A PAUCITY PROBLEM FOR CERTAIN TRIPLES OF DIAGONAL EQUATIONS

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Abstract. We consider certain systems of three linked simultaneous diagonal equations in ten variables with total degree exceeding five. By means of a complification argument, we obtain an asymptotic formula for the number of integral solutions of this system of bounded height that resolves the associated paucity problem.

1. Introduction

In this note we investigate the simultaneous Diophantine equations

$$\sum_{i=1}^{5}(x_i^k - y_i^k) = \sum_{i=1}^{3}(x_i^n - y_i^n) = \sum_{i=4}^{5}(x_i^m - y_i^m) = 0, \tag{1.1}$$

focusing our attention on the number $N_{k,m,n}(B)$ of integral solutions $\mathbf{x}, \mathbf{y}$ of this system satisfying $1 \leqslant x_i, y_i \leqslant B$ $(1 \leqslant i \leqslant 5)$. These equations admit the diagonal solutions with

$$\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\} \quad \text{and} \quad \{x_4, x_5\} = \{y_4, y_5\},$$

contributing an amount

$$T(B) = (3!B^3 + O(B^2))(2!B^2 + O(B)) = 12B^5 + O(B^4) \tag{1.2}$$

to the total count $N_{k,m,n}(B)$. Whether or not one should expect an abundance of non-diagonal solutions to the system (1.1) depends on the triple (k, m, n). Excluding from consideration the degenerate cases in which $k \in \{m, n\}$, the goal of this paper is the characterisation of the triples (k, m, n) for which there is a paucity of non-diagonal solutions.

Theorem 1.1. *Suppose that* $(k, m, n) \neq (3, 1, 1)$*, and further that neither* $(k, n) = (2, 1)$ *nor* $(k, n) = (1, 2)$*. Then, for any positive number* δ *with* $\delta < 1/12$*, one has*

$$N_{k,m,n}(B) = 12B^5 + O(B^{5-\delta}).$$

In §2 we show that when (k, n) is either $(2, 1)$ or $(1, 2)$, one has

$$N_{k,m,n}(B) \gg B^5 \log(2B). \tag{1.3}$$

Moreover, as a consequence of our earlier work [5], one may show that

$$N_{3,1,1}(B) - T(B) \gg B^5. \tag{1.4}$$

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For all other triples (k,m,n) with $k\notin\{m,n\}$, it follows from Theorem 1.1 that
$$N_{k,m,n}(B)=T(B)+o(T(B)),$$
whence there is a paucity of non-diagonal solutions in the system (1.1).

It would be possible to extend our methods from the counting problem of estimating $N_{k,m,n}(B)$ to the associated problem of estimating the quantity $N^{\pm}_{k,m,n}(B)$, wherein the solutions of (1.1) are counted with $|x_i|,|y_i|\leqslant B$. By weakening the condition $1\leqslant x_i,y_i\leqslant B$ so as to include also negative solutions of (1.1), one encounters additional linear spaces of solutions, and thus the asymptotic formula $N_{k,m,n}(B)=12B^5+O(B^{5-\delta})$ must be replaced by the relation
$$N^{\pm}_{k,m,n}(B)=\rho_{k,m,n}B^5+O(B^{5-\delta}),$$
where $\rho_{k,m,n}$ is a certain positive integer depending on the respective parities of k, m and n. The exposition of our ideas would be significantly complicated and lengthened by the associated combinatorial details, as much by additional notation as anything of substance. Dedicated readers may check the details for themselves.

Existing paucity results for a single equation in four variables, and for pairs of equations in six variables, play a role in our proof of Theorem 1.1. However, the ideas underlying such results would be insufficient by themselves to deliver the conclusion of our theorem. We instead reach for the strategy described in our recent work [5] concerning diagonal cubic equations with two linear slices. This work, which addresses the case $(k,m,n)=(3,1,1)$ of the system (1.1), and yields an asymptotic formula confirming the lower bound (1.4), involves an application of the Hardy-Littlewood method in combination with a certain complification argument. Our approach in the present note once again highlights the opportunity for powerful interplay between equations to be exploited when analysing systems of many diagonal equations. We refer the reader to [3] and [4] for earlier instances in which such an observation has been utilised.

This paper is organised as follows. In §2 we introduce the infrastructure required for the subsequent discussion, justifying *en passant* the relations (1.3) and (1.4). A paucity result involving four m-th powers in §3 handily disposes of triples (k,m,n) with $m\geqslant 3$. We examine in §4 an upper bound for the number of non-zero integers h represented by the trailing block
$$\begin{aligned}x_4^k-y_4^k+x_5^k-y_5^k&=h\\ x_4^m-y_4^m+x_5^m-y_5^m&=0\end{aligned}$$
in (1.1). Thus equipped, we dispose of triples (k,m,n) with $n\geqslant 3$. The complification process comes into play in §§5-7. Here, an application of Cauchy's inequality relates non-diagonal solutions in the system (1.1) to the number of solutions of a related system in 12 variables having respective degrees k, n and n. The simplest application of this idea handles triples (k,m,n) in §5 with $n=2$. Then, in §6, a similar argument takes care of triples (k,m,n) with $n=1$ and $k\geqslant 4$. Our final case awaits our attention in §7, namely that

with $(k,m,n)=(3,2,1)$. In this situation we are forced to apply a crude version of the Hardy-Littlewood method in concert with complification, drawing inspiration from aspects of our treatment of the case $(k,m,n)=(3,1,1)$ in [5].

