SUBCONVEXITY AND THE HILBERT-KAMKE PROBLEM

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Abstract. When $s \geqslant k \geqslant 3$ and $n_1, \ldots, n_k$ are large natural numbers, denote by $A_{s,k}(\mathbf{n})$ the number of solutions in non-negative integers $\mathbf{x}$ to the system

$$x_1^j + \ldots + x_s^j = n_j \quad (1 \leqslant j \leqslant k).$$

Under appropriate local solubility conditions on $\mathbf{n}$, we obtain an asymptotic formula for $A_{s,k}(\mathbf{n})$ when $s \geqslant k(k+1)$. This establishes a local-global principle in the Hilbert-Kamke problem at the convexity barrier. Our arguments involve minor arc estimates going beyond square-root cancellation.

1. Introduction

In this memoir we consider the asymptotic formula for the number of representations in the Hilbert-Kamke problem, our goal being to derive conclusions at or below the classical convexity barrier. When $n_1, \ldots, n_k$ are positive integers, we denote by $A_{s,k}(\mathbf{n})$ the number of solutions in non-negative integers $\mathbf{x}$ to the system of Diophantine equations

$$x_1^j + \ldots + x_s^j = n_j \quad (1 \leqslant j \leqslant k). \tag{1.1}$$

Motivated by his recent work on Waring's problem (see [5]), Hilbert posed the problem of determining suitable conditions on $\mathbf{n}$ that would guarantee, for an appropriate function $H(k)$, the non-vanishing of $A_{s,k}(\mathbf{n})$ for $s \geqslant H(k)$. This problem was taken up, first by Kamke [6], and subsequently by Mardzhanishvili [7] using Vinogradov's methods. The precise nature of the local conditions on $\mathbf{n}$ that must be imposed are quite complicated to describe, and we defer further discussion on this issue until later in this section.

In order to outline the current state of play, we recall the mean value

$$J_{t,k}(X) = \int_{[0,1)^k} \Bigl| \sum_{1 \leqslant x \leqslant X} e(\alpha_1 x + \ldots + \alpha_k x^k) \Bigr|^{2t} \, \mathrm{d}\boldsymbol{\alpha}, \tag{1.2}$$

in which $e(z)$ denotes $e^{2\pi i z}$. A consequence of the main conjecture in *Vinogradov's mean value theorem* asserts that when $k \in \mathbb{N}$ and $t \geqslant k(k+1)/2$, then for each $\varepsilon > 0$ one has

$$J_{t,k}(X) \ll X^{2t - \frac{1}{2}k(k+1) + \varepsilon}, \tag{1.3}$$

with the implicit constant depending at most on k, t and ε. Experts in the Hardy-Littlewood method will perceive that, provided the upper bound (1.3)

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is known to hold for $t \geqslant t_0(k)$, then one is able to derive an asymptotic formula for $A_{s,k}(\mathbf{n})$ whenever $s > 2t_0(k)$. This formula takes the shape

$$A_{s,k}(\mathbf{n}) = \mathfrak{J}_{s,k}(\mathbf{n})\mathfrak{S}_{s,k}(\mathbf{n})X^{s-k(k+1)/2} + o(X^{s-k(k+1)/2}), \tag{1.4}$$

where $X = \max\{n_1, n_2^{1/2}, \dots, n_k^{1/k}\}$ and $\mathfrak{J}_{s,k}(\mathbf{n})$ and $\mathfrak{S}_{s,k}(\mathbf{n})$ are respectively the *singular integral* and *singular series* associated with this problem. For now we defer explicit definition of these quantities, as well as discussion of conditions ensuring that $1 \ll \mathfrak{J}_{s,k}(\mathbf{n}) \ll 1$ and $1 \ll \mathfrak{S}_{s,k}(\mathbf{n}) \ll 1$.

Prior to 2010, in the classical version of the subject resolved by Arkhipov [1], the upper bound (1.3) was known to hold for $t \geqslant (2 + o(1))k^2 \log k$. This delivers (1.4) for $s \geqslant (4 + o(1))k^2 \log k$. The trivial lower bound $J_{t,k}(X) \gg X^t$ shows that (1.3) cannot hold when $t < k(k+1)/2$, and hence $t_0(k) \geqslant k(k+1)/2$. Consequently, the application of conventional methods can at best establish the asymptotic formula (1.4) when $s \geqslant k^2 + k + 1$. Decisive progress toward this convexity-limited bound was made by the author [9, Theorem 9.2] via the efficient congruencing method, establishing (1.4) for $s \geqslant 2k^2 + 2k + 1$. Finally, advances in the theory of Vinogradov's mean value theorem (see [4, 11, 13]) show that the conjectured estimate (1.3) holds for $t \geqslant k(k + 1)/2$, and hence the asymptotic formula (1.4) holds for $s \geqslant k^2 + k + 1$.

Our goal in the present paper is to go beyond this convexity-limited conclusion by confirming (1.4) for $s \geqslant k^2 + k$. In preparation for the statement of our new conclusion, we first introduce the generating functions

$$I(\boldsymbol{\beta}; X) = \int_0^X e(\beta_1\gamma + \ldots + \beta_k\gamma^k)\,\mathrm{d}\gamma \tag{1.5}$$

and

$$S(q, \mathbf{a}) = \sum_{r=1}^{q} e_q(a_1 r + \ldots + a_k r^k), \tag{1.6}$$

in which $e_q(u)$ denotes $e^{2\pi iu/q}$. Then, with $X = \max_{1\leqslant j\leqslant k} n_j^{1/j}$, we define

$$\mathfrak{J}_{s,k}(\mathbf{n}) = \int_{\mathbb{R}^k} I(\boldsymbol{\beta}; 1)^s e\left(-\beta_1\frac{n_1}{X} - \ldots - \beta_k\frac{n_k}{X^k}\right) \mathrm{d}\boldsymbol{\beta} \tag{1.7}$$

and

$$\mathfrak{S}_{s,k}(\mathbf{n}) = \sum_{q=1}^{\infty} \sum_{\substack{1\leqslant \mathbf{a}\leqslant q\\ (q,a_1,\dots,a_k)=1}} q^{-s}S(q, \mathbf{a})^s e_q(-a_1n_1 - \ldots - a_kn_k). \tag{1.8}$$

