SUBCONVEXITY IN INHOMOGENEOUS VINOGRADOV SYSTEMS

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ABSTRACT. When k and s are natural numbers and $\mathbf{h} \in \mathbb{Z}^k$, denote by $J_{s,k}(X;\mathbf{h})$ the number of integral solutions of the system

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = h_j \quad (1 \leqslant j \leqslant k),$$

with $1 \leqslant x_i, y_i \leqslant X$. When $s < k(k+1)/2$ and $(h_1, \ldots, h_{k-1}) \neq \mathbf{0}$, Brandes and Hughes have shown that $J_{s,k}(X;\mathbf{h}) = o(X^s)$. In this paper we improve on quantitative aspects of this result, and, subject to an extension of the main conjecture in Vinogradov's mean value theorem, we obtain an asymptotic formula for $J_{s,k}(X;\mathbf{h})$ in the critical case $s = k(k+1)/2$. The latter requires minor arc estimates going beyond square-root cancellation.

1. INTRODUCTION

In the analysis of Diophantine systems via the Hardy-Littlewood (circle) method, estimates are almost always limited by the convexity barrier, the most optimistic bound anticipated for error terms being given by the square-root of the number of choices for the variables available to the system. A recent exception to this rule involves inhomogeneous variants of Vinogradov's mean value theorem. When k and s are natural numbers and $\mathbf{h} \in \mathbb{Z}^k$, denote by $J_{s,k}(X;\mathbf{h})$ the number of integral solutions of the system

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = h_j \quad (1 \leqslant j \leqslant k), \tag{1.1}$$

with $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X$. Then Brandes and Hughes [4, Theorem 1] have shown that $J_{s,k}(X;\mathbf{h}) = o(X^s)$ when $s < k(k+1)/2$ and $h_j \neq 0$ for some index j with $j \leqslant k-1$. We emphasise that a consideration of diagonal solutions reveals that $J_{s,k}(X;\mathbf{0}) \gg X^s$, so one must certainly have $\mathbf{h} \neq \mathbf{0}$ in order to obtain a subconvex estimate for $J_{s,k}(X;\mathbf{h})$. Such estimates will also be inaccessible when $s \geqslant \frac{1}{2}k(k+1)+1$, since an averaging argument then confirms that there are numerous k-tuples $\mathbf{h}$ for which $J_{s,k}(X;\mathbf{h}) \gg X^{2s-k(k+1)/2} \gg X^{s+1}$.

Our goal in this paper is to sharpen the results of Brandes and Hughes both quantitatively, and in the range of s accessible to such conclusions. We seek also to establish an asymptotic formula for $J_{s,k}(X;\mathbf{h})$ in the critical case

2010 *Mathematics Subject Classification.* 11D45, 11L15, 11P05.
Key words and phrases. Subconvexity, Vinogradov's mean value theorem.
The author's work is supported by NSF grants DMS-2001549 and DMS-1854398.

$s = k(k+1)/2$, extending to exponents $k > 3$ our recent work [17] relevant to the cubic case, on the assumption of an extended version of the main conjecture in Vinogradov's mean value theorem. Our conclusions vary in type according to the regime of interest. We begin with the estimates simplest to state.

Theorem 1.1. *Suppose that $k \geqslant 3$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Let l be the smallest index with $h_l \neq 0$. Then, whenever $l < k$ and s is an integer with*

$$1 \leqslant s \leqslant \tfrac{1}{2}k(k+1) - \frac{k(k+1) - l(l+1)}{2(k-l)(k-l+1)}, \tag{1.2}$$

one has

$$J_{s,k}(X;\mathbf{h}) \ll X^{s-1/2+\varepsilon}. \tag{1.3}$$

In particular, this estimate holds when $1 \leqslant l \leqslant (k+1)/3$ and $s < k(k+1)/2$.

It may be illuminating to observe that the constraint (1.2) can be rewritten in the form

$$1 \leqslant s \leqslant \tfrac{1}{2}k(k+1) - \tfrac{1}{2} - \frac{l}{k-l+1}.$$

We note that [4, Corollary 2] obtains the estimate (1.3) in the shorter range

$$1 \leqslant l \leqslant k - \tfrac{1}{2}(\sqrt{2k^2+2k+1} - 1) \approx 0.29289k.$$

The case $k = 2$ is omitted from the statement of Theorem 1.1 because much stronger bounds are available in this case from the classical theory of quadratic polynomials. Thus, for example, the reader will have no difficulty in showing that when $\mathbf{h} \neq \mathbf{0}$, one has $J_{s,2}(X;\mathbf{h}) \ll X^{s-1+\varepsilon}$ $(s = 1, 2)$. Analogous estimates of similar strength to the latter may be obtained when $k \geqslant 3$ subject to suitable hypotheses concerning s and the k-tuple $\mathbf{h}$.

Theorem 1.2. *Let k be an integer with $k \geqslant 3$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Let l be the smallest index having the property that $h_l \neq 0$. Then, whenever $l < k$ and s is an integer with $1 \leqslant s \leqslant l(l+1)/2$, one has $J_{s,k}(X;\mathbf{h}) \ll X^{s-1+\varepsilon}$.*

The upper bound presented in this theorem saves a factor $X^{1-\varepsilon}$ beyond square-root cancellation, improving on the factor $X^{1/2-\varepsilon}$ visible in (1.3). Such a conclusion lies beyond any anticipated by Brandes and Hughes (see the discussion concluding [4]). Moreover, as we show in Theorem 7.1, there exist k-tuples $\mathbf{h} \neq \mathbf{0}$ having the property that $J_{s,k}(X;\mathbf{h}) \gg X^{s-1}$, so the conclusion of Theorem 1.2 is in some respects best possible.

Our next theorem shows that $J_{s,k}(X;\mathbf{h}) = o(X^s)$ whenever $s < \frac{1}{2}k(k+1)$ and $(h_1, \dots, h_{k-1}) \neq \mathbf{0}$, improving on an earlier result of [4].

Theorem 1.3. *Suppose that $k \geqslant 3$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Let l be the smallest index having the property that $h_l \neq 0$. Then whenever $l < k$ and s is a natural number satisfying $s < \frac{1}{2}k(k+1)$, one has*

$$J_{s,k}(X;\mathbf{h}) \ll X^{\varepsilon}\left(X^{s-1/2} + X^{s-\delta(s,k,l)}\right),$$

where

$$\delta(s,k,l) = \tfrac{1}{2}(k-l)(k-l+1)\left(\frac{k(k+1)-2s}{k(k+1)-l(l+1)}\right).$$

A conclusion analogous to that of Theorem 1.3 is obtained in [4, Theorem 1], though with the weaker exponent

$$\delta(s,k,l)=\tfrac{1}{2}(k-l)(k-l+1)\left(\frac{k(k+1)-2s}{k(k+1)}\right).$$

The conclusions of Theorems 1.1, 1.2 and 1.3 have nothing to say concerning $J_{s,k}(X;\mathbf{h})$ at the critical exponent $s=\frac{1}{2}k(k+1)$. Readers less familiar with the nuances of Vinogradov's mean value theorem may care to note in this context that when $s>\frac{1}{2}k(k+1)$, then an application of the circle method delivers an asymptotic formula of the shape $J_{s,k}(X;\mathbf{h})\sim C(\mathbf{h})X^{2s-k(k+1)/2}$, where $C(\mathbf{h})$ is positive provided that $\mathbf{h}$ satisfies appropriate local solubility conditions. Since the main term here is larger than the square-root of the number of available choices for the underlying variables, this situation with $s>\frac{1}{2}k(k+1)$ does not require subconvexity in its treatment. In contrast, when $s=\frac{1}{2}k(k+1)$, one requires subconvex minor arc estimates in order to show that the expected product of local densities delivers the anticipated asymptotic formula.

In recent work concerning the cubic case of the inhomogeneous Vinogradov system, the author applied the Hardy-Littlewood method to obtain an asymptotic formula for $J_{6,3}(X;\mathbf{h})$ when $h_1\neq 0$ (see [17, Theorem 1.1]). Moreover, when $h_1=0$ and $h_2\neq 0$, an asymptotic formula for $J_{6,3}(X;\mathbf{h})$ is obtained in [17, Theorem 1.2] provided that X is sufficiently large in terms of h_2. Both conclusions depend on minor arc estimates with better than square-root cancellation. When the degree k exceeds 3, such conclusions are beyond the reach of current technology. Nonetheless, by application of conjectural mean value estimates potentially within reach of efficient congruencing and decoupling methods, some progress is possible.

In order to describe the asymptotic formula associated with $J_{s,k}(X;\mathbf{h})$ at the critical point $s=k(k+1)/2$, we introduce some notation. We write $B_k(X;\mathbf{h})$ for $J_{k(k+1)/2,k}(X;\mathbf{h})$. Next, we introduce the generating functions

$$I(\boldsymbol{\beta})=\int_0^1 e(\beta_1\gamma+\ldots+\beta_k\gamma^k)\,\mathrm{d}\gamma \tag{1.4}$$

and

$$S(q,\mathbf{a})=\sum_{r=1}^{q} e_q(a_1r+\ldots+a_kr^k), \tag{1.5}$$

in which we write $e(z)$ for $e^{2\pi iz}$ and use $e_q(u)$ as shorthand for $e^{2\pi iu/q}$. Putting $n_j=h_jX^{-j}$ $(1\leqslant j\leqslant k)$, we define the *singular integral*

$$\mathfrak{J}_k(\mathbf{h})=\int_{\mathbb{R}^k}|I(\boldsymbol{\beta})|^{k(k+1)}e(-\boldsymbol{\beta}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\beta}, \tag{1.6}$$

in which $\boldsymbol{\beta}\cdot\mathbf{n}$ denotes $\beta_1n_1+\ldots+\beta_kn_k$. Finally, we define the *singular series*

$$\mathfrak{S}_k(\mathbf{h})=\sum_{q=1}^{\infty}\sum_{\substack{1\leqslant\mathbf{a}\leqslant q\\(q,a_1,\ldots a_k)=1}}\left|q^{-1}S(q,\mathbf{a})\right|^{k(k+1)}e_q(-\mathbf{a}\cdot\mathbf{h}). \tag{1.7}$$

We note that both the singular integral $\mathfrak{J}_k(\mathbf{h})$ and the singular series $\mathfrak{S}_k(\mathbf{h})$ are known to converge absolutely (see [1, Theorem 1] or [2, Theorem 3.7]).

Our progress is conditional on the extended main conjecture in Vinogradov's mean value theorem (Conjecture 8.1). Once again, the conclusion of the next theorem implicitly encodes a minor arc estimate beyond the convexity barrier.

Theorem 1.4. *Assume the extended main conjecture in Vinogradov's mean value theorem. Suppose that* $\mathbf{h} \in \mathbb{Z}^k$ *and* $h_l \neq 0$ *for some index* l *with* $1 \leqslant l < k$. *Then provided that* X *is sufficiently large in terms of* $\mathbf{h}$, *one has*

$$B_k(X;\mathbf{h}) = \mathfrak{J}_k(\mathbf{h})\mathfrak{S}_k(\mathbf{h})X^{k(k+1)/2} + o(X^{k(k+1)/2}),$$

in which $0 \leqslant \mathfrak{J}_k(\mathbf{h}) \ll 1$ *and* $0 \leqslant \mathfrak{S}_k(\mathbf{h}) \ll 1$.