Our basic parameter is B, a sufficiently large positive number. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon>0$. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε, k, m and n. We make frequent use of vector notation in the form $\mathbf{x}=(x_1,\dots,x_r)$. Here, the dimension r depends on the course of the argument. Finally, we write $e(z)$ for $e^{2\pi iz}$.

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2. Infrastructure and the excluded cases

We fix a triple (k,m,n) with $k\notin\{m,n\}$. Defining the exponential sum

$$f_{k_1,k_2}(\alpha_1,\alpha_2)=\sum_{1\leqslant x\leqslant B}e(\alpha_1x^{k_1}+\alpha_2x^{k_2}),$$

it follows via orthogonality that

$$N_{k,m,n}(B)=\int_{[0,1)^3}|f_{k,n}(\alpha,\beta)^6f_{k,m}(\alpha,\gamma)^4|\,\mathrm{d}\boldsymbol{\alpha},\tag{2.1}$$

where we use $\boldsymbol{\alpha}$ to denote (α,β,γ).

For the time-being, it suffices to decompose the mean value (2.1) by introducing the auxiliary integrals $u(h)=u_{k,n}(h)$ and $v(h)=v_{k,m}(h)$, defined by

$$u(h)=\int_{[0,1)^2}|f_{k,n}(\alpha,\beta)|^6e(-h\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta\tag{2.2}$$

and

$$v(h)=\int_{[0,1)^2}|f_{k,m}(\alpha,\gamma)|^4e(-h\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\gamma.\tag{2.3}$$

Here, by orthogonality, one sees that $u(h)$ counts the representations of the integer h in the form

$$\sum_{i=1}^3(x_i^k-y_i^k)=h\tag{2.4}$$

subject to

$$\sum_{i=1}^3(x_i^n-y_i^n)=0,\tag{2.5}$$

with $1\leqslant x_i,y_i\leqslant B$ $(1\leqslant i\leqslant 3)$. Likewise, we find from (2.3) that $v(h)$ counts the number of solutions of the system

$$x_1^k+x_2^k-y_1^k-y_2^k=h,\tag{2.6}$$

$$x_1^m+x_2^m-y_1^m-y_2^m=0,\tag{2.7}$$

with $1 \leqslant x_i, y_i \leqslant B$ $(i = 1, 2)$. Thus, we see that

$$N_{k,m,n}(B) = \sum_{|h| \leqslant 2B^k} u(h)v(h). \tag{2.8}$$

We pause at this point to remark that, as a consequence of the work of the first author joint with Blomer [1], one has the asymptotic formula

$$u_{2,1}(0) = u_{1,2}(0) = \frac{18}{\pi^2} B^3 \log B + O(B^3).$$

Moreover, when $m \neq k$, it follows that whenever $\mathbf{x}, \mathbf{y} \in \mathbb{N}^2$ and

$$\begin{aligned} x_1^k + x_2^k &= y_1^k + y_2^k \\ x_1^m + x_2^m &= y_1^m + y_2^m, \end{aligned}$$

then $\{x_1, x_2\} = \{y_1, y_2\}$. This assertion may be confirmed either by elementary arguments, or by reference to [10]. It follows that one has the asymptotic relation

$$v_{k,m}(0) = 2B^2 + O(B). \tag{2.9}$$

By substituting these estimates into (2.8), we conclude that

$$N_{2,m,1}(B) \geqslant u_{2,1}(0)v_{2,m}(0) \gg B^5 \log(2B),$$

and likewise

$$N_{1,m,2}(B) \geqslant u_{1,2}(0)v_{1,m}(0) \gg B^5 \log(2B).$$

The lower bound (1.3) follows.

The relation (1.4), though essentially immediate from [5], merits some discussion. In the latter source, it is shown that

$$N_{3,1,1}^{\pm}(B) = (45 + \mathcal{C})(2B)^5 + O(B^{5-1/200}),$$

where $\mathcal{C} > 0$ is a product of local densities. Here, the constant 45 is associated with the number of linear spaces of solutions of the system (1.1) in the case $(k, m, n) = (3, 1, 1)$ generalising the diagonal solutions relevant to our examination of $N_{3,1,1}(B)$. Excluding solutions of (1.1) involving negative integers simplifies the analysis of [5] somewhat, and thus one may proceed at a pedestrian pace to obtain the asymptotic formula

$$N_{3,1,1}(B) = (12 + \mathcal{C}')B^5 + O(B^{5-1/200}),$$

where $\mathcal{C}' > 0$ is the product of local densities associated with the system (1.1) in the positive sector. In particular, in view of (1.2), one has the relation

$$N_{3,1,1}(B) - T(B) \sim \mathcal{C}' B^5,$$

confirming the lower bound (1.4).

Having discussed the excluded cases, we proceed in the remainder of the paper under the assumption that

$$(k, n) \notin \{(2, 1), (1, 2)\} \quad \text{and} \quad (k, m, n) \neq (3, 1, 1). \tag{2.10}$$

Since also $k \notin \{m, n\}$, we may assume that one of the following holds:

(i) $m \geqslant 3$;
(ii) $m \in \{1, 2\}$ and $n \geqslant 3$;

(iii) $m \in \{1,2\}$, $n = 2$ and $k \geqslant 3$;
(iv) $m \in \{1,2\}$, $n = 1$ and $k \geqslant 4$;
(v) $(k,m,n) = (3,2,1)$.

