

PAIRS OF DIAGONAL QUARTIC FORMS: THE NON-SINGULAR HASSE PRINCIPLE

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ABSTRACT. We establish the non-singular Hasse Principle for pairs of diagonal quartic equations in 22 or more variables.

1. INTRODUCTION

Given integers A_j, B_j with $(A_j, B_j) \neq (0, 0)$ ($1 \leq j \leq s$), we consider the pair of Diophantine equations

$$A_1x_1^4 + A_2x_2^4 + \dots + A_sx_s^4 = B_1x_1^4 + B_2x_2^4 + \dots + B_sx_s^4 = 0. \quad (1.1)$$

Associated with the coefficients A_j, B_j is the number

$$q_0 = q_0(\mathbf{A}, \mathbf{B}) = \min_{(C,D) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \text{card}\{1 \leq j \leq s : CA_j + DB_j \neq 0\}.$$

Subject to the conditions $s \geq 22$ and $q_0 \geq s - 7$, our main result in [5] is a quantitative version of the non-singular Hasse principle for the pair of equations (1.1). This states that whenever the system (1.1) admits non-singular solutions in real numbers and in p -adic numbers for each prime p , then the number $\mathcal{N}(P)$ of solutions in integers x_j , with $|x_j| \leq P$ ($1 \leq j \leq s$), satisfies the lower bound $\mathcal{N}(P) \gg P^{s-8}$.

Although the rank condition on the coefficient matrix expressed through a lower bound for q_0 may appear unnatural and restrictive, it cannot be abandoned entirely. We demonstrate in §7 that whenever $s \geq 9$ the pair of equations

$$x_1^4 + x_2^4 - 6x_3^4 - 12x_4^4 = x_4^4 - 7x_5^4 - 5x_6^4 - 3x_7^4 - \sum_{j=8}^s x_j^4 = 0 \quad (1.2)$$

has non-singular solutions in all completions of the rationals, but only the zero solution in integers. Thus, the non-singular Hasse principle fails for this pair with $q_0 = 4$. As promised in [6], we return to the subject here and relax the condition on q_0 to one that is independent of s .

Theorem 1.1. *Let $s \geq 22$ and $q_0 \geq 12$. Then provided that the system (1.1) has non-singular solutions in each completion of the rational numbers, one has $\mathcal{N}(P) \gg P^{s-8}$.*

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For an assessment of the strength of this result, we refer to important work of Vaughan [9], implying that a single equation

$$A_1x_1^4 + A_2x_2^4 + \dots + A_t x_t^4 = 0$$

obeys the Hasse principle when $t \geq 12$. For such equations with $t = 11$ this is not yet known. Thus, as detailed in the introduction of the predecessor to this memoir [5], our result breaks with the established rule that systems of Diophantine equations deny treatment unless the number of variables is at least as large as the sum of the number of variables needed to solve each of the equations individually. Also, if the conclusion of Theorem 1.1 were to be true in all cases where $q_0 = 11$, then the Hasse principle would follow for diagonal quartic equations in 11 variables. To see this, consider the pair of equations

$$A_1x_1^4 + A_2x_2^4 + \dots + A_{11}x_{11}^4 = x_{11}^4 - x_{12}^4 + x_{13}^4 - x_{14}^4 + x_{15}^4 + \dots + x_{22}^4 = 0, \quad (1.3)$$

in which $A_j \neq 0$ for $1 \leq j \leq 11$. Then $q_0 = 11$. If (x_1, \dots, x_{11}) is a non-trivial solution to the equation

$$A_1x_1^4 + A_2x_2^4 + \dots + A_{11}x_{11}^4 = 0 \quad (1.4)$$

in a completion of the rationals, then in the above system we choose $x_{12} = x_{11}$, $x_{13} = x_{14} = 1$ and $x_j = 0$ for $15 \leq j \leq 22$ to produce a non-singular solution to the system (1.3). The putative extension of Theorem 1.1 to pairs of equations with $q_0 = 11$ implies that there are $\gg P^{14}$ solutions in integers x_j with $|x_j| \leq P$ ($1 \leq j \leq 22$). This is only possible if some of these solutions have some variable x_j with $1 \leq j \leq 11$ non-zero. Consequently, the quantitative non-singular Hasse principle for the above system implies the Hasse principle for the equation (1.4), as claimed. In light of this observation, it appears difficult to further relax the condition on q_0 in Theorem 1.1.

Three novel ingredients are required to establish Theorem 1.1, of which two were not available at the time when [5] was written. We proceed to describe discriminating invariants associated with the system (1.1) that we shall use to identify various cases that require treatment by three different methods. With each pair of coefficients (A_j, B_j) occurring in the system (1.1), we associate the linear form $\Lambda_j = \Lambda_j(\alpha, \beta)$ defined by

$$\Lambda_j = A_j\alpha + B_j\beta \quad (1 \leq j \leq s). \quad (1.5)$$

Recall in this context that $(A_j, B_j) \neq (0, 0)$ ($1 \leq j \leq s$). We refer to indices i and j as being *equivalent* when there exists a non-zero rational number λ with $\Lambda_i = \lambda\Lambda_j$. Suppose that the equivalence relation thus defined amongst the indices $1, 2, \dots, s$ has exactly t equivalence classes, and that the number of indices in these classes are $r_1 \geq r_2 \geq \dots \geq r_t$. If we assume that the system (1.1) admits a non-singular real solution, then we must have $t \geq 2$. In the interest of notational simplicity, we put $n = r_1$, $m = r_2$ and suppose that x_j ($1 \leq j \leq n$) are the variables counted by r_1 , and that $y_j = x_{n+j}$ ($1 \leq j \leq m$) are the variables counted by r_2 . The remaining variables (if any) we denote by $z_j = x_{n+m+j}$ ($1 \leq j \leq l$). Then, by taking suitable linear combinations of

the two equations in (1.1), we pass to an equivalent system of the shape

$$\left. \begin{aligned} a_1x_1^4 + \dots + a_nx_n^4 + c_1z_1^4 + \dots + c_lz_l^4 &= 0 \\ b_1y_1^4 + \dots + b_my_m^4 + d_1z_1^4 + \dots + d_lz_l^4 &= 0 \end{aligned} \right\}, \quad (1.6)$$

for suitable non-zero integers a_i, b_j, c_k, d_k . Note that q_0 is invariant with respect to this operation, so that $q_0 = s - n$. When m is not too small, it is possible to adopt just the simplest ideas from our work on similarly shaped systems of cubic forms [4] to derive a considerable strengthening of Theorem 1.1.

Theorem 1.2. *Suppose that $n \geq m \geq 6$ and $m + l \geq 12$, and that the pair of equations (1.6) has non-singular solutions in each completion of the rational numbers. Then $\mathcal{N}(P) \gg P^{s-8}$.*

Note that Theorem 1.2 applies to pairs of quartic forms with $n = m = l = 6$ and $s = 18$, for example. Evidently, Theorem 1.2 implies Theorem 1.1 in all cases where $m \geq 6$. Our approach in the remaining cases is radically different and depends on a new entangled two-dimensional moment estimate that involves 18 smooth Weyl sums (see Theorem 2.4 below). The arithmetic harmonic analysis provides an estimate in terms of eighth and tenth moments of smooth Weyl sums, and only the most recent bound for the latter [6, Theorem 1.3] is of strength sufficient for the application in this paper. The new 18th moment estimate replaces a similar 21st moment estimate in our previous work [5, Theorem 2.3], and demonstrates more flexibility in absorbing larger values of the parameters r_j introduced earlier. This suffices to deal with all cases of Theorem 1.1 where $q_0 \geq 13$. If one directs this second line of attack to the cases with $q_0 = 12$ then difficulties arise on which we comment later, in the course of the argument. This prompted the authors [7] to employ the second author's breaking convexity devices [12] to the installment of a minor arc moment estimate for biquadratic smooth Weyl sums. This, our third new tool, seems indispensable for a successful treatment of those cases of Theorem 1.1 where $q_0 = 12$ and m is small, but also helps us along to give a slick proof of Theorem 1.2.

Notation. Our basic parameter is P , a sufficiently large real number. Implicit constants in Vinogradov's familiar symbols \ll and \gg may depend on s and ε as well as ambient coefficients such as those in the system (1.1). In this paper, whenever ε appears in a statement we assert that the statement holds for each positive real value assigned to ε .

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2. MEAN VALUE ESTIMATES: OLD AND NEW

In this section we present the new entangled moment estimate, preceded by a summary of more familiar one dimensional mean values. We begin with some preparatory notation.

When P and R are real numbers with $1 \leq R \leq P$, we define the set of smooth numbers $\mathcal{A}(P, R)$ by

$$\mathcal{A}(P, R) = \{u \in \mathbb{Z} \cap [1, P] : p \text{ prime and } p|u \Rightarrow p \leq R\}.$$

We then define the Weyl sum $h(\alpha) = h(\alpha; P, R)$ by

$$h(\alpha; P, R) = \sum_{x \in \mathcal{A}(P, R)} e(\alpha x^4).$$

It is convenient to refer to an exponent Δ_t as *admissible* if there exists a positive number η such that, whenever $1 \leq R \leq P^\eta$, one has

$$\int_0^1 |h(\alpha; P, R)|^t d\alpha \ll P^{t-4+\Delta_t}. \quad (2.1)$$

Lemma 2.1. *The exponents $\Delta_6 = 1.1835$, $\Delta_8 = 0.5942$, $\Delta_{10} = 0.1992$ and $\Delta_t = 0$, for $t \geq 11.9560$, are admissible.*

Proof. For $t = 6$ and 8 , this follows from [2, Theorem 2]. For $t = 10$ see [6, Theorem 1.3]. The last statement is a consequence of [7, Theorem 1.2]. \square

Throughout the rest of this paper, we fix $\eta > 0$ to be in accordance with Lemma 2.1 and the estimate (2.1).