We turn now to the topic of paucity and its relation to inhomogeneous Vinogradov systems. When the number of variables in the Vinogradov system (1.1) is small, one may obtain estimates for $J_{s,k}(X;\mathbf{h})$ far below the convexity barrier. That such should be possible is apparent from recent work of the author [15] concerning paucity in relatives of Vinogradov's mean value theorem. Consider, by way of an illustrative example, the system of equations

$$x_1^j + \ldots + x_k^j = y_1^j + \ldots + y_k^j \quad (1 \leqslant j \leqslant k,\ j \neq k-d), \tag{1.8}$$

in which $k \geqslant 4$ and the non-negative integer d is fixed. Let $I^*_{k,d}(X)$ denote the number of integral solutions of (1.8) with $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X$ in which $(x_1, \ldots, x_k)$ is not a permutation of $(y_1, \ldots, y_k)$. Then [15, Corollary 1.2] shows that when $d = o(k^{1/4})$, one has $I^*_{k,d}(X) \ll X^{(2+o(1))\sqrt{k}}$. However, should one have

$$\sum_{i=1}^k (x_i^{k-d} - y_i^{k-d}) = h_{k-d} \neq 0,$$

then $(x_1, \ldots, x_k)$ cannot be a permutation of $(y_1, \ldots, y_k)$. Thus we conclude that when $\mathbf{h} = (0, \ldots, 0, h_{k-d}, 0, \ldots, 0)$, with $h_{k-d} \neq 0$, then

$$J_{k,k}(X;\mathbf{h}) \ll X^{(2+o(1))\sqrt{k}}.$$

By elaborating on these ideas, non-trivial estimates may be obtained without restriction on d.

Theorem 1.5. *Suppose that* $k \geqslant 3$ *and* $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. *Suppose further that for some index* l *with* $2 \leqslant l \leqslant k$ *one has* $h_l \neq 0$, *but that* $h_j = 0$ *when* $j \neq l$ *and* $1 \leqslant j \leqslant k$. *Then one has* $J_{k,k}(X;\mathbf{h}) \ll X^{k-l+1+\varepsilon}$.

The conclusion of this theorem yields stronger bounds than any supplied by Theorems 1.1 and 1.2 when $l \geqslant 3$.

This paper is organised as follows. In §2 we adapt the author's work on the asymptotic formula in Waring's problem [11] to bound Fourier coefficients associated with the inhomogeneous Vinogradov system (1.1). This approach has a significant advantage over the corresponding analysis of [4], which is that the mean values of interest may be restricted to subsets of $[0,1)^k$, such as sets of minor arcs of use in applications of the Hardy-Littlewood method. We apply this method in combination with Hölder's inequality, relating $J_{s,k}(X;\mathbf{h})$ to mixed mean value estimates more efficient than the simple ones considered in [4]. These mixed mean values are examined in §3, preparing the ground in §4 for the proof of our simplest subconvex bounds described in Theorems 1.1, 1.2 and 1.3. Preparations for the proof of Theorem 1.4 are presented in §5, where we apply the extended main conjecture in Vinogradov's mean value theorem as the key input to provide subconvex minor arc estimates. The application of the Hardy-Littlewood method itself is described in §6, where the proof of Theorem 1.4 is completed. In §7 we explore the application of ideas from the theory of paucity to bounds for $J_{s,k}(X;\mathbf{h})$, and in particular we prove Theorem 1.5. Finally, in the appendix attached as §8, we discuss the extended main conjecture in Vinogradov's mean value theorem and its immediate applications to generalisations of small cap estimates.

Our basic parameter is X, a sufficiently large positive number. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε, k and s. We make liberal use of vector notation in the form $\mathbf{x} = (x_1, \ldots, x_r)$, the dimension r depending on the course of the argument. Thus, for example, we write $1 \leqslant \mathbf{x} \leqslant X$ to denote that every coordinate x_i of $\mathbf{x}$ satisfies $1 \leqslant x_i \leqslant X$. We also write $(a_1, \ldots, a_s)$ for the greatest common divisor of the integers $a_1, \ldots, a_s$, any ambiguity between ordered s-tuples and corresponding greatest common divisors being easily resolved by context. Finally, we write $\|\theta\|$ for $\min\{|\theta - m| : m \in \mathbb{Z}\}$.

The author is grateful to both Igor Shparlinski and the referee for useful comments.

2. Auxiliary mean values utilising shifts

We first establish a reasonably flexible mean value estimate by applying ideas underlying our recent work on the Hilbert-Kamke problem, as modified to handle the cubic case of the inhomogeneous Vinogradov system (see [16, Theorem 2.1] and [17, Lemma 2.1]). This argument has its genesis in earlier work of the author concerning the asymptotic formula in Waring's problem (see [10, Lemma 10.1] and [11, Theorem 2.1]). Define $f(\boldsymbol{\alpha};X) = f_k(\boldsymbol{\alpha};X)$ by

$$f_k(\boldsymbol{\alpha};X) = \sum_{1\leqslant x\leqslant X} e(\alpha_1 x + \ldots + \alpha_k x^k). \tag{2.1}$$

Then, when $\mathbf{h} \in \mathbb{Z}^k$ and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, we put

$$I_s(\mathfrak{B};X;\mathbf{h}) = \int_{\mathfrak{B}} \int_{[0,1)^{k-1}} |f_k(\boldsymbol{\alpha};X)|^{2s} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}, \tag{2.2}$$

in which $\mathrm{d}\boldsymbol{\alpha}$ denotes $\mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_k$. We have been urged to emphasise that the latter convention implies that here (and elsewhere in this paper), the integration with respect to the variable α_k occurs last, and hence

$$I_s(\mathfrak{B};X;\mathbf{h}) = \int_{\mathfrak{B}} \left(\int_0^1 \cdots \int_0^1 |f_k(\boldsymbol{\alpha};X)|^{2s} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{k-1} \right) \mathrm{d}\alpha_k.$$

Note that by orthogonality, one then has
$$J_{s,k}(X;\mathbf{h})=I_s([0,1);X;\mathbf{h}). \tag{2.3}$$
We also make use of the generating function $g(\boldsymbol{\alpha},\theta;X)=g_k(\boldsymbol{\alpha},\theta;X)$ defined by putting
$$g_k(\boldsymbol{\alpha},\theta;X)=\sum_{1\leqslant y\leqslant X} e\left(y\theta+\nu_2(y;\mathbf{h})\alpha_2+\ldots+\nu_k(y;\mathbf{h})\alpha_k\right), \tag{2.4}$$
in which
$$\nu_j(y;\mathbf{h})=\sum_{i=0}^{j-1}\binom{j}{i}h_{j-i}y^i\quad (1\leqslant j\leqslant k). \tag{2.5}$$

Lemma 2.1. *Suppose that $s\in\mathbb{N}$, $\mathbf{h}\in\mathbb{Z}^k$ and $\mathfrak{B}\subseteq\mathbb{R}$ is measurable. Then*
$$I_s(\mathfrak{B};X;\mathbf{h})\ll X^{-1}(\log X)^{2s}\sup_{\Gamma\in[0,1)}\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|f_k(\boldsymbol{\alpha};2X)^{2s}g_k(\boldsymbol{\alpha},\Gamma;X)|\,\mathrm{d}\boldsymbol{\alpha}.$$

Proof. The argument we present here is very similar to that underlying the proof of [17, Lemma 2.1], though there are sufficiently many differences that a full account seems warranted. We first reformulate the mean value $I_s(\mathfrak{B};X;\mathbf{h})$ defined in (2.2) in preparation for the exploitation of a shift in the underlying variables. Write $\psi(u;\boldsymbol{\theta})=\theta_1u+\ldots+\theta_ku^k$. Then, as in the analogous argument of [17, Lemma 2.1], it follows via orthogonality that for every integral shift y with $1\leqslant y\leqslant X$, one has
$$f_k(\boldsymbol{\alpha};X)=\int_0^1\mathfrak{f}_y(\boldsymbol{\alpha};\gamma)K(\gamma)\,\mathrm{d}\gamma,$$
where
$$\mathfrak{f}_y(\boldsymbol{\alpha};\gamma)=\sum_{1\leqslant x\leqslant 2X}e\left(\psi(x-y;\boldsymbol{\alpha})+\gamma(x-y)\right) \tag{2.6}$$
and
$$K(\gamma)=\sum_{1\leqslant z\leqslant X}e(-\gamma z).$$
Write
$$\mathfrak{F}_y(\boldsymbol{\alpha};\boldsymbol{\gamma})=\prod_{i=1}^s\mathfrak{f}_y(\boldsymbol{\alpha};\gamma_i)\mathfrak{f}_y(-\boldsymbol{\alpha};-\gamma_{s+i}), \tag{2.7}$$
$$\widetilde{K}(\boldsymbol{\gamma})=\prod_{i=1}^sK(\gamma_i)K(-\gamma_{s+i}),$$
and
$$\mathcal{I}(\boldsymbol{\gamma};y;\mathbf{h})=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}\mathfrak{F}_y(\boldsymbol{\alpha};\boldsymbol{\gamma})e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}. \tag{2.8}$$
Then we infer from (2.2) that
$$I_s(\mathfrak{B};X;\mathbf{h})=\int_{[0,1)^{2s}}\mathcal{I}(\boldsymbol{\gamma};y;\mathbf{h})\widetilde{K}(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma}. \tag{2.9}$$

Write $\mathrm{d}\boldsymbol{\alpha}_{k-1}$ as shorthand for $\mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{k-1}$. Then, by orthogonality, we discern from (2.7) that

$$\int_{[0,1)^{k-1}} \mathfrak{F}_y(\boldsymbol{\alpha};\boldsymbol{\gamma}) e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}_{k-1} = \sum_{1\leqslant \mathbf{x}\leqslant 2X} \Delta(\alpha_k,\boldsymbol{\gamma};\mathbf{h},y), \tag{2.10}$$

where $\Delta(\alpha_k,\boldsymbol{\gamma};\mathbf{h},y)$ is equal to

$$e\biggl(\sum_{i=1}^s \left(\alpha_k\left((x_i-y)^k-(x_{s+i}-y)^k\right)+(\gamma_i(x_i-y)-\gamma_{s+i}(x_{s+i}-y))\right)-\alpha_k h_k\biggr),$$

when

$$\sum_{i=1}^s \left((x_i-y)^j-(x_{s+i}-y)^j\right)=h_j \quad (1\leqslant j\leqslant k-1), \tag{2.11}$$

and otherwise $\Delta(\alpha_k,\boldsymbol{\gamma};\mathbf{h},y)$ is equal to 0.

By applying the binomial theorem within (2.11), we obtain the relations

$$\sum_{i=1}^s (x_i^j-x_{s+i}^j)=\sum_{l=0}^{j-1}\binom{j}{l}h_{j-l}y^l \quad (1\leqslant j\leqslant k-1),$$

and

$$\sum_{i=1}^s (x_i^k-x_{s+i}^k)=\sum_{l=1}^{k-1}\binom{k}{l}h_{k-l}y^l+\sum_{i=1}^s\left((x_i-y)^k-(x_{s+i}-y)^k\right).$$

Define

$$G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})=\sum_{1\leqslant y\leqslant X} e\biggl(-\sum_{j=1}^k \alpha_j\sum_{l=0}^{j-1}\binom{j}{l}h_{j-l}y^l-y\Gamma(\boldsymbol{\gamma})\biggr), \tag{2.12}$$

where

$$\Gamma(\boldsymbol{\gamma})=\sum_{i=1}^s(\gamma_i-\gamma_{s+i}).$$

Then we find from (2.7) and (2.10) that

$$\sum_{1\leqslant y\leqslant X}\int_{[0,1)^{k-1}} \mathfrak{F}_y(\boldsymbol{\alpha};\boldsymbol{\gamma})e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}_{k-1}=\int_{[0,1)^{k-1}}\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\alpha}_{k-1}.$$

On substituting this last relation into (2.8) and thence into (2.9), we obtain

$$\begin{aligned} I_s(\mathfrak{B};X;\mathbf{h}) &= \lfloor X\rfloor^{-1}\sum_{1\leqslant y\leqslant X}\int_{[0,1)^{2s}}\mathcal{I}(\boldsymbol{\gamma};y;\mathbf{h})\widetilde{K}(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma} \\ &\ll X^{-1}\int_{[0,1)^{2s}}|H(\boldsymbol{\gamma})\widetilde{K}(\boldsymbol{\gamma})|\,\mathrm{d}\boldsymbol{\gamma}, \end{aligned} \tag{2.13}$$

where

$$H(\boldsymbol{\gamma})=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\alpha}. \tag{2.14}$$