Notice that the first condition in (2.10) ensures, via available paucity results, that

$$u_{k,n}(0) = 6B^3 + O(B^{8/3}). \tag{2.11}$$

A convenient reference for a result of this strength may be obtained by combining [12, Theorem 1.2], when $(k,n) = (3,1)$, with [13, Theorem 1], when $(k,n) = (3,2)$, and [8, Corollary 0.3], when $k \geqslant 4$. By combining this conclusion with (2.9), we see from (2.8) that

$$\begin{aligned} N_{k,m,n}(B) &= u_{k,n}(0)v_{k,m}(0) + \sum_{1\leqslant |h| \leqslant 2B^k} u_{k,n}(h)v_{k,m}(h) \\ &= 12B^5 + \sum_{1\leqslant |h| \leqslant 2B^k} u_{k,n}(h)v_{k,m}(h) + O(B^{14/3}). \end{aligned} \tag{2.12}$$

Our task in the remaining sections is to analyse the sum on the right hand side of (2.12). We claim that for the triples (k,m,n) classified in the cases (i) to (v) above, for any positive number $\eta < 1/12$, one has

$$\sum_{1\leqslant |h| \leqslant 2B^k} u_{k,n}(h)v_{k,m}(h) \ll B^{5-\eta}. \tag{2.13}$$

By substituting this estimate into (2.12), we infer that

$$N_{k,m,n}(B) = 12B^5 + O(B^{5-\eta}),$$

and the conclusion of Theorem 1.1 follows.

3. Paucity for four m-th powers

Our first step towards the proof of Theorem 1.1 is the discussion of triples (k,m,n) of type (i), with $m \geqslant 3$. Here we make use of available upper bounds for the number $w_m(B)$ of solutions $\mathbf{x}, \mathbf{y}$ of the equation

$$x_1^m + x_2^m = y_1^m + y_2^m,$$

with $\{x_1,x_2\} \neq \{y_1,y_2\}$ and $1 \leqslant x_i, y_i \leqslant B$ $(i = 1,2)$.

Lemma 3.1. *When $m \geqslant 3$, one has $w_m(B) \ll B^{5/3+\varepsilon}$.*

Proof. Perhaps the most convenient references for this conclusion are the papers [6] and [7], respectively dealing with odd and even exponents m. More recent developments can be perused in [9, Corollary 0.2] and the associated discussion. □

We are now equipped to establish the main conclusion of this section.

Lemma 3.2. *Suppose that $(k,n) \notin \{(2,1),(1,2)\}$ and $k \notin \{m,n\}$. Then whenever $m \geqslant 3$, one has*

$$N_{k,m,n}(B) - 12B^5 \ll B^{14/3+\varepsilon}.$$

Proof. Suppose that $\mathbf{x},\mathbf{y}$ is a solution of the equations (2.6) and (2.7) with $1\leqslant x_i,y_i\leqslant B$ $(i=1,2)$. When $\{x_1,x_2\}=\{y_1,y_2\}$, one must have $h=0$. Thus, when $h\ne 0$, it follows that $\{x_1,x_2\}\ne\{y_1,y_2\}$, whence $\mathbf{x},\mathbf{y}$ is counted by $w_m(B)$. In particular, one has

$$\sum_{1\leqslant |h|\leqslant 2B^k} v_{k,m}(h)\leqslant w_m(B)\ll B^{5/3+\varepsilon},$$

and consequently,

$$\sum_{1\leqslant |h|\leqslant 2B^k} u_{k,n}(h)v_{k,m}(h)\leqslant \Bigl(\sup_h u_{k,n}(h)\Bigr)\sum_{1\leqslant |h|\leqslant 2B^k} v_{k,m}(h) \ll B^{5/3+\varepsilon}\sup_h u_{k,n}(h). \tag{3.1}$$

By the triangle inequality, it follows from (2.2) that

$$\sup_h u_{k,n}(h)\leqslant \int_{[0,1)^2}|f_{k,n}(\alpha,\beta)|^6\,\mathrm{d}\alpha\,\mathrm{d}\beta=u_{k,n}(0).$$

Thus, on substituting this estimate into (3.1) and recalling (2.11), we find that

$$\sum_{1\leqslant |h|\leqslant 2B^k} u_{k,n}(h)v_{k,m}(h)\ll B^{5/3+\varepsilon}\cdot B^3=B^{14/3+\varepsilon}.$$

The conclusion of the lemma is now immediate from (2.12). ☐

4. An upper bound for $v(h)$

We next consider triples (k,m,n) of type (ii), with $m\in\{1,2\}$ and $n\geqslant 3$. Our strategy applies bounds for $v_{k,m}(h)$ going beyond square-root cancellation.

Lemma 4.1. *Suppose that $h\ne 0$. Then*

(i) *when $k>2$, one has $v_{k,1}(h)\ll |h|^\varepsilon$;*
(ii) *when $k\ne 2$, one has $v_{k,2}(h)\ll |h|^\varepsilon B^{1+\varepsilon}$;*
(iii) *one has $v_{2,1}(h)\ll |h|^\varepsilon B$.*