In the following chain of auxiliary estimates let \mathcal{Z} denote a set of Z integers and put

$$K(\alpha) = \sum_{z \in \mathcal{Z}} e(\alpha z). \quad (2.2)$$

For $\nu = 1$ or 2 put

$$J_\nu = \int_0^1 |h(\alpha)^{2\nu} K(\alpha)^2| d\alpha.$$

Lemma 2.2. *One has*

$$J_1 \ll PZ + P^{1/2+\varepsilon} Z^{3/2} \quad \text{and} \quad J_2 \ll P^3 Z + P^{2+\varepsilon} Z^{3/2}.$$

Proof. Both estimates are instances of [8, Lemma 6.1]. Here we note that, although in the proof of the latter it is supposed that $\mathcal{Z} \subseteq [1, P^4]$, it is readily seen that this constraint is redundant in the argument. \square

We next introduce a simple form of the entangled moment. For fixed integers a, b, c, d , define

$$I(a, b, c, d) = \int_0^1 \int_0^1 |h(a\alpha)h(b\beta)h(c\alpha + d\beta)|^6 d\alpha d\beta.$$

For the discussion to come, we fix choices for admissible exponents according to Lemma 2.1, and we fix a choice for the parameters τ and τ_1 satisfying

$$0 < \tau < \tau_1 < (1 - \Delta_8 - 2\Delta_{10})/4.$$

Thus, we may suppose that $0 < \tau < 0.00185$. We stress that the positivity of τ is assured only because $\Delta_8 + 2\Delta_{10} < 1$, an inequality obtained by the narrowest of margins by virtue of our recent work [6, Theorem 1.3].

Lemma 2.3. *Let a, b, c, d be non-zero integers. Then $I(a, b, c, d) \ll P^{21/2-2\tau}$.*

Proof. Let $\psi(n)$ denote the number of integers $x_j \in \mathcal{A}(P, P^n)$ ($1 \leq j \leq 6$) with

$$x_1^4 + x_2^4 + x_3^4 - x_4^4 - x_5^4 - x_6^4 = n.$$

Then

$$|h(\gamma)|^6 = \sum_{n \in \mathbb{Z}} \psi(n) e(\gamma n). \quad (2.3)$$

Note that $\psi(n) = 0$ holds for all $|n| > 3P^4$, so that the sum in (2.3) is in fact restricted to a finite range. Also, one has $\psi(n) = \psi(-n)$ for all n .

Write $I = I(a, b, c, d)$. Then, by orthogonality and (2.3), we see that

$$\begin{aligned} I &= \sum_{\substack{an_1 = cn_3 \\ bn_2 = dn_3}} \psi(n_1) \psi(n_2) \psi(n_3) \\ &\leq \sum_{\substack{an_1 = cn_3 \\ bn_2 = dn_3}} (\psi(n_1)^3 + \psi(n_2)^3 + \psi(n_3)^3) \leq 3 \sum_{n \in \mathbb{Z}} \psi(n)^3. \end{aligned} \quad (2.4)$$

Again by orthogonality, we have

$$\psi(n) = \int_0^1 |h(\alpha)|^6 e(-\alpha n) d\alpha. \quad (2.5)$$

Hence, by Lemma 2.1, for each $n \in \mathbb{Z}$ we have

$$\psi(n) \leq \psi(0) \ll P^{2+\Delta_6}. \quad (2.6)$$

Since $3(2 + \Delta_6) < 10$, we may combine the terms with n and $-n$ in (2.4) to deduce that

$$I \ll P^{10} + \sum_{n=1}^{\infty} \psi(n)^3.$$

Let $T \geq 1$, and define

$$\mathcal{Z}_T = \{n \in \mathbb{N} : T \leq \psi(n) < 2T\}.$$

When P is large, it follows from (2.6) that this set is empty unless $T \leq P^{3.185}$. Since all natural numbers n with $\psi(n) \neq 0$ belong to some one of these sets \mathcal{Z}_T as T runs through powers of 2, it follows that there is a choice for T with $1 \leq T \leq P^{3.185}$ for which

$$I \ll P^{10} + P^\varepsilon T^3 \text{card}(\mathcal{Z}_T). \quad (2.7)$$

Three different arguments are now required, depending on the size of T . We take $\mathcal{Z} = \mathcal{Z}_T$ in (2.2), write $Z = \text{card}(\mathcal{Z})$ as before, and then deduce from (2.2) and (2.5) that

$$TZ \leq \int_0^1 |h(\alpha)|^6 K(-\alpha) d\alpha. \quad (2.8)$$

Our first approach to bounding Z applies Hölder's inequality on the right hand side of (2.8) to obtain

$$TZ \leq J_2^{1/3} \left(\int_0^1 |K(\alpha)|^2 d\alpha \right)^{1/6} \left(\int_0^1 |h(\alpha)|^8 d\alpha \right)^{1/6} \left(\int_0^1 |h(\alpha)|^{10} d\alpha \right)^{1/3}.$$

Recalling Lemma 2.2, Parseval's identity and (2.1), we deduce that

$$TZ \ll (P^3Z + P^{2+\varepsilon}Z^{3/2})^{1/3}(Z)^{1/6}(P^{4+\Delta_8})^{1/6}(P^{6+\Delta_{10}})^{1/3}.$$

Since our hypothesis on τ_1 ensures that $\Delta_8 + 2\Delta_{10} < 1 - 4\tau_1$, we infer that

$$T^3Z \ll TP^{(23-4\tau_1)/3} + P^{21/2-2\tau_1}.$$

In view of (2.7), this bound provides an acceptable estimate for I should T be in the range $1 \leq T \leq P^{(17-4\tau_1)/6}$.

Alternatively, we may apply Schwarz's inequality to the right hand side of (2.8) to infer that

$$TZ \leq J_2^{1/2} \left(\int_0^1 |h(\alpha)|^8 d\alpha \right)^{1/2}.$$

By Lemma 2.2 and (2.1), this gives

$$TZ \ll (P^3Z + P^{2+\varepsilon}Z^{3/2})^{1/2}(P^{4+\Delta_8})^{1/2},$$

an estimate that disentangles to yield the bound

$$T^3Z \ll TP^{7+\Delta_8} + T^{-1}P^{12+2\Delta_8+\varepsilon}.$$

On recalling that Lemma 2.1 permits us the assumption that $\Delta_8 \leq 3/5$, we deduce by reference to (2.7) that our alternative bound for T^3Z produces an acceptable estimate for I whenever $P^{27/10+2\tau_1} \leq T \leq P^{7/2-\Delta_8-2\tau_1}$.

The first two approaches just described handle overlapping ranges for T , leaving only the range $P^{7/2-\Delta_8-2\tau_1} < T \leq P^{3.185}$ to be addressed. Our final method eliminates these large values of T from consideration. Again, we apply Schwarz's inequality to the right hand side of (2.8) to obtain

$$TZ \leq J_1^{1/2} \left(\int_0^1 |h(\alpha)|^{10} d\alpha \right)^{1/2}.$$

By Lemma 2.2 and (2.1), we infer that

$$TZ \ll (PZ + P^{1/2+\varepsilon}Z^{3/2})^{1/2}(P^{6+\Delta_{10}})^{1/2}.$$

On recalling our hypothesis that $\Delta_8 + 2\Delta_{10} < 1 - 4\tau_1$, this leads to the bound

$$T^3Z \ll TP^{7+\Delta_{10}} + T^{-1}P^{14-\Delta_8-4\tau_1}.$$

By Lemma 2.1, we may suppose that $\Delta_{10} + 3.185 < 3.4$, and thus in the final range $P^{7/2-\Delta_8-2\tau_1} < T \leq P^{3.185}$ we again obtain an acceptable upper bound for I . This completes the proof of the lemma. \square

Theorem 2.4. *Suppose that C_i and D_i ($1 \leq i \leq 3$) are integers having the property that any two of the three linear forms $M_i = C_i\alpha + D_i\beta$ ($1 \leq i \leq 3$) are linearly independent. Then*

$$\int_0^1 \int_0^1 |h(M_1)h(M_2)h(M_3)|^6 d\alpha d\beta \ll P^{21/2-2\tau}.$$

Proof. By (2.3) and orthogonality, the integral in question is equal to

$$\sum \psi(n_1)\psi(n_2)\psi(n_3),$$

with the sum extended over all $n_i \in \mathbb{Z}$ ($1 \leq i \leq 3$) satisfying

$$C_1n_1 + C_2n_2 + C_3n_3 = D_1n_1 + D_2n_2 + D_3n_3 = 0.$$

All 2×2 minors of the coefficient matrix associated with this pair of linear equations are non-singular, so there is an equivalent system $an_1 = cn_3$, $bn_2 = dn_3$ with non-zero integers a, b, c, d . The conclusion of the theorem now follows from Lemma 2.3 via (2.4). \square

3. THE CIRCLE METHOD

In this section we prepare the ground for a circle method approach to Theorems 1.1 and 1.2. We shall assume throughout, as we may, that the system (1.1) is already in the form (1.6). The linear forms $\Lambda_j = \Lambda_j(\alpha, \beta)$ defined by (1.5) can then be written also as

$$\begin{aligned} \Lambda_j &= a_j\alpha \quad (1 \leq j \leq n), & \Lambda_{n+j} &= b_j\beta \quad (1 \leq j \leq m), \\ \Lambda_{n+m+j} &= c_j\alpha + d_j\beta \quad (1 \leq j \leq l). \end{aligned} \tag{3.1}$$

We fix real numbers $\eta_j \in (0, 1]$ for $1 \leq j \leq s$, form the generating function

$$\mathcal{F}_\eta(\alpha, \beta) = \prod_{j=1}^s h(\Lambda_j; P, P^{\eta_j}) \tag{3.2}$$

and define

$$\mathcal{N}_\eta(P) = \int_0^1 \int_0^1 \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta. \tag{3.3}$$

By orthogonality, it follows from the definition of $\mathcal{N}(P)$ that

$$\mathcal{N}(P) \geq \mathcal{N}_\eta(P). \tag{3.4}$$

The specific choice of η will depend on the coefficients in (1.1) and (1.6). Here we analyse the contribution of the major arcs to the integral (3.3) for a generic choice of η . This is largely standard but the relatively low number of variables available to us in Theorem 1.2 calls for a brief account.