Note next from (2.1) and (2.6) that $\mathfrak{f}_0(\boldsymbol{\alpha};0)=f_k(\boldsymbol{\alpha};2X)$. Thus, from the elementary inequality $|z_1\cdots z_n|\leqslant |z_1|^n+\ldots+|z_n|^n$ and (2.7), we deduce that

$$|\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})|\leqslant \sum_{i=1}^{2s}|\mathfrak{f}_0(\boldsymbol{\alpha};\gamma_i)|^{2s}=\sum_{i=1}^{2s}|f_k(\alpha_k,\ldots,\alpha_2,\alpha_1+\gamma_i;2X)|^{2s}.$$

Moreover, in view of (2.5), we have $\nu_1(y;\mathbf{h})=h_1$, so that $\nu_1(y;\mathbf{h})$ is independent of y. Then, from (2.4), (2.5) and (2.12) we see that

$$|G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})|=|g_k(\boldsymbol{\alpha},\Gamma(\boldsymbol{\gamma});X)|.$$

Define

$$U_s(\mathfrak{B})=\sup_{\Gamma\in[0,1)}\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|f_k(\boldsymbol{\alpha};2X)^{2s}g_k(\boldsymbol{\alpha},\Gamma;X)|\,\mathrm{d}\boldsymbol{\alpha},$$

and observe that $|G(\boldsymbol{\alpha};\mathbf{h};\boldsymbol{\gamma})|$ is independent of α_1. Then a change of variable leads from (2.14) to the upper bound

$$|H(\boldsymbol{\gamma})|\leqslant \int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|f_k(\boldsymbol{\alpha};2X)^{2s}g_k(\boldsymbol{\alpha},\Gamma(\boldsymbol{\gamma});X)|\,\mathrm{d}\boldsymbol{\alpha}\leqslant U_s(\mathfrak{B}). \tag{2.15}$$

Recall next that

$$\int_0^1|K(\gamma)|\,\mathrm{d}\gamma\ll\int_0^1\min\{X,\|\gamma\|^{-1}\}\,\mathrm{d}\gamma\ll\log X.$$

Then we perceive from (2.13) and (2.15) that

$$I_s(\mathfrak{B};X;\mathbf{h})\ll X^{-1}U_s(\mathfrak{B})\biggl(\int_0^1|K(\gamma)|\,\mathrm{d}\gamma\biggr)^{2s}\ll X^{-1}(\log X)^{2s}U_s(\mathfrak{B}).$$

This completes the proof of the lemma. □

3. Mixed mean value estimates

In this section we derive mixed mean value estimates involving $f(\boldsymbol{\alpha};2X)$ and $g(\boldsymbol{\alpha},\Gamma;X)$. We apply these estimates in §4 to establish Theorems 1.1, 1.2 and 1.3. Throughout, we abbreviate $g_k(\boldsymbol{\alpha},0;X)$ to $g_k(\boldsymbol{\alpha};X)$.

Lemma 3.1. *Suppose that $k\geqslant 3$ and $\mathbf{h}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}$. Let l be the smallest index having the property that $h_l\neq 0$. Then whenever $l<k$, and u and r are non-negative integers with $u\leqslant l(l+1)/2$, one has*

$$\int_{[0,1)^k}|g_k(\boldsymbol{\alpha};X)^{2r}f_k(\boldsymbol{\alpha};2X)^{2u}|\,\mathrm{d}\boldsymbol{\alpha}\ll X^{u+\varepsilon}\left(X^r+X^{2r-(k-l)(k-l+1)/2}\right).$$

Proof. Suppose that u is an integer with $0\leqslant u\leqslant l(l+1)/2$, and define

$$I=\int_{[0,1)^k}|g_k(\boldsymbol{\alpha};X)^{2r}f_k(\boldsymbol{\alpha};2X)^{2u}|\,\mathrm{d}\boldsymbol{\alpha}.$$

Recall the definition (2.5) of the polynomial $\nu_j(y;\mathbf{h})$ $(1\leqslant j\leqslant k)$. Then, by orthogonality, the mean value I counts the integral solutions of the system

$$\sum_{i=1}^{u}(x_i^j-y_i^j)=\sum_{m=1}^{r}\left(\nu_j(w_m;\mathbf{h})-\nu_j(z_m;\mathbf{h})\right)\quad (1\leqslant j\leqslant k), \tag{3.1}$$

with $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant 2X$ and $1 \leqslant \mathbf{w}, \mathbf{z} \leqslant X$. Our hypothesis that $h_j = 0$ for $1 \leqslant j < l$ ensures that

$$\nu_j(w_m; \mathbf{h}) - \nu_j(z_m; \mathbf{h}) = 0 \quad (1 \leqslant j \leqslant l),$$

and so we deduce from (3.1) that

$$\sum_{i=1}^{u} (x_i^j - y_i^j) = 0 \quad (1 \leqslant j \leqslant l). \tag{3.2}$$

Since $u \leqslant l(l+1)/2$, we therefore deduce from the (now proven) main conjecture in Vinogradov's mean value theorem (see [3] and [13, 14]) that the number of choices I_0 for $\mathbf{x}$ and $\mathbf{y}$ satisfies

$$I_0 \leqslant J_{u,l}(2X; \mathbf{0}) \ll X^{u+\varepsilon}. \tag{3.3}$$

Fix any choice of $\mathbf{x}$ and $\mathbf{y}$ satisfying (3.2). Then by taking appropriate linear combinations of the equations (3.1), we find that there are integers $n_j = n_j(\mathbf{x}, \mathbf{y}; \mathbf{h})$ for which

$$\sum_{m=1}^{r} (w_m^j - z_m^j) = n_j(\mathbf{x}, \mathbf{y}; \mathbf{h}) \quad (1 \leqslant j \leqslant k - l). \tag{3.4}$$

For each fixed choice of $\mathbf{x}$ and $\mathbf{y}$, denote by $I_1 = I_1(\mathbf{x}, \mathbf{y})$ the number of solutions of the system (3.4) with $1 \leqslant \mathbf{z}, \mathbf{w} \leqslant X$. Then by orthogonality and the triangle inequality, we find that

$$\begin{aligned} I_1(\mathbf{x}, \mathbf{y}) &= \int_{[0,1)^{k-l}} |f_{k-l}(\boldsymbol{\alpha}; X)|^{2r} e(-\boldsymbol{\alpha} \cdot \mathbf{n}) \, \mathrm{d}\boldsymbol{\alpha} \\ &\leqslant \int_{[0,1)^{k-l}} |f_{k-l}(\boldsymbol{\alpha}; X)|^{2r} \, \mathrm{d}\boldsymbol{\alpha}. \end{aligned}$$

Thus, again applying the (now proven) main conjecture in Vinogradov's mean value theorem, we infer that

$$I_1(\mathbf{x}, \mathbf{y}) \leqslant J_{r,k-l}(X; \mathbf{0}) \ll X^{\varepsilon} \left(X^r + X^{2r-(k-l)(k-l+1)/2} \right).$$

On recalling (3.3), we therefore conclude that

$$I \leqslant I_0 \max_{\mathbf{x}, \mathbf{y}} I_1(\mathbf{x}, \mathbf{y}) \ll X^{u+\varepsilon} \left(X^r + X^{2r-(k-l)(k-l+1)/2} \right).$$

This completes the proof of the lemma. □

As an immediate consequence of the upper bound of Lemma 3.1, we record the following estimate in which $g_k(\boldsymbol{\alpha}, \Gamma; X)$ substitutes for $g_k(\boldsymbol{\alpha}; X)$.

Lemma 3.2. *Suppose that $k \geqslant 3$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Let l be the smallest index having the property that $h_l \neq 0$. Then whenever $l < k$, and u and r are non-negative integers with $u \leqslant l(l+1)/2$, one has*

$$\sup_{\Gamma \in [0,1)} \int_{[0,1)^k} |g_k(\boldsymbol{\alpha}, \Gamma; X)^{2r} f_k(\boldsymbol{\alpha}; 2X)^{2u}| \, \mathrm{d}\boldsymbol{\alpha} \ll X^{u+\varepsilon} \left(X^r + X^{2r-(k-l)(k-l+1)/2} \right).$$

Proof. The mean value

$$\Theta_{u,r}(\Gamma;X)=\int_{[0,1)^k}|g_k(\boldsymbol{\alpha},\Gamma;X)^{2r}f_k(\boldsymbol{\alpha};2X)^{2u}|\,\mathrm{d}\boldsymbol{\alpha}$$

counts the integral solutions $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}$ of the system (3.1) with weight

$$e\biggl(-\Gamma\sum_{m=1}^r(w_m-z_m)\biggr).$$

Since this weight is unimodular, we deduce via orthogonality that

$$\sup_{\Gamma\in[0,1)}\Theta_{u,r}(\Gamma;X)\leqslant\int_{[0,1)^k}|g_k(\boldsymbol{\alpha};X)^{2r}f_k(\boldsymbol{\alpha};2X)^{2u}|\,\mathrm{d}\boldsymbol{\alpha}.$$

The conclusion of the lemma is now immediate from that of Lemma 3.1. □

We next prepare for the proof of Theorem 1.2. When $s\in\mathbb{N}$, write

$$\Psi_s(\Gamma;X)=\int_{[0,1)^k}|f_k(\boldsymbol{\alpha};2X)^{2s}g_k(\boldsymbol{\alpha},\Gamma;X)|\,\mathrm{d}\boldsymbol{\alpha}. \tag{3.5}$$

Lemma 3.3. *Suppose that $k\geqslant 3$, and $\mathbf{h}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}$ satisfies the condition that $h_j=0$ for $j\leqslant k-2$, but $h_{k-1}\neq 0$. Then whenever s is an integer with $1\leqslant s\leqslant k(k-1)/2$, one has*

$$\sup_{\Gamma\in[0,1)}\Psi_s(\Gamma;X)\ll X^{s+\varepsilon}.$$

Proof. With the hypotheses on $\mathbf{h}$ available from the statement of the lemma, it follows from (2.4) and (2.5) that

$$\begin{aligned}|g_k(\boldsymbol{\alpha},\Gamma;X)|&=\biggl|\sum_{1\leqslant y\leqslant X}e\left((kh_{k-1}\alpha_k+\Gamma)y\right)\biggr|\\&\ll\min\{X,\|kh_{k-1}\alpha_k+\Gamma\|^{-1}\}.\end{aligned}$$

Since $|g_k(\boldsymbol{\alpha},\Gamma;X)|$ is independent of $\alpha_1,\dots,\alpha_{k-1}$, we therefore perceive via orthogonality that

$$\int_{[0,1)^{k-1}}|f_k(\boldsymbol{\alpha};2X)^{2s}g_k(\boldsymbol{\alpha},\Gamma;X)|\,\mathrm{d}\boldsymbol{\alpha}_{k-1}\ll T\min\{X,\|kh_{k-1}\alpha_k+\Gamma\|^{-1}\}, \tag{3.6}$$

where

$$T=\int_{[0,1)^{k-1}}|f_k(\boldsymbol{\alpha};2X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}_{k-1}$$

counts the number of integral solutions of the system of equations

$$\sum_{i=1}^s(x_i^j-y_i^j)=0\quad(1\leqslant j\leqslant k-1),$$

with $1\leqslant\mathbf{x},\mathbf{y}\leqslant X$, each solution being counted with weight

$$e\biggl(\alpha_k\sum_{i=1}^s(x_i^k-y_i^k)\biggr). \tag{3.7}$$

Since the weight (3.7) is unimodular, we see that $T \ll J_{s,k-1}(X;\mathbf{0})$. From the (now proven) main conjecture in Vinogradov's mean value theorem, we thus deduce from (3.5) and (3.6) via a change of variables that

$$\begin{aligned}\Psi_s(\Gamma;X) &\ll X^{s+\varepsilon}\int_0^1 \min\{X, \|kh_{k-1}\alpha_k+\Gamma\|^{-1}\}\,\mathrm{d}\alpha_k\\ &= X^{s+\varepsilon}\int_0^1 \min\{X,\|\beta\|^{-1}\}\,\mathrm{d}\beta.\end{aligned}$$

Thus we conclude that

$$\sup_{\Gamma\in[0,1)} \Psi_s(\Gamma;X) \ll X^{s+\varepsilon}\log X \ll X^{s+2\varepsilon},$$

and the proof of the lemma is complete. □

4. The simplest subconvex bounds

We now attend to the matter of converting the auxiliary estimates of §3, using the apparatus prepared in §2, so as to establish Theorems 1.1, 1.2 and 1.3. We begin with the proof of Theorems 1.1 and 1.3. We should emphasise here that our formulation of Lemma 2.1, which we will shortly wield in earnest, bounds $I_s(\mathfrak{B};X;\mathbf{h})$ for any measurable set $\mathfrak{B}$. In the proofs of Theorems 1.1, 1.2 and 1.3, we make use of this lemma only when $\mathfrak{B}=[0,1)$. In such circumstances, one could do away with the elaborate arguments employed in the proof of Lemma 2.1, making do only with simple arguments counting solutions of Diophantine systems. Indeed, this was the approach taken in the proof of [10, Lemma 10.1]. We will, however, need the full force of Lemma 2.1 in §5, and we express the hope that the greater flexibility of this lemma may inspire future refinement even to the results established in the present section.

