PAIRS OF DIAGONAL QUARTIC FORMS: THE ASYMPTOTIC FORMULAE

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Abstract. We establish an asymptotic formula for the number of integral solutions of bounded height for pairs of diagonal quartic equations in 26 or more variables. In certain cases, pairs in 25 variables can be handled.

1. Introduction

Once again we are concerned with the pair of Diophantine equations

$$a_1x_1^4 + a_2x_2^4 + \ldots + a_sx_s^4 = b_1x_1^4 + b_2x_2^4 + \ldots + b_sx_s^4 = 0, \tag{1.1}$$

wherein the given coefficients a_j, b_j satisfy $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ $(1 \leqslant j \leqslant s)$. While our focus was on the validity of the Hasse principle for such pairs in two precursors of this article [6, 9], we now investigate the asymptotic density of integral solutions. Denote by $\mathscr{N}(P)$ the number of solutions in integers x_j with $|x_j| \leqslant P$ $(1 \leqslant j \leqslant s)$ to this system. Then, subject to a natural rank condition on the coefficient matrix, one expects an asymptotic formula for $\mathscr{N}(P)$ to hold provided that s is not too small. Indeed, following Hardy and Littlewood [11] in spirit, the quantity $P^{8-s}\mathscr{N}(P)$ should tend to a limit that is itself a product of local densities. On a formal level, the densities are readily described. The real density, also known as the singular integral, is defined by

$$\mathfrak{J} = \lim_{T\to\infty} \int_{-T}^{T}\int_{-T}^{T}\prod_{j=1}^{s}\int_{-1}^{1} e\big((a_j\alpha + b_j\beta)t_j^4\big)\,\mathrm{d}t_j\,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{1.2}$$

whenever the limit exists. Let $M(q)$ denote the number of solutions $\mathbf{x}$ in $(\mathbb{Z}/q\mathbb{Z})^s$ satisfying (1.1). Then for primes p, the p-adic density is defined by

$$\mathfrak{s}_p = \lim_{h\to\infty} p^{(2-s)h}M(p^h), \tag{1.3}$$

assuming again that this limit exists. In case of convergence, the product $\mathfrak{S} = \prod_p \mathfrak{s}_p$ is referred to as the singular series, and the desired asymptotic relation can be presented as the limit formula

$$\lim_{P\to\infty} P^{8-s}\mathscr{N}(P) = \mathfrak{J}\mathfrak{S}. \tag{1.4}$$

Note that (1.4) can hold only when in each of the two equations comprising (1.1) there are sufficiently many non-zero coefficients. Of course one may pass

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from (1.1) to an equivalent system obtained by taking linear combinations of the two constituent equations. Thus, the invariant $q_0 = q_0(\mathbf{a}, \mathbf{b})$, defined by

$$q_0(\mathbf{a}, \mathbf{b}) = \min_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} \mathrm{card}\{1 \leqslant j \leqslant s : ca_j + db_j \neq 0\},$$

must be reasonably large. Indeed, it follows from Lemmata 3.1, 3.2 and 3.3 in our companion paper [9] that the conditions $s \geqslant 16$ and $q_0 \geqslant 12$ ensure that the limits (1.2) and (1.3) all exist, that the product $\mathfrak{S}$ is absolutely convergent, and that the existence of non-singular solutions to the system (1.1) in each completion of the rationals implies that $\mathfrak{J}\mathfrak{S} > 0$. A first result concerning the limit (1.4) is then obtained by introducing the moment estimate

$$\int_0^1 \Bigl| \sum_{x \leqslant P} e(\alpha x^4) \Bigr|^{14} \mathrm{d}\alpha \ll P^{10+\varepsilon}, \tag{1.5}$$

derived as the special case $u = 14$ of Lemma 5.3 below, to a familiar method of Cook [10] (see also [2]). Here we point out that the estimate (1.5) first occurs implicitly in the proof of [15, Theorem 4.1], conditional on the validity of the (now proven) main conjecture in Vinogradov's mean value theorem (for which see [1] and [17, Corollary 1.3]). In this way, one routinely confirms (1.4) when $s \geqslant 29$ and $q_0 \geqslant 15$. This result, although not explicitly mentioned in the literature, is certainly familiar to experts in the area, and has to be considered as the state of the art today. It seems worth remarking in this context that, at a time when the estimate (1.5) was not yet available, the authors [3, 5] handled the case $s \geqslant 29$ with more restrictive rank conditions. The main purpose of this memoir is to make three variables redundant.

Theorem 1.1. *For pairs of equations* (1.1) *with* $s \geqslant 26$ *and* $q_0 \geqslant 15$*, one has* $\mathscr{N}(P) \sim \mathfrak{J}\mathfrak{S}P^{s-8}$.

Relaxing the rank condition $q_0 \geqslant 15$ appears to be a difficult enterprise, as we now explain. Consider a pair of equations (1.1) with $s \geqslant 29$, and suppose that $b_i = a_j = 0$ for $1 \leqslant i \leqslant 14 < j \leqslant s$. These two equations are independent and thus $\mathscr{N}(P)$ factorises as $\mathscr{N}(P) = N_1(P)N_2(P)$, where $N_1(P)$ and $N_2(P)$ denote the number of integral solutions of the respective single equations

$$a_1x_1^4 + a_2x_2^4 + \ldots + a_{14}x_{14}^4 = 0, \tag{1.6}$$

with $|x_j| \leqslant P$ $(1 \leqslant j \leqslant 14)$, and

$$b_{15}y_1^4 + b_{16}y_2^4 + \ldots + b_sy_{s-14}^4 = 0, \tag{1.7}$$

with $|y_j| \leqslant P$ $(1 \leqslant j \leqslant s-14)$. The equation (1.7) has at least 15 non-zero coefficients, and so a straightforward application of the Hardy-Littlewood method using the mean value (1.5) shows that $P^{18-s}N_2(P)$ tends to a limit as $P \to \infty$, with this limit equal to a product of local densities analogous to $\mathfrak{J}$ and $\mathfrak{s}_p$. By choosing $b_j = (-1)^j$ for $15 \leqslant j \leqslant s$, we ensure that this limit is positive, and thus $P^{8-s}\mathscr{N}(P)$ tends to a limit as $P \to \infty$ if and only if $P^{-10}N_1(P)$ likewise tends to a limit. From the definitions (1.2) and (1.3), it is apparent that the local densities $\mathfrak{J}$ and $\mathfrak{s}_p$ factorise into components stemming from the equations underlying N_1 and N_2. The relation (1.4) therefore holds for this particular pair of equations if and only if $P^{-10}N_1(P)$ tends to the product of local densities associated with the equation (1.6). In particular, were (1.4) known to hold in any case where $q_0 = 14$ and s is large, then it would follow that $P^{-10}N_1(P)$ tends to the limit suggested by a formal application of the circle method, a result that is not yet known. This shows that relaxing the condition on q_0 would imply progress with single diagonal quartic equations.

The invariant q_0 is a very rough measure for the entanglement of the two equations present in (1.1). This can be refined considerably. The pairs (a_j, b_j) are all non-zero in $\mathbb{Z}^2$, so they define a point $(a_j : b_j) \in \mathbb{P}(\mathbb{Q})$. We refer to indices $i, j \in \{1, 2, \ldots, s\}$ as *equivalent* if $(a_i : b_i) = (a_j : b_j)$. This defines an equivalence relation on $\{1, 2, \ldots, s\}$. Suppose that there are ν equivalence classes with $r_1, \ldots, r_\nu$ elements, respectively, where $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_\nu$. On an earlier occasion [5] we named the tuple $(r_1, \ldots, r_\nu)$ the *profile* of the equations (1.1). Note that $q_0 = s - r_1$, whence our assumed lower bound $q_0 \geqslant 15$ implies that $r_1 \leqslant s - 15$ and $\nu \geqslant 2$. If more is known about the profile, then we can save yet another variable.

Theorem 1.2. *Suppose that $s = 25$ and that $(r_1, \ldots, r_\nu)$ is the profile of the pair of equations* (1.1). *If $q_0 \geqslant 16$ and $\nu \geqslant 5$, then $\mathscr{N}(P) \sim \mathfrak{J}\mathfrak{S}P^{s-8}$.*

For a pair (1.1) in "general position" one has $\nu = s$ and $r_1 = 1$, and in a quantitative sense easily made precise, such pairs constitute almost all such Diophantine systems. Hence, the conclusion of Theorem 1.2 applies to almost all pairs of equations of the shape (1.1).

We pointed out long ago [5] that a diffuse profile can be advantageous. However, even with the estimate (1.5) in hand, the method of [5] only handles cases where $s \geqslant 27$ and r_1 and r_2 are not too large. Thus our results improve on all previous work on the subject even if the input to the published versions is enhanced by the newer mean value bound (1.5).

It is time to describe the methods, and in particular the new ideas involved in the proofs. Our more recent results specific to systems of diagonal quartic forms [6, 8, 9] all depend on large values estimates for Fourier coefficients of powers of Weyl sums, and the current communication is no exception. The large values estimates provide upper bounds for higher moments of these Fourier coefficients, and these in turn yield mean value bounds for correlations of Weyl sums. We describe this link here in a setting appropriate for application to pairs of equations. Consider a 1-periodic twice differentiable function $h : \mathbb{R} \to \mathbb{R}$. Its Fourier expansion

$$h(\alpha) = \sum_{n\in\mathbb{Z}} \hat{h}(n)e(\alpha n) \tag{1.8}$$

converges uniformly and absolutely. Hence, by orthogonality, one has

$$\int_0^1\int_0^1 h(\alpha)h(\beta)h(-\alpha-\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta = \sum_{n\in\mathbb{Z}} \hat{h}(n)^3. \tag{1.9}$$

The methods of [6, 8, 9] rest on this and closely related identities, choosing $h(\alpha) = |g(\alpha)|^u$ with suitable quartic Weyl sums g and a positive real number u.

As a service to future scholars, we analyse in some detail the differentiability properties of functions like $|g(\alpha)|^u$ in §3. It transpires that when $u \geqslant 2$ then the relation (1.9) holds. We use (1.9) with $h(\alpha) = |f(\alpha)|^u$, where now

$$f(\alpha) = \sum_{x \leqslant P} e(\alpha x^4) \tag{1.10}$$

is the ordinary Weyl sum. We then obtain new entangled mean value estimates for smaller values of u. This alone is not of strength sufficient to reach the conclusions of Theorem 1.1.

As experts in the field will readily recognise, for larger values of u the quality of the aforementioned mean value estimates is diluted by major arc contributions, and one would therefore like to achieve their removal. Thus, if $\mathfrak{n}$ is a 1-periodic set of real numbers with $\mathfrak{n} \cap [0, 1)$ a classical choice of minor arcs and $\mathbf{1}_\mathfrak{n}$ is the indicator function of $\mathfrak{n}$, then one is tempted to apply the function $h(\alpha) = \mathbf{1}_\mathfrak{n}(\alpha)|f(\alpha)|^u$ in place of $|f(\alpha)|^u$ within (1.9). However, this function is no longer continuous. We bypass this difficulty by introducing a smoothed Farey dissection in §4. This is achieved by a simple and very familiar convolution technique that should be useful in other contexts, too. In this way, in §5 we obtain a minor arc variant of the cubic moment method developed in our earlier work [6]. Equipped with this and the mean value bounds that follow from it, one reaches the conclusions of Theorem 1.1 in the majority of cases under consideration. Unfortunately, some cases with exceptionally large values of r_j stubbornly deny treatment. To cope with these remaining cases, we develop a mixed moment method in §6.

The point of departure is a generalisation of (1.9). If h_1, h_2, h_3 are functions that qualify for the discussion surrounding (1.8) and (1.9), then by invoking orthogonality once again, we see that

$$\int_0^1 \int_0^1 h_1(\alpha) h_2(\beta) h_3(-\alpha - \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta = \sum_{n \in \mathbb{Z}} \hat{h}_1(n) \hat{h}_2(n) \hat{h}_3(n). \tag{1.11}$$

By Hölder's inequality, the right hand side here is bounded in terms of the three moments

$$\sum_{n \in \mathbb{Z}} |\hat{h}_j(n)|^3. \tag{1.12}$$

In all cases where $h_j(\alpha) = |f(\alpha)|^{u_j}$ for some even positive integral exponent u_j one has $\hat{h}_j(n) \geqslant 0$, so (1.9) can be used in reverse to interpret (1.12) in terms of the number of solutions of a pair of Diophantine equations. The purely analytic description of the method has several advantages. First and foremost, one can break away from even numbers u_j, and still estimate all three cubic moments (1.12). This paves the way to a complete treatment of pairs of equations (1.1) with $s \geqslant 26$ and $q_0 \geqslant 15$. Beyond this, the identity (1.11) offers extra flexibility for the arithmetic harmonic analysis. Instead of the homogeneous passage from (1.11) to (1.12) one could apply Hölder's inequality with differing weights. As an example of stunning simplicity, we note that the expression in (1.11) is bounded above by

$$\Big(\sum_{n\in\mathbb{Z}} |\hat{h}_1(n)|^2\Big)^{1/2}\Big(\sum_{n\in\mathbb{Z}} |\hat{h}_2(n)|^4\Big)^{1/4}\Big(\sum_{n\in\mathbb{Z}} |\hat{h}_3(n)|^4\Big)^{1/4}.$$

If we apply this idea with $h_j(\alpha) = |f(\alpha)|^{u_j}$ and u_j a positive even integer, then the first factor relates to a single diagonal Diophantine equation while the other two factors concern systems consisting of three diagonal Diophantine equations. This argument is dual (in the sense that we work with Fourier coefficients) to a method that we described as *complification* in our work on systems of cubic forms [7]. There is, of course, an obvious generalisation of (1.9) to higher dimensional integrals that has been used here. This points to a complex interplay between systems of diagonal equations in which the size parameters (number of variables and number of equations) vary, and need not be restricted to natural numbers. We have yet to explore the full potential of this observation.

We briefly comment on the role of the Hausdorff-Young inequality [18, Chapter XII, Theorem 2.3] within this circle of ideas. In the notation of (1.11) this asserts that

$$\sum_{n\in\mathbb{Z}} |\hat{h}_j(n)|^3 \leqslant \Big(\int_0^1 |h_j(\alpha)|^{3/2}\,\mathrm{d}\alpha\Big)^2.$$

Passing through (1.11) and (1.12), one then arrives at the estimate

$$\Big|\int_0^1\!\!\int_0^1 h_1(\alpha)h_2(\beta)h_3(-\alpha-\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\Big| \leqslant \prod_{j=1}^3\Big(\int_0^1 |h_j(\alpha)|^{3/2}\,\mathrm{d}\alpha\Big)^{2/3}. \tag{1.13}$$

However, by Hölder's inequality, one finds

$$\Big|\int_0^1\!\!\int_0^1 h_1(\alpha)h_2(\beta)h_3(-\alpha-\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\Big| \leqslant \prod_{1\leqslant i<j\leqslant 3}\Big(\int_0^1 |h_ih_j|^{3/2}\,\mathrm{d}\alpha\,\mathrm{d}\beta\Big)^{1/3},$$

where, on the right hand side, one should read $h_1 = h_1(\alpha)$, $h_2 = h_2(\beta)$ and $h_3 = h_3(-\alpha-\beta)$. By means of obvious linear substitutions, this also delivers the bound (1.13). This last method is essentially that of Cook [10]. Our approach is superior because the methods are designed to remember the arithmetic source of the Weyl sums when estimating moments of Fourier coefficients.