We begin by defining the singular integral associated with the system of equations (1.1). This features the integral

$$v(\gamma) = \int_0^P e(\gamma\xi^4) \, d\xi \tag{3.5}$$

from which we build the generating function

$$V(\alpha, \beta) = \prod_{j=1}^s v(\Lambda_j), \tag{3.6}$$

and, for $X \geq 1$, the truncated singular integral

$$\mathfrak{I}(X) = \int_{-XP^{-4}}^{XP^{-4}} \int_{-XP^{-4}}^{XP^{-4}} V(\alpha, \beta) \, d\alpha \, d\beta.$$

Lemma 3.1. *Suppose that $q_0(\mathbf{A}, \mathbf{B}) \geq 9$. Then V is integrable over \mathbb{R}^2 and*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |V(\alpha, \beta)| \, d\alpha \, d\beta \ll P^{s-8}. \quad (3.7)$$

Moreover, the limit

$$\mathfrak{I} = \lim_{X \rightarrow \infty} \mathfrak{I}(X)$$

exists, and whenever $X \geq 1$ one has

$$\mathfrak{I} - \mathfrak{I}(X) \ll P^{s-8} X^{-1/4}. \quad (3.8)$$

Finally, if the system (1.1) has a non-singular real solution, then $\mathfrak{I} \gg P^{s-8}$.

Proof. There are at least five disjoint subsets $\{i, j\} \subset \{1, 2, \dots, s\}$ having the property that the two linear forms Λ_i, Λ_j are linearly independent. To see this, let us suppose that at most four such pairs can be formed. Then at least $s - 8$ of the forms $\Lambda_1, \dots, \Lambda_s$ lie in a one-dimensional space, showing that $n \geq s - 8$. But $s = n + q_0$, so that $q_0 \leq 8$, which is not the case.

Based on this observation we temporarily (only within this proof) relabel the indices of the forms $\Lambda_1, \dots, \Lambda_s$ so as to arrange that for $1 \leq j \leq 5$, the pairs $\Lambda_{2j-1}, \Lambda_{2j}$ are linearly independent. We use the familiar inequality

$$|z_1 z_2 \cdots z_r| \ll |z_1|^r + \dots + |z_r|^r \quad (3.9)$$

with $r = 5$ and find from (3.6) that

$$V(\alpha, \beta) \ll P^{s-10} \sum_{j=1}^5 |v(\Lambda_{2j-1})v(\Lambda_{2j})|^5. \quad (3.10)$$

By (3.5) and integration by parts, we have

$$v(\gamma) \ll P(1 + P^4|\gamma|)^{-1/4}. \quad (3.11)$$

This shows that v^5 is integrable with

$$\int_{-\infty}^{\infty} |v(\gamma)|^5 \, d\gamma \ll P. \quad (3.12)$$

The linear change of variable from (α, β) to $(\Lambda_{2j-1}, \Lambda_{2j})$ shows $|v(\Lambda_{2j-1})v(\Lambda_{2j})|^5$ to be integrable over \mathbb{R}^2 . Since V is continuous, its integrability follows from dominated convergence and (3.10), while (3.12) in combination with (3.10) implies (3.7). The existence of the limit \mathfrak{I} is now immediate.

Next, with

$$\mathcal{B}(X) = \{(\alpha, \beta) \in \mathbb{R}^2 : \max\{|\alpha|, |\beta|\} \geq XP^{-4}\},$$

we have

$$\mathfrak{I} - \mathfrak{I}(X) \ll \int_{\mathcal{B}(X)} |V(\alpha, \beta)| \, d\alpha \, d\beta.$$

We apply (3.10) and observe that whenever $(\alpha, \beta) \in \mathcal{B}(X)$, one has

$$\max\{|\Lambda_{2j-1}|, |\Lambda_{2j}|\} \gg XP^{-4}.$$

Hence, by linear changes of variables as before, and with a suitable $C > 0$ depending only on \mathbf{A} and \mathbf{B} , we find that

$$\mathfrak{J} - \mathfrak{J}(X) \ll P^{s-10} \int_{-\infty}^{\infty} \int_{CXP^{-4}}^{\infty} |v(\gamma)v(\delta)|^5 d\gamma d\delta,$$

and (3.8) follows from (3.11).

Finally, in order to derive the lower bound for \mathfrak{J} , one first observes that if (1.1) has a non-singular real solution, then there is such a solution with all of its coordinates in the interval $(0, 1)$ (see the discussion on page 2894 of [5]). From here one may follow the argument used to prove [3, Lemma 13], *mutatis mutandis*, so as to establish the final conclusion of the lemma. \square

We now turn to the singular series. Its germ is the Gauss sum

$$S(q, a) = \sum_{r=1}^q e(ar^4/q)$$

allowing us to define the generating functions

$$T(q, a, b) = \prod_{j=1}^s S(q, \Lambda_j(a, b)),$$

$$U(q) = q^{-s} \sum_{\substack{a=1 \\ (a,b,q)=1}}^q \sum_{b=1}^q T(q, a, b) \quad \text{and} \quad U^\dagger(q) = q^{-s} \sum_{\substack{a=1 \\ (a,b,q)=1}}^q \sum_{b=1}^q |T(q, a, b)|.$$

Lemma 3.2. *Suppose that $s \geq 16$ and $q_0(\mathbf{A}, \mathbf{B}) \geq 12$. Then $U^\dagger(q) \ll q^{\varepsilon-2}$.*

Proof. From [10, Theorem 4.2] we deduce that

$$q^{-1}S(q, u) \ll q^{-1/4}(q, u)^{1/4},$$

so that

$$U^\dagger(q) \ll q^{-s/4} \sum_{\substack{a=1 \\ (a,b,q)=1}}^q \sum_{b=1}^q \prod_{j=1}^s (q, \Lambda_j(a, b))^{1/4}.$$

Recall the data r_1, \dots, r_t introduced in the preamble to Theorem 1.2. Now following through the argument on page 890 of [3] in combination with [3, Lemma 11] one finds that there is a number Δ depending only on the coefficients \mathbf{A} and \mathbf{B} such that

$$U^\dagger(q) \ll q^{2-s/4} \sum_{\substack{v_1, \dots, v_t \\ v_1 v_2 \cdots v_t | \Delta q}} v_1^{(r_1-4)/4} \cdots v_t^{(r_t-4)/4}.$$

But $r_j \leq r_1 = n \leq s - 12$ ($1 \leq j \leq t$) and $s \geq 16$, and thus

$$U^\dagger(q) \ll q^{2-s/4} \sum_{\substack{v_1, \dots, v_t \\ v_1 v_2 \cdots v_t | \Delta q}} (v_1 \cdots v_t)^{(s-16)/4} \ll q^{\varepsilon-2}.$$

This completes the proof of the lemma. \square

The truncated singular series is defined by

$$\mathfrak{S}(X) = \sum_{1 \leq q \leq X} U(q).$$

The preceding lemma has the following corollary.

Lemma 3.3. *Suppose that $s \geq 16$ and $q_0(\mathbf{A}, \mathbf{B}) \geq 12$. Then the series*

$$\mathfrak{S} = \sum_{q=1}^{\infty} U(q)$$

converges absolutely, and $\mathfrak{S}(X) - \mathfrak{S} \ll X^{\varepsilon-1}$. Moreover, if, for each prime p , the system of equations (1.1) has a non-singular solution in \mathbb{Q}_p , then $\mathfrak{S} \gg 1$.

Proof. Only the last statement requires justification, for which we may follow through the argument of the proof of [3, Lemma 12] with obvious adjustments. \square

Now is the time to define the major arcs in our Hardy-Littlewood dissection. We suppose now that $0 < \tau < 10^{-4}$ and put $Q = (\log P)^\tau$. We require dissections of dimension one and two. First, when $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ we define

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1] : |\alpha - a/q| \leq QP^{-4}\}.$$

We then take \mathfrak{M} to be the union of the intervals $\mathfrak{M}(q, a)$ with $0 \leq a \leq q \leq Q$ and $(q, a) = 1$, and we put $\mathfrak{m} = [0, 1] \setminus \mathfrak{M}$. Second, when $a, b \in \mathbb{Z}$ and $q \in \mathbb{N}$, we define

$$\mathfrak{N}(q, a, b) = \{(\alpha, \beta) \in [0, 1]^2 : |\alpha - a/q| \leq QP^{-4} \text{ and } |\beta - b/q| \leq QP^{-4}\}.$$

We take \mathfrak{N} to be the union of the rectangles $\mathfrak{N}(q, a, b)$ with $0 \leq a, b \leq q \leq Q$ and $(q, a, b) = 1$, and we put $\mathfrak{n} = [0, 1]^2 \setminus \mathfrak{N}$.

Lemma 3.4. *Suppose that $s \geq 16$ and $q_0(\mathbf{A}, \mathbf{B}) \geq 12$. Then there exists a number $\rho > 0$ such that both*

$$\iint_{\mathfrak{N}} \mathcal{F}_{\boldsymbol{\eta}}(\alpha, \beta) \, d\alpha \, d\beta = \rho \mathfrak{S} \mathfrak{J} + O(P^{s-8}(\log P)^{-\tau/4})$$

and

$$\int_{\mathfrak{M}} \int_{\mathfrak{M}} \mathcal{F}_{\boldsymbol{\eta}}(\alpha, \beta) \, d\alpha \, d\beta = \rho \mathfrak{S} \mathfrak{J} + O(P^{s-8}(\log P)^{-\tau/4}).$$

Moreover, if $\eta_j = 1$ for $1 \leq j \leq s$, then $\rho = 1$.