Proof. When $m=1$ and $k>2$, the validity of equations (2.6) and (2.7) implies first that

$$x_2=y_1+y_2-x_1, \tag{4.1}$$

and hence that

$$(y_1+y_2-x_1)^k-(y_1^k+y_2^k-x_1^k)=h. \tag{4.2}$$

The polynomial on the left hand side here has factors y_1-x_1 and y_2-x_1, and hence there is a polynomial $\Psi_1\in\mathbb{Z}[s_1,s_2,s_3]$ of degree $k-2$ for which

$$(y_1-x_1)(y_2-x_1)\Psi_1(y_1,y_2,x_1)=h.$$

We therefore see that y_1-x_1, y_2-x_1 and $\Psi_1(y_1,y_2,x_1)$ are all divisors of the non-zero integer h. There are $O(|h|^\varepsilon)$ such divisors, say $d_1=y_1-x_1$, $d_2=y_2-x_1$ and $d_3=\Psi_1(y_1,y_2,x_1)$, whence

$$y_1=x_1+d_1,\quad y_2=x_1+d_2\quad\text{and}\quad \Psi_1(x_1+d_1,x_1+d_2,x_1)=d_3. \tag{4.3}$$

An examination of (4.2) reveals that

$$(x_1+d_1+d_2)^k-(x_1+d_1)^k-(x_1+d_2)^k+x_1^k=d_1d_2\Psi_1(x_1+d_1,x_1+d_2,x_1),$$

and so a consideration of the second forward difference polynomial associated with x^k reveals that $\Psi_1(x_1+d_1,x_1+d_2,x_1)$ is non-constant as a polynomial in x_1. For each fixed one of the $O(|h|^\varepsilon)$ possible choices for d_1, d_2, d_3, it therefore follows from the final equation in (4.3) that there are $O(1)$ possible choices for x_1. From here, by back substituting first into (4.3), and thence into (4.1), we find that x_1, x_2, y_1, y_2 are all fixed. Thus indeed $v_{k,1}(h)\ll |h|^\varepsilon$, and the proof of the lemma is complete in case (i).

In case (iii) we may proceed in like manner, though in this case we find that $\Psi_1=2$. We therefore have as many as $O(B)$ choices remaining available for x_1, and so we arrive at the weaker upper bound $v_{2,1}(h)\ll |h|^\varepsilon B$.

Finally, we examine the situation with $m=2$. Notice first that when $x_1=x_2$ and $2x_1^k=h$, then equation (2.6) simplifies to $y_1^k+y_2^k=0$, and this is impossible because $y_1,y_2\in\mathbb{N}$. It follows that either $2x_1^k\neq h$ or $2x_2^k\neq h$, and we may assume the latter by symmetry. We now substitute the equation

$$x_2^2=y_1^2+y_2^2-x_1^2 \tag{4.4}$$

for (4.1), and thus infer that in place of (4.2) we have the equation

$$\begin{aligned}(y_1^2+y_2^2-x_1^2)^k-(y_1^k+y_2^k-x_1^k)^2&=(x_2^2)^k-(x_2^k-h)^2\\ &=h(2x_2^k-h).\end{aligned}$$

The polynomial on the left hand side here has factors y_1-x_1 and y_2-x_1, and hence there is a polynomial $\Psi_2\in\mathbb{Z}[s_1,s_2,s_3]$ of degree $2k-2$ for which

$$(y_1-x_1)(y_2-x_1)\Psi_2(y_1,y_2,x_1)=h(2x_2^k-h). \tag{4.5}$$

For each fixed choice of x_2 with $1\leqslant x_2\leqslant B$ in question, we may suppose that the right hand side of (4.5) is a fixed non-zero integer N with $N\ll |h|B^k$. The integers y_1-x_1, y_2-x_1 and $\Psi_2(y_1,y_2,x_1)$ are each divisors of N, and hence there are $O(|N|^\varepsilon)$ such divisors, say

$$d_1=y_1-x_1,\quad d_2=y_2-x_1\quad\text{and}\quad d_3=\Psi_2(y_1,y_2,x_1). \tag{4.6}$$

We now find from (4.4) that

$$(x_1+d_1)^2+(x_1+d_2)^2-x_1^2=x_2^2,$$

whence

$$(x_1+d_1+d_2)^2=x_2^2+2d_1d_2.$$

With x_2 already fixed, it follows that for each fixed one of the $O(|h|^\varepsilon B^\varepsilon)$ possible choices for d_1, d_2 and d_3, the choice for x_1 is fixed by this last equation. The variables y_1 and y_2 are then fixed via (4.6), and we conclude that $v_{k,2}(h)\ll |h|^\varepsilon B^{1+\varepsilon}$. This completes the proof of part (ii), and hence also the lemma. □

The conclusion of Theorem 1.1 in case (ii) is now obtained in a straightforward manner by appealing to Hua's lemma.

Lemma 4.2. *Suppose that $m \in \{1,2\}$, $k \notin \{m,n\}$ and $n \geqslant 3$. Then one has*

$$N_{k,m,n}(B) - 12B^5 \ll B^{14/3}.$$

Proof. It follows from Lemma 4.1 that

$$\max_{1\leqslant |h|\leqslant 2B^k} v_{k,m}(h) \ll B^{1+\varepsilon}.$$

On substituting this estimate into (2.12), we infer that

$$N_{k,m,n}(B) - 12B^5 \ll B^{14/3} + B^{1+\varepsilon}\sum_{h\in\mathbb{Z}} u_{k,n}(h). \tag{4.7}$$