The proof of Theorem 1.2 requires yet another tool that is a development of our multidimensional version of Hua's lemma [3]. This somewhat outdated work is based on Weyl differencing. An analysis of the method shows that whenever a new block of differenced Weyl sums enters the recursive process, a new entry r_j to the profile of the underlying Diophantine system is needed. It is here where one imports undesired constraints on the profile, as in Theorem 1.2. However, powered with the new upper bound (1.5), the method just described yields a bound for a two-dimensional entangled mean value over eighteen Weyl sums that outperforms the cubic moments technique by a factor $P^{1/6}$ (compare Theorem 6.1 with Theorem 7.2). Within a circle method approach, this mean value is introduced via Hölder's inequality. In the complementary factor, we have available an abundance of Weyl sums. Fortunately the cubic moments technique restricted to minor arcs presses the method home. We point out that our proof of Theorem 1.2 constitutes the first instance in which the cubic moments technique is successfully coupled with the differencing techniques derived from [3].

One might ask whether more restrictive conditions on the profile allow one to reduce the number of variables even further. As we demonstrate at the very end of this memoir it is indeed possible to accelerate the convergence in (1.4), but even the extreme condition $r_1 = 1$ seems insufficient to save a variable without another new idea.

Once the new moment estimates are established, our proofs of Theorems 1.1 and 1.2 are fairly concise. There are two reasons. First, we may import the major arc work, to a large extent, from [9]. Second, more importantly, our minor arc treatment rests on a new inequality (Lemma 2.3 below) that entirely avoids combinatorial difficulties associated with exceptional profiles. This allows us to reduce the minor arc work to a single profile with a certain maximality property. We expect this argument to become a standard preparation step in related work, and have therefore presented this material in broad generality. We refer to §2 where the reader will also find comment on previous attempts in this direction.

Notation. Our basic parameter is P, a sufficiently large real number. Implicit constants in Vinogradov's familiar symbols $\ll$ and $\gg$ may depend on s and ε as well as ambient coefficients such as those in the system (1.1). Whenever ε appears in a statement we assert that the statement holds for each positive real value assigned to ε. As usual, we write $e(z)$ for $e^{2\pi i z}$.

2. Some inequalities

This section belongs to real analysis. We discuss a number of inequalities for products. As is familiar for decades, in an attempt to prove results of the type described in Theorems 1.1 and 1.2 via harmonic analysis, it is desirable to simplify to a situation where the profile is extremal relative to the conditions in hand, that is, the multiplicities $r_1, r_2, \dots$ are as large as possible, and consequently ν is as small as is possible. In the past, most scholars have applied Hölder's inequality to achieve this objective, often by an *ad hoc* argument that led to the consideration of several cases separately. The purpose of this section is to make available general inequalities that encapsulate the reduction step in a single lemma of generality sufficient to include all situations that one encounters in practice.

The germ of our method is a classical estimate, sometimes referred to as Young's inequality: if p and q are real numbers with $p > 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then for all non-negative real numbers u and v one has

$$uv \leqslant \frac{u^p}{p} + \frac{v^q}{q}. \tag{2.1}$$

This includes the case $r = 2$ of the bound

$$|z_1 z_2 \cdots z_r| \leqslant \frac{1}{r}\big(|z_1|^r + \ldots + |z_r|^r\big) \tag{2.2}$$

which holds for all $r \in \mathbb{N}$ and all $z_j \in \mathbb{C}$ $(1 \leqslant j \leqslant r)$. Indeed, the general case of (2.2) follows from (2.1) by an easy induction on r.

In the following chain of lemmata we are given a number $\nu \in \mathbb{N}$ and integral exponents m_j, M_j $(1 \leqslant j \leqslant \nu)$ with

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_\nu \geqslant 0, \qquad M_1 \geqslant M_2 \geqslant \ldots \geqslant M_\nu \geqslant 0 \tag{2.3}$$

and

$$\sum_{l=1}^{L} m_l \leqslant \sum_{l=1}^{L} M_l \quad (1 \leqslant L < \nu), \qquad \sum_{l=1}^{\nu} m_l = \sum_{l=1}^{\nu} M_l. \tag{2.4}$$

We write S_ν for the group of permutations on ν elements. We refer to a function $w : S_\nu \to [0,1]$ with

$$\sum_{\sigma \in S_\nu} w(\sigma) = 1$$

as a *weight on* S_ν.

Lemma 2.1. *Suppose that the exponents m_j, M_j $(1 \leqslant j \leqslant \nu)$ satisfy* (2.3) *and* (2.4)*. Then there is a weight w on S_ν with the property that for all non-negative real numbers $u_1, u_2, \ldots, u_\nu$ one has*

$$u_1^{m_1} u_2^{m_2} \cdots u_\nu^{m_\nu} \leqslant \sum_{\sigma \in S_\nu} w(\sigma) u_{\sigma(1)}^{M_1} u_{\sigma(2)}^{M_2} \cdots u_{\sigma(\nu)}^{M_\nu}. \tag{2.5}$$

Proof. We define

$$D = \sum_{l=1}^{\nu} |M_l - m_l|$$

and proceed by induction on $\nu + D$. In the base case of the induction one has $\nu + D = 1$. In this situation $\nu = 1$ and $D = 0$, and the claim of the lemma is trivially true with $\sigma = \mathrm{id}$ and $w(\sigma) = 1$.

Now suppose that $\nu + D > 1$. We consider two cases. First we suppose that there is a number ν_1 with $1 \leqslant \nu_1 < \nu$ and

$$\sum_{l=1}^{\nu_1} m_l = \sum_{l=1}^{\nu_1} M_l.$$

We put

$$D_1 = \sum_{l=1}^{\nu_1} |M_l - m_l|, \quad D_2 = \sum_{l=\nu_1+1}^{\nu} |M_l - m_l|, \quad \nu_2 = \nu - \nu_1.$$

Then (2.3) and (2.4) are valid with ν_1 in place of ν, and one has $D_1 \leqslant D$. Hence $\nu_1 + D_1 < \nu + D$ so that we may invoke the inductive hypothesis to find a weight w_1 on S_{ν_1} with

$$u_1^{m_1} u_2^{m_2} \cdots u_{\nu_1}^{m_{\nu_1}} \leqslant \sum_{\sigma \in S_{\nu_1}} w_1(\sigma) u_{\sigma(1)}^{M_1} u_{\sigma(2)}^{M_2} \cdots u_{\sigma(\nu_1)}^{M_{\nu_1}}. \tag{2.6}$$

Similarly, in the current situation, the numbers m_{ν_1+j}, M_{ν_1+j} $(1 \leqslant j \leqslant \nu_2)$ may take the roles of m_j, M_j in (2.3) and (2.4) with ν_2 in place of ν. Again, we have $\nu_2 + D_2 < \nu + D$. Now writing τ for a permutation in S_{ν_2} acting on the set $\{\nu_1+1, \nu_1+2, \ldots, \nu\}$, we may invoke the inductive hypothesis again to find a weight w_2 on S_{ν_2} with

$$u_{\nu_1+1}^{m_{\nu_1+1}} u_{\nu_1+2}^{m_{\nu_1+2}} \cdots u_{\nu}^{m_\nu} \leqslant \sum_{\tau \in S_{\nu_2}} w_2(\tau) u_{\tau(\nu_1+1)}^{M_{\nu_1+1}} u_{\tau(\nu_1+2)}^{M_{\nu_1+2}} \cdots u_{\tau(\nu)}^{M_\nu}. \tag{2.7}$$

We multiply the inequalities (2.6) and (2.7). It is then convenient to read permutations σ on $1, 2, \ldots, \nu_1$ and τ on $\nu_1+1, \nu_1+2, \ldots, \nu$ as permutations on $1, 2, \ldots, \nu$ with $\sigma(j) = j$ for $j > \nu_1$ and $\tau(j) = j$ for $j \leqslant \nu_1$. Then, for permutations of the type $\sigma\tau$ in S_ν we put $w(\sigma\tau) = w_1(\sigma)w_2(\tau)$, and we put $w(\phi) = 0$ for the remaining permutations $\phi \in S_\nu$. With this function w the product of (2.6) and (2.7) becomes (2.5), completing the induction in the case under consideration.

In the complementary case we have

$$\sum_{l=1}^{L} m_l < \sum_{l=1}^{L} M_l \qquad (1 \leqslant L < \nu). \tag{2.8}$$

In particular, this shows that $m_1 < M_1$. Also, by comparing the case $L = \nu - 1$ of (2.8) with the equation corresponding to the case $L = \nu$ in (2.4), we see that $m_\nu > M_\nu$, as a consequence of which we have $m_\nu \geqslant 1$. We write $m_1 = m_\nu + r$. In view of (2.3), we see that $r \geqslant 0$, and so an application of (2.1) with $q = r+2$ leads to the inequality

$$u_1^{r+1} u_\nu \leqslant \frac{r+1}{r+2} u_1^{r+2} + \frac{1}{r+2} u_\nu^{r+2}.$$

Recall that $m_\nu \geqslant 1$, whence $m_1 - r - 1 = m_\nu - 1 \geqslant 0$. It follows that

$$u_1^{m_1} u_\nu^{m_\nu} \leqslant u_1^{m_1-r-1} u_\nu^{m_\nu-1} \Big(\frac{r+1}{r+2} u_1^{r+2} + \frac{1}{r+2} u_\nu^{r+2} \Big),$$

and thus

$$\begin{aligned} u_1^{m_1} \cdots u_\nu^{m_\nu} \leqslant \frac{r+1}{r+2} & u_1^{m_1+1} u_2^{m_2} u_3^{m_3} \cdots u_{\nu-1}^{m_{\nu-1}} u_\nu^{m_\nu-1} \\ & + \frac{1}{r+2} u_1^{m_\nu-1} u_2^{m_2} u_3^{m_3} \cdots u_{\nu-1}^{m_{\nu-1}} u_\nu^{m_1+1}. \end{aligned}$$

The chain of exponents $m_1+1, m_2, m_3, \ldots, m_{\nu-1}, m_\nu - 1$ is decreasing, and we have $m_1 + 1 \leqslant M_1$ and $m_\nu - 1 \geqslant 0$. Hence, in view of (2.8), the hypotheses (2.3) and (2.4) are still met when we put $m_1 + 1$ in place of m_1 and $m_\nu - 1$ in place of m_ν. However, $m_1 + 1$ is closer to M_1 than is m_1, and likewise $m_\nu - 1$

is closer to M_ν than is m_ν. The value of D associated with this new chain of exponents therefore decreases, and so we may apply the inductive hypothesis to find a weight W on S_ν with

$$u_1^{m_1+1}u_2^{m_2}u_3^{m_3}\cdots u_{\nu-1}^{m_{\nu-1}}u_\nu^{m_\nu-1} \leqslant \sum_{\sigma\in S_\nu} W(\sigma)u_{\sigma(1)}^{M_1}u_{\sigma(2)}^{M_2}\cdots u_{\sigma(\nu)}^{M_\nu}.$$

Interchanging the roles of u_1 and u_ν, and denoting by τ the transposition of 1 and ν, we obtain in like manner the bound

$$u_\nu^{m_1+1}u_2^{m_2}u_3^{m_3}\cdots u_{\nu-1}^{m_{\nu-1}}u_1^{m_\nu-1} \leqslant \sum_{\sigma\in S_\nu} W(\sigma\circ\tau)u_{\sigma(1)}^{M_1}u_{\sigma(2)}^{M_2}\cdots u_{\sigma(\nu)}^{M_\nu}.$$

If we now import the last two inequalities into the inequality preceding them, we find that (2.5) holds with

$$w(\sigma) = \frac{r+1}{r+2}W(\sigma) + \frac{1}{r+2}W(\sigma\circ\tau),$$

and w is a weight on S_ν. This completes the induction in the second case. □

Lemma 2.2. *Suppose that m_j, M_j $(1 \leqslant j \leqslant \nu)$ satisfy (2.3) and (2.4). For $1 \leqslant j \leqslant \nu$ let $h_j : \mathbb{R}^n \to [0,\infty)$ denote a Lebesgue measurable function. Then*

$$\int h_1^{m_1}h_2^{m_2}\cdots h_\nu^{m_\nu}\,\mathrm{d}\mathbf{x} \leqslant \max_{\sigma\in S_\nu}\int h_{\sigma(1)}^{M_1}h_{\sigma(2)}^{M_2}\cdots h_{\sigma(\nu)}^{M_\nu}\,\mathrm{d}\mathbf{x}.$$

Proof. Choose $u_j = h_j$ in Lemma 2.1 for $1 \leqslant j \leqslant \nu$ and integrate. □

For applications to systems of diagonal equations or inequalities, functions h_j come with an equivalence relation between them. This we encode as a partition of the set of indices j in the final lemma of this section.

Lemma 2.3. *Suppose that the exponents m_j, M_j $(1 \leqslant j \leqslant \nu)$ satisfy (2.3) and (2.4). Let $s = m_1 + m_2 + \ldots + m_\nu$, and for $1 \leqslant j \leqslant s$, let $h_j : \mathbb{R}^n \to [0,\infty)$ denote a Lebesgue measurable function. Finally, suppose that $J_1, J_2, \ldots, J_\nu$ are sets with respective cardinalities $m_1, m_2, \ldots, m_\nu$ that partition $\{1, 2, \ldots, s\}$. Then, there exists a tuple $(i_1, \ldots, i_\nu)$ and a permutation $\sigma \in S_\nu$, with $i_l \in J_{\sigma(l)}$ $(1 \leqslant l \leqslant \nu)$, having the property that*

$$\int h_1h_2\cdots h_s\,\mathrm{d}\mathbf{x} \leqslant \int h_{i_1}^{M_1}h_{i_2}^{M_2}\ldots h_{i_\nu}^{M_\nu}\,\mathrm{d}\mathbf{x}. \tag{2.9}$$

Proof. For each suffix l with $1 \leqslant l \leqslant \nu$, it follows from (2.2) that

$$\prod_{j\in J_l} h_j \leqslant \frac{1}{m_j}\sum_{j\in J_l} h_j^{m_j}.$$

Multiplying these inequalities together yields the bound

$$h_1h_2\cdots h_s \leqslant \frac{1}{m_1\cdots m_\nu}\sum_{j_1\in J_1}\cdots\sum_{j_\nu\in J_\nu} h_{j_1}^{m_1}h_{j_2}^{m_2}\cdots h_{j_\nu}^{m_\nu}.$$

Now integrate. One then finds that there exists a tuple $(j_1, \ldots, j_\nu)$, with $j_l \in J_l$ $(1 \leqslant l \leqslant \nu)$, for which

$$\int h_1 h_2 \cdots h_s \, \mathrm{d}\mathbf{x} \leqslant \int h_{j_1}^{m_1} h_{j_2}^{m_2} \ldots h_{j_\nu}^{m_\nu} \, \mathrm{d}\mathbf{x}.$$

Finally, we apply Lemma 2.2. One then finds that for some $\sigma \in S_\nu$ the upper bound (2.9) holds with $i_l = j_{\sigma(l)}$ $(1 \leqslant l \leqslant \nu)$. □

3. Smooth Farey dissections

In this section we describe a partition of unity that mimics the traditional Farey dissection. With other applications in mind, we work in some generality. Throughout this section we take X and Y to be real numbers with $1 \leqslant Y \leqslant \frac{1}{2}\sqrt{X}$, and then let $\mathfrak{N}(q,a)$ denote the interval of all real α satisfying $|q\alpha - a| \leqslant YX^{-1}$. Define $\mathfrak{N} = \mathfrak{N}_{X,Y}$ as the union of all $\mathfrak{N}(q,a)$ with $1 \leqslant q \leqslant Y$, $a \in \mathbb{Z}$ and $(a,q) = 1$. Note that the intervals $\mathfrak{N}(q,a)$ comprising $\mathfrak{N}$ are pairwise disjoint. We also write $\mathfrak{M} = \mathfrak{M}_{X,Y}$ for the set $\mathfrak{N} \cap [0,1]$. For appropriate choices of the parameter Y, the latter is a typical choice of major arcs in applications of the Hardy-Littlewood method.

