $\mathfrak{K} \setminus \mathfrak{c} = \{\alpha \in [0,1] : \|\alpha\| > CXY^{-k}\}$. The argument leading to (5.6) yields

$$\int_{\mathfrak{K}\setminus\mathfrak{c}} |w(\alpha)|^2\,\mathrm{d}\alpha \ll Y^{2-k}X^{-1},$$

so that a trivial bound for $|f(\alpha)u(\alpha)|^2$ provides the estimate

$$\int_{\mathfrak{K}\setminus\mathfrak{c}} |fwu|^2\,\mathrm{d}\alpha \ll XY^{2-k}Z^2 \ll XYZ. \tag{5.8}$$

It remains to examine the contribution from $\mathfrak{c}$. For $\alpha \in \mathfrak{c}$ we deduce from [27, Theorem 4.1 and Lemma 6.2] that

$$f(\alpha) \ll X(1+X^k\|\alpha\|)^{-1} + (X^k\|\alpha\|)^{1/2},$$

and hence,

$$|f(\alpha)w(\alpha)|^2 \ll X^2Y^2(1+X^k\|\alpha\|)^{-2} + X^kY^2\|\alpha\|(1+Y^k\|\alpha\|)^{-2}.$$

Since $\|\alpha\| \geqslant QX^{-k} \gg H^{1-1/k}Y^{-k}$, the previous bound implies that

$$|f(\alpha)w(\alpha)|^2 \ll X^2Y^2Q^{-1}(1+X^k\|\alpha\|)^{-1} + X^kY^{2-2k}\|\alpha\|^{-1}.$$

By applying a trivial bound for $u(\alpha)$, we may conclude that

$$\begin{aligned}\int_{\mathfrak{c}} |fwu|^2\,\mathrm{d}\alpha &\ll Z^2(X^{2-k+\varepsilon}Y^2Q^{-1} + X^{k+\varepsilon}Y^{2-2k})\\ &\ll XYZ(H^{2/k}X^{\varepsilon-1} + H^{-2+1/k}).\end{aligned} \tag{5.9}$$

The lemma now follows from (5.7), (5.8) and (5.9). □

We are ready to assemble the puzzle. By combining Lemmata 5.2, 5.3 and 5.4, we find from (5.1) that

$$\Upsilon_{\mathfrak{m}} \leqslant \int_0^1 (F-|f|^2)|wu|^2\,\mathrm{d}\alpha + O(X^{1+\varepsilon}YZ).$$

By applying orthogonality and reversing the transformation $h = x_1 - x_2$ and $x = x_2$ within (4.3), one finds that the main term here is a weighted count of the integral solutions of the equation

$$x_1^k - x_2^k = m_1 - m_2 + z_1 - z_2,$$

with $X < x_i \leqslant 2X$, $Y^k < m_i \leqslant (2Y)^k$ and $1 \leqslant z_i \leqslant Z$ $(i=1,2)$, subject to the condition $|x_1 - x_2| > H$. For each such putative solution, one has

$$|x_1^k - x_2^k| \geqslant kHX^{k-1} > (2Y)^k + Z > |m_1 - m_2 + z_1 - z_2|,$$

whence one infers that in fact no solutions exist. Thus we conclude that the contribution of F to $\Upsilon_{\mathfrak{m}}$ annihilates the anti-contribution of $|f|^2$, implying that $\Upsilon_{\mathfrak{m}} \ll X^{1+\varepsilon}YZ$. By combining this estimate with (3.10) and (2.7), we arrive at the bound

$$\Upsilon(N,Z) \ll X^{1+\varepsilon}YZ. \tag{5.10}$$

6. Deduction of the main results

The proof of Theorem 1.1. Recall that $\phi_k = (1-1/k)^2$, and that $E_k(N,Z)$ denotes the number of integers n with $N < n \leqslant 2N$ for which the interval $(n, n+Z]$ contains no integer that is the sum of two positive integral kth powers. For the latter integers n, one has $r(n;Z) = 0$. Therefore, when $N^{\theta_k} \leqslant Z \leqslant 2k^2N^{\phi_k}$, it follows from (2.3) and (5.10) that

$$E_k(N,Z)\left(k^{-1}N^{-1+1/k}YZ\right)^2 \leqslant \Upsilon(N,Z) \ll X^{1+\varepsilon}YZ,$$

whence

$$E_k(N,Z) \ll N^{2-2/k}X^{1+\varepsilon}(YZ)^{-1} \ll N^{1+\theta_k+\varepsilon}Z^{-1}.$$

When $Z > 2k^2N^{\phi_k}$, meanwhile, it follows via the greedy algorithm that $E_k(N,Z) = 0$ for large N. This completes the proof of Theorem 1.1. □

The proof of Theorem 1.2. Within this proof we abbreviate $s_{k,n}$ to s_n. For large N, it follows from (1.5) that whenever $s_{n+1} \leqslant N$, then $s_{n+1} - s_n \leqslant k^2N^{\phi_k}$. This shows that $E_k(N,Z) = 0$ whenever $Z > 2k^2N^{\phi_k}$. Let

$$\Xi(N,Z) = \text{card}\{N/2 < s_n \leqslant N : Z/2 < s_{n+1} - s_n \leqslant Z\},$$

and put $Z_0 = 4k^2N^{\phi_k}$. Then we have $\Xi(N,Z) = 0$ for $Z > Z_0$. Also, when Z is an even integer with $4 \leqslant Z \leqslant Z_0$ and $s_{n+1} - s_n > Z$, then each of the intervals $(s_n + m - 1, s_n + m + Z/2)$ $(1 \leqslant m \leqslant Z/2)$ contains no sum of two positive integral k-th powers. Hence

$$E_k(N, Z/2) \geqslant (Z/2)\Xi(N, 2Z),$$

and therefore, we deduce from Theorem 1.1 that

$$\Xi(N, 2Z) \ll Z^{-1}E_k(N, Z/2) \ll N^{1+\theta_k+\varepsilon}Z^{-2}.$$

We now conclude that

$$\sum_{N/2<s_n\leqslant N} (s_{n+1} - s_n)^2 \ll \sum_{\substack{j=0\\2^j\leqslant Z_0}}^{\infty} (2^{-j}Z_0)^2E_k(N, 2^{-j}Z_0) \ll N^{1+\theta_k+2\varepsilon}.$$

On summing over dyadic intervals, the conclusion of Theorem 1.2 follows. □

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