The last sum here counts the number of integral solutions of the equation

$$\sum_{i=1}^{3}(x_i^n - y_i^n) = 0,$$

with $1 \leqslant x_i, y_i \leqslant B$ $(1 \leqslant i \leqslant 3)$. By orthogonality, an application of Schwarz's inequality, and the invocation of Hua's lemma (see [11, Lemma 2.5]), we obtain the standard estimate

$$\int_0^1 \Bigl| \sum_{1\leqslant x\leqslant B} e(\alpha x^n)\Bigr|^6 \,\mathrm{d}\alpha \ll B^{7/2+\varepsilon}$$

for this quantity. On substituting this upper bound into (4.7), we conclude that

$$N_{k,m,n}(B) - 12B^5 \ll B^{14/3} + B^{1+\varepsilon}\cdot B^{7/2+\varepsilon} \ll B^{14/3},$$

and the proof of the lemma is complete. □

5. A cheap complification argument when $n = 2$

Our purpose in this section is to handle triples of type (iii), wherein we may suppose that $m \in \{1,2\}$, $n = 2$ and $k \geqslant 3$. This we achieve through a complification argument the prosecution of which requires several auxiliary mean value estimates. We now supply these estimates.

Lemma 5.1. *Suppose that $m \in \{1,2\}$ and $k \geqslant 3$. Then one has*

$$\sum_{1\leqslant |h|\leqslant 2B^k} v_{k,m}(h)^2 \ll B^{3+\varepsilon}.$$

Proof. By Lemma 4.1, one has

$$\sum_{1\leqslant |h|\leqslant 2B^k} v_{k,m}(h)^2 \ll B^{m-1+\varepsilon}\sum_{h\in\mathbb{Z}} v_{k,m}(h). \tag{5.1}$$

On recalling (2.6) and (2.7), we see that the sum on the right hand side here is bounded above by the number of solutions of the equation

$$x_1^m + x_2^m = y_1^m + y_2^m,$$

with $1 \leqslant x_i, y_i \leqslant B$. When $m = 1$ this is plainly $O(B^3)$, whilst for $m = 2$ it follows from Hua's lemma that the number of solutions is $O(B^{2+\varepsilon})$ (see [11, Lemma 2.5]). Thus, in either case, the number of solutions is $O(B^{4-m+\varepsilon})$, and we conclude from (5.1) that

$$\sum_{1\leqslant |h|\leqslant 2B^k} v_{k,m}(h)^2 \ll B^{m-1+\varepsilon}\cdot B^{4-m+\varepsilon} \ll B^{3+2\varepsilon}.$$

This completes the proof of the lemma. □

Next we record an upper bound available from recent work associated with Vinogradov's mean value theorem.

Lemma 5.2. *Suppose that $k\geqslant 3$. Then one has*

$$\int_0^1\int_0^1 |f_{k,2}(\alpha,\beta)|^{12}\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll B^{7+\varepsilon}.$$

Proof. This is a special case of [14, Theorem 14.1], though the proof is simple and transparent enough to provide here in full. Write

$$c(\boldsymbol{\alpha})=\sum_{1\leqslant x\leqslant B} e(\alpha x^k+\beta x^2+\gamma x).$$

Then we deduce via the triangle inequality and orthogonality that

$$\begin{aligned}\int_0^1\int_0^1 |f_{k,2}(\alpha,\beta)|^{12}\,\mathrm{d}\alpha\,\mathrm{d}\beta &= \sum_{|l|\leqslant 6B}\int_{[0,1)^3}|c(\boldsymbol{\alpha})|^{12}e(-l\gamma)\,\mathrm{d}\boldsymbol{\alpha}\\ &\ll B\int_{[0,1)^3}|c(\boldsymbol{\alpha})|^{12}\,\mathrm{d}\boldsymbol{\alpha}.\end{aligned}$$

By [14, Corollary 1.2], the last integral is $O(B^{6+\varepsilon})$, and so the desired conclusion follows at once. □

Now we come to the proof of Theorem 1.1 in the case (iii).

Lemma 5.3. *Suppose that $m\in\{1,2\}$ and $k\geqslant 3$. Then one has*

$$N_{k,m,2}(B)-12B^5 \ll B^{14/3+\varepsilon}.$$

Proof. An application of Cauchy's inequality in combination with Lemma 5.1 yields the bound

$$\begin{aligned}\sum_{1\leqslant |h|\leqslant 2B^k} u_{k,2}(h)v_{k,m}(h) &\leqslant \Big(\sum_{1\leqslant |h|\leqslant 2B^k} v_{k,m}(h)^2\Big)^{1/2}\Big(\sum_{h\in\mathbb{Z}} u_{k,2}(h)^2\Big)^{1/2}\\ &\ll (B^{3+\varepsilon})^{1/2}\Big(\sum_{h\in\mathbb{Z}} u_{k,2}(h)^2\Big)^{1/2}. \end{aligned}\tag{5.2}$$

On recalling (2.4) and (2.5), the sum on the right hand side here may be reinterpreted in terms of a Diophantine equation. Thus, it follows via orthogonality, Schwarz's inequality and symmetry that

$$\sum_{h\in\mathbb{Z}} u_{k,2}(h)^2 = \int_{[0,1)^3} |f_{k,2}(\alpha,\beta)f_{k,2}(\alpha,\gamma)|^6 \,\mathrm{d}\boldsymbol{\alpha}$$
$$\leqslant \int_{[0,1)^3} |f_{k,2}(\alpha,\beta)^8 f_{k,2}(\alpha,\gamma)^4| \,\mathrm{d}\boldsymbol{\alpha}. \tag{5.3}$$