The proof of Theorems 1.1 and 1.3. We suppose that $k \geqslant 3$, $\mathbf{h}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}$, and that l is the smallest index with $1\leqslant l\leqslant k$ having the property that $h_l\neq 0$. The hypotheses of Theorems 1.1 and 1.3 then permit us the assumption that $l<k$. On recalling (2.3), we see that Lemma 2.1 delivers the bound

$$J_{s,k}(X;\mathbf{h}) \ll X^{-1}(\log X)^{2s}\sup_{\Gamma\in[0,1)} V_s(\Gamma), \tag{4.1}$$

where

$$V_s(\Gamma)=\int_{[0,1)^k} |f_k(\boldsymbol{\alpha};2X)^{2s} g_k(\boldsymbol{\alpha},\Gamma;X)|\,\mathrm{d}\boldsymbol{\alpha}. \tag{4.2}$$

We pursue two different analyses of the mean value (4.2), the first of which delivers Theorem 1.1, and the second Theorem 1.3. Write $R=(k-l)(k-l+1)$,

$$u=\min\{sR, l(l+1)/2\} \quad\text{and}\quad v=\frac{s-u/R}{1-1/R}. \tag{4.3}$$

Then it follows from an application of Hölder's inequality in (4.2) that

$$V_s(\Gamma)\leqslant W_{u,1}(\Gamma)^{1/R} W_{v,2}^{1-1/R}, \tag{4.4}$$

where

$$W_{u,1}(\Gamma) = \int_{[0,1)^k} |g_k(\boldsymbol{\alpha}, \Gamma; X)^R f_k(\boldsymbol{\alpha}; 2X)^{2u}| \,\mathrm{d}\boldsymbol{\alpha} \tag{4.5}$$

and

$$W_{v,2} = \int_{[0,1)^k} |f_k(\boldsymbol{\alpha}; 2X)|^{2v} \,\mathrm{d}\boldsymbol{\alpha}. \tag{4.6}$$

Since $R = (k-l)(k-l+1)$ is an even integer and $1 \leqslant u \leqslant l(l+1)/2$, an application of Lemma 3.2 to (4.5) reveals that

$$\sup_{\Gamma\in[0,1)} W_{u,1}(\Gamma) \ll X^{u+\frac{1}{2}R+\varepsilon}. \tag{4.7}$$

Moreover, by applying the (now proven) main conjecture in Vinogradov's mean value theorem, it follows from (4.6) that

$$W_{v,2} \ll X^{\varepsilon}(X^v + X^{2v-k(k+1)/2}). \tag{4.8}$$

We therefore deduce from (4.3) and (4.4) that

$$\begin{aligned} V_s(\Gamma) &\ll X^{\varepsilon} \left(X^{u+R/2}\right)^{1/R} \left(X^v + X^{2v-k(k+1)/2}\right)^{1-1/R} \\ &\ll X^{s+\frac{1}{2}+\theta+\varepsilon}, \end{aligned}$$

where

$$\theta = (1-1/R)\max\{0, v - k(k+1)/2\}.$$

By substituting this estimate for $V_s(\Gamma)$ into (4.1), we conclude thus far that

$$J_{s,k}(X; \mathbf{h}) \ll X^{s-\frac{1}{2}+\theta+\varepsilon}. \tag{4.9}$$

One has $\theta = 0$ when $v \leqslant k(k+1)/2$, and by (4.3) such is the case so long as

$$s \leqslant \frac{u}{R} + \tfrac{1}{2}k(k+1)(1-1/R),$$

a constraint guaranteed to hold provided that the hypothesis (1.2) is in force. The conclusion (1.3) of Theorem 1.1 therefore follows at once from (4.9) in this situation in which $\theta = 0$. Notice here that

$$\tfrac{1}{2}k(k+1) - \frac{k(k+1) - l(l+1)}{2(k-l)(k-l+1)} \leqslant \tfrac{1}{2}k(k+1) - 1$$

if and only if $2(k-l)(k-l+1) \leqslant k(k+1) - l(l+1)$, confirming that the condition (1.2) is met for all $s < k(k+1)/2$ whenever $(k+1-3l)(k-l) \leqslant 0$. This confirms that the estimate (1.3) does indeed hold, when $s < k(k+1)/2$, provided that $1 \leqslant l \leqslant (k+1)/3$, completing the proof of Theorem 1.1.

We now turn to the proof of Theorem 1.3. Here, in view of the conclusion of Theorem 1.1, we already have the bound $J_{s,k}(X; \mathbf{h}) \ll X^{s-1/2+\varepsilon}$ when the constraint (1.2) is in force. We may therefore suppose henceforth that

$$\tfrac{1}{2}k(k+1) - \frac{k(k+1) - l(l+1)}{2R} < s < \tfrac{1}{2}k(k+1),$$

whence

$$\frac{k(k+1) - 2s}{k(k+1) - l(l+1)} < \frac{1}{R}. \tag{4.10}$$

Write

$$u=\tfrac{1}{2}l(l+1),\quad v=\tfrac{1}{2}k(k+1)\quad\text{and}\quad a=\frac{k(k+1)-2s}{2(v-u)},$$

and recall the notation introduced in (4.5) and (4.6). Then since it follows from (4.10) that $0\leqslant a<1/R$, an application of Hölder's inequality in (4.2) shows in this situation that

$$V_s(\Gamma)\leqslant\Bigl(\sup_{\boldsymbol{\alpha}\in[0,1)^k}|g_k(\boldsymbol{\alpha},\Gamma;X)|\Bigr)^{1-Ra}W_{u,1}(\Gamma)^aW_{v,2}^{1-a}.$$

Thus, we deduce from (4.7) and (4.8) that

$$\begin{aligned}\sup_{\Gamma\in[0,1)}V_s(\Gamma)&\ll X^{1-Ra+\varepsilon}\left(X^{u+R/2}\right)^a(X^v)^{1-a}\\&\ll X^{s+1-\frac{1}{2}Ra+\varepsilon}.\end{aligned}$$

We therefore conclude from (4.1) that in this scenario, one has

$$J_{s,k}(X;\mathbf{h})\ll X^{s-\delta(s,k,l)+\varepsilon},$$

where

$$\delta(s,k,l)=\tfrac{1}{2}Ra=\tfrac{1}{2}(k-l)(k-l+1)\frac{k(k+1)-2s}{k(k+1)-l(l+1)}.$$

This completes the proof of Theorem 1.3. □

We complete this section by establishing Theorem 1.2, exploiting the fact that when $h_j=0$ for $1\leqslant j\leqslant k-2$ and $h_{k-1}\neq 0$, then the generating function $g_k(\boldsymbol{\alpha},\theta;X)$ is a linear exponential sum in the underlying variable y.

The proof of Theorem 1.2. We begin with a preliminary simplification. We work under the hypotheses of the statement of Theorem 1.2, and consider the $(l+1)$-tuple $\mathbf{h}'=(h_1,\dots,h_{l+1})$, in which we may assume that $h_j=0$ for $1\leqslant j<l$ but $h_l\neq 0$. By discarding the equations in (1.1) of degree exceeding $l+1$, we see that $J_{s,k}(X;\mathbf{h})\leqslant J_{s,l+1}(X;\mathbf{h}')$. But when $1\leqslant s\leqslant l(l+1)/2$, it follows from (2.3) by applying Lemma 2.1 with $l+1$ in place of k that

$$J_{s,l+1}(X;\mathbf{h}')\ll X^{-1}(\log X)^{2s}\sup_{\Gamma\in[0,1)}\int_{[0,1)^{l+1}}|f_{l+1}(\boldsymbol{\alpha};2X)^{2s}g_{l+1}(\boldsymbol{\alpha},\Gamma;X)|\,\mathrm{d}\boldsymbol{\alpha}.$$

Hence, by availing ourselves of Lemma 3.3 we obtain the upper bound

$$J_{s,k}(X;\mathbf{h})\leqslant J_{s,l+1}(X;\mathbf{h}')\ll X^{s-1+\varepsilon}.$$

The conclusion of Theorem 1.2 follows at once. □

5. A conditional asymptotic formula, I: minor arcs

Our goal in this section is to indicate how, equipped with mean value estimates conjectured to hold that fall short of breaking the convexity barrier, one may achieve subconvex minor arc estimates that deliver asymptotic formulae for $J_{s,k}(X;\mathbf{h})$ when $s=k(k+1)/2$. Thereby, we prove Theorem 1.4.

We begin by extracting from §8 an estimate sufficient for our purposes. Here, in order to simplify our exposition, we introduce some notation. When A is a fixed positive number and $0 \leqslant j \leqslant k$, we denote by $\mathfrak{U}_j = \mathfrak{U}_j(A)$ the interval $\mathfrak{U}_j(A) = [-AX^{-j}, AX^{-j}]$. Then, when $0 \leqslant m \leqslant k$, we write

$$\mathfrak{V}_m(A) = \mathfrak{U}_m(A) \times \mathfrak{U}_{m-1}(A) \times \cdots \times \mathfrak{U}_1(A) \times [0,1)^{k-m}.$$

In what follows, we shall refer to Conjecture 8.1 as the *extended main conjecture in Vinogradov's mean value theorem.*

Lemma 5.1. *Assume the extended main conjecture in Vinogradov's mean value theorem. Let A be a fixed positive number. Suppose that s is a positive number and m is an integer satisfying $0 \leqslant m \leqslant k$. Then, if either*

$$s \geqslant \tfrac{1}{4}k(k+1)+1 \quad \text{or} \quad m(m+1) \leqslant \tfrac{1}{2}k(k+1)-2,$$

one has

$$\int_{\mathfrak{V}_m(A)} |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon}(X^{s-m(m+1)/2} + X^{2s-k(k+1)/2}).$$

Proof. It follows from the extended main conjecture in Vinogradov's mean value theorem that when either

$$s \geqslant \tfrac{1}{4}k(k+1)+1 \quad \text{or} \quad \mathrm{mes}(\mathfrak{V}_m(A)) \gg X^{1-k(k+1)/4},$$

one has

$$\int_{\mathfrak{V}_m(A)} |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon}(X^{s}\mathrm{mes}(\mathfrak{V}_m(A)) + X^{2s-k(k+1)/2}).$$

Such is immediate in the first case from Conjecture 8.1, and in the second case from Conjecture 8.2, which as explained in §8 is a consequence of Conjecture 8.1. The upper bound presented in the lemma follows on observing that one has $\mathrm{mes}(\mathfrak{V}_m(A)) \ll_A X^{-m(m+1)/2}$. □

We first apply this estimate to obtain a bound for a mixed mean value.