The set $\mathfrak{N}$ has period 1. Its indicator function $\mathbf{1}_\mathfrak{N}$ has finitely many discontinuities in $[0,1)$, implying unwanted delicacies concerning the convergence of the Fourier series of $\mathbf{1}_\mathfrak{N}$. We avoid complications associated with this feature by a familiar convolution trick, which we now describe.

Define the positive real number

$$\kappa = \int_{-1}^{1} \exp(1/(t^2-1)) \, \mathrm{d}t,$$

and the function $K : \mathbb{R} \to [0,\infty)$ by

$$K(t) = \begin{cases} \kappa^{-1} \exp(1/(t^2-1)) & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geqslant 1. \end{cases}$$

As is well known, the function $K(t)$ is smooth and even. We scale this function with the positive parameter X in the form

$$K_X(t) = 4X \, K(4Xt).$$

Then K_X is supported on the interval $|t| \leqslant 1/(4X)$ and satisfies the important relation

$$\int_{-\infty}^{\infty} K_X(t) \, \mathrm{d}t = \int_{-\infty}^{\infty} K(t) \, \mathrm{d}t = 1. \tag{3.1}$$

We now define the function $\mathsf{N}_{X,Y} : \mathbb{R} \to [0,1]$ by

$$\mathsf{N}_{X,Y}(\alpha) = \int_{-\infty}^{\infty} \mathbf{1}_\mathfrak{N}(\alpha - t) K_X(t) \, \mathrm{d}t = \int_{-\infty}^{\infty} \mathbf{1}_\mathfrak{N}(t) K_X(\alpha - t) \, \mathrm{d}t. \tag{3.2}$$

The main properties of this function $\mathsf{N} = \mathsf{N}_{X,Y}$ are listed in the next lemma.

Lemma 3.1. *The function* $\mathsf{N} = \mathsf{N}_{X,Y}$ *is smooth, and for all* $\alpha \in \mathbb{R}$ *one has* $\mathsf{N}(\alpha) \in [0,1]$. *Further, whenever* $2 \leqslant Y \leqslant \frac{1}{4}\sqrt{X}$, *the inequalities*

$$\mathbf{1}_{\mathfrak{N}_{X,Y/2}}(\alpha) \leqslant \mathsf{N}(\alpha) \leqslant \mathbf{1}_{\mathfrak{N}_{X,2Y}}(\alpha) \tag{3.3}$$

and

$$\mathsf{N}'(\alpha) \ll X, \quad \mathsf{N}''(\alpha) \ll X^2 \tag{3.4}$$

hold uniformly in $\alpha \in \mathbb{R}$.

Proof. The integrands in (3.2) are non-negative, so $\mathsf{N}(\alpha) \geqslant 0$, while (3.1) shows that $\mathsf{N}(\alpha) \leqslant 1$. Since K is smooth and compactly supported, the second integral formulation of N in (3.2) shows that N is smooth, and that the derivative is obtained by differentiating the integrand. Thus, we obtain

$$\mathsf{N}'(\alpha) = \int_{\mathfrak{N}} \frac{\partial}{\partial \alpha} K_X(\alpha - t)\,\mathrm{d}t,$$

whence

$$|\mathsf{N}'(\alpha)| \leqslant 4X \int_{-1}^{1} |K'(t)|\,\mathrm{d}t.$$

This confirms the inequality for the first derivative in (3.4). The bound for the second derivative follows in like manner by differentiating again.

We now turn to the task of establishing (3.3). First suppose that $\alpha \in \mathfrak{N}_{X,Y/2}$. Then, there is a unique pair of integers $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, $q \leqslant \frac{1}{2}Y$ and $|q\alpha - a| \leqslant \frac{1}{2}YX^{-1}$. For $|t| \leqslant (4X)^{-1}$ we then have

$$\Big|(\alpha - t) - \frac{a}{q}\Big| \leqslant \frac{1}{4X} + \frac{Y}{2qX} \leqslant \frac{Y}{qX}.$$

Thus $\alpha - t \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}_{X,Y}$. Since K_X is supported on $[-1/(4X), 1/(4X)]$, we deduce from (3.1) and (3.2) that

$$\mathsf{N}(\alpha) \geqslant \int_{-\infty}^{\infty} \mathbf{1}_{\mathfrak{N}(q,a)}(\alpha - t) K_X(t)\,\mathrm{d}t \geqslant \int_{-1}^{1} K_X(t)\,\mathrm{d}t = 1.$$

It follows that one has $\mathsf{N}(\alpha) = 1$ for all $\alpha \in \mathfrak{N}_{X,Y/2}$. However, we know already that $\mathsf{N}(\alpha)$ is non-negative for all $\alpha \in \mathbb{R}$, and thus we have proved the first of the two inequalities in (3.3).

We complete the proof of the lemma by addressing the second inequality in (3.3). Suppose that $\mathsf{N}(\alpha) > 0$. Then, it follows from (3.2) that for some $t \in \mathbb{R}$ with $|t| \leqslant (4X)^{-1}$, one has $\alpha - t \in \mathfrak{N}_{X,Y}$. Hence, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, $q \leqslant Y$ and $|\alpha - t - a/q| \leqslant Y/(qX)$. By the triangle inequality,

$$\Big|\alpha - \frac{a}{q}\Big| \leqslant \frac{Y}{qX} + \frac{1}{4X} \leqslant \frac{2Y}{qX}.$$

This shows that $\alpha \in \mathfrak{N}_{X,2Y}$. Since $0 \leqslant \mathsf{N}(\alpha) \leqslant 1$, the second of the inequalities in (3.3) also follows. □

We consider $\mathsf{N} = \mathsf{N}_{X,Y}$ as a smooth model of the major arcs $\mathfrak{N}_{X,Y}$. It is convenient to define corresponding minor arcs $\mathfrak{n} = \mathfrak{n}_{X,Y}$, with $\mathfrak{n}_{X,Y} = \mathbb{R} \setminus \mathfrak{N}_{X,Y}$, and to write $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$ for the set of minor arcs complementary to $\mathfrak{M}$. The smoothed version of $\mathfrak{n}_{X,Y}$ is the function $\mathsf{n}_{X,Y} : \mathbb{R} \to [0,1]$ defined by

$$\mathsf{n}(\alpha) = \int_{-\infty}^{\infty} \mathbf{1}_{\mathfrak{n}}(\alpha - t) K_X(t)\,\mathrm{d}t.$$

We trivially have $\mathbf{1}_{\mathfrak{N}}(\alpha) + \mathbf{1}_{\mathfrak{n}}(\alpha) = 1$ for all $\alpha \in \mathbb{R}$, so it is a consequence of (3.1) and (3.2) that $\mathsf{n} = \mathsf{n}_{X,Y}$ satisfies the identity

$$\mathsf{N}(\alpha) + \mathsf{n}(\alpha) = 1. \tag{3.5}$$

The properties of n can therefore be deduced from the corresponding facts concerning N. In particular, Lemma 3.1 translates as follows.

Lemma 3.2. *The function $\mathsf{n} = \mathsf{n}_{X,Y}$ is smooth, and for all $\alpha \in \mathbb{R}$ one has $\mathsf{n}(\alpha) \in [0,1]$. Further, whenever $2 \leqslant Y \leqslant \frac{1}{4}\sqrt{X}$, the inequalities*

$$\mathbf{1}_{\mathfrak{n}_{X,2Y}}(\alpha) \leqslant \mathsf{n}(\alpha) \leqslant \mathbf{1}_{\mathfrak{n}_{X,Y/2}}(\alpha)$$

and

$$\mathsf{n}'(\alpha) \ll X, \quad \mathsf{n}''(\alpha) \ll X^2$$

hold uniformly in $\alpha \in \mathbb{R}$.

4. Fractional powers of Weyl sums

In this section we consider a trigonometric polynomial

$$T(\alpha) = \sum_{M < n \leqslant M+N} c_n e(\alpha n) \tag{4.1}$$

with complex coefficients c_n. The associated ordinary polynomial

$$P(z) = \sum_{n=1}^{N} c_{M+n} z^n \tag{4.2}$$

is related to T via the identity

$$T(\alpha) = e(M\alpha)P(e(\alpha)). \tag{4.3}$$

Lemma 4.1. *Let $k \in \mathbb{N}$. Then, for any real number $u > k$, the real function $\Omega_u : \mathbb{R} \to \mathbb{R}$, defined by $\Omega_u(\alpha) = |T(\alpha)|^u$, is k times continuously differentiable.*

Proof. In view of (4.3), we see that it suffices to prove this result in the special case where $M = 0$. This reduction step noted, we proceed by a succession of elementary exercises.

Let $u \in \mathbb{R}$. We begin by considering the function $\theta_u : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $\theta_u(\alpha) = |\alpha|^u$. This function is differentiable on $\mathbb{R} \setminus \{0\}$, and one has

$$\theta_u'(\alpha) = u|\alpha|^u \alpha^{-1} = u\theta_u(\alpha)\alpha^{-1}.$$

By induction, it follows that for any $l \in \mathbb{N}$ the function θ_u is l times differentiable, and that the l-th derivative is

$$\theta_u^{(l)}(\alpha) = u(u-1)\cdots(u-l+1)\theta_u(\alpha)\alpha^{-l}. \tag{4.4}$$

Now suppose that $u > 0$. Then, by putting $\theta_u(0) = 0$ we extend θ_u to a continuous function on $\mathbb{R}$. More generally, whenever $u > l$, then

$$\lim_{\alpha \to 0} \frac{\theta_u(\alpha)}{\alpha^l} = 0.$$

By (4.4), this shows that whenever $u > l$ then $\theta_u^{(l)}$ extends to a continuous function on $\mathbb{R}$ by choosing $\theta_u^{(l)}(0) = 0$, and that $\theta_u^{(l-1)}$ is differentiable at 0 with derivative 0. We summarize this last statement as follows:

(a) *Let $k \in \mathbb{N}$ and $u > k$. Then θ_u is k times continuously differentiable on $\mathbb{R}$.*

Next, for $u > 0$, consider the function $\rho_u : \mathbb{R} \to \mathbb{R}$ defined by putting $\rho_u(\alpha) = |\sin \pi\alpha|^u$. For $\alpha \in (0, 1)$ one has $\sin \pi\alpha > 0$, whence $\rho_u(\alpha) = (\sin \pi\alpha)^u$. Thus ρ_u is smooth on $(0, 1)$. But ρ has period 1, so it suffices to examine its differentiability properties at $\alpha = 0$, a point at which ρ_u is continuous. For all real α we have $\sin \pi\alpha = \pi\alpha E(\alpha)$, where

$$E(\alpha) = \sum_{j=0}^{\infty} (-1)^j \frac{(\pi\alpha)^{2j}}{(2j+1)!}.$$

The function E is smooth on $\mathbb{R}$ with $E(0) = 1$. Hence $E(\alpha) > 0$ in a neighbourhood of 0 where we then also have

$$\rho_u(\alpha) = \pi^u |\alpha|^u E(\alpha)^u.$$

By applying the product rule in combination with our earlier conclusion (a), we therefore conclude as follows:

(b) *Let $k \in \mathbb{N}$ and $u > k$. Then ρ_u is k times continuously differentiable on $\mathbb{R}$.*

We now turn to the function T where we suppose that $M = 0$, as we may. The sum in (4.1) defines a holomorphic function of the complex variable α, and hence the function $T : \mathbb{R} \to \mathbb{C}$ is a smooth map of period 1. The sum

$$\bar{T}(\alpha) = \sum_{1 \leqslant n \leqslant N} \bar{c}_n e(-\alpha n)$$

defines another trigonometric polynomial, and for $\alpha \in \mathbb{R}$ we have $\overline{T(\alpha)} = \bar{T}(\alpha)$. Consequently, for real α we have

$$|T(\alpha)|^2 = T(\alpha)\bar{T}(\alpha), \tag{4.5}$$

whence the function $|T|^2 : \mathbb{R} \to \mathbb{C}$, given by $\alpha \mapsto |T(\alpha)|^2$, is smooth on $\mathbb{R}$ with

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} |T(\alpha)|^2 = T'(\alpha)\bar{T}(\alpha) + T(\alpha)\bar{T}'(\alpha). \tag{4.6}$$

On noting that $T(\alpha)^j$ is again a trigonometric polynomial for all $j \in \mathbb{N}$, we see that $|T(\alpha)|^{2j}$ is smooth. Hence, from now on, we may suppose that *u is a real number but not an even natural number.* Also, the conclusion of Lemma 4.1 is certainly true in the trivial case where $c_n = 0$ for all n. In the contrary case, the polynomial in (4.2) has at most finitely many zeros. Therefore, the set

$$Z = \{\alpha \in \mathbb{R} : T(\alpha) = 0\}$$

is 1-periodic with $Z \cap [0, 1)$ finite, and consequently $\mathbb{R} \setminus Z$ is open.

We next examine the function $|T|^u : \mathbb{R} \setminus Z \to \mathbb{C}$, given by $\alpha \mapsto |T(\alpha)|^u$.

(c) *When u is real but not an even natural number, the function $|T|^u$ is smooth.*

In order to confirm this assertion, note that $|T(\alpha)|^u = \theta_{u/2}(|T(\alpha)|^2)$. By applying the chain rule in combination with the preamble to conclusion (a) and (4.6), we find that $|T(\alpha)|^u$ is differentiable for $\alpha \in \mathbb{R} \setminus Z$. Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\,|T(\alpha)|^u &= \theta'_{u/2}(|T(\alpha)|^2)\big(T'(\alpha)\bar{T}(\alpha) + T(\alpha)\bar{T}'(\alpha)\big)\\ &= \frac{u}{2}|T(\alpha)|^{u-2}\big(T'(\alpha)\bar{T}(\alpha) + T(\alpha)\bar{T}'(\alpha)\big). \qquad (4.7)\end{aligned}$$

Since the final factor on the right hand side here is smooth, we may repeatedly apply the product rule to conclude that $|T(\alpha)|^u$ is smooth on $\mathbb{R} \setminus Z$, as claimed.

Finally, we consider any element $\alpha_0 \in Z$. Then one has $P(e(\alpha_0)) = 0$. Since P is not the zero polynomial, there exists $r \in \mathbb{N}$ and a polynomial $Q \in \mathbb{C}[z]$ with $Q(e(\alpha_0)) \neq 0$ such that $P(z) = (z - e(\alpha_0))^r Q(z)$. Write $U(\alpha) = Q(e(\alpha))$ for the trigonometric polynomial associated with Q. Then $T(\alpha) = \big(e(\alpha) - e(\alpha_0)\big)^r U(\alpha)$. For $u > 0$ and all real α we then have

$$|T(\alpha)|^u = |e(\alpha) - e(\alpha_0)|^{ru}|U(\alpha)|^u = |2\sin \pi(\alpha - \alpha_0)|^{ru}|U(\alpha)|^u.$$

There is an open neighbourhood of α_0 on which $U(\alpha)$ does not vanish. By our conclusion (c) it is apparent that $|U(\alpha)|^u$ is smooth on this neighbourhood. If $u > k$, then the conclusion (b) implies that the function $|2\sin \pi(\alpha - \alpha_0)|^{ru}$ is k times continuously differentiable. The conclusion of the lemma therefore follows by application of the product rule. □

We mention in passing that if more is known about the zeros of P, then the argument that we have presented shows more. For example, if all the zeros in Z are double zeros and $u > k$, then $|T(\alpha)|^u$ is $2k$ times differentiable.