Theorem 1.1. *Suppose that $k \geqslant 3$ and $s \geqslant k^2 + k$. Then, whenever $n_1, \dots, n_k$ are natural numbers sufficiently large in terms of s and k, one has*

$$A_{s,k}(\mathbf{n}) = \mathfrak{J}_{s,k}(\mathbf{n})\mathfrak{S}_{s,k}(\mathbf{n})X^{s-k(k+1)/2} + o(X^{s-k(k+1)/2}),$$

in which $0 \leqslant \mathfrak{J}_{s,k}(\mathbf{n}) \ll 1$ and $0 \leqslant \mathfrak{S}_{s,k}(\mathbf{n}) \ll 1$. If, moreover, the system (1.1) possesses a non-singular real solution with positive coordinates, then $\mathfrak{J}_{s,k}(\mathbf{n}) \gg 1$. Likewise, if the system (1.1) possesses primitive non-singular p-adic solutions for each prime number p, then $\mathfrak{S}_{s,k}(\mathbf{n}) \gg 1$.

We remark that the singular integral $\mathfrak{J}_{s,k}(\mathbf{n})$ is known to converge absolutely for $s > \frac{1}{2}k(k+1)+1$, and that the singular series $\mathfrak{S}_{s,k}(\mathbf{n})$ is known to converge absolutely for $s > \frac{1}{2}k(k+1)+2$ (see [1, Theorem 1] or [3, Theorem 3.7]).

The extensive theory of quadratic forms ensures that the situation with $k=2$ is simple to handle for $s \geqslant 5$, and indeed much can be said even when $s=4$. We remark also that two obvious local conditions are in play if one is to have solutions to the system (1.1). First, by applying Hölder's inequality on the left hand side of (1.1), one sees that for solutions to exist one must have

$$n_l^{j/l} \leqslant n_j \leqslant s^{1-j/l} n_l^{j/l} \quad (1 \leqslant j \leqslant l \leqslant k).$$

Second, from Fermat's theorem, for each prime p one must have $n_l \equiv n_j \pmod{p}$ whenever $l \equiv j \pmod{p-1}$ and $1 \leqslant j \leqslant l \leqslant k$. The latter observation plainly impacts the p-adic solubility conditions associated with the system (1.1), a matter rather complicated to analyse in full. Indeed p-adic solubility is not assured in general without at least $2^k - 1$ variables being available. See the excellent accounts of Arkhipov [1, 2] and [3, Chapter 8] for a comprehensive account of such issues. However, if one is permitted to assume the existence of non-singular primitive solutions at each local completion of $\mathbb{Q}$, then one obtains an immediate corollary to Theorem 1.1 via the methods of [1].

Corollary 1.2. *Suppose the system (1.1) has non-singular primitive solutions at each local completion of* $\mathbb{Q}$*, and* $s \geqslant k(k+1)$*. Then* $A_{s,k}(\mathbf{n}) \gg n_1^{s-k(k+1)/2}$.

We establish Theorem 1.1 by applying the Hardy-Littlewood method, utilising an estimate for the contribution of the minor arcs going beyond square-root cancellation. This estimate is derived in §2 by adapting the author's work on the asymptotic formula in Waring's problem [10], the failure of translation-dilation invariance in the system (1.1) permitting an additional variable to be extracted and then utilised. Familiar in the context of equations, this device is more challenging in the absence of an immediate Diophantine interpretation, as when deriving estimates restricted to the minor arcs in a Hardy-Littlewood dissection. We launch our application of the circle method in earnest in §3 with the discussion of a Hardy-Littlewood dissection. Then, in §4, we convert the raw subconvex minor arc estimate extracted in §3 from the work of §2 into one more directly applicable to the proof of Theorem 1.1. Following some pruning in §5, the proof of Theorem 1.1 is concluded in §6 with a brief analysis of the major arc contribution. Finally, in §7, we make some remarks concerning other cognate problems to which our methods are applicable.

Our basic parameter is X, a sufficiently large positive number. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε, k and s. We make use of vector notation in the form $\mathbf{x} = (x_1, \dots, x_r)$, the dimension r depending on the course of the argument. We also write $(a_1, \dots, a_s)$ for the greatest common divisor of the integers $a_1, \dots, a_s$, any ambiguity between ordered s-tuples and corresponding greatest common divisors being easily resolved by context. Finally, we write $\|\theta\|$ for $\min\{|\theta - m| : m \in \mathbb{Z}\}$.

2. Mean value estimates via shifts

We begin by preparing the infrastructure required to describe our novel mean value estimate. When $k \geqslant 2$, define $f(\boldsymbol{\alpha}) = f_k(\boldsymbol{\alpha}; X)$ by putting

$$f_k(\boldsymbol{\alpha}; X) = \sum_{0 \leqslant x \leqslant X} e(\alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_k x^k). \tag{2.1}$$

Then, when $\mathbf{h} \in \mathbb{Z}^k$ and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, we introduce the mean value

$$I_s(\mathfrak{B}; X; \mathbf{h}) = \int_{\mathfrak{B}} \int_{[0,1)^{k-1}} f_k(\boldsymbol{\alpha}; X)^s e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, \mathrm{d}\boldsymbol{\alpha}, \tag{2.2}$$

in which $\boldsymbol{\alpha} \cdot \mathbf{h} = \alpha_1 h_1 + \ldots + \alpha_k h_k$ and $\mathrm{d}\boldsymbol{\alpha}$ denotes $\mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \cdots \mathrm{d}\alpha_k$. Provided that we take $X \geqslant \max\{n_1, n_2^{1/2}, \ldots, n_k^{1/k}\}$, it follows by orthogonality that $A_{s,k}(\mathbf{n}) = I_s([0,1); X; \mathbf{n})$. We make use of technology associated with Vinogradov's mean value theorem. With this in mind, when $t, k \in \mathbb{N}$, the parameter X is a positive real number, and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, we define

$$J_{t,k}^*(\mathfrak{B}; X) = \int_{\mathfrak{B}} \int_{[0,1)^{k-1}} |f_k(\boldsymbol{\alpha}; X)|^t \, \mathrm{d}\boldsymbol{\alpha}. \tag{2.3}$$

Finally, we adopt the convention of writing $s\mathfrak{B}$ for the set $\{s\alpha : \alpha \in \mathfrak{B}\}$.