Lemma 5.2. *Assume the extended main conjecture in Vinogradov's mean value theorem, and suppose that $k \geqslant 2$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Let l be the smallest index with $l \leqslant k$ for which $h_l \neq 0$, and write $v = (k-l)(k-l+1)/2$. Then, provided that s is a natural number with $s - v \geqslant \frac{1}{4}k(k+1)+1$, one has*

$$\int_{[0,1)^k} |f_k(\boldsymbol{\alpha};2X)^{2s-2v} g_k(\boldsymbol{\alpha};X)^{2v}|\,\mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon}(X^{s} + X^{2s-k(k+1)/2}). \tag{5.1}$$

Proof. It follows from orthogonality that the mean value I on the left hand side of (5.1) counts the number of integral solutions of the system of equations

$$\sum_{i=1}^{s-v}(x_i^j - y_i^j) = \sum_{m=1}^{v}(\nu_j(z_m;\mathbf{h}) - \nu_j(t_m;\mathbf{h})) \quad (1 \leqslant j \leqslant k), \tag{5.2}$$

with $1 \leqslant \mathbf{x},\mathbf{y} \leqslant 2X$ and $1 \leqslant \mathbf{z},\mathbf{t} \leqslant X$. When $1 \leqslant j \leqslant k$ and $n_j \in \mathbb{Z}$, denote by $\rho(\mathbf{n})$ the number of solutions of the system of equations

$$\sum_{m=1}^{v}(\nu_j(z_m;\mathbf{h}) - \nu_j(t_m;\mathbf{h})) = n_j \quad (1 \leqslant j \leqslant k),$$

with $1 \leqslant \mathbf{z}, \mathbf{t} \leqslant X$. Then, by orthogonality, one has

$$\rho(\mathbf{n}) = \int_{[0,1)^k} |g_k(\boldsymbol{\alpha}; X)|^{2v} e(-\boldsymbol{\alpha} \cdot \mathbf{n}) \, \mathrm{d}\boldsymbol{\alpha}.$$

Since the hypotheses of the lemma imply that $h_j = 0$ for $1 \leqslant j < l$, and $h_l \neq 0$, we find from (2.5) via the triangle inequality and a change of variables that

$$\rho(\mathbf{n}) \leqslant \int_{[0,1)^k} |g_k(\boldsymbol{\alpha}; X)|^{2v} \, \mathrm{d}\boldsymbol{\alpha} = \int_{[0,1)^{k-l}} |f_{k-l}(\boldsymbol{\alpha}; X)|^{2v} \, \mathrm{d}\boldsymbol{\alpha}. \tag{5.3}$$

In order to explain the origin of the rightmost mean value in (5.3), observe that since $h_j = 0$ for $1 \leqslant j < l$, the polynomial $\nu_j(z; \mathbf{h})$ is non-zero only when $j \geqslant l$, in which case its leading term is $\binom{j}{l} h_l z^{j-l}$. Thus, the first integral in (5.3) counts the number of integral solutions of the system of equations

$$\sum_{m=1}^{v} (z_m^j - t_m^j) = 0 \quad (1 \leqslant j \leqslant k - l),$$

with $1 \leqslant \mathbf{z}, \mathbf{t} \leqslant X$. By orthogonality, the second integral in (5.3) counts precisely these solutions, justifying the conclusion.

We thus have $\rho(\mathbf{n}) \leqslant J_{k-l,v}(X; \mathbf{0})$. Hence, by the (now confirmed) main conjecture in Vinogradov's mean value theorem, we deduce that $\rho(\mathbf{n}) \ll X^{v+\varepsilon}$. We now return to (5.2) and note that when $1 \leqslant \mathbf{z}, \mathbf{t} \leqslant X$, one has

$$\sum_{m=1}^{v} (\nu_j(z_m; \mathbf{h}) - \nu_j(t_m; \mathbf{h})) \ll_{\mathbf{h}} X^{j-l} \quad (l + 1 \leqslant j \leqslant k)$$

and

$$\sum_{m=1}^{v} (\nu_j(z_m; \mathbf{h}) - \nu_j(t_m; \mathbf{h})) = 0 \quad (1 \leqslant j \leqslant l).$$

Hence, in each solution $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}$ counted by I, there are positive numbers $C_j(\mathbf{h})$ for which

$$\left| \sum_{i=1}^{s-v} (x_i^j - y_i^j) \right| \leqslant C_j(\mathbf{h}) X^{j-l} \quad (l + 1 \leqslant j \leqslant k) \tag{5.4}$$

and

$$\sum_{i=1}^{s-v} (x_i^j - y_i^j) = 0 \quad (1 \leqslant j \leqslant l). \tag{5.5}$$

Let I_1 denote the number of integral solutions of the system (5.4) and (5.5) with $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant 2X$. Then we deduce that

$$I \leqslant I_1 \max_{\mathbf{n}} \rho(\mathbf{n}) \ll X^{v+\varepsilon} I_1. \tag{5.6}$$

Next we examine the system (5.4) and (5.5). A standard argument (see for example [9, Lemma 2.1]) shows that

$$I_1 \ll \left(\prod_{j=l+1}^{k} C_j(\mathbf{h}) X^{j-l} \right) \int_{\mathfrak{V}_{k-l}(A)} |f_k(\boldsymbol{\alpha}; X)|^{2s-2v} \, \mathrm{d}\boldsymbol{\alpha},$$

in which $\mathfrak{V}_{k-l}(A)$ is defined as in the preamble to the statement of Lemma 5.1, and $A = \max\{C_{l+1}(\mathbf{h}), \dots, C_k(\mathbf{h})\}$. Thus we deduce that

$$I_1 \ll_{\mathbf{h}} X^v \int_{\mathfrak{V}_{k-l}(A)} |f_k(\boldsymbol{\alpha}; X)|^{2s-2v}\, \mathrm{d}\boldsymbol{\alpha},$$

whence, in view of (5.6),

$$I \ll X^{2v+\varepsilon} \int_{\mathfrak{V}_{k-l}(A)} |f_k(\boldsymbol{\alpha}; X)|^{2s-2v}\, \mathrm{d}\boldsymbol{\alpha}.$$

We now invoke Lemma 5.1 to bound the mean value on the right hand side here. On noting that $\mathrm{mes}(\mathfrak{V}_{k-l}(A)) \ll X^{-v}$, we thus deduce that

$$I \ll X^{2v+\varepsilon}(X^{s-2v} + X^{2s-2v-k(k+1)/2}) \ll X^{\varepsilon}(X^s + X^{2s-k(k+1)/2}).$$

This completes the proof of the lemma. □

We apply this estimate in combination with Lemma 2.1 to obtain an acceptable minor arc bound of use in our application of the Hardy-Littlewood method. When Q is a real parameter with $1 \leqslant Q \leqslant X$, we define the set of major arcs $\mathfrak{M}(Q)$ to be the union of the arcs

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant QX^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$. We then define the complementary set of minor arcs $\mathfrak{m}(Q) = [0,1) \setminus \mathfrak{M}(Q)$.

Lemma 5.3. *Assume the extended main conjecture in Vinogradov's mean value theorem, and suppose that $k \geqslant 3$ and $\mathbf{h} \in \mathbb{Z}^k$. Let s be a natural number with $s \geqslant k(k+1)/2$, and put $\delta = 2/(k^2(k-1)^2)$. Then provided that $h_l \neq 0$ for some index l with $1 \leqslant l < k$, one has*

$$I_s(\mathfrak{m}(Q); X; \mathbf{h}) \ll X^{2s-\frac{1}{2}k(k+1)+\varepsilon} Q^{-\delta}.$$

Proof. We find from Lemma 2.1 that

$$I_s(\mathfrak{m}(Q); X; \mathbf{h}) \ll X^{\varepsilon-1} \sup_{\Gamma \in [0,1)} V_s(X; \mathbf{h}), \tag{5.7}$$

where

$$V_s(X; \mathbf{h}) = \int_{\mathfrak{m}(Q)} \int_{[0,1)^{k-1}} |f_k(\boldsymbol{\alpha}; 2X)^{2s} g_k(\boldsymbol{\alpha}, \Gamma; X)|\, \mathrm{d}\boldsymbol{\alpha}.$$

Write $v = (k-l)(k-l+1)/2$ and $u = k(k+1)/2$, and put

$$\omega_0 = 2s - k(k+1) + 1/v, \quad \omega_1 = 1 - 1/(2v) \quad \text{and} \quad \omega_2 = 1/(2v).$$

Then for $s \geqslant k(k+1)/2$, an application of Hölder's inequality yields the bound

$$V_s(X; \mathbf{h}) \leqslant V_0^{\omega_0} V_1^{\omega_1} V_2^{\omega_2}, \tag{5.8}$$

where

$$V_0 = \sup_{\boldsymbol{\alpha}\in[0,1)^{k-1}\times\mathfrak{m}(Q)} |f_k(\boldsymbol{\alpha};2X)|,$$

$$V_1 = \int_{[0,1)^k} |f_k(\boldsymbol{\alpha};2X)|^{2u}\,\mathrm{d}\boldsymbol{\alpha},$$

$$V_2 = \int_{[0,1)^k} |f_k(\boldsymbol{\alpha};2X)^{2u-2} g_k(\boldsymbol{\alpha},\Gamma;X)^{2v}|\,\mathrm{d}\boldsymbol{\alpha}.$$

It follows from [16, Lemma 2.2] that $V_0 \ll X^{1+\varepsilon}Q^{-\sigma}$, where $\sigma = 1/(k(k-1))$. Also, from the (now confirmed) main conjecture in Vinogradov's mean value theorem, we have $V_1 \ll X^{u+\varepsilon}$, whilst Lemma 5.2 delivers the conditional bound $V_2 \ll X^{2v+u-2+\varepsilon}$ by the now familiar routine. Thus we deduce from (5.8) that

$$V_s(X;\mathbf{h}) \ll X^{2s+1-u+\varepsilon}Q^{-\omega_0\sigma} \ll X^{2s+1-\frac{1}{2}k(k+1)+\varepsilon}Q^{-\delta}, \tag{5.9}$$

where

$$\delta = \frac{1}{vk(k-1)} \geqslant \frac{2}{k^2(k-1)^2}.$$

The proof of the lemma is completed by substituting (5.9) into (5.7). □

6. A conditional asymptotic formula, II: the endgame

Equipped now with the conditional minor arc estimate supplied by Lemma 5.3, our proof of Theorem 1.4 follows the argument applied in our previous work [17, §§5-7] concerning the cubic case of the inhomogeneous Vinogradov system. There are few if any complications. When $\mathfrak{A} \subseteq [0,1)^k$ is measurable, we define the mean value $T_s(\mathfrak{A}) = T_s(\mathfrak{A};X;\mathbf{h})$ by putting

$$T_s(\mathfrak{A};X;\mathbf{h}) = \int_{\mathfrak{A}} |f_k(\boldsymbol{\alpha};X)|^{2s} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha}. \tag{6.1}$$

We formulate Hardy-Littlewood dissections of the unit cube $[0,1)^k$ suitable for our purpose. When Z is a real parameter with $1 \leqslant Z \leqslant X$, we define the set of major arcs $\mathfrak{K}(Z)$ to be the union of the arcs

$$\mathfrak{K}(q,\mathbf{a};Z) = \{\boldsymbol{\alpha} \in [0,1)^k : |\alpha_j - a_j/q| \leqslant ZX^{-j}\ (1 \leqslant j \leqslant k)\},$$

with $1 \leqslant q \leqslant Z$, $0 \leqslant a_j \leqslant q$ $(1 \leqslant j \leqslant k)$ and $(q,a_1,\dots,a_k) = 1$. We then define the complementary set of minor arcs $\mathfrak{k}(Z) = [0,1)^k \setminus \mathfrak{K}(Z)$.