Proof. We begin by establishing the first asymptotic relation asserted in the statement. Let $(\alpha, \beta) \in \mathfrak{N}(q, a, b)$ and $q \leq (\log P)^{1/10}$. Repeated use of [11, Lemma 8.5] shows that there is a number $\rho > 0$ with

$$\mathcal{F}_{\boldsymbol{\eta}}(\alpha, \beta) = \rho q^{-s} T(q, a, b) V(\alpha - a/q, \beta - b/q) + O(P^s (\log P)^{-1/2}), \quad (3.13)$$

while [10, Theorem 4.1] shows that $\rho = 1$ in case we have $\eta_j = 1$ for all j . Integrating over $(\alpha, \beta) \in \mathfrak{N}$ now delivers the asymptotic relation

$$\iint_{\mathfrak{N}} \mathcal{F}_{\boldsymbol{\eta}}(\alpha, \beta) \, d\alpha \, d\beta = \rho \mathfrak{S}(Q) \mathfrak{J}(Q) + O(P^{s-8}(\log P)^{-1/4}),$$

and the desired conclusion follows by reference to Lemmata 3.1 and 3.3.

Turning our attention now to the second asymptotic relation asserted in the statement, let \mathfrak{N}^* denote the union of the rectangles $\mathfrak{N}(q, a, b)$ with

$$0 \leq a, b \leq q, \quad 1 \leq q \leq Q^2 \quad \text{and} \quad (a, b, q) = 1.$$

Then $\mathfrak{N} \subset \mathfrak{M} \times \mathfrak{M} \subset \mathfrak{N}^*$ and \mathfrak{N}^* has measure $O(Q^8 P^{-8})$. By (3.13), we have

$$|\mathcal{F}_\eta(\alpha, \beta)| \ll \rho q^{-s} |T(q, a, b) V(\alpha - a/q, \beta - b/q)| + P^s (\log P)^{-1/2}.$$

Thus, by integrating over $(\alpha, \beta) \in \mathfrak{N}^* \setminus \mathfrak{N}$ and applying the bound (3.7) of Lemma 3.1 together with Lemma 3.2, we obtain

$$\iint_{\mathfrak{N}^* \setminus \mathfrak{N}} |\mathcal{F}_\eta(\alpha, \beta)| \, d\alpha \, d\beta \ll P^{s-8} \left(\sum_{Q < q \leq Q^2} U^\dagger(q) + (\log P)^{-1/4} \right) \ll P^{s-8} Q^{\varepsilon-1}.$$

Since

$$\int_{\mathfrak{M}} \int_{\mathfrak{M}} \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta - \iint_{\mathfrak{N}} \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta \ll \iint_{\mathfrak{N}^* \setminus \mathfrak{N}} |\mathcal{F}_\eta(\alpha, \beta)| \, d\alpha \, d\beta,$$

the second asymptotic relation asserted in the statement now follows from the first. \square

We close this section with a related auxiliary estimate for use in the next section.

Lemma 3.5. *Suppose that b is a non-zero integer and $0 < \eta \leq 1$. Then*

$$\int_{\mathfrak{M}} |h(b\alpha; P, P^\eta)|^6 \, d\alpha \ll P^2.$$

Proof. We abbreviate $h(\alpha; P, P^\eta)$ to $h(\alpha)$. Let $\alpha \in \mathfrak{M}(q, a)$ with $q \leq Q$. Then, from [11, Lemma 8.5] one finds that there is number $\theta > 0$ with

$$|h(b\alpha)|^6 = \theta q^{-6} |S(q, ab) v(b\alpha - ab/q)|^6 + O(P^6 (\log P)^{-1/4}).$$

Since \mathfrak{M} has measure $O(Q^3 P^{-4})$, one sees that

$$\int_{\mathfrak{M}} |h(b\alpha)|^6 \, d\alpha \ll \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q |S(q, ab)|^6 \int_{-\infty}^{\infty} |v(\gamma)|^6 \, d\gamma + P^2 (\log P)^{-1/8}.$$

By (3.11), the integral on the right hand side here is $O(P^2)$. Meanwhile, the remaining sum is readily estimated by an obvious adjustment in the proof of [10, Lemma 4.9], as given for example in the proof of [9, Lemma 5.1]. In this way, one finds that the sum on the right hand side is $O(1)$, and the conclusion of the lemma follows. \square

4. THE PROOF OF THEOREM 1.2

In this section we choose η in accordance with (2.1) and Lemma 2.1, and we take $\eta_j = \eta$ for all $1 \leq j \leq s$ in the generating function (3.2). We consider a system of the shape (1.1), and suppose that it is given in the form (1.6) in line with the hypotheses of Theorem 1.2. In particular, we suppose that this system has non-singular solutions in each completion of the rational numbers.

We continue to abbreviate $h(\alpha; P, P^\eta)$ to $h(\alpha)$ and put

$$F_1(\alpha) = h(a_1\alpha) \cdots h(a_6\alpha), \quad F_2(\beta) = h(b_1\beta) \cdots h(b_6\beta),$$

$$H(\alpha, \beta) = \prod_{i=7}^n h(a_i\alpha) \prod_{j=7}^m h(b_j\beta) \prod_{k=1}^l h(c_k\alpha + d_k\beta).$$

Note that $H(\alpha, \beta)$ is a product of $s - 12$ Weyl sums, and that

$$\mathcal{F}_\eta(\alpha, \beta) = F_1(\alpha)F_2(\beta)H(\alpha, \beta). \quad (4.1)$$

Further, our assumptions currently in play allow us to combine the conclusions of Lemmata 3.1, 3.3 and 3.4 so as to infer that

$$\int_{\mathfrak{M}} \int_{\mathfrak{M}} \mathcal{F}_\eta(\alpha, \beta) d\alpha d\beta \gg P^{s-8}. \quad (4.2)$$

The argument that we now explain is based on the unorthodox treatment of certain cubic forms in our memoir [4]. Section 3 of that work serves as a blueprint, but in the current context the arithmetic mollifier is more radical than its predecessor so as to cope with smooth variables more easily.

Let u and v be integers, and let $\varrho(u, v)$ denote the number of solutions of the system of equations

$$\sum_{i=7}^n a_i x_i^4 + \sum_{k=1}^l c_k z_k^4 = u, \quad (4.3)$$

$$\sum_{j=7}^m b_j y_j^4 + \sum_{k=1}^l d_k z_k^4 = v, \quad (4.4)$$

in integers

$$x_i, y_j, z_k \in \mathcal{A}(P, P^\eta) \quad (7 \leq i \leq n, 7 \leq j \leq m, 1 \leq k \leq l). \quad (4.5)$$

Further, let $\varrho_1(u)$ denote the number of solutions of (4.3) satisfying (4.5), and let $\varrho_2(v)$ denote the number of solutions of (4.4) satisfying (4.5). Then

$$\varrho_1(u) = \sum_{v \in \mathbb{Z}} \varrho(u, v) \quad \text{and} \quad \varrho_2(v) = \sum_{u \in \mathbb{Z}} \varrho(u, v). \quad (4.6)$$

Note that $\varrho(u, v) = 0$ unless u and v simultaneously satisfy the inequalities $|u| \leq CP^4$, $|v| \leq CP^4$, with $C = C(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ a sufficiently large constant.

Now let $L = \log \log P$ and $M = P^{s-16}L$. We partition \mathbb{Z}^2 into the three sets

$$\begin{aligned}\mathfrak{X} &= \{(u, v) \in \mathbb{Z}^2 : \varrho_1(u) \leq M \text{ and } \varrho_2(v) \leq M\}, \\ \mathfrak{Y}_1 &= \{(u, v) \in \mathbb{Z}^2 : \varrho_1(u) > M \text{ and } \varrho_2(v) \leq M\}, \\ \mathfrak{Y}_2 &= \{(u, v) \in \mathbb{Z}^2 : \varrho_2(v) > M\}.\end{aligned}$$

The next lemma shows that the sets \mathfrak{Y}_1 and \mathfrak{Y}_2 are slim.

Lemma 4.1. *For $i = 1$ and 2 one has*

$$\sum_{(u,v) \in \mathfrak{Y}_i} \varrho(u, v) \ll P^{s-12}L^{-1}.$$

Proof. We begin by observing that, as a consequence of (4.6), we have

$$\sum_{(u,v) \in \mathfrak{Y}_1} \varrho(u, v) \leq \sum_{\varrho_1(u) > M} \sum_{v \in \mathbb{Z}} \varrho(u, v) \leq M^{-1} \sum_{u \in \mathbb{Z}} \varrho_1(u)^2. \quad (4.7)$$

Next, by orthogonality, we see that

$$\sum_{u \in \mathbb{Z}} \varrho_1(u)^2 = \int_0^1 |H(\alpha, 0)|^2 d\alpha.$$

Now $n - 6 + l$ of the variables (4.5) appear in (4.3) with a non-zero coefficient. The hypotheses of Theorem 1.2 ensure, moreover, that

$$n - 6 + l \geq m + l - 6 \geq 6.$$

We apply (3.9) to exactly six of the respective factors in the product defining $H(\alpha, \beta)$, and estimate the remaining Weyl sums trivially. Thus, on recalling (2.1) and the conclusion of Lemma 2.1, and applying a change of variables, we conclude that for some natural number c one has

$$\sum_{u \in \mathbb{Z}} \varrho_1(u)^2 \ll P^{2(s-18)} \int_0^1 |h(c\alpha)|^{12} d\alpha \ll P^{2s-28}. \quad (4.8)$$

Hence, by means of (4.7), one is led to the upper bound

$$\sum_{(u,v) \in \mathfrak{Y}_1} \varrho(u, v) \ll M^{-1} P^{2s-28} \ll P^{s-12}L^{-1}.$$

This confirms the case $i = 1$ of the lemma, and a symmetrical argument delivers the case $i = 2$. \square

Let \mathfrak{A}_1 and \mathfrak{A}_2 be measurable subsets of $[0, 1]$, and put

$$R_j(w; \mathfrak{A}_j) = \int_{\mathfrak{A}_j} F_j(\alpha) e(\alpha w) d\alpha \quad (j = 1, 2). \quad (4.9)$$

The definition of $H(\alpha, \beta)$ shows that

$$H(\alpha, \beta) = \sum_{(u,v) \in \mathbb{Z}^2} \varrho(u, v) e(\alpha u + \beta v),$$

and so it follows from (4.1) that

$$\int_{\mathfrak{A}_2} \int_{\mathfrak{A}_1} \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta = \sum_{(u,v) \in \mathbb{Z}^2} \varrho(u, v) R_1(u; \mathfrak{A}_1) R_2(v; \mathfrak{A}_2). \quad (4.10)$$

Write

$$N(\mathfrak{A}_1, \mathfrak{A}_2) = \sum_{(u,v) \in \mathfrak{X}} \varrho(u, v) R_1(u; \mathfrak{A}_1) R_2(v; \mathfrak{A}_2). \quad (4.11)$$

Then, on comparing (3.3) and (4.10), it follows from (4.11) via orthogonality that $\mathcal{N}_\eta(P) \geq N([0, 1], [0, 1])$. Recalling the definitions of \mathfrak{M} and \mathfrak{m} from the preamble to Lemma 3.4, we may therefore disassemble the unit square to obtain the lower bound

$$\mathcal{N}_\eta(P) \geq N(\mathfrak{M}, \mathfrak{M}) + N(\mathfrak{M}, \mathfrak{m}) + N(\mathfrak{m}, \mathfrak{M}) + N(\mathfrak{m}, \mathfrak{m}). \quad (4.12)$$

We now consider the terms on the right hand side of (4.12) in turn.