Observe that by orthogonality in league with the triangle inequality,

$$\sup_{\alpha\in\mathbb{R}} \int_0^1 |f_{k,2}(\alpha,\gamma)|^4 \,\mathrm{d}\gamma \leqslant \int_0^1 |f_{k,2}(0,\gamma)|^4 \,\mathrm{d}\gamma \ll B^{2+\varepsilon},$$

wherein we interpreted the second integral as the number of solutions of the equation $x_1^2 + x_2^2 = y_1^2 + y_2^2$ with $1 \leqslant x_i, y_i \leqslant B$, and applied Hua's lemma. Returning to (5.3) and applying Hölder's inequality, therefore, we find that

$$\sum_{h\in\mathbb{Z}} u_{k,2}(h)^2 \ll B^{2+\varepsilon} \int_0^1 \int_0^1 |f_{k,2}(\alpha,\beta)|^8 \,\mathrm{d}\alpha \,\mathrm{d}\beta$$
$$\ll B^{2+\varepsilon} I_6^{2/3} I_{12}^{1/3},$$

where

$$I_t = \int_0^1 \int_0^1 |f_{k,2}(\alpha,\beta)|^t \,\mathrm{d}\alpha \,\mathrm{d}\beta.$$

By orthogonality, we see from (2.11) that

$$I_6 = u_{k,2}(0) \ll B^3,$$

whilst Lemma 5.2 delivers the bound $I_{12} \ll B^{7+\varepsilon}$. Thus we deduce that

$$\sum_{h\in\mathbb{Z}} u_{k,2}(h)^2 \ll B^{2+\varepsilon}(B^3)^{2/3}(B^{7+\varepsilon})^{1/3} \ll B^{19/3+2\varepsilon}.$$

Finally, by substituting the last bound into (5.2), we arrive at the estimate

$$\sum_{1\leqslant |h|\leqslant 2B^k} u_{k,2}(h)v_{k,m}(h) \ll (B^{3+\varepsilon})^{1/2}(B^{19/3+\varepsilon})^{1/2} \ll B^{14/3+\varepsilon}.$$

This, when substituted into (2.12), delivers the relation

$$N_{k,m,2}(B) - 12B^5 \ll B^{14/3+\varepsilon},$$

and this completes the proof of the lemma. □

6. A cheap complification argument when $n = 1$ and $k \geqslant 4$

The analysis of triples of type (iv) is similar to that applied in the previous section for triples of type (iii). We now suppose that $n = 1$ and $k \geqslant 4$, however, which prevents appeal to the relatively powerful mean value estimates for quadratic Weyl sums available when $n = 2$. We again begin with an auxiliary mean value estimate.

Lemma 6.1. *Suppose that $k \geqslant 4$. Then one has*
$$\int_{[0,1)^3} |f_{k,1}(\alpha,\beta)^6 f_{k,1}(\alpha,\gamma)^{14}| \,\mathrm{d}\boldsymbol{\alpha} \ll B^{14+\varepsilon}.$$

Proof. Write
$$F(\boldsymbol{\alpha}) = |f_{k,1}(\alpha,\beta)^6 f_{k,1}(\alpha,\gamma)^{14}|.$$
Then by applying the elementary inequality
$$|z_1 \dots z_r| \leqslant |z_1|^r + \dots + |z_r|^r,$$
we see that
$$F(\boldsymbol{\alpha}) \ll |f_{k,1}(\alpha,\beta)^{18} f_{k,1}(\alpha,\gamma)^2| + |f_{k,1}(\alpha,\beta)^2 f_{k,1}(\alpha,\gamma)^{18}|.$$
Thus, by symmetry and orthogonality, we find that
$$\begin{aligned}\int_{[0,1)^3} F(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} &\ll \int_0^1 \int_0^1 |f_{k,1}(\alpha,\beta)|^{18} \int_0^1 |f_{k,1}(\alpha,\gamma)|^2 \,\mathrm{d}\gamma\,\mathrm{d}\beta\,\mathrm{d}\alpha \\ &\leqslant B \int_0^1 \int_0^1 |f_{k,1}(\alpha,\beta)|^{18}\,\mathrm{d}\alpha\,\mathrm{d}\beta.\end{aligned}$$
The last integral is the subject of [2, Lemma 5], which shows that
$$\int_0^1 \int_0^1 |f_{k,1}(\alpha,\beta)|^{2^j+2}\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll B^{2^j-j+1+\varepsilon} \quad (2 \leqslant j \leqslant k).$$
Thus, by applying this estimate with $j = 4$, we deduce that
$$\int_{[0,1)^3} |f_{k,1}(\alpha,\beta)^6 f_{k,1}(\alpha,\gamma)^{14}| \,\mathrm{d}\boldsymbol{\alpha} \ll B \cdot B^{13+\varepsilon} = B^{14+\varepsilon}.$$
This completes the proof of the lemma. □

We may now tackle the main conclusion of this section.