Lemma 4.2. *Let $W : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function of period 1, and let $u \geqslant 2$. For $l \in \mathbb{Z}$ let*

$$b_l = \int_0^1 W(\alpha)|T(\alpha)|^u e(-\alpha l)\,\mathrm{d}\alpha. \qquad (4.8)$$

Then, for all $l \in \mathbb{Z} \setminus \{0\}$, one has

$$|b_l| \leqslant \frac{1}{(2\pi l)^2}\int_0^1 \Big|\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\,W(\alpha)|T(\alpha)|^u\Big|\,\mathrm{d}\alpha. \qquad (4.9)$$

Moreover, for all $\alpha \in \mathbb{R}$ one has the Fourier series expansion

$$W(\alpha)|T(\alpha)|^u = \sum_{l\in\mathbb{Z}} b_l e(\alpha l), \qquad (4.10)$$

in which the right hand side converges absolutely and uniformly on $\mathbb{R}$.

Proof. By (4.5) and Lemma 4.1, the condition $u \geqslant 2$ ensures that $W(\alpha)|T(\alpha)|^u$ is twice continuously differentiable. Hence, the integral on the right hand side of (4.9) exists, and the upper bound (4.9) follows from (4.8) by integrating by parts two times. Furthermore, the upper bound (4.9) ensures that the series in (4.10) converges absolutely and uniformly on $\mathbb{R}$. Thus, by [18, Chapter II, Theorem 8.14], this Fourier series sums to $W(\alpha)|T(\alpha)|^u$. ☐

In this paper Lemmata 4.1 and 4.2 will only be used with the quartic Weyl sum f, as defined in (1.10), in the role of T. The weight W will be either constantly 1 or a smooth minor arc. Let $u > 0$ and define the Fourier coefficient

$$\psi_u(n) = \int_0^1 |f(\alpha)|^u e(-\alpha n)\,\mathrm{d}\alpha. \tag{4.11}$$

Also, with a parameter Y at our disposal within the range $1 \leqslant Y \leqslant \frac{1}{4}P^2$, we consider the smooth minor arcs $\mathsf{n}(\alpha) = \mathsf{n}_{P^4,Y}(\alpha)$ and introduce the related Fourier coefficient

$$\phi_u(n) = \int_0^1 \mathsf{n}(\alpha)|f(\alpha)|^u e(-\alpha n)\,\mathrm{d}\alpha. \tag{4.12}$$

Lemma 4.3. *Suppose that* $u \geqslant 2$ *and* $1 \leqslant Y \leqslant \frac{1}{4}P^2$. *Then, for all* $n \in \mathbb{Z}\setminus\{0\}$, *one has*

$$|\phi_u(n)| + |\psi_u(n)| \ll P^{u+8}n^{-2}.$$

Proof. We first compute the derivatives of $|f(\alpha)|^u$. Suppose temporarily that u is not an even natural number. By (4.7), whenever $f(\alpha) \neq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}|f(\alpha)|^u = \frac{u}{2}|f(\alpha)|^{u-2}\left(f'(\alpha)\bar{f}(\alpha) + f(\alpha)\bar{f}'(\alpha)\right),$$

and we may differentiate again to confirm the identity

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}|f(\alpha)|^u =& \frac{u(u-2)}{4}|f(\alpha)|^{u-4}\big(f'(\alpha)\bar{f}(\alpha) + f(\alpha)\bar{f}'(\alpha)\big)^2 \\ &+ \frac{u}{2}|f(\alpha)|^{u-2}\big(f''(\alpha)\bar{f}(\alpha) + 2f'(\alpha)\bar{f}'(\alpha) + f(\alpha)\bar{f}''(\alpha)\big).\end{aligned}$$

These formulae hold for all $\alpha \in \mathbb{R}$ when u is an even natural number, and thus

$$\left|\frac{\mathrm{d}}{\mathrm{d}\alpha}|f(\alpha)|^u\right| \leqslant u|f(\alpha)|^{u-1}|f'(\alpha)|$$

and

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}|f(\alpha)|^u\right| \leqslant u(u-1)|f(\alpha)|^{u-2}|f'(\alpha)|^2 + u|f(\alpha)|^{u-1}|f''(\alpha)|.$$

Hence, the trivial estimates $f(\alpha) \ll P$, $f'(\alpha) \ll P^5$ and $f''(\alpha) \ll P^9$ suffice to conclude that the upper bounds

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}|f(\alpha)|^u \ll P^{u+4} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}|f(\alpha)|^u \ll P^{u+8} \tag{4.13}$$

hold for all $\alpha \in \mathbb{R}$ when either $u = 2$ or $f(\alpha) \neq 0$. However, when $u > 2$ these derivatives will be zero whenever $f(\alpha) = 0$, so the inequalities (4.13) hold uniformly in $\alpha \in \mathbb{R}$. The upper bound $\psi_u(n) \ll P^{u+8}n^{-2}$ is now immediate from Lemma 4.2. Furthermore, an application of the product rule in combination with Lemma 3.2 and (4.13) shows that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,\mathsf{n}(\alpha)|f(\alpha)|^u \ll P^{u+4} \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\,\mathsf{n}(\alpha)|f(\alpha)|^u \ll P^{u+8}.$$

The estimate $\phi_u(n) \ll P^{u+8}n^{-2}$ therefore follows by invoking Lemma 4.2 once again, and this completes the proof of the lemma. □

5. Cubic moments of Fourier coefficients

The principal results in this section are the upper bounds for cubic moments of $\phi_u(n)$ and $\psi_u(n)$ embodied in Theorem 5.1 below. The proof of these estimates involves a development of the ideas underpinning the main line of thought in our earlier paper [6]. For $u > 0$ it is convenient to define

$$\delta(u) = (25 - 3u)/6. \tag{5.1}$$

In many of the computations later it is useful to note that

$$3u - 8 + \delta(u) = \frac{5}{2}u - \frac{23}{6}. \tag{5.2}$$

Theorem 5.1. *Let u be a real number with $6 \leqslant u \leqslant 25/3$. Then*

$$\sum_{n \in \mathbb{Z}} |\psi_u(n)|^3 \ll P^{3u-8+\delta(u)+\varepsilon}. \tag{5.3}$$

Further, when $2P^{4/15} \leqslant Y \leqslant P/16$ and $6 \leqslant u \leqslant 11$, one has

$$\sum_{n \in \mathbb{Z}} |\phi_u(n)|^3 \ll P^{3u-8+\delta(u)+\varepsilon}. \tag{5.4}$$

When $u \geqslant 6$, the contribution from the major arcs to the sum in (5.3) is easily seen to be of order P^{3u-8}. Since $\delta(u)$ is negative for $u > 25/3$, we cannot expect that the upper bound (5.3) holds for such u. However, as is evident from (5.4), a minor arcs version remains valid for $u \leqslant 11$. Before we embark on the proof of this theorem, we summarize some mean value estimates related to the Weyl sum (1.10). In the following two lemmata, we assume that $1 \leqslant Y \leqslant P/8$ and write $\mathfrak{M} = \mathfrak{M}_{P^4,Y}$ and $\mathfrak{m} = \mathfrak{m}_{P^4,Y}$. It is useful to note that $\mathfrak{m}_{P^4,Y} = \mathfrak{m}_{P^4,P/8} \cup \mathfrak{K}$, where $\mathfrak{K} = \mathfrak{M}_{P^4,P/8} \setminus \mathfrak{M}_{P^4,Y}$. Then, from [13, Lemma 5.1], we have the bounds

$$\int_{\mathfrak{M}} |f(\alpha)|^6 \,\mathrm{d}\alpha \ll P^2 \quad \text{and} \quad \int_{\mathfrak{K}} |f(\alpha)|^6 \,\mathrm{d}\alpha \ll P^2 Y^{\varepsilon - 1/4}. \tag{5.5}$$

Lemma 5.2. *Suppose that $P^{4/15} \leqslant Y \leqslant P/8$. Then*

$$\int_{\mathfrak{m}} |f(\alpha)|^{20} \,\mathrm{d}\alpha \ll P^{15+\varepsilon}.$$

Proof. For $Y = P/8$, the desired estimate is the case $k = 4$, $w = 20$ of Wooley [16, Lemma 3.1]. For smaller values of Y, we make use of the case $Y = P/8$ and apply the second bound of (5.5). On combining [14, Theorem 4.1] with [14, Lemma 2.8 and Theorem 4.2], moreover, one readily confirms that the upper bound $f(\alpha) \ll PY^{-1/4}$ holds uniformly for $\alpha \in \mathfrak{K}$. Consequently, one has the estimate

$$\int_{\mathfrak{K}} |f(\alpha)|^{20}\,\mathrm{d}\alpha \ll P^{16}Y^{\varepsilon-15/4},$$

and the conclusion of the lemma follows. □

Lemma 5.3. *When* $8 \leqslant u \leqslant 14$*, one has*

$$\int_0^1 |f(\alpha)|^u\,\mathrm{d}\alpha \ll P^{\frac{5}{6}u-\frac{5}{3}+\varepsilon}. \tag{5.6}$$

Meanwhile, when $8 \leqslant u \leqslant 20$*, then uniformly in* $P^{4/15} \leqslant Y \leqslant P/8$*, one has*

$$\int_{\mathfrak{m}} |f(\alpha)|^u\,\mathrm{d}\alpha \ll P^{\frac{5}{6}u-\frac{5}{3}+\varepsilon}. \tag{5.7}$$

Proof. It is a consequence of Hua's Lemma [14, Lemma 2.5] that

$$\int_{\mathfrak{m}} |f(\alpha)|^8\,\mathrm{d}\alpha \leqslant \int_0^1 |f(\alpha)|^8\,\mathrm{d}\alpha \ll P^{5+\varepsilon}. \tag{5.8}$$

One interpolates linearly between this estimate and the bound established in Lemma 5.2 via Hölder's inequality to confirm the upper bound (5.7) for $8 \leqslant u \leqslant 20$. The upper bound (5.6) then follows on noting that for $6 \leqslant u \leqslant 14$, it follows from (5.5) that

$$\int_{\mathfrak{M}} |f(\alpha)|^u\,\mathrm{d}\alpha \ll P^{u-4} \ll P^{\frac{5}{6}u-\frac{5}{3}}.$$

Since $[0,1] = \mathfrak{M} \cup \mathfrak{m}$, the desired conclusion follows at once. □

In the special case $u = 14$, the first conclusion of Lemma 5.3 assumes the simple form already announced in (1.5).

Lemma 5.4. *Let* $\mathscr{Z}$ *be a set of* Z *integers. Then*

$$\int_0^1 \Big|\sum_{z\in\mathscr{Z}} e(\alpha z)\Big|^2 |f(\alpha)|^2\,\mathrm{d}\alpha \ll PZ + P^{1/2+\varepsilon}Z^{3/2}$$

and

$$\int_0^1 \Big|\sum_{z\in\mathscr{Z}} e(\alpha z)\Big|^2 |f(\alpha)|^4\,\mathrm{d}\alpha \ll P^3Z + P^{2+\varepsilon}Z^{3/2}.$$

Proof. This is essentially contained in [12, Lemma 6.1], where these estimates are established in the case when $\mathscr{Z}$ is contained in $[0, P^4]$. As pointed out in [9, Lemma 2.2] this condition is not required. □

We now have available sufficient infrastructure to derive upper bounds for cubic moments of $\phi_u(n)$ and $\psi_u(n)$.

The proof of Theorem 5.1. Let $\vartheta_u(n)$ denote one of $\psi_u(n)$, $\phi_u(n)$. On examining the statement of the theorem, it is apparent that we may assume that in the former case we have $6 \leqslant u \leqslant 25/3$, and in the latter case $6 \leqslant u \leqslant 11$ and $2P^{4/15} \leqslant Y \leqslant P/16$. We begin with the observation that, by Lemma 4.3, one has $\vartheta_u(n) \ll P^{u+8}n^{-2}$. Consequently, when $u \geqslant 6$, one has

$$\sum_{|n|>P^7} |\vartheta_u(n)|^3 + \sum_{\substack{|n|\leqslant P^7\\ |\vartheta_u(n)|\leqslant 1}} |\vartheta_u(n)|^3 \ll P^7 + P^{3u+24} \sum_{|n|>P^7} n^{-6} \ll P^{3u-11}.$$

It remains to consider the contribution of those integers n with $|n| \leqslant P^7$ and $|\vartheta_u(n)| > 1$. We put $\Theta(\alpha) = 1$ when $\vartheta_u = \psi_u$, and $\Theta(\alpha) = \mathsf{n}(\alpha)$ when $\vartheta_u = \phi_u$. Then the definitions (4.11) and (4.12) take the common form

$$\vartheta_u(n) = \int_0^1 \Theta(\alpha)|f(\alpha)|^u e(-\alpha n)\,\mathrm{d}\alpha. \tag{5.9}$$

By Lemma 3.2, it follows that $\Theta(\alpha) \in [0,1]$. Thus, by Lemma 5.3, one finds that

$$|\vartheta_u(n)| \leqslant \vartheta_u(0) \leqslant \psi_u(0) \ll P^{\frac{5}{6}u-\frac{5}{3}+\varepsilon} \quad (8 \leqslant u \leqslant 11).$$

In the missing cases where $6 \leqslant u < 8$ one interpolates between (5.8) and the elementary inequality

$$\int_0^1 |f(\alpha)|^4\,\mathrm{d}\alpha \ll P^{2+\varepsilon}, \tag{5.10}$$

also a consequence of Hua's Lemma [14, Lemma 2.5], to conclude that

$$|\vartheta_u(n)| \leqslant \vartheta_u(0) \leqslant \psi_u(0) \ll P^{2+\frac{3}{4}(u-4)+\varepsilon}.$$

Fix a number τ with $0 < \tau < 10^{-10}$ and define T_0 by

$$T_0 = \begin{cases} P^{\frac{3}{4}u-1+\tau}, & \text{when } 6 \leqslant u < 8,\\ P^{\frac{5}{6}u-\frac{5}{3}+\tau}, & \text{when } 8 \leqslant u \leqslant 11.\end{cases}$$

Then, on recalling the upper bounds for $\vartheta_u(n)$ just derived, a familiar dyadic dissection argument shows that there is a number $T \in [1, T_0]$ with the property that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\vartheta_u(n)|^3 &\ll P^{3u-11} + (\log P) \sum_{\substack{|n|\leqslant P^7\\ T<|\vartheta_u(n)|\leqslant 2T}} |\vartheta_u(n)|^3\\ &\ll P^{3u-11} + P^{\varepsilon}T^3 Z,\end{aligned} \tag{5.11}$$

where Z denotes the number of elements in the set

$$\mathscr{Z} = \{n \in \mathbb{Z} : |n| \leqslant P^7 \text{ and } T < |\vartheta_u(n)| \leqslant 2T\}.$$

For each $n \in \mathscr{Z}$ there is a complex number η_n, with $|\eta_n| = 1$, for which $\eta_n\vartheta_u(n)$ is a positive real number. Write

$$K(\alpha) = \sum_{n\in\mathscr{Z}} \eta_n e(-\alpha n). \tag{5.12}$$

Then one concludes from (5.9) and orthogonality that

$$TZ < \sum_{n\in\mathscr{Z}} \eta_n\vartheta_u(n) = \int_0^1 \Theta(\alpha)K(\alpha)|f(\alpha)|^u\,\mathrm{d}\alpha. \tag{5.13}$$

Beyond this point our argument depends on the size of T. Our first argument handles the small values $T \leqslant P^{\frac{5}{6}u-\frac{35}{18}}$. By (5.13) and Hölder's inequality, we obtain the bound

$$TZ \leqslant I^{1/2}\Bigl(\int_0^1 |K(\alpha)^2 f(\alpha)^4|\,\mathrm{d}\alpha\Bigr)^{1/3}\Bigl(\int_0^1 |K(\alpha)|^2\,\mathrm{d}\alpha\Bigr)^{1/6}, \tag{5.14}$$

where

$$I = \int_0^1 \Theta(\alpha)^2|f(\alpha)|^{2u-\frac{8}{3}}\,\mathrm{d}\alpha.$$

By orthogonality, one has

$$\int_0^1 |K(\alpha)|^2\,\mathrm{d}\alpha = Z,$$

and by a consideration of the underlying Diophantine equations, one deduces via Lemma 5.4 that

$$\int_0^1 |K(\alpha)^2 f(\alpha)^4|\,\mathrm{d}\alpha \ll P^3 Z + P^{2+\varepsilon}Z^{3/2}. \tag{5.15}$$

Next we confirm the bound $I \ll P^{\frac{5}{3}u-\frac{35}{9}+\varepsilon}$. Indeed, in the case where $\Theta = 1$ we have $6 \leqslant u \leqslant 25/3$. In such circumstances $8 < 2u - 8/3 \leqslant 14$, and so (5.6) applies and yields the claimed bound. In the case $\Theta = \mathsf{n}$ we have $u \leqslant 11$, and hence $2u - 8/3 < 20$. Write $\mathfrak{m} = \mathfrak{m}_{P^4,Y/2}$. Then by Lemma 3.2, we have $0 \leqslant \mathsf{n}(\alpha) \leqslant \mathbf{1}_{\mathfrak{m}}$. We therefore deduce that in this second case we have

$$I \leqslant \int_0^1 \mathsf{n}(\alpha)|f(\alpha)|^{2u-\frac{8}{3}}\,\mathrm{d}\alpha \leqslant \int_{\mathfrak{m}} |f(\alpha)|^{2u-\frac{8}{3}}\,\mathrm{d}\alpha,$$

and (5.7) confirms our claimed bound for I.