Recall the one-dimensional Hardy-Littlewood dissection of $[0,1)$ into sets of major arcs $\mathfrak{M}(Q)$ and minor arcs $\mathfrak{m}(Q)$ introduced in the preamble to Lemma 5.3. We now fix $L = X^{1/(8k^2)}$ and $Q = L^k$, and we define a k-dimensional set of arcs by taking $\mathfrak{N} = \mathfrak{K}(Q^2)$ and $\mathfrak{n} = \mathfrak{k}(Q^2)$. This intermediate Hardy-Littlewood dissection may be refined to obtain the narrow set of major arcs $\mathfrak{P} = \mathfrak{K}(L)$ and the corresponding set of minor arcs $\mathfrak{p} = \mathfrak{k}(L)$. It is useful then to write $\mathfrak{P}(q,\mathbf{a}) = \mathfrak{K}(q,\mathbf{a};L)$. One readily confirms that $\mathfrak{P} \subseteq [0,1)^{k-1} \times \mathfrak{M}$, and hence the set of points $(\alpha_1,\ldots,\alpha_k)$ lying in $[0,1)^k$ may be partitioned into the four disjoint subsets

$$\begin{aligned}\mathfrak{W}_1 &= [0,1)^{k-1}\times \mathfrak{m},\\ \mathfrak{W}_2 &= \big([0,1)^{k-1}\times \mathfrak{M}\big)\cap \mathfrak{n},\\ \mathfrak{W}_3 &= \big([0,1)^{k-1}\times \mathfrak{M}\big)\cap (\mathfrak{N}\setminus \mathfrak{P}),\\ \mathfrak{W}_4 &= \mathfrak{P}.\end{aligned}$$

Thus, in view of (2.3) and (6.1), one sees that

$$J_{s,k}(X;\mathbf{h})=T_s([0,1)^k)=\sum_{i=1}^4 T_s(\mathfrak{W}_i). \tag{6.2}$$

We assume throughout the extended main conjecture in Vinogradov's mean value theorem. Then by substituting $Q=X^{1/(8k)}$ into Lemma 5.3, we deduce that when $\mathbf{h}\in\mathbb{Z}^k$ and $h_l\ne 0$ for some index l with $1\leqslant l<k$, one has

$$T_s(\mathfrak{W}_1)=I_s(\mathfrak{m}(Q);X;\mathbf{h})\ll X^{2s-\frac{1}{2}k(k+1)-1/(4k^5)}. \tag{6.3}$$

Our definition of the sets of arcs $\mathfrak{N}$, $\mathfrak{n}$, $\mathfrak{P}$ and $\mathfrak{p}$ in the present memoir is identical with that employed in [16, §§3-6]. Thus, the analysis applied in [16, §4] may be employed without material alteration in present circumstances to obtain the upper bound

$$T_s(\mathfrak{W}_2)\ll X^{2s-\frac{1}{2}k(k+1)-1/(16k)}. \tag{6.4}$$

Likewise, the analysis of [16, §5] applies, mutatis mutandis, to reveal that

$$T_s(\mathfrak{W}_3)\ll X^{2s-\frac{1}{2}k(k+1)-1/(12k^3)}. \tag{6.5}$$

Finally, the discussion of [16, §6] provides a template for the analysis of the major arcs in the present circumstances that may be applied almost without modification. Recall the definitions (1.4) and (1.5), and in addition the notational device of writing $n_j=h_jX^{-j}$ $(1\leqslant j\leqslant k)$. Then one finds that when $2s>\frac{1}{2}k(k+1)+2$, one has

$$T_s(\mathfrak{W}_4)=T_s(\mathfrak{P})=\mathfrak{S}_{s,k}(\mathbf{h})\mathfrak{J}_{s,k}(\mathbf{h})X^{2s-k(k+1)/2}+o(X^{2s-k(k+1)/2}), \tag{6.6}$$

where

$$\mathfrak{J}_{s,k}(\mathbf{h})=\int_{\mathbb{R}^k}|I(\boldsymbol{\beta})|^{2s}e(-\boldsymbol{\beta}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\beta} \tag{6.7}$$

and

$$\mathfrak{S}_{s,k}(\mathbf{h})=\sum_{q=1}^\infty \sum_{\substack{1\leqslant \mathbf{a}\leqslant q\\ (q,a_1,\ldots,a_k)=1}} |q^{-1}S(q,\mathbf{a})|^{2s}e_q(-\mathbf{a}\cdot\mathbf{h}). \tag{6.8}$$

By substituting the relations (6.3) to (6.6) into (6.2), we conclude that

$$J_{s,k}(X;\mathbf{h})=\mathfrak{S}_{s,k}(\mathbf{h})\mathfrak{J}_{s,k}(\mathbf{h})X^{2s-k(k+1)/2}+o(X^{2s-k(k+1)/2}).$$

Take $s=k(k+1)/2$, and recall the notation (1.6) and (1.7). Then we obtain the relation

$$B_k(X;\mathbf{h})=\mathfrak{S}_k(\mathbf{h})\mathfrak{J}_k(\mathbf{h})X^{k(k+1)/2}+o(X^{k(k+1)/2}),$$

confirming the principal conclusion of Theorem 1.4.

The observation that $0 \leqslant \mathfrak{J}_k(\mathbf{h}) \ll 1$ and $0 \leqslant \mathfrak{S}_k(\mathbf{h}) \ll 1$ follows from the absolute convergence of $\mathfrak{S}_{s,k}(\mathbf{h})$ and $\mathfrak{J}_{s,k}(\mathbf{h})$ when $2s > \frac{1}{2}k(k+1)+2$, combined with the standard theory of the singular series and singular integral described in [2, Theorem 3.7]. This completes the proof of Theorem 1.4.

7. Paucity and subconvexity

Our objective in this section is, not only to establish Theorem 1.5, but also to illustrate the role that paucity phenomena play in subconvexity results associated with inhomogeneous Vinogradov systems. This circle of ideas is relevant in investigations of $J_{s,k}(X;\mathbf{h})$ when s is small, which is to say, no larger than k or thereabouts. We begin with two almost trivial observations. The first shows that one cannot in general expect to obtain upper bounds in which one saves more than a factor X over the convexity limited estimates exhibiting square-root cancellation.

Theorem 7.1. *Suppose that $s,k \in \mathbb{N}$. Then*

$$\max_{\mathbf{h}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}} J_{s,k}(X;\mathbf{h}) \gg X^{s-1}.$$

Proof. We fix integers a and b with $a > b$, say $a = 2$ and $b = 1$, and then fix $h_j = a^j - b^j$ $(1 \leqslant j \leqslant k)$. Thus we have $\mathbf{h} \neq \mathbf{0}$ and, for any $(s-1)$-tuple $\mathbf{t}$ with $1 \leqslant \mathbf{t} \leqslant X$, the system (1.1) has the solution

$$\mathbf{x} = (t_1,\dots,t_{s-1},a), \quad \mathbf{y} = (t_1,\dots,t_{s-1},b).$$

In this way, we see that when $X \in \mathbb{N}$ one has $J_{s,k}(X;\mathbf{h}) \geqslant X^{s-1}$. □

This theorem shows that the conclusion of Theorem 1.2 may be regarded as close to best possible. Moreover, any improvement in the upper bound $J_{s,k}(X;\mathbf{h}) \ll X^{s-1}$ must account for the special subvarieties of the complete intersection defined by (1.1) containing subdiagonal solutions, in the appropriate sense. The most extreme such situation is addressed in the second of these almost trivial conclusions.

Theorem 7.2. *Suppose that $s,k \in \mathbb{N}$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Suppose that $h_j = 0$ for precisely t indices, say $j \in \{j_1,\dots,j_t\}$ with $1 \leqslant j_1 < j_2 < \dots < j_t \leqslant k$. Then provided that $1 \leqslant s \leqslant t$, one has $J_{s,k}(X;\mathbf{h}) = 0$.*

Proof. The hypothesis on $\mathbf{h}$ in the statement of the theorem ensures that whenever $\mathbf{x}$ and $\mathbf{y}$ satisfy the system (1.1), then one has

$$\sum_{i=1}^s x_i^{j_l} = \sum_{i=1}^s y_i^{j_l} \quad (1 \leqslant l \leqslant t). \tag{7.1}$$

When $\mathbf{x},\mathbf{y} \in \mathbb{N}^s$ and $1 \leqslant s \leqslant t$, it follows from [7] that in all solutions of the system (7.1), the s-tuple $(x_1,\dots,x_s)$ is a permutation of $(y_1,\dots,y_s)$. For any such solution, we find from (1.1) that $\mathbf{h} = \mathbf{0}$, contradicting the hypothesis from the statement of the theorem. Thus we conclude that $J_{s,k}(X;\mathbf{h}) = 0$. □

We now turn to the proof of Theorem 1.5. This we view as establishing the principle that when there are few indices l for which $h_l \neq 0$, then $J_{k,k}(X;\mathbf{h})$ may be expected to be very small, and indeed far smaller than would be implied by the convexity limited bound. Our proof of this conclusion makes heavy use of our earlier work on paucity in relatives of Vinogradov systems [15].

The proof of Theorem 1.5. We work under the hypotheses of the statement of the theorem. Let $J^*_{k,k}(X;\mathbf{h})$ denote the number of solutions of the system (1.1) with $s=k$ in which $x_i=y_m$ for no indices i and m with $1\leqslant i,m\leqslant t$. Consider a solution $\mathbf{x},\mathbf{y}$ of the system (1.1) with $s=k$ counted by $J_{k,k}(X;\mathbf{h})$ but not by $J^*_{k,k}(X;\mathbf{h})$. By relabelling variables, we may suppose that $x_k=y_k$, and hence we deduce from (1.1) that

$$J_{k,k}(X;\mathbf{h})-J^*_{k,k}(X;\mathbf{h})\ll XJ_{k-1,k}(X;\mathbf{h}).$$

However, since $h_j=0$ for the $k-1$ indices j with $j\neq l$, it is apparent from Theorem 7.2 that $J_{k-1,k}(X;\mathbf{h})=0$. Thus we conclude that

$$J_{k,k}(X;\mathbf{h})=J^*_{k,k}(X;\mathbf{h}).$$

Consider next a solution $\mathbf{x},\mathbf{y}$ of (1.1) with $s=k$ counted by $J^*_{k,k}(X;\mathbf{h})$. Define the elementary symmetric polynomials $\sigma_j(\mathbf{z})\in\mathbb{Z}[z_1,\ldots,z_k]$ via the generating function identity

$$\sum_{j=0}^k \sigma_j(\mathbf{z})t^j=\prod_{i=1}^k(1+tz_i),$$

and write, further,

$$s_j(\mathbf{z})=z_1^j+\ldots+z_k^j\quad (1\leqslant j\leqslant k).$$

When $n\geqslant 1$, a familiar formula (see [15, equation (2.2)]) delivers the relation

$$\sigma_n(\mathbf{z})=(-1)^n\sum_{\substack{m_1+2m_2+\ldots+nm_n=n\\ m_i\geqslant 0}}\prod_{i=1}^n\frac{(-s_i(\mathbf{z}))^{m_i}}{i^{m_i}m_i!}.$$

The system of equations

$$s_j(\mathbf{x})=s_j(\mathbf{y})+h_j\quad (1\leqslant j\leqslant k),$$

is tantamount to (1.1) with $s=k$. Since $h_j=0$ for $j\neq l$, we deduce that

$$\sigma_n(\mathbf{x})=(-1)^n\sum_{\substack{m_1+2m_2+\ldots+nm_n=n\\ m_i\geqslant 0}}\frac{(-s_l(\mathbf{y})-h_l)^{m_l}}{l^{m_l}m_l!}\prod_{\substack{1\leqslant i\leqslant n\\ i\neq l}}\frac{(-s_i(\mathbf{y}))^{m_i}}{i^{m_i}m_i!}.$$

We conclude that

$$\sigma_n(\mathbf{x})=\sigma_n(\mathbf{y})\quad (1\leqslant n<l),\tag{7.2}$$

and, when $n\geqslant l$, that there is a weighted homogeneous polynomial $\Psi_n(h;\mathbf{y})$ having rational coefficients and satisfying the property that

$$\sigma_n(\mathbf{x})-\sigma_n(\mathbf{y})=h_l\Psi_n(h_l;\mathbf{y}).\tag{7.3}$$

Here, if a monomial term of $\Psi_n(h;\mathbf{y})$ has total degree d in terms of $\mathbf{y}$ and degree e in terms of h, then one has $d+le=n-l$. It is evident, moreover, that there is a non-zero integer A_n, with $A_n=O_k(1)$, having the property that $A_n\Psi_n(h;\mathbf{y})$ has integer coefficients all of size $O_k(1)$. In particular, in each solution $\mathbf{x},\mathbf{y}$ of (1.1) counted by $J^*_{k,k}(X;\mathbf{h})$, we have $|\Psi_n(h_l;\mathbf{y})|\ll X^{n-l}$.