Lemma 6.2. *Suppose that $m \in \{1,2\}$ and $k \geqslant 4$. Then one has*
$$N_{k,m,1}(B) - 12B^5 \ll B^{49/10+\varepsilon}.$$

Proof. Just as in the initial stages of the proof of Lemma 5.3, an application of Cauchy's inequality in combination with Lemma 5.1 yields the bound
$$\sum_{1 \leqslant |h| \leqslant 2B^k} u_{k,1}(h) v_{k,m}(h) \ll (B^{3+\varepsilon})^{1/2} \Big(\sum_{h \in \mathbb{Z}} u_{k,1}(h)^2 \Big)^{1/2}. \tag{6.1}$$
The sum on the right hand side here may be again reinterpreted as the number of solutions of a Diophantine system, and thence by orthogonality and Hölder's inequality we obtain
$$\sum_{h \in \mathbb{Z}} u_{k,1}(h)^2 = \int_{[0,1)^3} |f_{k,1}(\alpha,\beta) f_{k,1}(\alpha,\gamma)|^6 \,\mathrm{d}\boldsymbol{\alpha} \leqslant T_1^{4/5} T_2^{1/5}, \tag{6.2}$$
where
$$T_1 = \int_{[0,1)^3} |f_{k,1}(\alpha,\beta)^6 f_{k,1}(\alpha,\gamma)^4| \,\mathrm{d}\boldsymbol{\alpha}$$

and

$$T_2 = \int_{[0,1)^3} |f_{k,1}(\alpha,\beta)^6 f_{k,1}(\alpha,\gamma)^{14}| \, \mathrm{d}\boldsymbol{\alpha}.$$

By orthogonality, one sees that $T_1 = N_{k,1,1}(B)$, whilst by Lemma 6.1 we have $T_2 \ll B^{14+\varepsilon}$. On substituting these estimates into (6.2) and thence into (6.1), we see that

$$\sum_{1\leqslant |h|\leqslant 2B^k} u_{k,1}(h)v_{k,m}(h) \ll (B^{3+\varepsilon})^{1/2} \left(N_{k,1,1}(B)\right)^{2/5} (B^{14+\varepsilon})^{1/10},$$

so that, as a consequence of (2.12),

$$N_{k,m,1}(B) - 12B^5 \ll B^{14/3} + B^{29/10+\varepsilon} \left(N_{k,1,1}(B)\right)^{2/5}. \tag{6.3}$$

This estimate applies when $m = 1$, and hence in particular one finds that

$$N_{k,1,1}(B) \ll B^5 + B^{29/10+\varepsilon} \left(N_{k,1,1}(B)\right)^{2/5},$$

whence $N_{k,1,1}(B) \ll B^5$. By substituting this upper bound back into (6.3), we infer that

$$N_{k,m,1}(B) - 12B^5 \ll B^{14/3} + B^{29/10+\varepsilon}(B^5)^{2/5} \ll B^{49/10+\varepsilon}.$$

This completes the proof of the lemma. □

7. An application of the Hardy-Littlewood method

The final case (v) concerns the only remaining triple not already covered in cases (i) to (iv), namely the triple $(k, m, n) = (3, 2, 1)$. For this we must modify the treatment of §6 by introducing some crude estimates pertaining to the minor arcs of a Hardy-Littlewood dissection.

We define our Hardy-Littlewood dissection as follows. Take δ to be any positive number with $\delta < 1/3$, and let $\mathfrak{M}$ denote the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant B^{\delta-3}\}, \tag{7.1}$$

with $0 \leqslant a \leqslant q \leqslant B^\delta$ and $(a,q) = 1$. The complementary set of minor arcs is then $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. On writing

$$N(B;\mathfrak{B}) = \int_{\mathfrak{B}} \int_0^1 \int_0^1 |f_{3,1}(\alpha,\beta)^6 f_{3,2}(\alpha,\gamma)^4| \, \mathrm{d}\gamma \, \mathrm{d}\beta \, \mathrm{d}\alpha, \tag{7.2}$$

we see that

$$N_{3,2,1}(B) = N(B;\mathfrak{M}) + N(B;\mathfrak{m}). \tag{7.3}$$

We also define the auxiliary integral

$$u(h;\mathfrak{B}) = \int_{\mathfrak{B}} \int_0^1 |f_{3,1}(\alpha,\beta)|^6 e(-h\alpha) \, \mathrm{d}\beta \, \mathrm{d}\alpha.$$

In view of the definition of $v(h) = v_{3,2}(h)$ via (2.6) and (2.7), we then have

$$\sum_{|h|\leqslant 2B^3} u(h;\mathfrak{B})v(h) = \sum_{\mathbf{x},\mathbf{y}} \int_{\mathfrak{B}} \int_0^1 |f_{3,1}(\alpha,\beta)|^6 e(-(x_1^3 + x_2^3 - y_1^3 - y_2^3)\alpha) \, \mathrm{d}\beta \, \mathrm{d}\alpha,$$

where the summation over $\mathbf{x}$ and $\mathbf{y}$ is subject to the conditions $1 \leqslant x_i, y_i \leqslant B$ $(i = 1, 2)$ and $x_1^2+x_2^2 = y_1^2+y_2^2$. Thus, by employing orthogonality and recalling (7.2), we discern that

$$N(B;\mathfrak{B}) = \sum_{|h|\leqslant 2B^3} u(h;\mathfrak{B})v(h). \tag{7.4}$$

In general terms, our strategy makes use of a complification step resembling that used in both §§5 and 6. However, our use of a Hardy-Littlewood dissection necessitates that special attention be paid to the diagonal contribution restricted to minor arcs.

Lemma 7.1. *One has* $u(0;\mathfrak{m}) = 6B^3 + O(B^{8/3})$.