Theorem 2.1. *Suppose that $\mathbf{h} \in \mathbb{Z}^k$ and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable. Then one has*

$$I_s(\mathfrak{B}; X; \mathbf{h}) \ll X^{-1} (\log X)^s J_{s+1,k}^*(\mathfrak{B}; 2X)^{s/(s+1)} J_{s+1,k}^*(s\mathfrak{B}; X)^{1/(s+1)}.$$

Proof. Our strategy is based on that underlying the proof of [10, Theorem 2.1], in which the potential for translation-invariance is exploited in order to generate an additional variable. Write

$$\psi(u; \boldsymbol{\theta}) = \theta_1 u + \theta_2 u^2 + \ldots + \theta_k u^k.$$

Then for every integral shift y with $0 \leqslant y \leqslant X$, one has

$$f_k(\boldsymbol{\alpha}; X) = \sum_{y \leqslant x \leqslant X+y} e\left(\psi(x-y; \boldsymbol{\alpha})\right). \tag{2.4}$$

Next write

$$\mathfrak{f}_y(\boldsymbol{\alpha}; \gamma) = \sum_{0 \leqslant x \leqslant 2X} e\left(\psi(x-y; \boldsymbol{\alpha}) + \gamma(x-y)\right) \tag{2.5}$$

and

$$K(\gamma) = \sum_{0 \leqslant z \leqslant X} e(-\gamma z).$$

Then we deduce from (2.4) via orthogonality that when $0 \leqslant y \leqslant X$, one has

$$f_k(\boldsymbol{\alpha}; X) = \int_0^1 \mathfrak{f}_y(\boldsymbol{\alpha}; \gamma) K(\gamma) \, \mathrm{d}\gamma. \tag{2.6}$$

We move next to substitute (2.6) into (2.2) so as to exploit the shift of variables by y. Define

$$\mathfrak{F}_y(\boldsymbol{\alpha}; \boldsymbol{\gamma}) = \prod_{i=1}^s \mathfrak{f}_y(\boldsymbol{\alpha}; \gamma_i), \quad \widetilde{K}(\boldsymbol{\gamma}) = \prod_{i=1}^s K(\gamma_i), \tag{2.7}$$

and

$$\mathcal{I}(\boldsymbol{\gamma};y;\mathbf{h})=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}\mathfrak{F}_y(\boldsymbol{\alpha};\boldsymbol{\gamma})e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}. \tag{2.8}$$

Then, when $0\leqslant y\leqslant X$, it follows that

$$I_s(\mathfrak{B};X;\mathbf{h})=\int_{[0,1)^s}\mathcal{I}(\boldsymbol{\gamma};y;\mathbf{h})\widetilde{K}(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma}. \tag{2.9}$$

For the sake of concision, write $\mathrm{d}\boldsymbol{\alpha}_{k-1}$ for $\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\cdots\,\mathrm{d}\alpha_{k-1}$. Then by orthogonality, one finds from (2.5) and (2.7) that

$$\int_{[0,1)^{k-1}}\mathfrak{F}_y(\boldsymbol{\alpha};\boldsymbol{\gamma})e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}_{k-1}=\sum_{0\leqslant\mathbf{x}\leqslant 2X}\Delta(\alpha_k,\boldsymbol{\gamma};\mathbf{h},y), \tag{2.10}$$

where $\Delta(\alpha_k,\boldsymbol{\gamma};\mathbf{h},y)$ is equal to

$$e\biggl(\sum_{i=1}^s\bigl(\alpha_k(x_i-y)^k+\gamma_i(x_i-y)\bigr)-\alpha_kh_k\biggr),$$

when

$$\sum_{i=1}^s(x_i-y)^j=h_j\quad(1\leqslant j\leqslant k-1), \tag{2.11}$$

and otherwise $\Delta(\alpha_k,\boldsymbol{\gamma};\mathbf{h},y)$ is equal to 0.

By applying the binomial theorem, one sees that whenever the system (2.11) is satisfied for the s-tuple $\mathbf{x}$, then

$$\sum_{i=1}^s x_i^j=sy^j+\sum_{l=0}^{j-1}\binom{j}{l}h_{j-l}y^l\quad(1\leqslant j\leqslant k-1),$$

and

$$\sum_{i=1}^s x_i^k=sy^k+\sum_{l=1}^{k-1}\binom{k}{l}h_{k-l}y^l+\sum_{i=1}^s(x_i-y)^k.$$

Adopt the convention that $h_0=s$ and write

$$\mathfrak{g}_y(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})=e\biggl(-\sum_{j=1}^k\alpha_j\sum_{l=0}^j\binom{j}{l}h_{j-l}y^l-y\sum_{i=1}^s\gamma_i\biggr).$$

Then it follows from (2.7) and (2.10) that

$$\int_{[0,1)^{k-1}}\mathfrak{F}_y(\boldsymbol{\alpha};\boldsymbol{\gamma})e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}_{k-1}=\int_{[0,1)^{k-1}}\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})\mathfrak{g}_y(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\alpha}_{k-1}. \tag{2.12}$$

Observe next that as a consequence of (2.9), when $X\in\mathbb{N}$, one has

$$(X+1)I_s(\mathfrak{B};X;\mathbf{h})=\sum_{0\leqslant y\leqslant X}\int_{[0,1)^s}\mathcal{I}(\boldsymbol{\gamma};y;\mathbf{h})\widetilde{K}(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma}.$$