Lemma 4.2. *One has*

$$N(\mathfrak{M}, \mathfrak{M}) = \int_{\mathfrak{M}} \int_{\mathfrak{M}} \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta + O(P^{s-8}L^{-1}).$$

Proof. In (4.10) and (4.11) we take $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{M}$. By considering the difference of these expressions, we see that

$$N(\mathfrak{M}, \mathfrak{M}) = \int_{\mathfrak{M}} \int_{\mathfrak{M}} \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta - \mathcal{E}, \quad (4.13)$$

where

$$\mathcal{E} = \sum_{(u,v) \in \mathfrak{Y}_1 \cup \mathfrak{Y}_2} \varrho(u, v) R_1(u; \mathfrak{M}) R_2(v; \mathfrak{M}).$$

By (3.9), (4.9) and Lemma 3.5, it is immediate that the upper bound

$$R_j(w; \mathfrak{M}) \ll P^2 \quad (4.14)$$

holds uniformly in $w \in \mathbb{Z}$ for $j = 1$ and 2 . Thus, in view of Lemma 4.1, we have

$$\mathcal{E} \ll P^4 \sum_{(u,v) \in \mathfrak{Y}_1 \cup \mathfrak{Y}_2} \varrho(u, v) \ll P^{s-8}L^{-1},$$

and the desired conclusion follows from (4.13). \square

Lemma 4.3. *One has*

$$N(\mathfrak{m}, \mathfrak{M}) + N(\mathfrak{M}, \mathfrak{m}) \ll P^{s-8}L^{-1}.$$

Proof. By taking $\mathfrak{A}_1 = \mathfrak{m}$ and $\mathfrak{A}_2 = \mathfrak{M}$ in (4.11), we deduce via (4.14) that

$$N(\mathfrak{m}, \mathfrak{M}) \ll P^2 \sum_{(u,v) \in \mathfrak{X}} \varrho(u, v) |R_1(u; \mathfrak{m})| \ll P^2 \sum_{u \in \mathbb{Z}} \varrho_1(u) |R_1(u; \mathfrak{m})|.$$

By (4.8) and Cauchy's inequality, we infer that

$$N(\mathfrak{m}, \mathfrak{M}) \ll P^{s-12} \left(\sum_{u \in \mathbb{Z}} |R_1(u; \mathfrak{m})|^2 \right)^{1/2}. \quad (4.15)$$

Meanwhile, from the argument of the proof of [13, Lemma 8.1], we find that for any fixed integer $c \neq 0$ one has $h(c\alpha) \ll P(\log P)^{-\tau^2}$ uniformly for $\alpha \in \mathfrak{m}$. Hence, by Lemma 2.1,

$$\int_{\mathfrak{m}} |h(c\alpha)|^{12} d\alpha \ll P^{0.04}(\log P)^{-\tau^3} \int_0^1 |h(c\alpha)|^{11.96} d\alpha \ll P^8(\log P)^{-\tau^3}. \quad (4.16)$$

By Bessel's inequality, Hölder's inequality and (4.16), we thus obtain the bound

$$\sum_{u \in \mathbb{Z}} |R_1(u; \mathfrak{m})|^2 \leq \int_{\mathfrak{m}} |F_1(\alpha)|^2 d\alpha \ll P^8(\log P)^{-\tau^3}. \quad (4.17)$$

The bound $N(\mathfrak{m}, \mathfrak{M}) \ll P^{s-8}(\log P)^{-\tau^4}$ follows by substituting (4.17) into (4.15), and by symmetry, an upper bound of the same strength holds also for $N(\mathfrak{M}, \mathfrak{m})$. The conclusion of the lemma is now immediate. \square

Lemma 4.4. *One has*

$$N(\mathfrak{m}, \mathfrak{m}) \ll P^{s-8}L^{-1}.$$

Proof. Taking $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{m}$ in (4.11), we infer via (3.9) that

$$N(\mathfrak{m}, \mathfrak{m}) \leq \sum_{(u,v) \in \mathfrak{X}} \varrho(u, v) (|R_1(u; \mathfrak{m})|^2 + |R_2(v; \mathfrak{m})|^2).$$

But in view of (4.6), one has

$$\sum_{(u,v) \in \mathfrak{X}} \varrho(u, v) |R_1(u; \mathfrak{m})|^2 \leq \sum_{\varrho_1(u) \leq M} \varrho_1(u) |R_1(u; \mathfrak{m})|^2 \leq M \sum_{u \in \mathbb{Z}} |R_1(u; \mathfrak{m})|^2.$$

We may now apply (4.17) and observe that a symmetrical argument provides an estimate of the same strength when R_1 is replaced by R_2 . Since we have $M \ll P^{s-16}(\log P)^\varepsilon$, we conclude that

$$N(\mathfrak{m}, \mathfrak{m}) \ll P^{s-8}(\log P)^{\varepsilon-\tau^3} \ll P^{s-8}L^{-1}.$$

This completes the proof of the lemma. \square

Now suppose that the equations (1.6) have non-singular solutions in all completions of the rationals. Then, by (4.2) and Lemma 4.2 we see that $N(\mathfrak{M}, \mathfrak{M}) \gg P^{s-8}$. By (4.12) and Lemmata 4.3 and 4.4, we have

$$\mathcal{N}_\eta(P) \geq N(\mathfrak{M}, \mathfrak{M}) + O(P^{s-8}L^{-1}).$$

Hence, in view of (3.4), we conclude that $\mathcal{N}(P) \geq \mathcal{N}_\eta(P) \gg P^{s-8}$. This completes the proof of Theorem 1.2.

5. THE PROOF OF THEOREM 1.1: LARGE n

In this section and the next we suppose that the hypotheses of Theorem 1.1 are satisfied, that the system (1.1) is given in the form (1.6), and that η is again chosen in accordance with (2.1) and Lemma 2.1. Since all cases of Theorem 1.1 with $m \geq 6$ are covered by Theorem 1.2, we may assume in addition that $m \leq 5$. Our focus in the current section is the situation in which $n \geq 8$. We defer to the next section the corresponding scenario in which $n \leq 7$.

Suppose then that $n \geq 8$. We take

$$\eta_j = 1 \quad (1 \leq j \leq 4) \quad \text{and} \quad \eta_j = \eta \quad (5 \leq j \leq s).$$

The generating function \mathcal{F}_η is then defined by means of the formula (3.2). Provided that the equations (1.1) have non-singular solutions in all completions of the rational numbers, it now follows from Lemmata 3.1, 3.3 and 3.4 that

$$\iint_{\mathfrak{N}} \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta \gg P^{s-8}. \quad (5.1)$$

We imminently show that

$$\iint_{\mathfrak{N}} \mathcal{F}_\eta(\alpha, \beta) \, d\alpha \, d\beta \ll P^{s-8} L^{-1}. \quad (5.2)$$

The sum of the integrals in (5.1) and (5.2) is $\mathcal{N}_\eta(P)$. Thus, granted the validity of the upper bound (5.2), we conclude from (3.4) that $\mathcal{N}(P) \geq \mathcal{N}_\eta(P) \gg P^{s-8}$, so this will establish Theorem 1.1 for $n \geq 8$.

Before launching the proof of Theorem 1.1 for $n \geq 8$ in earnest, we prepare an auxiliary estimate of use in both this and the next section. In this context, we recall the notation introduced in (1.5), and write

$$f(\alpha) = h(\alpha; P, P) \quad \text{and} \quad h(\alpha) = h(\alpha; P, P^n). \quad (5.3)$$

Then, by (3.1) and (3.2), we have

$$\mathcal{F}_\eta(\alpha, \beta) = f(a_1\alpha) f(a_2\alpha) f(a_3\alpha) f(a_4\alpha) \prod_{j=5}^s h(\Lambda_j). \quad (5.4)$$

Lemma 5.1. *Suppose that Λ_i and Λ_j are linearly independent. Then, provided that P is sufficiently large, and one has both*

$$|h(\Lambda_i)| \geq PL^{-300} \quad \text{and} \quad |h(\Lambda_j)| \geq PL^{-300},$$

it follows that $(\alpha, \beta) \in \mathfrak{N} + \mathbb{Z}^2$.