We deduce from (7.2) and (7.3) that for the indeterminate z, one has

$$\prod_{i=1}^k(z-x_i)-\prod_{i=1}^k(z-y_i)=(-1)^k\sum_{n=0}^k(\sigma_n(\mathbf{x})-\sigma_n(\mathbf{y}))(-z)^{k-n}$$
$$=(-1)^kh_l\sum_{n=l}^k\Psi_n(h_l;\mathbf{y})(-z)^{k-n}.$$

By substituting $z=y_j$, we therefore infer that there is a polynomial $\tau(\mathbf{y};y_j;h)$ for which one has

$$A_l\dots A_k\prod_{i=1}^k(y_j-x_i)=h_l\tau(\mathbf{y};y_j;h_l).$$

This polynomial $\tau(\mathbf{y};y_j;h)$ has integer coefficients and is weighted homogeneous of total degree $k-l$, with each variable y_i carrying weight 1, and the variable h carrying weight l. In particular, with the choices for $\mathbf{y},y_j,h_l$ associated with the solution $\mathbf{x},\mathbf{y}$ counted by $J^*_{k,k}(X;\mathbf{h})$ currently under consideration, we may assume that $\tau(\mathbf{y};y_j;h_l)$ is an integer of size $O(X^{k-l})$.

There are $O(X^{k-l})$ possible choices for the integer $\tau(\mathbf{y};y_j;h_l)$, and hence also for $h_l\tau(\mathbf{y};y_j;h_l)$. None of these are zero, for this would contradict the non-vanishing of y_j-x_i $(1\leqslant i,j\leqslant k)$. Then, for each of the $O(X^{k-l})$ possible choices for $N(\mathbf{y})=h_l\tau(\mathbf{y};y_j;h_l)$, we see that the integers y_j-x_i $(1\leqslant i\leqslant k)$ are divisors of $N(\mathbf{y})$. Since $N(\mathbf{y})=O(X^k)$, a standard divisor function estimate reveals that there are $O(X^\varepsilon)$ possible choices for these divisors. Fixing any one of these choices and one of the $O(X)$ possible choices for y_j, it follows that $x_1,\dots,x_k$ and y_j are now all fixed. Next, interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$ in the argument just described, we find that

$$A_l\dots A_k\prod_{i=1}^k(x_j-y_i)=-h_l\tau(\mathbf{x};x_j;-h_l).$$

Since $\mathbf{x}$ is already fixed, it follows that the integers x_j-y_i are all divisors of the fixed non-zero integer $N'(\mathbf{x})=h_l\tau(\mathbf{x},x_j;-h_l)$. As in the situation just discussed, there are $O(X^\varepsilon)$ possible choices for these divisors. Fixing any one of these choices, and noting that x_j is already fixed, we find that the integers $y_1,\dots,y_k$ are now also fixed. The total number of choices for $\mathbf{x}$ and $\mathbf{y}$ is consequently $O(X^{k-l+1+\varepsilon})$. This confirms that

$$J_{k,k}(X;\mathbf{h})=J^*_{k,k}(X;\mathbf{h})\ll X^{k-l+1+\varepsilon}$$

and completes the proof of the theorem. □

8. Appendix: the extended main conjecture in Vinogradov's mean value theorem

The purpose of this section is to discuss an extension to the main conjecture in Vinogradov's mean value theorem previously announced in 2019 by the author at a workshop in Oberwolfach. Since this conjecture offers numerous consequences, including but not limited to Theorem 1.4, we take the opportunity to discuss its origin, nature, and its implications relevant herein.

We begin by recalling the main conjecture in Vinogradov's mean value theorem, proved in work of Bourgain, Demeter and Guth [3] and in work of the author [13, 14]. A brief account of the history of Vinogradov's mean value theorem, and developments at the cusp of the proof of the main conjecture, is offered in [12]. For the present discussion, we write $J_{s,k}(X)$ for $J_{s,k}(X;\mathbf{0})$, which counts the number of integral solutions of the system of equations

$$\sum_{i=1}^{s}(x_i^j-y_i^j)=0\quad (1\leqslant j\leqslant k),$$

with $1\leqslant \mathbf{x},\mathbf{y}\leqslant X$. The main conjecture in Vinogradov's mean value theorem asserts that

$$J_{s,k}(X)\ll X^{s+\varepsilon}+X^{2s-k(k+1)/2}. \tag{8.1}$$

In order to motivate the formulation of the extended main conjecture, we briefly sketch how the two terms on the right hand side of (8.1) arise from the application of the circle method. Consider a Hardy-Littlewood dissection of the unit cube $[0,1)^k$ into sets of major and minor arcs of the type $\mathfrak{K}(Y)$ and $\mathfrak{k}(Y)$, for a suitable parameter Y, as defined in §6. Define the generating functions $I(\boldsymbol{\beta})$ and $S(q,\mathbf{a})$ as in (1.4) and (1.5), and put

$$\begin{aligned}I(\boldsymbol{\beta};X)&=\int_0^X e(\beta_1\gamma+\beta_2\gamma^2+\ldots+\beta_k\gamma^k)\,\mathrm{d}\gamma\\&=XI(\beta_1X,\beta_2X^2,\ldots,\beta_kX^k).\end{aligned}$$

When $\boldsymbol{\alpha}\in\mathfrak{K}(q,\mathbf{a};Y)\subseteq\mathfrak{K}(Y)$, the exponential sum $f_k(\boldsymbol{\alpha};X)$ is closely approximated by $q^{-1}S(q,\mathbf{a})I(\boldsymbol{\alpha}-\mathbf{a}/q;X)$. Until recently, it was widely believed by many experts in the Hardy-Littlewood method that when $\boldsymbol{\alpha}\in[0,1)^k$, there should exist $q\in\mathbb{N}$ and $\mathbf{a}\in\mathbb{Z}^k$ with $(q,a_1,\ldots,a_k)=1$ satisfying

$$f_k(\boldsymbol{\alpha};X)-q^{-1}S(q,\mathbf{a})I(\boldsymbol{\alpha}-\mathbf{a}/q;X)\ll X^{1/2+\varepsilon}. \tag{8.2}$$

Recent work of Brandes et al. [5] shows that such a strong relation cannot be true in full generality when $\boldsymbol{\alpha}$ is very close to $\mathbf{a}/q$. However, any failure of this relation is expected to produce a small number of secondary terms also of major arc type, and hence is not expected to have any material impact on the outcome of the ensuing discussion.

Recalling the definitions (6.7) and (6.8) of $\mathfrak{J}_{s,k}(\mathbf{h})$ and $\mathfrak{S}_{s,k}(\mathbf{h})$, one finds that the contribution of the term $q^{-1}S(q,\mathbf{a})I(\boldsymbol{\alpha}-\mathbf{a}/q;X)$ from (8.2) within the mean value $J_{s,k}(X)$ is at most $X^{2s-k(k+1)/2}I_0$, where

$$I_0 = \sum_{q=1}^{\infty} \sum_{\substack{1\leqslant a_1,\dots,a_k\leqslant q\\ (q,a_1,\dots,a_k)=1}} |q^{-1}S(q,\mathbf{a})|^{2s} \int_{-\infty}^{\infty} |I(\boldsymbol{\beta})|^{2s}\,\mathrm{d}\boldsymbol{\beta} = \mathfrak{S}_{s,k}(\mathbf{0})\mathfrak{J}_{s,k}(\mathbf{0}).$$

The details of the argument here are familiar from the analysis of the major arc contribution in an application of the circle method to the problem. We note that the singular series $\mathfrak{S}_{s,k}(\mathbf{0})$ converges absolutely for $2s > \frac{1}{2}k(k+1)+2$, and the singular integral $\mathfrak{J}_{s,k}(\mathbf{0})$ converges absolutely for $2s > \frac{1}{2}k(k+1)+1$ (see [1, Theorem 1]). Thus, under the first of these conditions on s, we obtain

$$J_{s,k}(X) = \int_{[0,1)^k} |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \ll X^{2s-k(k+1)/2} + \int_{[0,1)^k} (X^{1/2+\varepsilon})^{2s}\,\mathrm{d}\boldsymbol{\alpha},$$

and thereby we recover the main conjecture (8.1) established in [3, 13, 14]. For smaller values of s, the same conclusion follows by application of Hölder's inequality, since the term $X^{s+\varepsilon}$ dominates, so (8.1) follows for all $s \in \mathbb{N}$. As a final comment relevant to this preliminary discussion, we remark that the aforementioned deviations from this model suggested by work of [5] would, at worst, inflate the above estimates by a factor of X^ε for smaller values of s, and this has no impact in our wider discusion.

The question now arises concerning what outcome is to be expected when we integrate, not over the whole unit cube $[0,1)^k$, but instead over a subset $\mathfrak{V}$. The same philosophy demonstrates that when s is any real number satisfying $2s > \frac{1}{2}k(k+1)+2$, one should have the estimate

$$\begin{aligned}\int_{\mathfrak{V}} |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} &\ll X^{2s-k(k+1)/2}I_0 + \int_{\mathfrak{V}} (X^{1/2+\varepsilon})^{2s}\,\mathrm{d}\boldsymbol{\alpha}\\ &\ll X^{2s-k(k+1)/2} + X^{s+\varepsilon}\mathrm{mes}(\mathfrak{V}).\end{aligned}$$

This is tantamount to the extended main conjecture in Vinogradov's mean value theorem.

Conjecture 8.1. *Suppose that $k \in \mathbb{N}$ and $\mathfrak{V} \subseteq [0,1)^k$ is measurable. Then whenever s is a real number with $s \geqslant \frac{1}{4}k(k+1)+1$, one has*

$$\int_{\mathfrak{V}} |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \ll X^\varepsilon \left(X^s mes(\mathfrak{V}) + X^{2s-k(k+1)/2}\right).$$

We have limited the values of s admissible in this conclusion to the range $s \geqslant \frac{1}{4}k(k+1)+1$ in order that appropriate convergence of the singular series $\mathfrak{S}_{s,k}(\mathbf{0})$ and singular integral $\mathfrak{J}_{s,k}(\mathbf{0})$ be guaranteed. For larger values of s absolute convergence follows from [1, Theorem 1], and an application of Hölder's inequality delivers similar conclusions at the cost of inflating bounds by a factor of X^ε when $s = \frac{1}{4}k(k+1)+1$. For smaller values of s, the conjecture should be modified to reflect a larger secondary term arising from the potential divergence of these quantities. However, one may recover a cheap but useful version of the conjecture applicable for all s provided that $\mathrm{mes}(\mathfrak{V})$ is not too small.

Conjecture 8.2. *Suppose that $k \in \mathbb{N}$ and $\mathfrak{V} \subseteq [0,1)^k$ is measurable. Then whenever s is a positive number and*

$$\text{mes}(\mathfrak{V}) \gg X^{1-k(k+1)/4}, \tag{8.3}$$

one has

$$\int_{\mathfrak{V}} |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon}\left(X^s \text{mes}(\mathfrak{V}) + X^{2s-k(k+1)/2}\right). \tag{8.4}$$

To see that Conjecture 8.2 follows from Conjecture 8.1, note first that the conclusion (8.4) of the former is immediate from the latter in the situation wherein $s \geqslant \frac{1}{4}k(k+1)+1$. Suppose then that t is a positive number with $t < \frac{1}{4}k(k+1)+1$, and put $K = \frac{1}{2}k(k+1)+2$. Then on assuming the validity of Conjecture 8.1, an application of Hölder's inequality yields

$$\begin{aligned}\int_{\mathfrak{V}} |f_k(\boldsymbol{\alpha};X)|^{2t}\,\mathrm{d}\boldsymbol{\alpha} &\leqslant \left(\int_{\mathfrak{V}} \mathrm{d}\boldsymbol{\alpha}\right)^{1-2t/K}\left(\int_{\mathfrak{V}} |f_k(\boldsymbol{\alpha};X)|^{K}\,\mathrm{d}\boldsymbol{\alpha}\right)^{2t/K}\\ &\ll X^{\varepsilon}\,(\text{mes}(\mathfrak{V}))^{1-2t/K}\left(X^{K/2}\text{mes}(\mathfrak{V}) + X^{K-k(k+1)/2}\right)^{2t/K}\\ &\ll X^{\varepsilon}\left(X^t\text{mes}(\mathfrak{V}) + (\text{mes}(\mathfrak{V}))^{1-2t/K}X^{4t/K}\right).\end{aligned}$$

The second term here is asymptotically majorised by the first provided that $(\text{mes}(\mathfrak{V}))^{2t/K} \gg X^{4t/K-t}$, a condition that is satisfied when $\text{mes}(\mathfrak{V}) \gg X^{2-K/2}$. The conclusion (8.4) therefore follows provided that (8.3) holds.