Collecting these estimates together within (5.14), we now have

$$TZ \ll P^{\varepsilon}\left(P^3 Z + P^2 Z^{3/2}\right)^{1/3} Z^{1/6}\bigl(P^{\frac{5}{3}u-\frac{35}{9}}\bigr)^{1/2}.$$

On recalling (5.2), we find that this relation disentangles to yield the bound

$$\begin{aligned} T^3 Z &\ll P^{2+\frac{3}{2}(\frac{5}{3}u-\frac{35}{9})+\varepsilon} + TP^{2+\frac{5}{3}u-\frac{35}{9}+\varepsilon} \\ &= P^{3u-8+\delta(u)+\varepsilon} + TP^{\frac{5}{3}u-\frac{17}{9}+\varepsilon}. \end{aligned}$$

It transpires that in the range $T \leqslant P^{\frac{5}{6}u-\frac{35}{18}}$ the first term on the right hand side dominates, so that we finally reach the desired conclusion $T^3 Z \ll P^{3u-8+\delta(u)+\varepsilon}$. In view of (5.11), this is enough to complete the proof of Theorem 5.1 in the case that T is small.

Our second approach is suitable for T of medium size, with

$$P^{\frac{5}{6}u-\frac{35}{18}} < T \leqslant P^{\frac{5}{6}u-\frac{11}{6}}. \tag{5.16}$$

We apply Schwarz's inequality to (5.13), obtaining the bound

$$TZ \leqslant \biggl(\int_0^1 |K(\alpha)^2 f(\alpha)^4| \, \mathrm{d}\alpha \biggr)^{1/2} \biggl(\int_0^1 \Theta(\alpha)^2 |f(\alpha)|^{2u-4} \, \mathrm{d}\alpha \biggr)^{1/2}.$$

Note that when $6 \leqslant u \leqslant 11$, one has $8 \leqslant 2u - 4 \leqslant 18$, and when instead $u \leqslant 25/3$, we have $2u - 4 < 14$. Hence, as in the proof of our earlier estimate for I, it follows from Lemma 5.3 that

$$\int_0^1 \Theta(\alpha)^2 |f(\alpha)|^{2u-4} \, \mathrm{d}\alpha \ll P^{\frac{5}{3}u - 5 + \varepsilon}.$$

Applying this estimate in combination with (5.15), we conclude that

$$TZ \ll P^{\varepsilon} (P^3 Z + P^2 Z^{3/2})^{1/2} (P^{\frac{5}{3}u - 5})^{1/2}.$$

This bound disentangles to deliver the relation

$$T^3 Z \ll T P^{\frac{5}{3}u - 2 + \varepsilon} + T^{-1} P^{\frac{10}{3}u - 6 + \varepsilon}.$$

On recalling (5.2), we find that our present assumptions (5.16) concerning the size of T deliver the estimate

$$T^3 Z \ll P^{\frac{5}{2}u - \frac{23}{6} + \varepsilon} + P^{\frac{5}{2}u - \frac{73}{18} + \varepsilon} \ll P^{3u - 8 + \delta(u) + \varepsilon}.$$

The conclusion of Theorem 5.1 again follows in this case, by virtue of (5.11).

The analysis of the large values T satisfying $P^{\frac{5}{6}u - \frac{11}{6}} < T \leqslant T_0$ is more subtle. Suppose temporarily that $\vartheta_u = \psi_u$, and hence that $u \leqslant 25/3$. Then, by (3.5) and (5.13),

$$TZ \leqslant \int_0^1 \mathsf{N}(\alpha) K(\alpha) |f(\alpha)|^u \, \mathrm{d}\alpha + \int_0^1 \mathsf{n}(\alpha) K(\alpha) |f(\alpha)|^u \, \mathrm{d}\alpha.$$

By hypothesis, we have $u \geqslant 6$. Also, from Lemma 3.1, we have $\mathsf{N} \leqslant \mathbf{1}_{\mathfrak{N}_{P^4, P/8}}$, so that (5.5) yields the bound

$$\int_0^1 \mathsf{N}(\alpha) K(\alpha) |f(\alpha)|^u \, \mathrm{d}\alpha \leqslant Z \int_{\mathfrak{M}_{P^4, P/8}} |f(\alpha)|^u \, \mathrm{d}\alpha \ll Z P^{u-4}.$$

Since $u - 4 < \frac{5}{6}u - \frac{11}{6}$, for large enough P one has $ZP^{u-4} < \frac{1}{2} TZ$. Thus

$$TZ \ll \int_0^1 \mathsf{n}(\alpha) K(\alpha) |f(\alpha)|^u \, \mathrm{d}\alpha. \tag{5.17}$$

Note that this is exactly the inequality (5.13) in the case where $\vartheta_u = \phi_u$. Consequently, the upper bound (5.17) holds for the large values of T currently under consideration, irrespective of the choice of ϑ_u. Now apply Schwarz's inequality to (5.17). Then, by Lemma 3.2, we deduce that

$$TZ \leqslant \biggl(\int_0^1 |K(\alpha) f(\alpha)|^2 \, \mathrm{d}\alpha \biggr)^{1/2} \biggl(\int_{\mathfrak{m}} |f(\alpha)|^{2u-2} \, \mathrm{d}\alpha \biggr)^{1/2},$$

where again we write $\mathfrak{m} = \mathfrak{m}_{P^4, Y/2}$. Note here that $u \leqslant 11$, so that $2u - 2 \leqslant 20$. Hence, by Lemmata 5.3 and 5.4, we have

$$TZ \ll P^{\varepsilon} \bigl(PZ + P^{\frac{1}{2}} Z^{\frac{3}{2}} \bigr)^{1/2} \bigl(P^{\frac{5}{3}u - \frac{10}{3}} \bigr)^{1/2}.$$

Consequently, our assumptions concerning the size of T reveal that

$$\begin{aligned} T^3Z &\ll TP^{\frac{5}{3}u-\frac{7}{3}+\varepsilon}+T^{-1}P^{\frac{10}{3}u-\frac{17}{3}+\varepsilon} \\ &\ll T_0P^{\frac{5}{3}u-\frac{7}{3}+\varepsilon}+P^{\frac{5}{2}u-\frac{23}{6}+\varepsilon}. \end{aligned} \tag{5.18}$$

When $6\leqslant u<8$, one has

$$\big(\tfrac{3}{4}u-1\big)+\big(\tfrac{5}{3}u-\tfrac{7}{3}\big)=\tfrac{29}{12}u-\tfrac{10}{3}\leqslant \tfrac{5}{2}u-\tfrac{23}{6},$$

whilst for $8\leqslant u\leqslant 11$,

$$\big(\tfrac{5}{6}u-\tfrac{5}{3}\big)+\big(\tfrac{5}{3}u-\tfrac{7}{3}\big)=\tfrac{5}{2}u-4<\tfrac{5}{2}u-\tfrac{23}{6}.$$

Then in either case one finds from (5.18) via (5.2) that $T^3Z\ll P^{3u-8+\delta(u)+2\tau}$, and the conclusion of Theorem 5.1 follows in this final case, again by (5.11), on taking τ sufficiently small. $\square$

We close this section with a related but simpler result.

Theorem 5.5. *One has*

$$\sum_{n\in\mathbb{Z}}\psi_4(n)^3\ll P^{13/2+\varepsilon}.$$

Proof. By (4.11) and orthogonality, the Fourier coefficient $\psi_4(n)$ has a Diophantine interpretation that shows on the one hand that $\psi_4(n)\in\mathbb{N}_0$, and on the other that $\psi_4(n)=0$ for all $n\in\mathbb{Z}$ with $|n|>2P^4$. By (4.11) and (5.10), we also have the bound $\psi_4(n)\leqslant\psi_4(0)\ll P^{2+\varepsilon}$. The argument leading to (5.11) now shows that there is a number T with $1\leqslant T\leqslant P^{2+\varepsilon}$ having the property that

$$\begin{aligned} \sum_{n\in\mathbb{Z}}\psi_4(n)^3 &\ll P^{6+\varepsilon}+P^{\varepsilon}\sum_{\substack{|n|\leqslant 2P^4\\ T\leqslant\psi_4(n)\leqslant 2T}}\psi_4(n)^3 \\ &\ll P^{6+\varepsilon}+P^{\varepsilon}T^3Z, \end{aligned} \tag{5.19}$$

where Z denotes the number of elements in the set

$$\mathscr{Z}=\{n\in\mathbb{Z}:|n|\leqslant 2P^4 \text{ and } T<|\psi_4(n)|\leqslant 2T\}.$$

As in the corresponding analysis within the proof of Theorem 5.1, we next find that there are unimodular complex numbers η_n $(n\in\mathscr{Z})$ having the property that, with $K(\alpha)$ defined via (5.12), one has

$$TZ<\int_0^1 K(\alpha)|f(\alpha)|^4\,\mathrm{d}\alpha.$$

We first handle small values of T. Here, an application of Schwarz's inequality leads via (5.8) to the bound

$$TZ\leqslant\bigg(\int_0^1|f(\alpha)|^8\,\mathrm{d}\alpha\bigg)^{1/2}\bigg(\int_0^1|K(\alpha)|^2\,\mathrm{d}\alpha\bigg)^{1/2}\ll P^{5/2+\varepsilon}Z^{1/2}.$$

This disentangles to yield $T^3Z\ll TP^{5+\varepsilon}$, proving the theorem for $T\leqslant P^{3/2}$.

Next, when T is large, we apply Hölder's inequality in a manner similar to that employed in the large values analysis of the proof of Theorem 5.1. Thus

$$TZ \leqslant \left(\int_0^1 |K(\alpha)^2 f(\alpha)^2|\,\mathrm{d}\alpha\right)^{1/2}\left(\int_0^1 |f(\alpha)|^4\,\mathrm{d}\alpha\right)^{1/4}\left(\int_0^1 |f(\alpha)|^8\,\mathrm{d}\alpha\right)^{1/4},$$

and hence

$$TZ \ll P^{\varepsilon}(PZ + P^{1/2}Z^{3/2})^{1/2}P^{7/4}.$$

We now obtain the bound

$$T^3 Z \ll TP^{9/2+\varepsilon} + T^{-1}P^{8+\varepsilon},$$

and in view of (5.19), this proves Theorem 5.5 in the complementary case $P^{3/2} \leqslant T \leqslant P^{2+\varepsilon}$. □

6. Mean values of quartic Weyl sums

In this section we estimate certain entangled moments of quartic Weyl sums, and then apply them to obtain minor arc estimates for use within the proofs of Theorems 1.1 and 1.2. Throughout this section and the next, let the pair of integers c_i, d_i $(1 \leqslant i \leqslant 5)$ satisfy the condition that the points $(c_i : d_i) \in \mathbb{P}^1(\mathbb{Q})$ are distinct. Define the linear forms $\mathrm{M}_i = \mathrm{M}_i(\alpha, \beta)$ $(1 \leqslant i \leqslant 5)$ by

$$\mathrm{M}_i(\alpha,\beta) = c_i\alpha + d_i\beta. \tag{6.1}$$

Let $u > 0$, and recall the definition of the exponent $\delta(u)$ from (5.1). Then, with $2P^{4/15} \leqslant Y \leqslant P/16$ and $\mathsf{n} = \mathsf{n}_{P^4,Y}$, we consider the mean values

$$I_u = \int_0^1\int_0^1 |f(\mathrm{M}_1)f(\mathrm{M}_2)f(\mathrm{M}_3)|^u\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$
$$J_u = \int_0^1\int_0^1 \mathsf{n}(\mathrm{M}_1)\mathsf{n}(\mathrm{M}_2)\mathsf{n}(\mathrm{M}_3)|f(\mathrm{M}_1)f(\mathrm{M}_2)f(\mathrm{M}_3)|^u\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Theorem 6.1. *One has $I_4 \ll P^{13/2+\varepsilon}$ and $I_u \ll P^{3u-8+\delta(u)+\varepsilon}$ $(6 \leqslant u \leqslant 25/3)$. Also, when $6 \leqslant u \leqslant 11$, one has $J_u \ll P^{3u-8+\delta(u)+\varepsilon}$.*

Proof. It follows from Lemmata 3.2 and 4.2 that the function $\mathsf{n}(\gamma)|f(\gamma)|^u$ has a uniformly convergent Fourier series with coefficients $\phi_u(n)$. By orthogonality, we conclude that

$$J_u = \sum_{(n_1,n_2,n_3)\in N} \phi_u(n_1)\phi_u(n_2)\phi_u(n_3),$$

where N is the set of solutions in integers n_1, n_2, n_3 of the linear system

$$c_1n_1 + c_2n_2 + c_3n_3 = d_1n_1 + d_2n_2 + d_3n_3 = 0.$$

Since the projective points $(c_i : d_i)$ are distinct, there exist non-zero integers l_i, depending only on the c_i, d_i, having the property that the solutions of this system are precisely the triples $(n_1, n_2, n_3) = m(l_1, l_2, l_3)$ $(m \in \mathbb{Z})$. It therefore follows from (2.2) that

$$J_u \leqslant \frac{1}{3}\sum_{m\in\mathbb{Z}} \big(|\phi_u(l_1m)|^3 + |\phi_u(l_2m)|^3 + |\phi_u(l_3m)|^3\big) \leqslant \sum_{n\in\mathbb{Z}} |\phi_u(n)|^3.$$

The desired bound for J_u now follows from Theorem 5.1. The bounds for I_4 and I_u follow in the same way, but the argument has to be built on the cubic moment estimates for $\psi_u(n)$ that are provided by Theorems 5.1 and 5.5. □