Proof. Let $\nu \in \{i, j\}$ and suppose that P is sufficiently large. From the argument of the proof of [13, Lemma 8.1] it follows that there exist $a_\nu \in \mathbb{Z}$ and $q_\nu \in \mathbb{N}$ with $q_\nu \leq L^{3000}$ and $|\Lambda_\nu - a_\nu/q_\nu| \leq L^{3000} P^{-4}$. By eliminating β , or α , respectively, from the two inequalities here, we see that there exist $a, b \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(q, a, b) = 1$ and $q \leq L^{6001}$ satisfying the property that

$$|\alpha - a/q| \leq L^{3001} P^{-4} \quad \text{and} \quad |\beta - b/q| \leq L^{3001} P^{-4}.$$

Thus, on noting that $L^{6001} \leq (\log P)^\tau$ when P is sufficiently large, we conclude that $(\alpha, \beta) \in \mathfrak{N} + \mathbb{Z}^2$. This completes the proof of the lemma. \square

It is now the moment to launch the proof of Theorem 1.1 in the case $n \geq 8$. Consider the linear forms Λ_j with $s - 17 \leq j \leq s$, presented simultaneously in the formats (1.5) and (3.1). Then the hypothesis $q_0 \geq 12$ implies that $B_j \neq 0$ for $s - 11 \leq j \leq s$. Hence, amongst the indices j with $s - 17 \leq j \leq s$, we see that the linear form Λ_j can be independent of β no more than six times. Restricting attention to the forms that depend explicitly on β , the number of repetitions amongst the numbers A_j/B_j is at most $m \leq 5$. It follows that

the 18 forms Λ_j with $s - 17 \leq j \leq s$ can be grouped into six disjoint subsets $\{\Lambda_\kappa, \Lambda_\mu, \Lambda_\nu\}$ satisfying the condition that $\Lambda_\kappa, \Lambda_\mu$ and Λ_ν are pairwise linearly independent. By (3.9) with $r = 6$ we then see that

$$\int_0^1 \int_0^1 \prod_{j=s-17}^s |h(\Lambda_j)| \, d\alpha \, d\beta \ll \sum'_{\kappa < \mu < \nu} \int_0^1 \int_0^1 |h(\Lambda_\kappa)h(\Lambda_\mu)h(\Lambda_\nu)|^6 \, d\alpha \, d\beta, \quad (5.5)$$

where the sum is over all triples κ, μ, ν satisfying the condition that $\Lambda_\kappa, \Lambda_\mu, \Lambda_\nu$ are pairwise linearly independent. By Theorem 2.4, it follows that the integral on the right hand side of (5.5) is $O(P^{21/2-2\tau})$. But $s - 17 \geq 5$ so trivial estimates for the remaining exponential sums suffice to conclude that

$$\int_0^1 \int_0^1 \prod_{j=5}^s |h(\Lambda_j)| \, d\alpha \, d\beta \ll P^{s-12-2\tau+1/2}. \quad (5.6)$$

We apply the estimate (5.6) first to reduce the integration over α to a broad set of major arcs. Let \mathfrak{K} denote the union of the intervals

$$\mathfrak{K}(q, a) = \{\alpha \in [0, 1] : |q\alpha - a| \leq P^{-3}\},$$

with $0 \leq a \leq q \leq P$ and $(a, q) = 1$. Also, let $\mathfrak{k} = [0, 1] \setminus \mathfrak{K}$. By Weyl's inequality (see [10, Lemma 2.4]) we see that the bound $f(a_j\alpha) \ll P^{7/8+\varepsilon}$ ($1 \leq j \leq 4$) holds uniformly for $\alpha \in \mathfrak{k}$. Hence, by (5.6), we discern that

$$\int_0^1 \int_{\mathfrak{k}} |\mathcal{F}_\eta(\alpha, \beta)| \, d\alpha \, d\beta \ll P^{7/2+\varepsilon} \cdot P^{s-12-2\tau+1/2} \ll P^{s-8-\tau}. \quad (5.7)$$

Next we seek to prune the remaining domain $\mathfrak{K} \times [0, 1]$ down to the narrow set of major arcs \mathfrak{N} . For $5 \leq k \leq s$, let

$$\mathfrak{n}_k = \{(\alpha, \beta) \in \mathfrak{K} \times [0, 1] : |h(\Lambda_k)| \leq PL^{-300}\}.$$

Also, put

$$J_k = \iint_{\mathfrak{n}_k} |\mathcal{F}_\eta(\alpha, \beta)| \, d\alpha \, d\beta \quad (5 \leq k \leq s).$$

Suppose that $s-11 \leq k \leq s$, so that $B_k \neq 0$. In order to initiate the estimation of J_k , we temporarily write

$$F(\alpha) = |f(a_1\alpha) \cdots f(a_4\alpha)| \quad \text{and} \quad G(\alpha) = |h(a_5\alpha) \cdots h(a_8\alpha)|.$$

Then, since in this section we work under the assumption that $n \geq 8$, we deduce from (5.4) that whenever $(\alpha, \beta) \in \mathfrak{n}_k$, one has

$$\begin{aligned} |\mathcal{F}_\eta(\alpha, \beta)| &\leq P^{s-20} F(\alpha) G(\alpha) \prod_{j=s-11}^s |h(\Lambda_j)| \\ &\leq P^{s-20} F(\alpha) G(\alpha) (P^{12} L^{-300})^{1/300} \prod_{j=s-11}^s |h(\Lambda_j)|^{299/300}. \end{aligned}$$

By integrating over $(\alpha, \beta) \in \mathfrak{n}_k$, we deduce that

$$J_k \leq P^{s-20+1/25} L^{-1} \int_{\mathfrak{K}} F(\alpha) G(\alpha) \int_0^1 \prod_{j=s-11}^s |h(\Lambda_j)|^{299/300} \, d\beta \, d\alpha. \quad (5.8)$$

In the inner integral on the right hand side of (5.8), we apply Hölder's inequality to obtain the bound

$$\int_0^1 \prod_{j=s-11}^s |h(\Lambda_j)|^{299/300} d\beta \leq \prod_{j=s-11}^s \left(\int_0^1 |h(\Lambda_j)|^{11.96} d\beta \right)^{1/12}. \quad (5.9)$$

Recall that $B_j \neq 0$ for $s-11 \leq j \leq s$. In particular, the exponential sum $h(\Lambda_j)$ has period $1/|B_j|$ in β , and thus

$$\int_0^1 |h(\Lambda_j)|^{11.96} d\beta = \int_0^1 |h(B_j\beta)|^{11.96} d\beta = \int_0^1 |h(\beta)|^{11.96} d\beta.$$

Consequently, an application of Lemma 2.1 reveals that

$$\int_0^1 |h(\Lambda_j)|^{11.96} d\beta \ll P^{7.96}.$$

On substituting this estimate into (5.9), we obtain a bound for the inner integral in (5.8) that is independent of α . Hence we arrive at the relation

$$J_k \ll P^{s-12} L^{-1} \int_{\mathfrak{K}} F(\alpha) G(\alpha) d\alpha. \quad (5.10)$$

It remains to estimate the integral on the right hand side of (5.10). By an argument largely identical to the one that yielded Lemma 3.5, but this time invoking [10, Theorem 4.1], one readily confirms the estimate

$$\int_{\mathfrak{K}} |f(a\alpha)|^6 d\alpha \ll P^2 \quad (5.11)$$

that is valid for any fixed choice of $a \in \mathbb{Z} \setminus \{0\}$. Hence, by Hölder's inequality,

$$\int_{\mathfrak{K}} F(\alpha)^{3/2} d\alpha \ll P^2.$$

By Hölder's inequality again, Lemma 2.1 delivers the bound

$$\int_0^1 G(\alpha)^3 d\alpha \leq \prod_{j=5}^8 \left(\int_0^1 |h(a_j\alpha)|^{12} d\alpha \right)^{1/4} \ll P^8.$$

Another application of Hölder's inequality therefore leads us to the estimate

$$\int_{\mathfrak{K}} F(\alpha) G(\alpha) d\alpha \ll \left(\int_{\mathfrak{K}} F(\alpha)^{3/2} d\alpha \right)^{2/3} \left(\int_0^1 G(\alpha)^3 d\alpha \right)^{1/3} \ll P^4.$$

By substituting this estimate into (5.10) and recalling the definition of J_k , we obtain the bound

$$\iint_{\mathfrak{n}_k} |\mathcal{F}_{\eta}(\alpha, \beta)| d\alpha d\beta \ll P^{s-8} L^{-1} \quad (s-11 \leq k \leq s). \quad (5.12)$$

Now suppose that $(\alpha, \beta) \in [0, 1]^2$ is neither in $\mathfrak{k} \times [0, 1]$ nor in one of the sets \mathfrak{n}_k with $s-11 \leq k \leq s$. Then $|h(\Lambda_k)| \geq PL^{-300}$ for $s-11 \leq k \leq s$. But $m \leq 5$, so among the forms Λ_j with $s-11 \leq j \leq s$ we can find a pair of

linearly independent forms. By Lemma 5.1, this implies that $(\alpha, \beta) \in \mathfrak{N} + \mathbb{Z}^2$. We therefore have the relation

$$\mathfrak{n} \subseteq (\mathfrak{k} \times [0, 1]) \cup \mathfrak{n}_{s-11} \cup \dots \cup \mathfrak{n}_s,$$

and consequently it follows from (5.7) and (5.12) that

$$\begin{aligned} \iint_{\mathfrak{n}} |\mathcal{F}_{\eta}(\alpha, \beta)| \, d\alpha \, d\beta &\leq \int_0^1 \int_{\mathfrak{k}} |\mathcal{F}_{\eta}(\alpha, \beta)| \, d\alpha \, d\beta + \sum_{k=s-11}^s \iint_{\mathfrak{n}_k} |\mathcal{F}_{\eta}(\alpha, \beta)| \, d\alpha \, d\beta \\ &\ll P^{s-8} L^{-1}. \end{aligned}$$

This confirms (5.2) and completes the proof of Theorem 1.1 when $n \geq 8$.

We briefly pause to indicate the relevance of our estimates from [7] to the progress in this section. This concerns the cases where $q_0 = 12$, in which case $n \geq 10$. In such circumstances exactly 12 Weyl sums $h(\Lambda_j)$ depend on β , and some minor arc information has to be extracted from the β -integration. This would be straightforward were one or two of the relevant Weyl sums to have smoothness parameter η_j equal to 1, but this would be in conflict with the requirements in (5.5) where all Weyl sums must be smooth. Thus we are fortunate to have optimal control on a moment smaller than the twelfth of a smooth biquadratic Weyl sum.