Thus, from (2.8) and (2.12) we obtain the relation

$$I_s(\mathfrak{B};X;\mathbf{h})\ll X^{-1}\int_{[0,1)^s}|H(\boldsymbol{\gamma})\widetilde{K}(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma}, \tag{2.13}$$

where
$$H(\boldsymbol{\gamma})=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\alpha},\tag{2.14}$$
and
$$G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})=\sum_{0\leqslant y\leqslant X}\mathfrak{g}_y(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma}).\tag{2.15}$$

We begin the investigation of the relation (2.13) by bounding $H(\boldsymbol{\gamma})$. Thus, by applying Hölder's inequality on the right hand side of (2.14), we deduce that
$$H(\boldsymbol{\gamma})\ll I_1^{s/(s+1)}I_2^{1/(s+1)},\tag{2.16}$$
where
$$I_1=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|\mathfrak{F}_0(\boldsymbol{\alpha};\gamma)|^{1+1/s}\,\mathrm{d}\boldsymbol{\alpha}\tag{2.17}$$
and
$$I_2=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}.\tag{2.18}$$

On recalling (2.7), an application of Hölder's inequality to (2.17) yields
$$I_1\leqslant\prod_{i=1}^s\biggl(\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|\mathfrak{f}_0(\boldsymbol{\alpha};\gamma_i)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}\biggr)^{1/s}.$$
Moreover, it follows from (2.1) and (2.5) via a change of variable that
$$\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|\mathfrak{f}_0(\boldsymbol{\alpha};\gamma_i)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|f_k(\boldsymbol{\alpha};2X)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}.$$
Thus, in view of (2.3), we have $I_1\leqslant J^*_{s+1,k}(\mathfrak{B};2X)$. Also, for suitable polynomials $\nu_j(y;\mathbf{h})$ with leading term sy^j $(1\leqslant j\leqslant k)$, we find from (2.15) that
$$G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})=\sum_{0\leqslant y\leqslant X}e\biggl(-\biggl(\alpha_1\nu_1(y;\mathbf{h})+\ldots+\alpha_k\nu_k(y;\mathbf{h})+y\sum_{i=1}^s\gamma_i\biggr)\biggr).$$
Then it follows from (2.18) via a change of variable that
$$I_2=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|f_k(s\boldsymbol{\alpha};X)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}=s^{-1}J^*_{s+1,k}(s\mathfrak{B};X).$$
By substituting these estimates for I_1 and I_2 into (2.16), we obtain the bound
$$H(\boldsymbol{\gamma})\ll\bigl(J^*_{s+1,k}(\mathfrak{B};2X)\bigr)^{s/(s+1)}\bigl(J^*_{s+1,k}(s\mathfrak{B};X)\bigr)^{1/(s+1)}.\tag{2.19}$$

We are almost at the end of the proof. All that remains is to recall that
$$\int_0^1|K(\gamma)|\,\mathrm{d}\gamma\ll\int_0^1\min\{X,\|\gamma\|^{-1}\}\,\mathrm{d}\gamma\ll\log X,$$
whence by (2.7), we see that
$$\int_{[0,1)^s}|\widetilde{K}(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma}\ll(\log X)^s.$$

On substituting the latter bound with (2.19) into (2.13), we conclude that

$$\begin{aligned} I_s(\mathfrak{B};X;\mathbf{h}) &\ll X^{-1}\Big(\sup_{\boldsymbol{\gamma}\in[0,1)^s}|H(\boldsymbol{\gamma})|\Big)\int_{[0,1)^s}|\widetilde{K}(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma} \\ &\ll X^{-1}J^*_{s+1,k}(\mathfrak{B};2X)^{s/(s+1)}J^*_{s+1,k}(s\mathfrak{B};X)^{1/(s+1)}(\log X)^s. \end{aligned}$$

This completes the proof of the theorem. □

We now extract from Theorem 2.1 an estimate of minor arc type. When $1\leqslant Q\leqslant X$, we define a one-dimensional Hardy-Littlewood dissection as follows. We define the set of major arcs $\mathfrak{M}(Q)$ to be the union of the arcs

$$\mathfrak{M}(q,a)=\{\alpha\in[0,1):|q\alpha-a|\leqslant QX^{-k}\},$$

with $0\leqslant a\leqslant q\leqslant Q$ and $(a,q)=1$, and then write $\mathfrak{m}(Q)=[0,1)\setminus\mathfrak{M}(Q)$ for the corresponding set of minor arcs.

Before announcing the next lemma, we introduce a Weyl exponent relevant to our discussion. When k is an integer with $k\geqslant 2$, we define the exponent $\sigma=\sigma(k)$ by taking

$$\sigma(k)^{-1}=\begin{cases}2^{k-1}, & \text{when } 2\leqslant k\leqslant 5,\\ k(k-1), & \text{when } k\geqslant 6.\end{cases}$$

Lemma 2.2. *Suppose that $k\geqslant 2$ and $1\leqslant Q\leqslant X$. Then uniformly in $(\alpha_1,\dots,\alpha_{k-1})\in\mathbb{R}^{k-1}$, one has*

$$\sup_{\alpha_k\in\mathfrak{m}(Q)}|f_k(\boldsymbol{\alpha};X)|\ll X^{1+\varepsilon}Q^{-\sigma}.$$

Proof. Let $\alpha_k\in\mathfrak{m}(Q)$. Then by Dirichlet's approximation theorem, there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ with $0\leqslant a\leqslant q\leqslant Q^{-1}X^k$ and $(a,q)=1$ for which $|q\alpha_k-a|\leqslant QX^{-k}$. Since $\alpha_k\notin\mathfrak{M}(Q)$, we may assume that $q>Q$. When $2\leqslant k\leqslant 5$, it now follows from Weyl's inequality (see [8, Lemma 2.4]) that

$$f_k(\boldsymbol{\alpha};X)\ll X^{1+\varepsilon}(q^{-1}+X^{-1}+qX^{-k})^{2^{1-k}}\ll X^{1+\varepsilon}Q^{-\sigma(k)}.$$