We now turn to related, less balanced mixed moments. With u and Y as before, we define

$$K_u = \int_0^1 \int_0^1 |f(\mathrm{M}_1)f(\mathrm{M}_2)|^u |f(\mathrm{M}_3)|^6 \,\mathrm{d}\alpha\,\mathrm{d}\beta,$$
$$L_u = \int_0^1 \int_0^1 \mathsf{n}(\mathrm{M}_1)\mathsf{n}(\mathrm{M}_2)|f(\mathrm{M}_1)f(\mathrm{M}_2)|^u |f(\mathrm{M}_3)|^6 \,\mathrm{d}\alpha\,\mathrm{d}\beta,$$

and put

$$\eta(u) = \frac{19}{6} - \frac{u}{3}.$$

Theorem 6.2. *Subject to the hypotheses of this section, one has*

$$K_u \ll P^{2u-2+\eta(u)+\varepsilon} \quad (6 \leqslant u \leqslant 19/2),$$
$$L_u \ll P^{2u-2+\eta(u)+\varepsilon} \quad (6 \leqslant u \leqslant 11).$$

Proof. We proceed as in the initial phase of the proof of Theorem 6.1. Using the same notation, we obtain

$$L_u = \sum_{(n_1,n_2,n_3)\in N} \phi_u(n_1)\phi_u(n_2)\psi_6(n_3).$$

Note here that $\psi_6(m)$ counts solutions of a Diophantine equation, and consequently is a non-negative integer. Hence

$$L_u \leqslant \frac{1}{2} \sum_{(n_1,n_2,n_3)\in N} \psi_6(n_3)\big(|\phi_u(n_2)|^2 + |\phi_u(n_1)|^2\big).$$

By symmetry, we may therefore suppose that for appropriate non-zero integers l_2 and l_3, depending at most on $\mathbf{c}$ and $\mathbf{d}$, one has

$$L_u \leqslant \sum_{(n_1,n_2,n_3)\in N} \psi_6(n_3)|\phi_u(n_2)|^2 = \sum_{m\in\mathbb{Z}} \psi_6(l_3 m)|\phi_u(l_2 m)|^2. \tag{6.2}$$

Next, first applying Hölder's inequality, and then Theorem 5.1 and (5.2), we obtain the bound

$$L_u \leqslant \Big(\sum_{n\in\mathbb{Z}} \psi_6(n)^3\Big)^{1/3} \Big(\sum_{m\in\mathbb{Z}} |\phi_u(m)|^3\Big)^{2/3}$$
$$\ll P^\varepsilon \big(P^{15-\frac{23}{6}}\big)^{1/3} \Big(P^{\frac{5}{2}u-\frac{23}{6}}\Big)^{2/3}.$$

The estimate for L_u recorded in Theorem 6.2 therefore follows on recalling the definition of $\eta(u)$.

The initial steps in the estimation of K_u are the same, and one reaches a bound for K_u identical to (6.2) except that ϕ_u now becomes ψ_u. We split into major and minor arcs by inserting the relation $1 = \mathsf{N}(\alpha) + \mathsf{n}(\alpha)$, with parameters $X = P^4$ and $Y = P^{1/3}$, into (4.11). From (5.5) we obtain

$$\left| \int_0^1 \mathsf{N}(\alpha)|f(\alpha)|^u e(-\alpha n)\,\mathrm{d}\alpha \right| \leqslant \int_{\mathfrak{M}_{P^4,P}} |f(\alpha)|^u\,\mathrm{d}\alpha \ll P^{u-4}.$$

Hence, we discern from (4.11) and (4.12) that

$$|\psi_u(n)|^2 \ll |\phi_u(n)|^2 + P^{2u-8},$$

and so,

$$K_u \ll \sum_{m\in\mathbb{Z}} \psi_6(l_3 m)|\phi_u(l_2 m)|^2 + P^{2u-8}\sum_{m\in\mathbb{Z}} \psi_6(l_3 m).$$

Here the first sum over m is the same as that occurring in the estimation of L_u in (6.2), and has already been estimated above. Thus, since

$$\sum_{n\in\mathbb{Z}} \psi_6(n) = |f(0)|^6 \ll P^6,$$

we conclude that

$$K_u \ll P^{2u-2+\eta(u)+\varepsilon} + P^{2u-8}\sum_{n\in\mathbb{Z}} \psi_6(n) \ll P^{2u-2+\eta(u)+\varepsilon} + P^{2u-2}.$$

Provided that $u \leqslant 19/2$, which guarantees $\eta(u)$ to be non-negative, this estimate confirms the upper bound for K_u claimed in the theorem. $\square$

Note that the mean values I_u and J_u involve $s = 3u$ Weyl sums, at least for integral values of u. By comparison, the number of Weyl sums in K_u and L_u is $s = 2u + 6$. A short calculation shows that when applied with the same value of s, with $s \geqslant 18$, the exponents of P in Theorems 6.1 and 6.2 coincide. Since almost all of Theorem 6.1 may be recovered from Theorem 6.2 via Hölder's inequality, and since for fixed values of s the exponent u in Theorem 6.2 is at least as large, Theorem 6.2 is morally the stronger result. In our later application of the circle method, this allows for larger values of r_j in the profiles associated to the simultaneous equations (1.1), and this is essential for our method to succeed. Another advantage is that in L_u only two of the forms M_i are on minor arcs, while in the mean value J_u all three are constrained to minor arcs.

We continue with another result in which the profile is even farther out of balance. We consider the integral

$$M = \int_0^1\int_0^1 \mathsf{n}(\mathrm{M}_1)\mathsf{n}(\mathrm{M}_2)|f(\mathrm{M}_1)^{11}f(\mathrm{M}_2)^{11}f(\mathrm{M}_3)^4|\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Theorem 6.3. *Given the hypotheses of this section, one has* $M \ll P^{18-1/18+\varepsilon}$.

Proof. We again traverse the initial phase of the proof of Theorem 6.1 to confirm the relation

$$M = \sum_{(n_1,n_2,n_3)\in N} \phi_{11}(n_1)\phi_{11}(n_2)\psi_4(n_3).$$

Then, just as in the argument of the proof of Theorem 6.2 leading to (6.2), we find that for appropriate non-zero integers l_2 and l_3, depending at most on $\mathbf{c}$ and $\mathbf{d}$, one has

$$M \leqslant \sum_{m\in\mathbb{Z}} \psi_4(l_3m)|\phi_{11}(l_2m)|^2.$$

Thus, an application of Hölder's inequality in combination with Theorems 5.1 and 5.5, together with (5.2), yields the bound

$$M \leqslant \Big(\sum_{n\in\mathbb{Z}} \psi_4(n)^3\Big)^{1/3}\Big(\sum_{n\in\mathbb{Z}} |\phi_{11}(n)|^3\Big)^{2/3} \ll P^{\varepsilon}\big(P^{13/2}\big)^{1/3}\big(P^{71/3}\big)^{2/3}.$$

The desired conclusion follows a rapid computation. □

Finally, we transform the estimates for L_u and M into proper minor arc estimates. In the interest of brevity we write $\mathfrak{M} = \mathfrak{M}_{P^4,P^{1/3}}$ and put

$$\mathfrak{p} = [0,1]^2 \setminus (\mathfrak{M}\times\mathfrak{M}). \tag{6.3}$$

Theorem 6.4. *Suppose that* $19/2 < u \leqslant 11$. *Then*

$$\iint_{\mathfrak{p}} |f(\mathrm{M}_1)f(\mathrm{M}_2)|^u|f(\mathrm{M}_3)|^6\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{2u-2+\eta(u)+\varepsilon}. \tag{6.4}$$

Further, one has

$$\iint_{\mathfrak{p}} |f(\mathrm{M}_1)^{11}f(\mathrm{M}_2)^{11}f(\mathrm{M}_3)^4|\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{18-1/18+\varepsilon}. \tag{6.5}$$

Proof. Let $\mathsf{N} = \mathsf{N}_{P^4,P^{2/7}}$ and $\mathsf{n} = 1-\mathsf{N}$. Then

$$1 = \big(\mathsf{N}(\mathrm{M}_1)+\mathsf{n}(\mathrm{M}_1)\big)\big(\mathsf{N}(\mathrm{M}_2)+\mathsf{n}(\mathrm{M}_2)\big). \tag{6.6}$$

We note at once that whenever $(\alpha,\beta)\in\mathfrak{p}$, one has $\mathsf{N}(\mathrm{M}_1)\mathsf{N}(\mathrm{M}_2) = 0$. The explanation for this observation is that whenever $\mathsf{N}(\mathrm{M}_1)\mathsf{N}(\mathrm{M}_2) > 0$, then it follows from Lemma 3.1 that $\mathrm{M}_j \in \mathfrak{N}_{P^4,2P^{2/7}}$ $(j=1,2)$. By taking suitable linear combinations of M_1 and M_2 we find that α and β lie in $\mathfrak{N}_{P^4,AP^{2/7}}$, with some $A \geqslant 2$ depending only on the coefficients of M_1 and M_2. But $(\alpha,\beta)\in[0,1]^2$, and so $(\alpha,\beta)\in\mathfrak{M}\times\mathfrak{M}$ for large enough P. This is not the case when $(\alpha,\beta)\in\mathfrak{p}$, as claimed.

With this observation in hand, we apply (6.6) within the integral on the left hand side of (6.5) to conclude that

$$\iint_{\mathfrak{p}} |f(\mathrm{M}_1)^{11}f(\mathrm{M}_2)^{11}f(\mathrm{M}_3)^4|\,\mathrm{d}\alpha\,\mathrm{d}\beta \leqslant M + M_{\mathsf{Nn}} + M_{\mathsf{nN}}, \tag{6.7}$$

where

$$M_{\mathsf{Nn}} = \int_0^1\!\!\int_0^1 \mathsf{N}(\mathrm{M}_1)\mathsf{n}(\mathrm{M}_2)|f(\mathrm{M}_1)^{11}f(\mathrm{M}_2)^{11}f(\mathrm{M}_3)^4|\,\mathrm{d}\alpha\,\mathrm{d}\beta \tag{6.8}$$

and M_{nN} is the integral in (6.8) with M_1, M_2 interchanged.

By symmetry in M_1 and M_2, it now suffices to estimate M_{Nn}. Recalling the definition (6.1) of the linear forms M_i, we put $D = |c_1d_2 - c_2d_1|$ and note that $D > 0$. Consider the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, with $(\alpha, \beta) \mapsto (\alpha', \beta')$, defined by means of the relation

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = D^{-1} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{6.9}$$

Then $\mathrm{M}_1 = D\alpha'$, $\mathrm{M}_2 = D\beta'$, and α and β are linear forms in α' and β' with integer coefficients. By applying the transformation formula as a change of variables, one finds that

$$M_{\mathsf{Nn}} = \iint_{\mathfrak{B}} \mathsf{N}(D\alpha')\mathsf{n}(D\beta')|f(D\alpha')^{11} f(D\beta')^{11} f(A\alpha' + B\beta')^4| \,\mathrm{d}\alpha' \,\mathrm{d}\beta',$$

wherein A, B are non-zero integers and $\mathfrak{B}$ is the image of $[0,1]^2$ under the transformation (6.9). The parallelogram $\mathfrak{B}$ is covered by finitely many sets $[0,1]^2 + \mathbf{t}$, with $\mathbf{t} \in \mathbb{Z}^2$. Since the integrand in the last expression for M_{Nn} is $\mathbb{Z}^2$-periodic it follows that

$$M_{\mathsf{Nn}} \ll \int_0^1 \int_0^1 \mathsf{N}(D\alpha)\mathsf{n}(D\beta)|f(D\alpha)^{11} f(D\beta)^{11} f(A\alpha + B\beta)^4| \,\mathrm{d}\alpha \,\mathrm{d}\beta.$$

Here we have removed decorations from the variables of integration for notational simplicity.

We now inspect all factors of the integrand in the latter upper bound that depend on β. By Hölder's inequality, Lemma 5.3 and obvious changes of variable, one obtains the estimate

$$\begin{aligned} &\int_0^1 \mathsf{n}(D\beta)|f(D\beta)^{11} f(A\alpha + B\beta)^4| \,\mathrm{d}\beta \\ &\qquad \ll \Big(\int_0^1 \mathsf{n}(D\beta)|f(D\beta)|^{77/5} \,\mathrm{d}\beta \Big)^{5/7} \Big(\int_0^1 |f(A\alpha + B\beta)|^{14} \,\mathrm{d}\beta \Big)^{2/7} \\ &\qquad \ll P^{\varepsilon} \big(P^{67/6}\big)^{5/7} (P^{10})^{2/7} = P^{65/6+\varepsilon}, \end{aligned}$$

uniformly in $\alpha \in \mathbb{R}$. Consequently, applying (5.5) in combination with yet another change of variable, we finally arrive at the bound

$$M_{\mathsf{Nn}} \ll P^{65/6+\varepsilon} \int_0^1 \mathsf{N}(D\alpha)|f(D\alpha)|^{11} \,\mathrm{d}\alpha \ll P^{18-1/6+\varepsilon}.$$

We may infer thus far that $M_{\mathsf{Nn}} + M_{\mathsf{nN}} \ll P^{18-1/6+\varepsilon}$. On substituting this estimate into (6.7), noting also the bound $M \ll P^{18-1/18+\varepsilon}$ supplied by Theorem 6.3, the conclusion (6.5) is confirmed.

The proof of (6.4) is essentially the same, and we economise by making similar notational conventions. The exponents 11 and 4 that occur in (6.5) must now be replaced by u and 6, respectively. The initial phase of the preceding argument then remains valid, and an appeal to Theorem 6.2 delivers the bound

$$\iint_{\mathfrak{p}} |f(\mathrm{M}_1) f(\mathrm{M}_2)|^u |f(\mathrm{M}_3)|^6 \,\mathrm{d}\alpha \,\mathrm{d}\beta \ll L_{\mathsf{Nn}} + L_{\mathsf{nN}} + P^{2u-2+\eta(u)+\varepsilon}, \tag{6.10}$$

where

$$L_{\mathsf{Nn}} \ll \int_0^1 \int_0^1 \mathsf{N}(D\alpha)\mathsf{n}(D\beta)|f(D\alpha)f(D\beta)|^u |f(A\alpha+B\beta)|^6 \,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Here, we isolate factors of the integrand that depend on β and apply Hölder's inequality. Note that since $u \leqslant 11$ we have $7u/4 < 20$. Thus, by Lemma 5.3,

$$\begin{aligned}\int_0^1 \mathsf{n}(D\beta)|f(D\beta)^u f(A\alpha+B\beta)^6|\,\mathrm{d}\beta \\ &\ll \left(\int_0^1 \mathsf{n}(D\beta)|f(D\beta)|^{7u/4}\,\mathrm{d}\beta\right)^{4/7} \left(\int_0^1 |f(A\alpha+B\beta)|^{14}\,\mathrm{d}\beta\right)^{3/7} \\ &\ll P^{\varepsilon}\big(P^{\frac{35}{24}u-\frac{5}{3}}\big)^{4/7}\big(P^{10}\big)^{3/7}.\end{aligned}$$

Applying this bound, which is uniform in $\alpha \in \mathbb{R}$, together with (5.5), we arrive at the estimate

$$L_{\mathsf{Nn}} \ll P^{\frac{5}{6}u+\frac{10}{3}+\varepsilon} \int_0^1 \mathsf{N}(D\alpha)|f(D\alpha)|^u \,\mathrm{d}\alpha \ll P^{\frac{11}{6}u-\frac{2}{3}+\varepsilon}.$$

When $u \leqslant 11$, the definition of $\eta(u)$ ensures that $\frac{11}{6}u - \frac{2}{3} \leqslant 2u-2+\eta(u)$, and hence $L_{\mathsf{Nn}} + L_{\mathsf{nN}} \ll P^{2u-2+\eta(u)+\varepsilon}$. The conclusion (6.4) now follows by substituting this estimate into (6.10). □

7. Another mean value estimate

This section is an update for quartic Weyl sums of our earlier work [3] on highly entangled mean values. We now attempt to avoid independence conditions on linear forms as far as the argument allows while incorporating the consequences of the recent bound (1.5). We emphasise that throughout this section, we continue to work subject to the overall assumptions made at the outset of the previous section. We begin by examining the mean value

$$G_1 = \int_0^1 \int_0^1 |f(\mathrm{M}_1)^2 f(\mathrm{M}_2)^4 f(\mathrm{M}_3)^4|\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{7.1}$$

Lemma 7.1. *One has* $G_1 \ll P^{5+\varepsilon}$.