6. THE PROOF OF THEOREM 1.1: SMALL n

We devote this section to a discussion of the situation in which $n \leq 7$. Our previous work [5] in fact handles all cases in which $n \leq 7$. However, equipped with the new mean value estimate provided by Theorem 2.4, we are able now to offer a streamlined treatment. We include the present discussion of the situation with $n \leq 7$, therefore, in order that our account be self-contained. This treatment may also offer inspiration for future investigations concerning the solubility of systems of diagonal equations. We now have

$$r_1 = n \leq 7, \quad r_3 \leq r_2 = m \leq 5 \quad \text{and} \quad s \geq 22,$$

whence necessarily $r_4 \geq 1$. For $1 \leq \nu \leq 4$ let $i_{\nu} \in \{1, 2, \dots, s\}$ be any index counted by r_{ν} . Thus, for example, we may take $i_1 = 1$ and $i_2 = n + 1$. Let

$$I = \{1, 2, \dots, s\} \setminus \{i_1, i_2, i_3, i_4\}.$$

It transpires that a combinatorial observation concerning the set I greatly facilitates our analysis of certain auxiliary mean values, and this we formulate in the next lemma. In this context, we recall the notion of equivalence of indices introduced following the definition of Λ_j in equation (1.5).

Lemma 6.1. *Subject to the hypotheses in the preamble, fix indices ν with $1 \leq \nu \leq 4$ and $k \in I$, and suppose that the linear forms $\Lambda_{i_{\nu}}$ and Λ_k are linearly independent. Then I has two disjoint subsets I_0 and I_1 , with eight elements each, and possessing the following properties:*

- (i) *one has $k \in I_0$, and for all $j \in I_0$ the forms Λ_j and $\Lambda_{i_{\nu}}$ are linearly independent;*
- (ii) *the maximal number of mutually equivalent indices $j \in I_1$ is four.*

Proof. First suppose that $\nu = 1$. Then there are $r_1 - 1$ indices $j \in I$ that are counted by r_1 , and $r_1 = n \leq 7$. We put $\min\{4, r_1 - 1\}$ of these indices into the set I_1 and discard the remaining ones (if any). Note that we discard at most two indices in this first phase of the assignment process.

If $r_2 = m = 5$ then four indices $j \in I$ are counted by r_2 . If the index k is among them, then we put it into the set I_0 and put the remaining three indices into the set I_1 . If the index k is not among them, then we put all four indices into the set I_1 , and then place the index k into the set I_0 . Under the current assumptions, we have $r_1 \geq r_2 = 5$, so following these two rounds of allocations, the subset I_1 already has 7 or 8 elements. Since we discarded at most two indices earlier, we have sufficient indices not already allocated from I that we may assign the remaining ones to I_0 and I_1 arbitrarily so that each subset emerges with eight elements apiece. It is then apparent that properties (i) and (ii) are satisfied for this assignment.

If, meanwhile, one has $r_2 = m \leq 4$, then we adjust the second phase of the assignment process by putting the index k into the set I_0 , but otherwise we assign indices not already allocated from I to I_0 and I_1 arbitrarily so that each subset again emerges with eight elements apiece. It is then again apparent that properties (i) and (ii) are satisfied for this assignment. This completes the proof of the lemma when $\nu = 1$.

Now suppose that $\nu \geq 2$. Then there are $r_\nu - 1 \leq r_2 - 1 = m - 1 \leq 4$ indices $j \in I$ that are counted by r_ν , and these we insert into the set I_1 . All of the $r_1 - 1$ indices $j \in I$ counted by r_1 we put into the set I_0 . If the index k is not yet in I_0 then we insert it into this set. So far, we have at most $(r_1 - 1) + 1 \leq 7$ elements assigned to I_0 and at most 4 elements assigned to I_1 .

If there is a suffix $\kappa \notin \{1, \nu\}$ for which $r_\kappa = 5$ and the index k is not counted by r_κ , then all five indices $j \in I$ counted by r_κ are not yet distributed, and we put one into I_0 and four into I_1 . Both in this situation, and when there is no such suffix κ , we may assign the remaining indices not already allocated from I to I_0 and I_1 arbitrarily so that each subset once more emerges with eight elements apiece. On this occasion as in earlier cases, it is again apparent that properties (i) and (ii) are satisfied for this assignment. This completes the proof of the lemma when $\nu \geq 2$. \square

We now embark on the proof of Theorem 1.1 when $n \leq 7$. We take

$$\eta_j = \eta \quad (j \in I) \quad \text{and} \quad \eta_{i_\nu} = 1 \quad (1 \leq \nu \leq 4).$$

We then define \mathcal{F}_η by means of (3.2). Again adopting the notation introduced in (5.3), and writing

$$f_\nu = f(\Lambda_{i_\nu}) \quad (1 \leq \nu \leq 4) \quad \text{and} \quad H_\Theta = \prod_{j \in \Theta} h(\Lambda_j) \quad (\Theta \subseteq I),$$

we have $\mathcal{F}_\eta = f_1 f_2 f_3 f_4 H_1$. It then follows as before that the proof of Theorem 1.1 when $n \leq 7$ is completed by confirming the upper bound (5.2).

Let $1 \leq \nu \leq 4$, and let $\mathbf{a} \subset [0, 1]^2$ be measurable. Put

$$I_\nu(\mathbf{a}) = \iint_{\mathbf{a}} |f_\nu|^4 |H_1| d\alpha d\beta. \quad (6.1)$$

Then, an application of (3.9) reveals that

$$\iint_{\mathbf{n}} |\mathcal{F}_\eta| d\alpha d\beta \ll I_1(\mathbf{n}) + I_2(\mathbf{n}) + I_3(\mathbf{n}) + I_4(\mathbf{n}). \quad (6.2)$$

We fix $\nu \in \{1, 2, 3, 4\}$ and estimate $I_\nu(\mathbf{n})$ by a method that is similar to that applied in the situation with $n \geq 8$ analysed in the previous section. We begin by defining the sets

$$\begin{aligned} \mathfrak{p} &= \{(\alpha, \beta) \in [0, 1]^2 : \Lambda_{i_\nu} \in \mathfrak{k} + \mathbb{Z}\}, \\ \mathfrak{P} &= \{(\alpha, \beta) \in [0, 1]^2 : \Lambda_{i_\nu} \in \mathfrak{K} + \mathbb{Z}\} \end{aligned}$$

and

$$\mathfrak{P}_k = \{(\alpha, \beta) \in \mathfrak{P} : |h(\Lambda_k)| \leq PL^{-300}\} \quad (k \in \mathbb{I}).$$

Observe that $\text{card}(\mathbb{I}) \geq 18$. Moreover, under the current hypotheses, at most six of the forms Λ_j with $j \in \mathbb{I}$ can be multiples of one another. It follows that there are indices k_1 and k_2 in \mathbb{I} having the property that the linear forms Λ_{k_1} , Λ_{k_2} and Λ_{i_ν} are pairwise linearly independent. Hence, if $(\alpha, \beta) \in [0, 1]^2$ is neither in \mathfrak{p} nor in $\mathfrak{P}_{k_1} \cup \mathfrak{P}_{k_2}$, then $|h(\Lambda_{k_\iota})| > PL^{-300}$ for both $\iota = 1$ and $\iota = 2$, and hence Lemma 5.1 implies that $(\alpha, \beta) \in \mathfrak{N}$. We therefore deduce that for these indices k_1 and k_2 , we have $\mathbf{n} \subseteq \mathfrak{p} \cup \mathfrak{P}_{k_1} \cup \mathfrak{P}_{k_2}$. Consequently, the upper bound (5.2) follows from (6.2) once we confirm that for $1 \leq \nu \leq 4$, and all $k \in \mathbb{I}$ for which Λ_k is independent of Λ_{i_ν} , one has the estimates

$$I_\nu(\mathfrak{p}) \ll P^{s-8}L^{-1} \quad \text{and} \quad I_\nu(\mathfrak{P}_k) \ll P^{s-8}L^{-1}. \quad (6.3)$$

We begin by estimating $I_\nu(\mathfrak{p})$. Since i_1 is an index counted by r_1 , the current assumptions show that among the indices $j \in \mathbb{I}$, no more than six are mutually equivalent. Hence, an appropriate reinterpretation of the argument leading to (5.6) yields the bound

$$\int_0^1 \int_0^1 |H_1(\alpha, \beta)| d\alpha d\beta \ll P^{s-12-2\tau+1/2}.$$

Further, Weyl's inequality [10, Lemma 2.4] shows that $f_\nu \ll P^{7/8+\varepsilon}$ for $(\alpha, \beta) \in \mathfrak{p}$. Hence, as an echo of (5.7), we deduce from (6.1) that

$$I_\nu(\mathfrak{p}) \ll P^{s-8-\tau}. \quad (6.4)$$

The treatment of the sets \mathfrak{P}_k makes use of Lemma 6.1. In the notation of this lemma, we have $|H_1| \leq P^{s-20}|H_{1_0}H_{1_1}|$. Thus, by applying Hölder's inequality to (6.1), we deduce that

$$I_\nu(\mathfrak{P}_k) \leq P^{s-20}T_1^{1/3}T_2^{2/3}, \quad (6.5)$$

where

$$T_1 = \int_0^1 \int_0^1 |H_{1_1}|^3 d\alpha d\beta \quad \text{and} \quad T_2 = \iint_{\mathfrak{P}_k} |f_\nu|^6 |H_{1_0}|^{3/2} d\alpha d\beta. \quad (6.6)$$