We next consider some consequences and limitations of these conjectures.

Theorem 8.3. *Assume the extended main conjecture in Vinogradov's mean value theorem. Consider positive numbers $\theta_1,\ldots,\theta_k$ and the box*

$$\mathfrak{V}(\boldsymbol{\theta}) = [-X^{-\theta_1}, X^{-\theta_1}] \times \cdots \times [-X^{-\theta_k}, X^{-\theta_k}].$$

Then whenever $s \geqslant \frac{1}{4}k(k+1)+1$, one has

$$\int_{\mathfrak{V}(\boldsymbol{\theta})} |f_k(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon}\left(X^{s-\theta_1-\ldots-\theta_k} + X^{2s-k(k+1)/2}\right).$$

The same conclusion also holds without condition on s provided that

$$\theta_1 + \ldots + \theta_k \leqslant \tfrac{1}{4}k(k+1) - 1.$$

Proof. The conclusions of the theorem are immediate from Conjectures 8.1 and 8.2 on observing that $\text{mes}(\mathfrak{V}(\boldsymbol{\theta})) \asymp X^{-\theta_1-\ldots-\theta_k}$. □

The cap sets $\mathfrak{V} = [0,1]^{k-1} \times [0, X^{-\theta_k}]$ considered by Demeter, Guth and Wang [6] are addressed by the special case $\boldsymbol{\theta} = (0,\ldots,0,\theta_k)$ of this theorem. Indeed, the reader will find that [6, Conjecture 2.5] asserts that the conclusion of Theorem 8.3 should hold in the special case $\mathfrak{V}(\boldsymbol{\theta}) = [0,1]^{k-1} \times [0, X^{-\theta_k}]$, provided that $0 \leqslant \theta_k \leqslant k-1$ and $k \geqslant 2$. These authors proved this conjecture when $k=3$ and $0 \leqslant \theta_3 \leqslant 3/2$ in the special case $s = 6 - \theta_3$ (see [6, Theorem 3.3]). Plainly, the (conditional) conclusion of Theorem 8.3 is decidedly more general in scope, and Conjecture 8.1 is of even wider generality.

We remark that in the situation wherein $\mathfrak{V}$ is restricted to the kind of generalised cap sets that are the subject of Theorem 8.3, the major arc analysis implicit in the formulation of the conjecture may presumably be improved. The absolute convergence of the singular series $\mathfrak{S}_{s,k}(\mathbf{0})$ requires that one have $s > \frac{1}{4}k(k+1)+1$ owing in part to the extra divergence arising from the sum over $a_1, \ldots, a_k$ implicit in the definition (6.8). If one or more of the variables α_j is restricted in a manner ensuring that a_j is limited to a much smaller range than the interval $[1, q]$, as is the case in cap set problems, then presumably there is scope for improving the condition on s towards the less onerous constraint $s > \frac{1}{4}k(k+1) + \frac{1}{2}$.

We finish this appendix by noting that the conclusion of Conjecture 8.1 cannot hold as stated when $s < \frac{1}{4}k(k+1)$. In order to confirm this assertion, consider the system of inequalities

$$\left|\sum_{i=1}^{s}(x_i^j - y_i^j)\right| \leqslant sX^{j-1} \quad (1 \leqslant j \leqslant k). \tag{8.5}$$

Denote by $\Omega_1(X)$ the number of solutions of the system of inequalities (8.5) with $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X$. We observe that when $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X^{1-1/k}$, then the system (8.5) is satisfied whenever

$$\left|\sum_{i=1}^{s}(x_i^j - y_i^j)\right| \leqslant s(X^{1-1/k})^j X^{-(k-j)/k} \quad (1 \leqslant j \leqslant k-1). \tag{8.6}$$

Notice, in particular, that the inequality in (8.5) corresponding to the exponent $j = k$ is automatically satisfied in these circumstances. We denote by $\Omega_2(X)$ the number of solutions of (8.6) subject to this condition $1 \leqslant \mathbf{x}, \mathbf{y} \leqslant X^{1-1/k}$. Thus we have the lower bound $\Omega_1(X) \geqslant \Omega_2(X)$.

Write

$$\mathfrak{D} = \bigtimes_{j=1}^{k-1}\Big[-\frac{1}{2s}(X^{1-1/k})^{-j}X^{(k-j)/k}, \frac{1}{2s}(X^{1-1/k})^{-j}X^{(k-j)/k}\Big].$$

Then a standard argument (see for example [9, Lemma 2.1]) shows that

$$\Omega_2(X) \gg \left(\prod_{j=1}^{k-1}(X^{1-1/k})^j X^{-(k-j)/k}\right)\int_{\mathfrak{D}} |f_{k-1}(\boldsymbol{\alpha}; X^{1-1/k})|^{2s}\,\mathrm{d}\boldsymbol{\alpha}. \tag{8.7}$$

Next define a narrow major arc around $\mathbf{0}$ by taking $\tau > 0$ sufficiently small in terms of s and k, and putting

$$\mathfrak{D}_0 = \bigtimes_{j=1}^{k-1}\Big[-\tau(X^{1-1/k})^{-j}, \tau(X^{1-1/k})^{-j}\Big].$$

Standard arguments from the theory of Vinogradov's mean value theorem (see [8, Chapter 7]) show that for $\boldsymbol{\alpha} \in \mathfrak{D}_0$ one has $|f_{k-1}(\boldsymbol{\alpha}; X^{1-1/k})| \gg X^{1-1/k}$, whence

$$\begin{aligned}\int_{\mathfrak{D}_0} |f_{k-1}(\boldsymbol{\alpha}; X^{1-1/k})|^{2s}\,\mathrm{d}\boldsymbol{\alpha} &\gg (X^{1-1/k})^{2s}\mathrm{mes}(\mathfrak{D}_0)\\ &\gg (X^{1-1/k})^{2s-k(k-1)/2}.\end{aligned}$$

Since we may assume that $\mathfrak{D}_0 \subseteq \mathfrak{D}$, we deduce from (8.7) that

$$\begin{aligned}\Omega_2(X) &\gg X^{(k-1)(k-2)/2} \int_{\mathfrak{D}_0} |f_{k-1}(\boldsymbol{\alpha}; X^{1-1/k})|^{2s} \, \mathrm{d}\boldsymbol{\alpha} \\ &\gg X^{(k-1)(k-2)/2} \cdot X^{2s(1-1/k)-(k-1)^2/2}.\end{aligned}$$

Thus, the lower bound $\Omega_1(X) \geqslant \Omega_2(X)$ leads us to the conclusion

$$\Omega_1(X) \gg X^{2s(1-1/k)-(k-1)/2}. \tag{8.8}$$

On the other hand, again employing the same standard argument (see [9, Lemma 2.1]), we find that

$$\Omega_1(X) \ll \left(\prod_{j=1}^k X^{j-1} \right) \int_{\mathfrak{D}_1} |f_k(\boldsymbol{\alpha}; X)|^{2s} \, \mathrm{d}\boldsymbol{\alpha},$$

where

$$\mathfrak{D}_1 = \bigtimes_{j=1}^k \Big[-\frac{1}{2s} X^{1-j}, \frac{1}{2s} X^{1-j} \Big].$$

Thus

$$\Omega_1(X) \ll X^{k(k-1)/2} \int_{\mathfrak{D}_1} |f_k(\boldsymbol{\alpha}; X)|^{2s} \, \mathrm{d}\boldsymbol{\alpha}.$$

Assuming the validity of Conjecture 8.1 without constraint on s, it follows that

$$\begin{aligned}\Omega_1(X) &\ll X^{k(k-1)/2+\varepsilon} \left(X^s \mathrm{mes}(\mathfrak{D}_1) + X^{2s-k(k+1)/2} \right) \\ &\ll X^{\varepsilon}(X^s + X^{2s-k}).\end{aligned}$$

We therefore conclude from (8.8) that

$$X^{2s(1-1/k)-(k-1)/2} \ll \Omega_1(X) \ll X^{\varepsilon}(X^s + X^{2s-k}).$$

This is tenable only when

$$2s(1-1/k) - (k-1)/2 \leqslant \max\{s, 2s-k\},$$

which is to say that either $s \leqslant \frac{1}{2}(k+1) + 1/(k-2)$, or $2s \geqslant k(k+1)/2$. Thus we find that the upper bound asserted in Conjecture 8.1 cannot hold in general in the absence of a condition at least as strong as $s \geqslant \frac{1}{4}k(k+1)$.

References

[1] G. I. Arkhipov, *The Hilbert-Kamke problem*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 1, 3–52.

[2] G. I. Arkhipov, V. N. Chubarikov and A. A. Karatsuba, *Trigonometric sums in number theory and analysis*, De Gruyter Expositions in Mathematics, **39**, Walter de Gruyter, Berlin, 2004.

[3] J. Bourgain, C. Demeter and L. Guth, *Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three*, Ann. of Math. (2) **184** (2016), no. 2, 633-682.

[4] J. Brandes and K. Hughes, *On the inhomogeneous Vinogradov system*, Bull. Aust. Math. Soc. (in press); arxiv:2110.02366, doi:10.1017/S0004972722000284.

[5] J. Brandes, S. T. Parsell, C. Poulias, G. Shakan and R. C. Vaughan, *On generating functions in additive number theory, II: lower-order terms and applications to PDEs*, Math. Ann. **379** (2021), no. 1–2, 347–376.

[6] C. Demeter, L. Guth and H. Wang, *Small cap decouplings*, Geom. Funct. Anal. **30** (2020), no. 4, 989–1062.
[7] J. Steinig, *On some rules of Laguerre's, and systems of equal sums of like powers*, Rend. Mat. (6) **4** (1971), 629–644.
[8] R. C. Vaughan, *The Hardy-Littlewood method*, 2nd edition, Cambridge University Press, Cambridge, 1997.
[9] N. Watt, *Exponential sums and the Riemann zeta-function II*, J. London Math. Soc. (2) **39** (1989), no. 3, 385–404.
[10] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing*, Ann. of Math. (2) **175** (2012), no. 3, 1575–1627.
[11] T. D. Wooley, *The asymptotic formula in Waring's problem*, Internat. Math. Res. Notices **2012** (2012), no. 7, 1485–1504.
[12] T. D. Wooley, *Translation invariance, exponential sums, and Waring's problem*, Proceedings of the International Congress of Mathematicians, August 13–21, 2014, Seoul, Korea, Volume II, Kyung Moon Sa Co. Ltd., Seoul, Korea, 2014, pp. 505–529.
[13] T. D. Wooley, *The cubic case of the main conjecture in Vinogradov's mean value theorem*, Adv. Math. **294** (2016), 532–561.
[14] T. D. Wooley, *Nested efficient congruencing and relatives of Vinogradov's mean value theorem*, Proc. London Math. Soc. (3) **118** (2019), no. 4, 942–1016.
[15] T. D. Wooley, *Paucity problems and some relatives of Vinogradov's mean value theorem*, submitted, 17pp; arxiv:2107.12238.
[16] T. D. Wooley, *Subconvexity and the Hilbert-Kamke problem*, submitted, 13pp; arxiv:2201.02699.
[17] T. D. Wooley, *Subconvexity in the inhomogeneous cubic Vinogradov system*, submitted, 18pp; arxiv:2202.05804.

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