On the other hand, when $k\geqslant 6$, we may apply Vinogradov's methods in their most modern incarnations to see that

$$f_k(\boldsymbol{\alpha};X)\ll X^{1+\varepsilon}(q^{-1}+X^{-1}+qX^{-k})^{1/(k(k-1))}\ll X^{1+\varepsilon}Q^{-\sigma(k)}.$$

The reader may extract this last conclusion from [8, Theorem 5.2] by following the proof of [9, Theorem 1.5], substituting the now confirmed bound (1.3) for the version of Vinogradov's mean value theorem utilised in the latter. □

Lemma 2.3. *Suppose that $\mathbf{h}\in\mathbb{Z}^k$ and $1\leqslant Q\leqslant X$. Then for $s\geqslant k(k+1)$,*

$$I_s(\mathfrak{m}(Q);X;\mathbf{h})\ll X^{s-\frac{1}{2}k(k+1)+\varepsilon}Q^{-\sigma(k)}.$$

Proof. Recall the definition (1.2) and write $w=k(k+1)/2$. Then, provided that $s\geqslant k(k+1)$, it follows from (2.3) via a trivial estimate for $f_k(\boldsymbol{\alpha};X)$ that

$$J^*_{s+1,k}(\mathfrak{m}(Q);2X)\ll X^{s-k(k+1)}\Big(\sup_{\alpha_k\in\mathfrak{m}(Q)}|f_k(\boldsymbol{\alpha};2X)|\Big)J_{w,k}(X).$$

Since the now confirmed bound (1.3) shows that $J_{w,k}(X) \ll X^{w+\varepsilon}$, we deduce from Lemma 2.2 that

$$J^*_{s+1,k}(\mathfrak{m}(Q); 2X) \ll X^{1+\varepsilon} Q^{-\sigma(k)} \cdot X^{s-\frac{1}{2}k(k+1)+\varepsilon}. \tag{2.20}$$

We bound $J^*_{s+1,k}(s\mathfrak{m}(Q); X)$ in a similar manner, noting that an elementary exercise confirms that $s\mathfrak{m}(Q) \subseteq \mathfrak{m}(Q/s) \pmod 1$. Thus we find that

$$\sup_{\alpha_k \in s\mathfrak{m}(Q)} |f_k(\boldsymbol{\alpha}; X)| \ll X^{1+\varepsilon} Q^{-\sigma(k)},$$

and just as in the previous discussion, we infer that

$$J^*_{s+1,k}(s\mathfrak{m}(Q); X) \ll X^{1+\varepsilon} Q^{-\sigma(k)} \cdot X^{s-\frac{1}{2}k(k+1)+\varepsilon}. \tag{2.21}$$

By substituting the estimates (2.20) and (2.21) into Theorem 2.1, we obtain

$$I_s(\mathfrak{m}(Q); X; \mathbf{h}) \ll X^{-1}(\log X)^s X^{s+1-\frac{1}{2}k(k+1)+\varepsilon} Q^{-\sigma(k)}.$$

The conclusion of the lemma follows on noting our convention concerning ε. $\square$

3. The Hardy-Littlewood dissection

In this section we explain our application of the Hardy-Littlewood method in pursuit of the asymptotic formula delivered by Theorem 1.1. We now fix k and $X = 2\max\{n_1, n_2^{1/2}, \ldots, n_k^{1/k}\}$, and henceforth abbreviate the exponential sum $f_k(\boldsymbol{\alpha}; X)$ introduced in (2.1) simply to $f(\boldsymbol{\alpha})$. When $\mathfrak{A} \subseteq [0,1)^k$ is measurable, we define the mean value $T_s(\mathfrak{A}) = T_s(\mathfrak{A}; X; \mathbf{n})$ by

$$T_s(\mathfrak{A}; X; \mathbf{n}) = \int_{\mathfrak{A}} f(\boldsymbol{\alpha})^s e(-\boldsymbol{\alpha} \cdot \mathbf{n}) \, \mathrm{d}\boldsymbol{\alpha}. \tag{3.1}$$

Our application of the Hardy-Littlewood method requires some discussion concerning the associated infrastructure. When $1 \leqslant Z \leqslant X$, we denote by $\mathfrak{K}(Z)$ the union of the arcs

$$\mathfrak{K}(q, \mathbf{a}; Z) = \{\boldsymbol{\alpha} \in [0,1)^k : |\alpha_j - a_j/q| \leqslant Z X^{-j} \ (1 \leqslant j \leqslant k)\},$$

with $1 \leqslant q \leqslant Z$, $0 \leqslant a_j \leqslant q$ $(1 \leqslant j \leqslant k)$ and $(q, a_1, \ldots, a_k) = 1$, and we put $\mathfrak{k}(Z) = [0,1)^k \setminus \mathfrak{K}(Z)$. We have already defined a one-dimensional Hardy-Littlewood dissection of $[0,1)$ into sets of arcs $\mathfrak{M} = \mathfrak{M}(Q)$ and $\mathfrak{m} = \mathfrak{m}(Q)$. We now fix $L = X^{1/(8k^2)}$ and $Q = L^k$, and we define a k-dimensional set of arcs by taking $\mathfrak{N} = \mathfrak{K}(Q^2)$ and $\mathfrak{n} = \mathfrak{k}(Q^2)$. This intermediate Hardy-Littlewood dissection can be refined to obtain a dissection into a narrower set of major arcs $\mathfrak{P} = \mathfrak{K}(L)$ and a corresponding set of minor arcs $\mathfrak{p} = \mathfrak{k}(L)$. In this latter dissection, for the sake of concision, it is useful to write $\mathfrak{P}(q, \mathbf{a}) = \mathfrak{K}(q, \mathbf{a}; L)$.