It follows from the property (ii) associated with the set I_1 in Lemma 6.1 that this set contains four disjoint subsets $\{i, j\}$ having the property that Λ_i and Λ_j are linearly independent. We therefore see from (3.9) with $r = 4$ that

$$|H_{I_1}(\alpha, \beta)|^3 \ll \sum |h(\Lambda_i)h(\Lambda_j)|^{12}, \quad (6.7)$$

where the summation is taken over the four pairs i, j comprising these subsets. A change of variables in combination with orthogonality and Lemma 2.1 delivers the bound

$$\int_0^1 \int_0^1 |h(\Lambda_i)h(\Lambda_j)|^{12} d\alpha d\beta = \int_0^1 \int_0^1 |h(\alpha)h(\beta)|^{12} d\alpha d\beta \ll P^{16}$$

for each of the four pairs i, j . We therefore deduce from (6.6) and (6.7) that

$$T_1 \ll P^{16}. \quad (6.8)$$

Next we estimate the mean value T_2 . By applying Hölder's inequality in (6.6), we obtain the bound

$$T_2 \leq \prod_{j \in I_0} \left(\iint_{\mathfrak{P}_k} |f_\nu|^6 |h(\Lambda_j)|^{12} d\alpha d\beta \right)^{1/8}. \quad (6.9)$$

It follows from the property (i) associated with the set I_0 in Lemma 6.1 that the forms Λ_{i_ν} and Λ_j are linearly independent when $j \in I_0$. By the transformation formula applied to the non-singular linear map $(\alpha, \beta) \rightarrow (\Lambda_{i_\nu}, \Lambda_j)$, it follows via (5.11) and Lemma 2.1 that

$$\iint_{\mathfrak{P}_k} |f_\nu|^6 |h(\Lambda_j)|^{12} d\alpha d\beta \ll \int_{\mathfrak{R}} |f(\xi)|^6 d\xi \int_0^1 |h(\zeta)|^{12} d\zeta \ll P^{10}. \quad (6.10)$$

Here we made use of the fact that \mathfrak{P}_k maps to a compact set, and in addition that f and h are functions of period 1. It follows from the property (i) associated with the set I_0 in Lemma 6.1, moreover, that $k \in I_0$, and in this case the last estimate can be improved. Since one has the upper bound $|h(\Lambda_k)| \leq PL^{-300}$ for $(\alpha, \beta) \in \mathfrak{P}_k$, we see that

$$\iint_{\mathfrak{P}_k} |f_\nu|^6 |h(\Lambda_k)|^{12} d\alpha d\beta \leq P^{0.04} L^{-12} \iint_{\mathfrak{P}_k} |f_\nu|^6 |h(\Lambda_k)|^{11.96} d\alpha d\beta.$$

We may now apply the transformation formula as before, and conclude via Lemma 2.1 that

$$\iint_{\mathfrak{P}_k} |f_\nu|^6 |h(\Lambda_k)|^{12} d\alpha d\beta \ll P^{10} L^{-12}. \quad (6.11)$$

By substituting (6.10) and (6.11) into (6.9), we arrive at the upper bound

$$T_2 \ll P^{10} L^{-3/2}. \quad (6.12)$$

We are now equipped to derive the conclusion we have sought. By substituting (6.8) and (6.12) into (6.5), we infer the estimate

$$I_\nu(\mathfrak{P}_k) \ll P^{s-20} (P^{16})^{1/3} (P^{10} L^{-3/2})^{2/3} = P^{s-8} L^{-1}$$

that is valid for all $k \in I$ satisfying the condition that Λ_k is linearly independent of Λ_{i_ν} . This, together with (6.4), confirms the bounds (6.3). Hence, as we explained in the preamble to (6.3), the upper bound (5.2) does indeed hold. Since the bound (5.1) holds in the present circumstances, once more as a consequence of Lemmata 3.1, 3.3 and 3.4, we again conclude from (3.4) that $\mathcal{N}(P) \geq \mathcal{N}_\eta(P) \gg P^{s-8}$. This completes the proof of Theorem 1.1 for $n \leq 7$.

7. OBSTRUCTIONS

In this section we discuss the pair of equations (1.2) and substantiate our introductory claims concerning some of its properties. Bright [1] has analysed in detail the validity of the Hasse principle for diagonal quartic forms in four variables. The following example (see [1, Example 2.3]), is just one of the cases where the Hasse principle fails.

Lemma 7.1. *Let p be a prime. Then the equation $x_1^4 + x_2^4 - 6x_3^4 - 12x_4^4 = 0$ has a non-trivial solution in \mathbb{Q}_p , but only the trivial solution in \mathbb{Z} .*

We now study the form $y_1^4 + y_2^4 + 3y_3^4 + 5y_4^4 + 7y_5^4$. It turns out that this form is universal over all \mathbb{Q}_p , in a strong sense. This is the content of Lemma 7.4 below. We begin with the distribution of the above form in residue classes.

Lemma 7.2. *Let p be an odd prime, and let $a \in \mathbb{Z}$. Then, the congruence*

$$y_1^4 + y_2^4 + 3y_3^4 + 5y_4^4 + 7y_5^4 \equiv a \pmod{p} \quad (7.1)$$

has a solution with not all the variables divisible by p . If $p = 3, 5$ or 7 , then the solution can be so chosen that the variable with coefficient p is zero.

Proof. The set $\{y^4 : 1 \leq y \leq p-1\}$ has at least $(p-1)/4$ elements. We write

$$b_1 = b_2 = 1, \quad b_3 = 3, \quad b_4 = 5, \quad b_5 = 7. \quad (7.2)$$

Then, by repeated use of the Cauchy-Davenport Theorem [10, Lemma 2.14], for $p > 7$ and $1 \leq \nu \leq 5$, one finds that the set

$$\left\{ \sum_{j=1}^{\nu} b_j y_j^4 : 1 \leq y_1 \leq p-1 \text{ and } 1 \leq y_j \leq p \text{ (} 2 \leq j \leq \nu \text{)} \right\}$$

contains at least $\min\{\nu(p-1)/4, p\}$ residue classes modulo p . Consequently, when $p \geq 11$ and $\nu = 5$, all congruence classes modulo p are covered, and in particular the class a modulo p . This proves the lemma when $p \geq 11$.

If $p = 7$ then there are 3 fourth power residues modulo 7, and the above argument shows that when $1 \leq \nu \leq 3$, the above set contains at least $\min\{3\nu, 7\}$ residue classes modulo 7. In particular, the values of the form $y_1^4 + y_2^4 + 3y_3^4$, with $1 \leq y_1 \leq 6$ and $1 \leq y_2, y_3 \leq 7$, range over all residue classes modulo 7, and in particular the class a modulo 7. In the cases $p = 3$ and 5 one has $y_i^4 \in \{0, 1\}$ modulo p , and a trivial check of cases verifies that all residue classes a modulo p are indeed covered in the prescribed manner. \square

Lemma 7.3. *Let $a \in \mathbb{Z}$. Then the congruence*

$$y_1^4 + y_2^4 + 3y_3^4 + 5y_4^4 + 7y_5^4 \equiv a \pmod{16} \quad (7.3)$$

has a solution in which one at least of the variables is equal to 1.

Proof. One has $y_i^4 \in \{0, 1\}$ modulo 16, and so a direct check of cases verifies that all residues a modulo 16 possess a representation of the desired type. \square

Lemma 7.4. *Let p be a prime and $a \in \mathbb{Z}$. Then the equation*

$$z_1^4 + z_2^4 + 3z_3^4 + 5z_4^4 + 7z_5^4 = a$$

has a solution in \mathbb{Z}_p in which not all of the variables are zero.

Proof. Let p be an odd prime and adopt the notation (7.2). Then it follows from Lemma 7.2 that there is a solution $\mathbf{y} \in \mathbb{Z}^5$ of the congruence (7.1) having the property that for some index j one has $p \nmid 4b_j y_j^3$. Fixing $z_i = y_i$ for $i \neq j$, it is a consequence of Hensel's lemma that there exists $z_j \in \mathbb{Z}_p \setminus \{0\}$ with $z_j \equiv y_j \pmod{p}$ and $b_1 z_1^4 + \dots + b_5 z_5^4 = a$. Thus, indeed, one has $\mathbf{z} \neq \mathbf{0}$.

When $p = 2$ only modest adjustments are required in this argument. Here one observes that the structure of the group of reduced residues modulo 2^h for $h \geq 3$ ensures that if b is an integer with $b \equiv 1 \pmod{16}$, then there is a 2-adic integer β with $\beta^4 = b$. Since Lemma 7.3 ensures that the congruence (7.3) has a solution $\mathbf{y} \in \mathbb{Z}^5$ having the property that for some index j one has $y_j = 1$, we may proceed as before to show that there is a solution $\mathbf{z} \in \mathbb{Z}_2^5 \setminus \{\mathbf{0}\}$ to the equation $b_1 z_1^4 + \dots + b_5 z_5^4 = a$. \square

Now we turn to the system of equations (1.2). Let p be any prime number. It follows from Lemma 7.1 that there is a solution $(x_1, \dots, x_4) \in \mathbb{Z}_p^4 \setminus \{\mathbf{0}\}$ of the equation $x_1^4 + x_2^4 - 6x_3^4 - 12x_4^4 = 0$. By multiplying the coordinates of this solution by an appropriate unit, moreover, there is no loss of generality in assuming that x_4 is a rational integer. Fixing this value of x_4 , there is a solution $(x_5, \dots, x_9) \in \mathbb{Z}_p^5 \setminus \{\mathbf{0}\}$ of the equation

$$x_4^4 = 7x_5^4 + 5x_6^4 + 3x_7^4 + x_8^4 + x_9^4.$$

This follows from Lemma 7.4. With $x_j = 0$ for $j \geq 10$, this provides us with a non-singular solution of (1.2) with coordinates in \mathbb{Z}_p . The existence of non-singular real solutions to the system (1.2) is plain. This confirms that the system of equations (1.2) has non-singular solutions in all completions of the rationals, as claimed in the preamble to the statement of Theorem 1.1.

However, as a consequence of Lemma 7.1, any solution of (1.2) in rational integers must have $x_1 = x_2 = x_3 = x_4 = 0$. But then one has

$$7x_5^4 + 5x_6^4 + 3x_7^4 + x_8^4 + \dots + x_s^4 = 0.$$

Since the form on the left hand side of this equation is positive definite, we are forced to conclude that $x_j = 0$ for $1 \leq j \leq s$, whence (1.2) has no solution in the integers other than the trivial solution $\mathbf{x} = \mathbf{0}$. As claimed in the introduction, this shows that the Hasse principle fails for the system of equations (1.2).

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