Proof. This is essentially contained in [4, Section 2], but we give a proof for completeness. Recall the definition (6.1) of the linear forms M_i. By orthogonality, the integral G_1 is equal to the number of solutions of an associated pair of quartic equations. By taking suitable integral linear combinations of these two equations, we may assume that they take the shape

$$a(x_1^4 - x_2^4) = b(x_3^4 + x_4^4 - x_5^4 - x_6^4) = c(x_7^4 + x_8^4 - x_9^4 - x_{10}^4), \tag{7.2}$$

for suitable natural numbers a, b, c. Thus, we see that G_1 is equal to the number of solutions of the Diophantine system (7.2) with $x_i \leqslant P$. For each of the $O(P)$ possible choices for x_1 and x_2 with $x_1 = x_2$, it follows via orthogonality and

(5.10) that the number of solutions of this system in the remaining variables $x_3, \ldots, x_{10}$ is equal to

$$\left(\int_0^1 |f(\alpha)|^4 \, \mathrm{d}\alpha \right)^2 \ll P^{4+\varepsilon}.$$

Consequently, the contribution to G_1 from this first class of solutions is $O(P^{5+\varepsilon})$. Now consider solutions of (7.2) in which $x_1 \neq x_2$. By orthogonality, the total number of choices for $x_3, \ldots, x_{10}$ satisfying the rightmost equation in (7.2) is

$$\int_0^1 |f(b\alpha)f(c\alpha)|^4 \, \mathrm{d}\alpha.$$

Schwarz's inequality in combination with (5.8) shows this integral to be $O(P^{5+\varepsilon})$. However, for any fixed choice of $x_3, \ldots, x_{10}$ in this second class of solutions, one has $x_1 \neq x_2$, and hence the fixed integer $N = b(x_3^4 + x_4^4 - x_5^4 - x_6^4)$ is non-zero. But it follows from (7.2) that $x_1^2 - x_2^2$ and $x_1^2 + x_2^2$ are each divisors of N. Thus, a standard divisor function estimate shows that the number of choices for x_1 and x_2 is $O(P^\varepsilon)$, and we conclude that the contribution to G_1 from this second class of solutions is $O(P^{5+\varepsilon})$. Adding these two contributions, we obtain the bound claimed in the statement of the lemma. □

We next examine the mean value

$$G_2 = \int_0^1 \int_0^1 |f(\mathrm{M}_1)^2 f(\mathrm{M}_2)^4 f(\mathrm{M}_3)^4 f(\mathrm{M}_4)^4 f(\mathrm{M}_5)^4| \, \mathrm{d}\alpha \, \mathrm{d}\beta. \tag{7.3}$$

Theorem 7.2. *One has $G_2 \ll P^{11+\varepsilon}$.*

Note that in this result we require the five linear forms M_j to be pairwise independent. Therefore, the result will be of use only in cases where the profile of (1.1) has $r_5 \geqslant 1$. The mean value in Theorem 7.2 involves 18 Weyl sums and should therefore be compared with the bound $I_6 \ll P^{67/6+\varepsilon}$ provided by Theorem 6.1. The extra savings that we obtain here are the essential stepping stone toward Theorem 1.2.

The proof of Theorem 7.2. As in the proof of Lemma 7.1, it follows from orthogonality that the integral G_2 is equal to the number of solutions of an associated pair of quartic equations. Taking suitable integral linear combinations of these two equations, we reduce to the situation where $c_4 = d_5 = 0$, and consequently $\mathrm{M}_4 = d_4\beta$ and $\mathrm{M}_5 = c_5\alpha$. Motivated by this observation, we begin our deliberations by estimating the auxiliary mean value

$$G_3 = \int_0^1 \int_0^1 |f(\mathrm{M}_1)^2 f(\mathrm{M}_2)^4 f(\mathrm{M}_3)^4 f(d_4\beta)^4| \, \mathrm{d}\alpha \, \mathrm{d}\beta.$$

The Weyl differencing argument [14, Lemma 2.3] shows that there are real numbers u_h with $u_h \ll P^\varepsilon$ for which

$$|f(\gamma)|^4 \ll P^3 + P \sum_{1 \leqslant |h| \leqslant 2P^4} u_h e(\gamma h). \tag{7.4}$$

We apply this relation with $\gamma = \mathrm{M}_4$ to the mean value G_3 and infer that

$$G_3 \ll P^3 G_1 + P G_4, \tag{7.5}$$

where G_1 is the mean value defined in (7.1), and

$$G_4 = \sum_{1\leqslant |h| \leqslant 2P^4} u_h \int_0^1 \int_0^1 |f(\mathrm{M}_1)^2 f(\mathrm{M}_2)^4 f(\mathrm{M}_3)^4| e(d_4 h\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

By orthogonality, the double integral on the right hand side here is equal to the number of solutions of the system of Diophantine equations

$$\begin{aligned} c_1(x_1^4 - y_1^4) + c_2(x_2^4 + x_3^4 - y_2^4 - y_3^4) + c_3(x_4^4 + x_5^4 - y_4^4 - y_5^4) \quad &= 0 \quad (7.6)\\ d_1(x_1^4 - y_1^4) + d_2(x_2^4 + x_3^4 - y_2^4 - y_3^4) + d_3(x_4^4 + x_5^4 - y_4^4 - y_5^4) + d_4 h &= 0 \end{aligned}$$

with $x_i \leqslant P$ and $y_i \leqslant P$. We may sum over $h \neq 0$ and replace u_h by its upper bound. Then we find that $G_4 \ll P^\varepsilon G_5$, where G_5 is the number of solutions of the equation (7.6) with the same conditions on x_i and y_i. By orthogonality again, we deduce that

$$G_5 = \int_0^1 |f(c_1\alpha)^2 f(c_2\alpha)^4 f(c_3\alpha)^4|\,\mathrm{d}\alpha.$$

For $1 \leqslant i \leqslant 3$ the linear form M_i is linearly independent of $\mathrm{M}_4 = d_4\beta$, and thus $c_1c_2c_3 \neq 0$. The trivial bound $|f(c_1\alpha)|^2 \ll P^2$ therefore combines with Schwarz's inequality and (5.8) to award us the bound

$$G_5 \ll P^2 \int_0^1 |f(\gamma)|^8\,\mathrm{d}\gamma \ll P^{7+\varepsilon}.$$

We therefore deduce that $G_4 \ll P^{7+2\varepsilon}$. Meanwhile, the estimate $G_1 \ll P^{5+\varepsilon}$ is available from Lemma 7.1. On substituting these bounds into (7.5), we conclude thus far that $G_3 \ll P^{8+\varepsilon}$.

We now repeat this argument with $\gamma = \mathrm{M}_5$ in (7.4), applying the resulting inequality within the integral G_2 defined in (7.3). Thus we obtain

$$G_2 \ll P^3 G_3 + P^{1+\varepsilon} G_6, \tag{7.7}$$

where G_6 denotes the number of solutions of the Diophantine equation

$$d_1(x_1^4-y_1^4)+d_2(x_2^4+x_3^4-y_2^4-y_3^4)+d_3(x_4^4+x_5^4-y_4^4-y_5^4)+d_4(x_6^4+x_7^4-y_6^4-y_7^4) = 0,$$

with $x_i \leqslant P$ and $y_i \leqslant P$. By orthogonality,

$$G_6 = \int_0^1 |f(d_1\alpha)^2 f(d_2\alpha)^4 f(d_3\alpha)^4 f(d_4\alpha)^4|\,\mathrm{d}\alpha.$$

One may confirm that $d_1d_2d_3d_4 \neq 0$ by arguing as above, and so an application of (2.2) in combination with (1.5) reveals that

$$G_6 \leqslant \sum_{i=1}^4 \int_0^1 |f(d_i\alpha)|^{14}\,\mathrm{d}\alpha = 4\int_0^1 |f(\gamma)|^{14}\,\mathrm{d}\gamma \ll P^{10+\varepsilon}.$$

The conclusion of the theorem now follows on substituting this bound together with our earlier estimate for G_3 into (7.7). □

8. The circle method

In this section we prepare the ground to advance to the proofs of Theorems 1.1 and 1.2. A preliminary manœuvre is in order. Let $k = 0$ or 1, and let $N_k(P) = N_k$ denote the number of solutions of the system (1.1) with $k \leqslant x_j \leqslant P$ $(1 \leqslant j \leqslant s)$. Note that the equations (1.1) are invariant under the s mappings $x_j \mapsto -x_j$. This observation shows that

$$2^s N_1(P) \leqslant \mathscr{N}(P) \leqslant 2^s N_0(P). \tag{8.1}$$

The goal is then to establish the formulae

$$\lim_{P\to\infty} 2^s P^{8-s} N_k(P) = \mathfrak{J}\mathfrak{S} \quad (k = 0, 1), \tag{8.2}$$

since then (1.4) follows immediately from (8.1) and the sandwich principle. Thus, we now launch the Hardy-Littlewood method to evaluate the counting functions $N_k(P)$. This involves the exponential sum

$$f_k(\alpha) = \sum_{k\leqslant x\leqslant P} e(\alpha x^4). \tag{8.3}$$

This sum is, of course, an instance of the sum (1.10), where we have been deliberately imprecise about the lower end of the interval of summation. The results we have formulated so far are indeed independent of the choice of k, and it is only now and temporarily where this detail matters. We require the linear forms $\Lambda_j = \Lambda_j(\alpha, \beta)$, defined by

$$\Lambda_j(\alpha, \beta) = a_j\alpha + b_j\beta \quad (1 \leqslant j \leqslant s)$$

that are associated with the equations (1.1). We then put

$$\mathscr{F}_k(\alpha, \beta) = f_k(\Lambda_1)f_k(\Lambda_2)\cdots f_k(\Lambda_s), \tag{8.4}$$

and observe that, by orthogonality, one has

$$N_k(P) = \int_0^1\int_0^1 \mathscr{F}_k(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{8.5}$$

Subject to conditions milder than those imposed in Theorems 1.1 and 1.2 we reduce the evaluation of the integral (8.5) to the estimation of its minor arc part. With this end in mind we define the major arcs $\mathfrak{V}$ as the union of the rectangles

$$\mathfrak{V}(q, a, b) = \{(\alpha, \beta) \in [0, 1]^2 : |\alpha - a/q| \leqslant P^{-31/8} \text{ and } |\beta - b/q| \leqslant P^{-31/8}\},$$

with $0 \leqslant a, b \leqslant q$, $(a, b, q) = 1$ and $1 \leqslant q \leqslant P^{1/8}$.

Define the generating functions

$$S(q, c) = \sum_{x=1}^{q} e(cx^4/q) \quad \text{and} \quad v(\gamma) = \int_0^P e(\gamma t^4)\,\mathrm{d}t.$$

Then, given $(\alpha, \beta) \in [0, 1]^2$, if we put $\gamma = \alpha - a/q$ and $\delta = \beta - b/q$ for some $a, b \in \mathbb{Z}$ and $q \in \mathbb{N}$, one concludes from (8.3) and [14, Theorem 4.1] that

$$f_k(\Lambda_j) = q^{-1}S\left(q, \Lambda_j(a, b)\right) v\left(\Lambda_j(\gamma, \delta)\right) + O\left(q^{1/2+\varepsilon}(1 + P^4|\Lambda_j(\gamma, \delta)|)^{1/2}\right). \tag{8.6}$$

Note that the right hand side here is independent of k. We multiply these approximations for $1 \leqslant j \leqslant s$. This brings into play the expressions

$$\mathscr{S}(q,a,b) = q^{-s}\prod_{j=1}^{s} S\left(q, \Lambda_j(a,b)\right) \quad \text{and} \quad \mathscr{V}(\gamma,\delta) = \prod_{j=1}^{s} v\left(\Lambda_j(\gamma,\delta)\right).$$

If $(\alpha,\beta) \in \mathfrak{V}(q,a,b) \subseteq \mathfrak{V}$ then the error term in (8.6) is $O(P^{1/8+\varepsilon})$, and we infer that

$$\mathscr{F}_k(\alpha,\beta) = \mathscr{S}(q,a,b)\mathscr{V}(\gamma,\delta) + O(P^{s-7/8+\varepsilon}).$$

Since $\mathfrak{V}$ is a set of measure $O(P^{-59/8})$, when we integrate this formula for $\mathscr{F}_k(\alpha,\beta)$ over $\mathfrak{V}$, we obtain the asymptotic relation

$$\iint_{\mathfrak{V}} \mathscr{F}_k(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta = \mathfrak{S}(P^{1/8})\mathfrak{J}^*(P^{1/8}) + O(P^{s-33/4+\varepsilon}),$$

where, for $1 \leqslant Q \leqslant P$ we define

$$\mathfrak{S}(Q) = \sum_{q\leqslant Q}\sum_{a=1}^{q}\sum_{\substack{b=1\\(a,b,q)=1}}^{q}\mathscr{S}(q,a,b),$$

$$\mathfrak{J}^*(Q) = \iint_{\mathfrak{U}(Q)} \mathscr{V}(\gamma,\delta)\,\mathrm{d}\gamma\,\mathrm{d}\delta,$$

and $\mathfrak{U}(Q) = [-QP^{-4}, QP^{-4}]^2$.

At this point, we require some more information concerning the matrix of coefficients, and we shall suppose that $q_0 \geqslant 15$. Then $s \geqslant 16$, and we may apply [9, Lemma 3.3] to conclude that $\mathfrak{S}(Q) = \mathfrak{S} + O(Q^{\varepsilon-1})$. Further, we have

$$\int_{-P}^{P} e(\gamma t^4)\,\mathrm{d}t = 2v(\gamma),$$

and thus [9, Lemma 3.1] shows that the limit (1.2) exists, and that we have $2^s\mathfrak{J}^*(Q) = P^{s-8}\mathfrak{J} + O(P^{s-8}Q^{-1/4})$. We summarise these deliberations in the following lemma.