We partition the set of points $(\alpha_1, \ldots, \alpha_k)$ lying in $[0,1)^k$ into four disjoint subsets, namely

$$\begin{aligned}
\mathfrak{W}_1 &= [0,1)^{k-1} \times \mathfrak{m}, \\
\mathfrak{W}_2 &= \big([0,1)^{k-1} \times \mathfrak{M}\big) \cap \mathfrak{n} \\
\mathfrak{W}_3 &= \big([0,1)^{k-1} \times \mathfrak{M}\big) \cap (\mathfrak{N} \setminus \mathfrak{P}), \\
\mathfrak{W}_4 &= \mathfrak{P}.
\end{aligned}$$

We note in this context that $\mathfrak{P} \subseteq [0,1)^{k-1} \times \mathfrak{M}$, since whenever
$$|\alpha_k - a_k/q| \leqslant LX^{-k} \quad \text{and} \quad 1 \leqslant q \leqslant L,$$
one has
$$|q\alpha_k - a_k| \leqslant L^2 X^{-k} < QX^{-k} \quad \text{and} \quad 1 \leqslant q < Q.$$
Thus $\mathfrak{W}_4 = \big([0,1)^{k-1} \times \mathfrak{m}\big) \cap \mathfrak{P}$, and it follows that $[0,1)^k = \mathfrak{W}_1 \cup \ldots \cup \mathfrak{W}_4$. We therefore deduce via orthogonality that
$$A_{s,k}(\mathbf{n}) = T_s\left([0,1)^k\right) = \sum_{i=1}^4 T_s(\mathfrak{W}_i). \tag{3.2}$$

The work of §2 already permits us to announce a satisfactory upper bound for the contribution of the set of arcs $\mathfrak{W}_1$ within (3.2).

Lemma 3.1. *Whenever* $\mathbf{n} \in \mathbb{Z}^k$ *and* $s \geqslant k(k+1)$*, one has*
$$T_s(\mathfrak{W}_1) \ll X^{s-\frac{1}{2}k(k+1)-1/(8k^3)}.$$

Proof. By substituting $Q = X^{1/(8k)}$ into Lemma 2.3, noting that $\sigma(k) > 1/k^2$ for $k \geqslant 3$, the conclusion of the lemma is immediate. □

4. Another minor arc estimate

Our next task is to bound the contribution of the set of arcs $\mathfrak{W}_2$ within (3.2). We begin by providing a familiar estimate of Weyl type for the exponential sum $f(\boldsymbol{\alpha})$ of strength sufficient for our purposes.

Lemma 4.1. *One has*
$$\sup_{\boldsymbol{\alpha} \in \mathfrak{n}} |f(\boldsymbol{\alpha})| \ll X^{1-1/(6k^2)} \quad \textit{and} \quad \sup_{\boldsymbol{\alpha} \in \mathfrak{p}} |f(\boldsymbol{\alpha})| \ll X^{1-1/(12k^3)}.$$

Proof. In order to confirm the first bound, we put $\tau = 1/(6k^2)$ and $\delta = 1/(4k)$. Since $\tau^{-1} > 4k(k-1)$ and $\delta > k\tau$, we find from [9, Theorem 1.6] that whenever $|f(\boldsymbol{\alpha})| \geqslant X^{1-\tau}$, there exist integers $q, a_1, \ldots, a_k$ such that $1 \leqslant q \leqslant X^\delta$ and $|q\alpha_j - a_j| \leqslant X^{\delta-j}$ $(1 \leqslant j \leqslant k)$. In particular, we see that $q \leqslant Q^2$ and $|\alpha_j - a_j/q| \leqslant Q^2 X^{-j}$ $(1 \leqslant j \leqslant k)$, and hence $\boldsymbol{\alpha} \in \mathfrak{N}$ (mod 1). We therefore infer that whenever X is sufficiently large in terms of k, and $\boldsymbol{\alpha} \in \mathfrak{n}$, then one must have $|f(\boldsymbol{\alpha})| \leqslant X^{1-\tau}$, and the first conclusion of the lemma follows.

For the second bound we put $\tau = 1/(12k^3)$ and $\delta = 1/(8k^2)$. We again have $\tau^{-1} > 4k(k-1)$ and $\delta > k\tau$, and so the same argument applies *mutatis mutandis*. We now find that whenever $|f(\boldsymbol{\alpha})| \geqslant X^{1-\tau}$, then $\boldsymbol{\alpha} \in \mathfrak{P}$ (mod 1). Thus, when X is sufficiently large in terms of k, and $\boldsymbol{\alpha} \in \mathfrak{p}$, then one must have $|f(\boldsymbol{\alpha})| \leqslant X^{1-\tau}$, and the second conclusion of the lemma follows. □

In order to obtain a satisfactory bound for the contribution of $\mathfrak{W}_2$, it suffices to combine the Weyl estimate provided by Lemma 4.1 with the observation that $\mathfrak{W}_2$ has small measure.

Lemma 4.2. *Whenever* $\mathbf{n} \in \mathbb{Z}^k$ *and* $s \geqslant k(k+1)$*, one has*
$$T_s(\mathfrak{W}_2) \ll X^{s-\frac{1}{2}k(k+1)-1/(16k)}.$$

Proof. We begin with an auxiliary estimate. Recall the definition (1.2) and write $v = k(k-1)/2$. Then, by orthogonality, the mean value

$$I(\alpha_k) = \int_{[0,1)^{k-1}} |f_k(\boldsymbol{\alpha};X)|^{2v} \, \mathrm{d}\boldsymbol{\alpha}_{k-1}$$

counts the integral solutions of the system of equations

$$\sum_{i=1}^{v} (x_i^j - y_i^j) = 0 \quad (1 \leqslant j \leqslant k-1),$$

with $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X$, each solution being counted with the unimodular weight $e\left(\alpha_k(x_1^k - y_1^k + \ldots + x_v^k - y_v^k)\right)$. By making use of the bound (1.3) (see [4, 11, 13]), we therefore find that $I(\alpha_k) \ll J_{v,k-1}(X) \ll X^{v+\varepsilon}$, uniformly in α_k.