Proof. On recalling (2.11), we find that

$$u(0;[0,1)) = u_{3,1}(0) = 6B^3 + O(B^{8/3}).$$

In view of (7.1), we see that $\mathrm{mes}(\mathfrak{M}) = O(B^{2\delta-3})$, meanwhile, and hence we deduce via orthogonality that

$$\begin{aligned} u(0;\mathfrak{M}) &= \int_{\mathfrak{M}} \sum_{\substack{1\leqslant x_i,y_i\leqslant B\\ x_1+x_2+x_3=y_1+y_2+y_3}} e(\alpha(x_1^3+x_2^3+x_3^3-y_1^3-y_2^3-y_3^3))\,\mathrm{d}\alpha \\ &\ll B^5\mathrm{mes}(\mathfrak{M}) \ll B^{2+2\delta}. \end{aligned}$$

Thus we conclude that

$$u(0;\mathfrak{m}) = u(0;[0,1)) - u(0;\mathfrak{M}) = 6B^3 + O(B^{8/3}).$$

This completes the proof of the lemma. □

A similarly crude estimate for the major arc contribution handles $N(B;\mathfrak{M})$.

Lemma 7.2. *One has* $N(B;\mathfrak{M}) \ll B^{4+2\delta+\varepsilon}$.

Proof. By orthogonality, it follows from (7.2) that

$$N(B;\mathfrak{M}) = \sum_{\mathbf{x},\mathbf{y}} \int_{\mathfrak{M}} e\Big(\alpha\sum_{i=1}^{5}(x_i^3-y_i^3)\Big)\,\mathrm{d}\alpha,$$

where the summation is over 5-tuples $\mathbf{x}$, $\mathbf{y}$ with $1 \leqslant x_i, y_i \leqslant B$ subject to the conditions

$$\sum_{i=1}^{3}(x_i-y_i) = \sum_{i=4}^{5}(x_i^2-y_i^2) = 0.$$

The number of choices for x_i and y_i $(1 \leqslant i \leqslant 3)$ is plainly $O(B^5)$. Meanwhile, by applying Hua's lemma (see [11, Lemma 2.5]) on a by now well-trodden path, the number of choices for x_j, y_j $(j = 4, 5)$ is $O(B^{2+\varepsilon})$. Thus we deduce via the triangle inequality that

$$N(B;\mathfrak{M}) \ll B^5 \cdot B^{2+\varepsilon}\mathrm{mes}(\mathfrak{M}) \ll B^{7+\varepsilon}\cdot B^{2\delta-3},$$

and the conclusion of the lemma follows. □

We are now equipped to establish the final case of Theorem 1.1.

Lemma 7.3. *One has*
$$N_{3,2,1}(B)-12B^5\ll B^{5-\delta/4+\varepsilon}.$$

Proof. In view of (7.3) and Lemma 7.2, we have
$$N_{3,2,1}(B)=N(B;\mathfrak{m})+O(B^{4+2\delta+\varepsilon}).$$
Then by (7.4), we deduce that
$$N_{3,2,1}(B)=u(0;\mathfrak{m})v(0)+\Xi+O(B^{4+2\delta+\varepsilon}),$$
where
$$\Xi=\sum_{1\leqslant|h|\leqslant 2B^3}u(h;\mathfrak{m})v(h).$$
By wielding (2.9) in combination with Lemma 7.1, we may conclude thus far that
$$N_{3,2,1}(B)=(6B^3+O(B^{8/3}))(2B^2+O(B))+O(B^{4+2\delta+\varepsilon})+\Xi,$$
whence
$$N_{3,2,1}(B)-12B^5\ll B^{14/3}+\Xi. \tag{7.5}$$

Next we recall Lemma 5.1 and apply the inequalities of Cauchy and Bessel to obtain the upper bound
$$\Xi^2\ll B^{3+\varepsilon}\sum_{|h|\leqslant 2B^3}|u(h;\mathfrak{m})|^2\ll B^{3+\varepsilon}\int_{\mathfrak{m}}\Bigl(\int_0^1|f_{3,1}(\alpha,\beta)|^6\,\mathrm{d}\beta\Bigr)^2\,\mathrm{d}\alpha. \tag{7.6}$$
As a consequence of Weyl's inequality (see [11, Lemma 2.4]), one has
$$\sup_{\alpha\in\mathfrak{m}}\sup_{\beta\in\mathbb{R}}|f_{3,1}(\alpha,\beta)|\ll B^{1-\delta/4+\varepsilon}.$$
Thus, by making use of orthogonality and [5, Theorem 1.1], we obtain the bound
$$\begin{aligned}\int_{\mathfrak{m}}\Bigl(\int_0^1|f_{3,1}(\alpha,\beta)|^6\,\mathrm{d}\beta\Bigr)^2\,\mathrm{d}\alpha&\ll(B^{1-\delta/4+\varepsilon})^2\int_{[0,1)^3}|f_{3,1}(\alpha,\beta)^6f_{3,1}(\alpha,\gamma)^4|\,\mathrm{d}\boldsymbol{\alpha}\\&=B^{2-\delta/2+2\varepsilon}N_{3,1,1}(B)\\&\ll B^{7-\delta/2+2\varepsilon}.\end{aligned}$$
By substituting this estimate into (7.6), we arrive at the bound $\Xi\ll B^{5-\delta/4+\varepsilon}$, and hence (7.5) delivers the relation
$$N_{3,2,1}(B)-12B^5\ll B^{14/3}+B^{5-\delta/4+\varepsilon}.$$
The conclusion of the lemma follows on recalling our hypothesis that δ is any positive number smaller than 1/3. □

This completes the proof of the last case of Theorem 1.1, namely case (v). We now discern via Lemmata 3.2, 4.2, 5.3, 6.2 and 7.3 that in cases (i) to (v) we have the estimate (2.13). Thus, as discussed in the sequel to that equation, the conclusion of Theorem 1.1 is confirmed.

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