Lemma 8.1. *Suppose that $q_0 \geqslant 15$ and that $k \in \{0,1\}$. Then*

$$\iint_{\mathfrak{V}} \mathscr{F}_k(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta = 2^{-s}P^{s-8}\mathfrak{S}\mathfrak{J} + O(P^{s-8-1/32}).$$

The major arcs in Lemma 8.1 are certainly too slim for efficient use of Weyl type inequalities on the complementary set. A pruning argument allows us to enlarge the major arcs considerably. Let $\mathfrak{W}$ denote the union of the rectangles

$$\mathfrak{W}(q,a,b) = \{(\alpha,\beta) \in [0,1]^2 : |q\alpha - a| \leqslant P^{-3} \text{ and } |q\beta - b| \leqslant P^{-3}\},$$

with $1 \leqslant q \leqslant P$, $0 \leqslant a,b \leqslant q$ and $(a,b,q) = 1$. Then $\mathfrak{V} \subset \mathfrak{W}$, and we proceed to estimate the contribution from $\mathfrak{W} \setminus \mathfrak{V}$ to the integral (8.5). A careful application of [14, Theorem 4.2] shows that $S(q,c) \ll q^{3/4}(q,c)^{1/4}$. Further, if $V(\gamma) = P(1+P^4|\gamma|)^{-1/4}$, then by [14, Theorem 7.3], one has $v(\gamma) \ll V(\gamma)$. Hence, whenever $(\alpha,\beta) \in \mathfrak{W}(q,a,b)$ with $q \leqslant P$, one deduces from (8.6) that

$$f_k(\Lambda_j) \ll q^{-1/4}\left(q, \Lambda_j(a,b)\right)^{1/4} V\left(\Lambda_j(\alpha - a/q, \beta - b/q)\right) + P^{1/2+\varepsilon}.$$

It is immediate that the first term on the right hand side here always dominates the second, and therefore,

$$\mathscr{F}_k(\alpha,\beta) \ll q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a,b))^{1/4}\, V\left(\Lambda_j(\alpha - a/q, \beta - b/q)\right).$$

We integrate over $\mathfrak{W} \setminus \mathfrak{V}$. The result is a sum over $q \leqslant P$ in which we consider the portion $q \leqslant P^{1/8}$ separately. This yields the bound

$$\iint_{\mathfrak{W}\setminus\mathfrak{V}} \mathscr{F}_k(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll K_1(P^{1/8}) + K_2(P^{1/8}), \tag{8.7}$$

where for $1 \leqslant Q \leqslant P$, we write

$$K_1(Q) = \sum_{q\leqslant Q} \sum_{\substack{a=1 \\ (a,b,q)=1}}^{q} \sum_{b=1}^{q} q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a,b))^{1/4} \iint_{\mathfrak{B}(Q)} \prod_{j=1}^{s} V(\Lambda_j)\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$

with $\mathfrak{B}(Q) = [-1,1]^2 \setminus \mathfrak{U}(Q)$, and

$$K_2(Q) = \sum_{Q<q\leqslant P} \sum_{\substack{a=1 \\ (a,b,q)=1}}^{q} \sum_{b=1}^{q} q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a,b))^{1/4} \iint_{[-1,1]^2} \prod_{j=1}^{s} V(\Lambda_j)\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Still subject to the condition $q_0 \geqslant 15$, the proof of [9, Lemma 3.2] shows that

$$\sum_{q>Q} \sum_{\substack{a=1 \\ (a,b,q)=1}}^{q} \sum_{b=1}^{q} q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a,b))^{1/4} \ll \sum_{q>Q} q^{\varepsilon-2} \ll Q^{\varepsilon-1},$$

and similarly, the proof of [9, Lemma 3.1] delivers the bound

$$\iint_{\mathfrak{B}(Q)} \prod_{j=1}^{s} V(\Lambda_j)\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{s-8} Q^{-1/4}.$$

Thus we deduce that $K_1(P^{1/8}) + K_2(P^{1/8}) \ll P^{s-8-1/32}$. Substituting this estimate into (8.7), and then recalling Lemma 8.1, we see that in the latter lemma we may replace $\mathfrak{V}$ by $\mathfrak{W}$. This establishes the following theorem.

Theorem 8.2. *Suppose that $q_0 \geqslant 15$ and that $k \in \{0,1\}$. Then*

$$\iint_{\mathfrak{W}} \mathscr{F}_k(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta = 2^{-s} P^{s-8} \mathfrak{S}\mathfrak{J} + O(P^{s-8-1/32}).$$

Let $\mathfrak{w} = [0,1]^2 \setminus \mathfrak{W}$ denote the minor arcs. Then, in view of (8.2), (8.5) and Theorem 8.2, whenever $q_0 \geqslant 15$, the asymptotic relation (1.4) is equivalent to the minor arc estimate

$$\iint_{\mathfrak{w}} \mathscr{F}_k(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta = o(P^{s-8}), \tag{8.8}$$

as $P \to \infty$, and in the next two sections we shall confirm this subject to the hypotheses imposed in Theorems 1.1 and 1.2.

9. The proof of Theorem 1.1

At the core of the proof of Theorem 1.1 we require two minor arc estimates.

Lemma 9.1. *Let* $c_1, c_2, d_1, d_2 \in \mathbb{Z}$*, and suppose that* $\mathrm{M}_j = c_j\alpha + d_j\beta$ $(j = 1, 2)$ *are linearly independent. Then*

$$\iint_{\mathfrak{w}} |f(\mathrm{M}_1)f(\mathrm{M}_2)|^{15}\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{22-1/6+\varepsilon}.$$

Proof. It is immediate from (6.3) that $\mathfrak{w} \subset \mathfrak{p}$. Recall the initial argument within the proof of Theorem 6.4. This shows that for $(\alpha, \beta) \in \mathfrak{p}$, the forms M_1 and M_2 cannot be in $\mathfrak{N}_{P^4,P^{2/7}}$ simultaneously. By symmetry we may therefore suppose that $\mathrm{M}_1 \in \mathfrak{n}_{P^4,P^{2/7}}$. Now apply the transformation formula as in (6.9). One finds that for an appropriate non-zero integer D, depending at most on $\mathbf{c}$ and $\mathbf{d}$, one has

$$\iint_{\mathfrak{w}} |f(\mathrm{M}_1)f(\mathrm{M}_2)|^{15}\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll \int_0^1\int_{\mathfrak{m}} |f(D\alpha)f(D\beta)|^{15}\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$

where $\mathfrak{m} = \mathfrak{m}_{P^4,P^{2/7}}$. Thus, applying a trivial estimate for one factor $f(D\beta)$, we deduce via Lemma 5.3 that

$$\iint_{\mathfrak{w}} |f(\mathrm{M}_1)f(\mathrm{M}_2)|^{15}\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{\varepsilon}\left(P^{65/6}\right)\left(P^{11}\right) \ll P^{22-1/6+\varepsilon}.$$

This completes the proof of the lemma. □

Lemma 9.2. *Suppose that any two of the binary linear forms* M_1, M_2, M_3 *are linearly independent. Then*

$$\iint_{\mathfrak{w}} |f(\mathrm{M}_1)^{11}f(\mathrm{M}_2)^{11}f(\mathrm{M}_3)^4|\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{18-1/18+\varepsilon}.$$

Proof. On recalling that $\mathfrak{w} \subset \mathfrak{p}$, the lemma is immediate from Theorem 6.4. □

We are now fully equipped to complete the proof of Theorem 1.1. Suppose that we are given a pair of equations (1.1) with $s \geqslant 26$, $q_0 \geqslant 15$ and profile $(r_1, r_2, \dots, r_\nu)$. The parameter $l = s - r_1 - r_2$ determines our argument. In the notation of Section 7, we let $\mathscr{F} = \mathscr{F}_k$ with $k = 0$ or 1 be the generating function defined in (8.4).

Small values of l call for special attention. Initially, we consider the situation with $0 \leqslant l \leqslant 3$. We apply Lemma 2.3 with J_1 and J_2 the subsets of the set of indices $\{1, 2, \dots, s\}$ counted by r_1 and r_2, respectively, and with J_3 the subset consisting of the remaining indices. Then $\mathrm{card}(J_3) = l$. We also choose

$$M_\nu = \ldots = M_4 = 0, \quad M_3 = l, \quad M_2 = 15 - l \quad \text{and} \quad M_1 = s - 15.$$

The condition $q_0 \geqslant 15$ ensures that $r_1 \leqslant s - 15$, and $r_1 + r_2 = s - l = M_1 + M_2$. Also, we have $M_1 = s - 15 \geqslant 15 - l = M_2$ because $r_1 \geqslant r_2 \geqslant 15 - l$ and $s = r_1 + r_2 + l \geqslant 2r_2 + l \geqslant 30 - l$. Finally, since $0 \leqslant l \leqslant 3$ it is apparent that $M_2 = 15 - l \geqslant l = M_3$. Therefore, Lemma 2.3 is indeed applicable and delivers the bound

$$\iint_{\mathfrak{w}} \mathscr{F}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll \iint_{\mathfrak{w}} |f(\mathrm{M}_1)^{s-15} f(\mathrm{M}_2)^{15-l} f(\mathrm{M}_3)^{l}|\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$

where each of the M_j is one of the linear forms Λ_i, and any two of the M_j are linearly independent. We now reduce the exponent $s-15$ to $15-l$ and then apply Hölder's inequality. Thus

$$\begin{aligned}\iint_{\mathfrak{w}} \mathscr{F}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta &\ll P^{s-30+l} \iint_{\mathfrak{w}} |f(\mathrm{M}_1)^{15-l} f(\mathrm{M}_2)^{15-l} f(\mathrm{M}_3)^{l}|\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &\ll \Upsilon_1^{l/4}\Upsilon_2^{1-l/4},\end{aligned}$$

where

$$\begin{aligned}\Upsilon_1 &= \iint_{\mathfrak{w}} |f(\mathrm{M}_1)^{11} f(\mathrm{M}_2)^{11} f(\mathrm{M}_3)^{4}|\,\mathrm{d}\alpha\,\mathrm{d}\beta, \\ \Upsilon_2 &= \iint_{\mathfrak{w}} |f(\mathrm{M}_1) f(\mathrm{M}_2)|^{15}\,\mathrm{d}\alpha\,\mathrm{d}\beta.\end{aligned}$$

In this scenario, therefore, we deduce from Lemmata 9.1 and 9.2 that

$$\begin{aligned}\iint_{\mathfrak{w}} \mathscr{F}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta &\ll P^{s-30+l+\varepsilon}\left(P^{18-1/18}\right)^{l/4}\left(P^{22-1/6}\right)^{1-l/4} \\ &\ll P^{s-8-1/18+\varepsilon}. \end{aligned} \tag{9.1}$$

We may now suppose that $l \geqslant 4$. Then $r_1 \leqslant s-15$ and $r_1 + r_2 \leqslant s-4$. In Lemma 2.3 we now take J_j to be the subset of the set of indices $\{1, 2, \ldots, s\}$ counted by r_j. We also choose

$$M_\nu = \ldots = M_4 = 0, \quad M_3 = 4, \quad M_2 = 11 \quad \text{and} \quad M_1 = s-15,$$

and note that the hypothesis $s \geqslant 26$ ensures that $M_1 \geqslant M_2$. The conditions required to apply Lemma 2.3 are consequently in play, and we deduce that

$$\iint_{\mathfrak{w}} \mathscr{F}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll \iint_{\mathfrak{w}} |f(\mathrm{M}_1)|^{s-15} |f(\mathrm{M}_2)|^{11} |f(\mathrm{M}_3)|^{4}\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$

where again each of the M_j is one of the linear forms Λ_i, and any two of the M_j are linearly independent. Here $s - 15 \geqslant 11$ by the hypothesis $s \geqslant 26$, and we may estimate excessive copies of $f(\mathrm{M}_1)$ trivially and apply Lemma 9.2. This confirms that (9.1) also holds for $l \geqslant 4$. In particular, we have (8.8) subject to the hypotheses of Theorem 1.1. This completes the proof of Theorem 1.1.

10. The proof of theorem 1.2

We continue to use the notation introduced in §§8 and 9, but now suppose that the hypotheses of Theorem 1.2 are met. Hence $s = 25$ and $r_1 \leqslant s - q_0 \leqslant 9$. We also assume that $r_5 \geqslant 1$. Our goal on this occasion is the estimate

$$\iint_{\mathfrak{w}} \mathscr{F}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{17-1/24+\varepsilon}. \tag{10.1}$$

Once this is established, Theorem 1.2 follows in the same way as Theorem 1.1 was deduced from (9.1).

We apply Lemma 2.3 with J_j the subset of the set of indices $\{1, 2, \ldots, s\}$ counted by r_j for $1 \leqslant j \leqslant \nu$. Also, we put $m_j = r_j$ for each j and

$$M_\nu = \ldots = M_6 = 0, \quad M_5 = M_4 = 1, \quad M_3 = 5 \quad \text{and} \quad M_2 = M_1 = 9.$$

On recalling that $r_1 \leqslant 9$, it is immediate that (2.3) and (2.4) hold. Hence, Lemma 2.3 is applicable, and yields linear forms $\mathrm{M}_1, \ldots, \mathrm{M}_5$ that are linearly independent in pairs, where each M_j is one of the Λ_i, and where

$$\iint_{\mathfrak{w}} \mathscr{F}(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \leqslant \iint_{\mathfrak{w}} |f(\mathrm{M}_1)^9 f(\mathrm{M}_2)^9 f(\mathrm{M}_3)^5 f(\mathrm{M}_4) f(\mathrm{M}_5)|\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

By Hölder's inequality, we find that

$$\iint_{\mathfrak{w}} \mathscr{F}(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \leqslant \Upsilon_3^{1/4} \Upsilon_4^{3/4},$$

where

$$\Upsilon_3 = \int_0^1 \int_0^1 |f(\mathrm{M}_1) f(\mathrm{M}_2) f(\mathrm{M}_4) f(\mathrm{M}_5)|^4 |f(\mathrm{M}_3)|^2\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$
$$\Upsilon_4 = \iint_{\mathfrak{w}} |f(\mathrm{M}_1) f(\mathrm{M}_2)|^{32/3} |f(\mathrm{M}_3)|^6\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Making use of the bounds supplied by Theorem 7.2 and Theorem 6.4 with $u = 32/3$, we therefore infer that

$$\iint_{\mathfrak{w}} \mathscr{F}(\alpha, \beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll P^{\varepsilon} \left(P^{11}\right)^{1/4} \left(P^{19-1/18}\right)^{3/4} \ll P^{17-1/24+\varepsilon}.$$

Thus the bound (10.1) is confirmed, and the proof of Theorem 1.2 is complete.

Finally, we briefly comment on the prospects of reducing the number of variables further. Note that the estimates for the minor arcs and for the whole unit square in Theorem 6.1 coincide for $u = 25/3$. Since $\delta(25/3) = 0$, therefore, when $s = 25$ our basic method narrowly fails to be applicable to the system of equations (1.1). Further, it transpires that each additional variable contributes a factor P to the major arc contribution, but only $P^{5/6}$ to the minor arc versions of Theorems 6.1 and 6.2. As indicated in §1 already, it is worth comparing the 18th moment ($u = 6$) in Theorem 6.1 with that in Theorem 7.2, the latter being superior by a factor $P^{1/6}$. It transpires that even if it were possible to propagate this saving through the moment method, then we would still fail to handle cases of (1.1) with $s = 24$, but only by a factor P^{ε}. However, at this stage, the only workable compromise seems to be to apply Theorem 7.2 in conjunction with Theorems 6.1 or 6.4, via Hölder's inequality. If the profile of the equations (1.1) is even more illustrious than in Theorem 1.2, then one can put more weight on the bound stemming from Theorem 7.2. For example, if we suppose that $s = 24$ and $r_1 \leqslant 5$, then $\nu \geqslant 5$ and $r_5 \leqslant 4$, so that in hopefully self-explanatory notation, the minor arc contribution can be reduced to something of the shape

$$\iint_{\mathfrak{w}} \mathscr{F}(\alpha,\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta \ll \iint_{\mathfrak{w}} |f(\mathrm{M}_1)^5 f(\mathrm{M}_2)^5 f(\mathrm{M}_3)^5 f(\mathrm{M}_4)^5 f(\mathrm{M}_5)^4|\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

One may then introduce the identity (3.5) with $\alpha = \mathrm{M}_j$ for all $1 \leqslant j \leqslant 5$ simultaneously. The most difficult term that then arises is that weighted with $\mathsf{n}(\mathrm{M}_1)\cdots\mathsf{n}(\mathrm{M}_5)$. A cascade of applications of Hölder's inequality together with Theorem 6.1 shows this term to be bounded by

$$(\Upsilon_3)^{3/5}(J_{11})^{2/5} \ll P^{16+1/15+\varepsilon},$$

which is quite far from saving another variable.

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