Observe next that since $\mathfrak{W}_2 = \left([0,1)^{k-1} \times \mathfrak{M}\right) \cap \mathfrak{n}$, it follows from (3.1) via the triangle inequality that

$$T_s(\mathfrak{W}_2) \ll \Big(\sup_{\boldsymbol{\alpha}\in\mathfrak{n}} |f(\boldsymbol{\alpha})|\Big)^{s-k(k-1)} \int_{\mathfrak{M}} I(\alpha_k) \, \mathrm{d}\alpha_k.$$

Since $\mathrm{mes}(\mathfrak{M}) \ll Q^2 X^{-k}$, we conclude from Lemma 4.1 that

$$\begin{aligned} T_s(\mathfrak{W}_2) &\ll X^{s-k(k+1)} \Big(X^{1-1/(6k^2)}\Big)^{2k} X^{v+\varepsilon} \mathrm{mes}(\mathfrak{M}) \\ &\ll \left(Q^2 X^{-1/(3k)}\right) X^{s-\frac{1}{2}k(k+1)+\varepsilon}. \end{aligned}$$

Since $Q^2 = X^{1/(4k)}$, the conclusion of the lemma is immediate. □

5. Pruning: the analysis of $\mathfrak{W}_3$

We take an economical approach to the analysis of the term $T_s(\mathfrak{W}_3)$. We begin by announcing a mean value estimate of major arc type.

Lemma 5.1. *Suppose that $u > \frac{1}{2}k(k+1) + 2$. Then one has*

$$\int_{\mathfrak{N}} |f(\boldsymbol{\alpha})|^u \, \mathrm{d}\boldsymbol{\alpha} \ll_u X^{u-k(k+1)/2}.$$

Proof. This is essentially [12, Lemma 7.1]. Our definition of the major arcs $\mathfrak{N}$ sets the parameter Q equal to $X^{1/(8k)}$, whereas in the source cited the analogous definition is tantamount to setting $Q = X^{1/(2k)}$. The conclusion presented here is therefore immediate from the latter source, since our major arcs are contained in those employed therein. □

Lemma 5.2. *When $\mathbf{n} \in \mathbb{Z}^3$ and $s \geqslant k(k+1)$, one has*

$$T_s(\mathfrak{W}_3) \ll X^{s-\frac{1}{2}k(k+1)-1/(12k^3)}.$$

Proof. Since $\mathfrak{W}_3 \subseteq \mathfrak{N} \setminus \mathfrak{P}$, we have

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{W}_3} |f(\boldsymbol{\alpha})| \leqslant \sup_{\boldsymbol{\alpha}\in\mathfrak{p}} |f(\boldsymbol{\alpha})|.$$

Thus, taking $u=\frac{1}{2}k(k+1)+3$, it follows from the triangle inequality that

$$T_s(\mathfrak{W}_3)\leqslant X^{s-u-1}\Bigl(\sup_{\boldsymbol{\alpha}\in\mathfrak{p}}|f(\boldsymbol{\alpha})|\Bigr)\int_{\mathfrak{N}}|f(\boldsymbol{\alpha})|^u\,\mathrm{d}\boldsymbol{\alpha}.$$

Hence, by employing Lemmata 4.1 and 5.1 we see that

$$T_s(\mathfrak{W}_3)\ll X^{s-1-\frac{1}{2}k(k+1)}\cdot X^{1-1/(12k^3)},$$

and the conclusion of the lemma follows. □

6. The major arc contribution

By substituting the conclusions of Lemmata 3.1, 4.2 and 5.2 into (3.2), we find that whenever $s\geqslant k(k+1)$, one has

$$A_{s,k}(\mathbf{n})=T_s(\mathfrak{W}_4)+o(X^{s-\frac{1}{2}k(k+1)}).\tag{6.1}$$

The proof of Theorem 1.1 will be completed by an analysis of $T_s(\mathfrak{W}_4)=T_s(\mathfrak{P})$.

Recall (1.5) and (1.6). Then, when $\boldsymbol{\alpha}\in\mathfrak{P}(q,\mathbf{a})\subseteq\mathfrak{P}$, write

$$V(\boldsymbol{\alpha};q,\mathbf{a})=q^{-1}S(q,\mathbf{a})I(\boldsymbol{\alpha}-\mathbf{a}/q;X).$$

Define the function $V(\boldsymbol{\alpha})$ to be $V(\boldsymbol{\alpha};q,\mathbf{a})$ when $\boldsymbol{\alpha}\in\mathfrak{P}(q,\mathbf{a})\subseteq\mathfrak{P}$, and to be 0 otherwise. Then, when $\boldsymbol{\alpha}\in\mathfrak{P}(q,\mathbf{a})\subseteq\mathfrak{P}$, we see from [8, Theorem 7.2] that

$$f(\boldsymbol{\alpha})-V(\boldsymbol{\alpha};q,\mathbf{a})\ll q+X|q\alpha_1-a_1|+\ldots+X^k|q\alpha_k-a_k|\ll L^2,$$

whence, uniformly for $\boldsymbol{\alpha}\in\mathfrak{P}$, we have the bound

$$f(\boldsymbol{\alpha})^s-V(\boldsymbol{\alpha})^s\ll X^{s-1+1/(4k^2)}.$$

Thus, since $\mathrm{mes}(\mathfrak{P})\ll L^{2k+1}X^{-k(k+1)/2}$, we deduce that

$$\int_{\mathfrak{P}}f(\boldsymbol{\alpha})^se(-\boldsymbol{\alpha}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\alpha}=\int_{\mathfrak{P}}V(\boldsymbol{\alpha})^se(-\boldsymbol{\alpha}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\alpha}+o(X^{s-\frac{1}{2}k(k+1)}).\tag{6.2}$$

Next, write

$$\Omega=[-LX^{-1},LX^{-1}]\times\cdots\times[-LX^{-k},LX^{-k}].$$

Then one finds that

$$\int_{\mathfrak{P}}V(\boldsymbol{\alpha})^se(-\boldsymbol{\alpha}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\alpha}=\mathfrak{S}(X)\mathfrak{J}(X),\tag{6.3}$$

where

$$\mathfrak{J}(X)=\int_{\Omega}I(\boldsymbol{\beta};X)^se(-\boldsymbol{\beta}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\beta}$$

and

$$\mathfrak{S}(X)=\sum_{1\leqslant q\leqslant L}\ \sum_{\substack{1\leqslant\mathbf{a}\leqslant q\\(q,a_1,\ldots,a_k)=1}}q^{-s}S(q,\mathbf{a})^se_q(-\mathbf{a}\cdot\mathbf{n}).$$

When $s>\frac{1}{2}k(k+1)+1$, the singular integral $\mathfrak{J}_{s,k}(\mathbf{n})$ defined in (1.7) converges absolutely (see [3, Theorem 1.3]), and in particular $\mathfrak{J}_{s,k}(\mathbf{n})\ll 1$. Also, by [8, Theorem 7.3], one has

$$I(\boldsymbol{\beta};X)\ll X(1+|\beta_1|X+\ldots+|\beta_k|X^k)^{-1/k}.$$

Write $h_j = n_j X^{-j}$ $(1 \leqslant j \leqslant k)$. Then after two changes of variable, we obtain

$$\begin{aligned}\mathfrak{J}(X) &= X^{s-\frac{1}{2}k(k+1)} \int_{\mathbb{R}^k} I(\boldsymbol{\beta};1)^s e(-\boldsymbol{\beta}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\beta} + o(X^{s-\frac{1}{2}k(k+1)})\\ &= (\mathfrak{J}_{s,k}(\mathbf{n}) + o(1))\,X^{s-\frac{1}{2}k(k+1)}.\end{aligned} \tag{6.4}$$

Similarly, by reference to [3, Theorem 2.4], one sees that the singular series $\mathfrak{S}_{s,k}(\mathbf{n})$ defined in (1.8) converges absolutely for $s > \frac{1}{2}k(k+1)+2$, and in particular $\mathfrak{S}_{s,k}(\mathbf{n}) \ll 1$. Also, it follows from [8, Theorem 7.1] that when $(q,a_1,\dots,a_k)=1$, one has $S(q,\mathbf{a}) \ll q^{1-1/k+\varepsilon}$. Thus,

$$\mathfrak{S}(X) = \mathfrak{S}_{s,k}(\mathbf{n}) + o(1). \tag{6.5}$$

By substituting (6.4) and (6.5) into (6.3), and thence into (6.2), we obtain

$$\int_{\mathfrak{P}} f(\boldsymbol{\alpha})^s e(-\boldsymbol{\alpha}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\alpha} = \mathfrak{S}_{s,k}(\mathbf{n})\mathfrak{J}_{s,k}(\mathbf{n})X^{s-\frac{1}{2}k(k+1)} + o(X^{s-\frac{1}{2}k(k+1)}).$$

Thus, on recalling (3.1), we find that

$$T_s(\mathfrak{W}_4) = T_s(\mathfrak{P}) = \mathfrak{S}_{s,k}(\mathbf{n})\mathfrak{J}_{s,k}(\mathbf{n})X^{s-\frac{1}{2}k(k+1)} + o(X^{s-\frac{1}{2}k(k+1)}),$$

and by substituting this relation into (6.1), we conclude that

$$A_{s,k}(\mathbf{n}) = \mathfrak{S}_{s,k}(\mathbf{n})\mathfrak{J}_{s,k}(\mathbf{n})X^{s-\frac{1}{2}k(k+1)} + o(X^{s-\frac{1}{2}k(k+1)}).$$

This confirms the conclusion of Theorem 1.1.

7. Other applications

The method underlying the proof of Theorem 2.1 may be applied in many similar situations. Thus, the system of equations (1.1) may be replaced by

$$\sum_{i=1}^{l} x_i^j - \sum_{i=l+1}^{l+m} x_i^j = n_j \quad (1 \leqslant j \leqslant k), \tag{7.1}$$

provided that $l \neq m$ and $s = l+m \geqslant k(k+1)$. The Hilbert-Kamke problem corresponds to the situation here where $m = 0$. What is critical is that the underlying system (7.1) is *not* translation-dilation invariant, even in the special situation in which $\mathbf{n} = \mathbf{0}$. Let $B_{l,m,k}(\mathbf{n};X)$ denote the number of solutions of the system (7.1) with $1 \leqslant x_i \leqslant X$ $(1 \leqslant i \leqslant s)$. Then provided that $s \geqslant k(k+1)$, methods almost identical to those of this paper establish an asymptotic formula

$$B_{l,m,k}(\mathbf{n};X) \sim CX^{s-\frac{1}{2}k(k+1)},$$

in which C is an appropriate product of local densities. Here, the major innovation lies with the subconvex minor arc estimate provided by an analogue of Theorem 2.1. The major arc analysis is handled in earlier work [1].

When $l = m = k(k+1)/2$, the system (7.1) lacks translation-dilation invariance when $(n_1,\dots,n_{k-1}) \neq \mathbf{0}$, though a shadow of this invariance property remains. The latter complicates any attempt to derive an analogue of Theorem 2.1. We have more to say concerning such systems in the memoir [14].

Finally, when the coefficients $c_i \in \mathbb{Z} \setminus \{0\}$ $(1 \leqslant i \leqslant k(k+1))$ satisfy the condition $c_1 + \ldots + c_s \neq 0$, the methods underlying the proof of Theorem 2.1 remain in play when one examines the system

$$c_1 x_1^j + \ldots + c_s x_s^j = n_j \quad (1 \leqslant j \leqslant k).$$

Indeed, this scenario was discussed in work of the author dating from 2015 that was the subject of talks presented in Göteborg, Oxford and Strobl-am-Wolfgangsee (see [15]).

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