ON WARING'S PROBLEM: BEYOND FREĬMAN'S THEOREM

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ABSTRACT. Let $k_i \in \mathbb{N}$ $(i \geqslant 1)$ satisfy $2 \leqslant k_1 \leqslant k_2 \leqslant \ldots$. Freĭman's theorem shows that when $j \in \mathbb{N}$, there exists $s = s(j) \in \mathbb{N}$ such that all large integers n are represented in the form $n = x_1^{k_j} + x_2^{k_{j+1}} + \ldots + x_s^{k_{j+s-1}}$, with $x_i \in \mathbb{N}$, if and only if $\sum k_i^{-1}$ diverges. We make this theorem effective by showing that, for each fixed j, it suffices to impose the condition

$$\sum_{i=j}^{\infty} k_i^{-1} \geqslant 2\log k_j + 4.71.$$

More is established when the sequence of exponents forms an arithmetic progression. Thus, for example, when $k \in \mathbb{N}$ and $s \geqslant 100(k+1)^2$, all large integers n are represented in the form $n = x_1^k + x_2^{k+1} + \ldots + x_s^{k+s-1}$, with $x_i \in \mathbb{N}$.

1. Introduction

Recent advances in the smooth number technology associated with Waring's problem (see [3]) make possible an investigation of the cognate problem to which Freĭman's theorem provides a qualitative answer. Consider then natural numbers k_i $(i \geqslant 1)$ satisfying $2 \leqslant k_1 \leqslant k_2 \leqslant \ldots$. We address the problem of determining circumstances in which, given $j \in \mathbb{N}$, there exists a natural number $s = s(j)$ such that all large integers n are represented in the form

$$x_1^{k_j} + x_2^{k_{j+1}} + \ldots + x_s^{k_{j+s-1}} = n,$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$. Freĭman's theorem, announced in 1949 (see [6]), asserts that such holds if and only if the infinite series $\sum k_i^{-1}$ diverges. Although Freĭman sketches a proof of this claim in [7], his argument contains a number of obscurities. A detailed proof of this conclusion was subsequently given by Scourfield in 1960 (see [12, Theorem 1]). We now provide an effective version of this conclusion.

Theorem 1.1. *Let $k_i \in \mathbb{N}$ $(i \geqslant 1)$ satisfy $2 \leqslant k_1 \leqslant k_2 \leqslant \ldots$. Suppose that s is a natural number for which*

$$\sum_{i=3}^{s} \frac{1}{k_i} > 2\log k_1 + \frac{1}{k_2} + 3.20032.$$

Then all sufficiently large natural numbers n are represented in the form

$$x_1^{k_1} + x_2^{k_2} + \ldots + x_s^{k_s} = n,$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$.

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Since the hypotheses of Theorem 1.1 impose the condition $k_2 \geqslant k_1 \geqslant 2$, one obtains an immediate consequence of this theorem that implies Freĭman's theorem.

Corollary 1.2. *Let $k_i \in \mathbb{N}$ $(i \geqslant 1)$ satisfy $2 \leqslant k_1 \leqslant k_2 \leqslant \dots$, and suppose that $j \in \mathbb{N}$. Then whenever s is a natural number for which*

$$\sum_{i=j}^{j+s-1} \frac{1}{k_i} \geqslant 2\log k_j + 4.71, \tag{1.1}$$

all sufficiently large natural numbers n are represented in the form

$$x_1^{k_j} + x_2^{k_{j+1}} + \dots + x_s^{k_{j+s-1}} = n,$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$.

By making better use of sharper Weyl exponents available for smaller exponents, most particularly in the situation in which one or more of the k_i are equal to 2, it would not be difficult to reduce the number 4.71 occurring in the lower bound (1.1) of the statement of Corollary 1.2. Back of the envelope computations suggest that a number comfortably below 3.5 should be accessible. For larger values of k_1, and k_2 large compared to k_1, on the other hand, the conclusion of Theorem 1.1 has strength reflecting the limits of current technology. Standard heuristics from the circle method, meanwhile, suggest that the conclusion of Corollary 1.2 should remain valid provided only that

$$\sum_{i=j}^{j+s-1} \frac{1}{k_i} > 4.$$

If one is prepared to accept a local solubility condition, then the assumption of square-root cancellation for the mean values of exponential sums encountered in the application of the circle method would reduce the lower bound 4 here to 2, while the most optimistic heuristics would reduce this number further to 1.

We now turn to the special case of this variant of Waring's problem involving mixed powers in which the exponents consist of consecutive terms of an arithmetic progression. Thus, when k and r are non-negative integers with $k \geqslant 2$, we consider the representation of large positive integers n in the shape

$$x_1^k + x_2^{k+r} + \dots + x_s^{k+r(s-1)} = n, \tag{1.2}$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$. We denote by $R(k, r)$ the least number s having the property that all large integers n are represented in the form (1.2). In particular, the important number $G(k)$ familiar to aficionados of Waring's problem is equal to $R(k, 0)$. Moreover, the pioneering work of Roth [11, Theorem 2] shows that $R(2, 1) \leqslant 50$, which is to say that all large enough integers n have a representation in the shape

$$n = x_1^2 + x_2^3 + \dots + x_{50}^{51},$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant 50)$.

Theorem 1.3. *Let k and r be natural numbers with $k \geqslant 2$. Then, uniformly in k and r one has $R(k, r) \leqslant A(r)(k+1)^{r+1}$, where $A(r) = r^{-1}25^r(r+1)^{r+1}$. Meanwhile, when $r \geqslant k$ one has $R(k, r) \leqslant (6k + 6)^{2r}$.*

It would appear that the only previous work concerning this problem of such generality hitherto available in the literature is that due to Scourfield [12, Theorem 2]. The latter work shows that when $k \geqslant 12$, one has

$$R(k,r) \leqslant C(r)k^{4r+1}(\log k)^{2r},$$

in which $C(r)$ is a quantity depending at most on r, but apparently growing somewhat more rapidly than $\exp(8r^2)$. Meanwhile, the early work of Roth [11] showing that $R(2,1) \leqslant 50$ has been improved by a sequence of authors over the past seven decades (see [1, 2, 4, 5, 13, 14, 15, 16, 17, 18]). Most recently, Liu and Zhao [9] have shown that $R(2,1) \leqslant 13$. As r increases in equation (1.2), the number of summands required to apply available technology increases rapidly. Thus, recent work of Kuan, Lesesvre and Xiao [8, Theorem 2] asserts that $R(2,2) \leqslant 133$.

We isolate two cases of the representation problem (1.2) for special attention. First, in the case $r = 1$, we note that Ford [5, Theorems 2 and 3] has shown that $R(3,1) \leqslant 72$, and that for large values of k one has $R(k,1) \ll k^2 \log k$. A corollary of Theorem 1.3 improves the order of magnitude of the latter bound.

Corollary 1.4. *When k is an integer with $k \geqslant 2$, one has $R(k,1) \leqslant 100(k+1)^2$.*

Thus, when $k \geqslant 2$ and $s \geqslant 100(k+1)^2$, all large integers n possess a representation in the shape

$$x_1^k + x_2^{k+1} + \ldots + x_s^{k+s-1} = n,$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$. The cognate problem in which one seeks representations of large integers n in the shape

$$x_1^k + x_2^{2k} + \ldots + x_s^{sk} = n,$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$, is considerably more difficult. Here, by taking $r = k$ in Theorem 1.3 we obtain the following conclusion.

Corollary 1.5. *Let k be an integer with $k \geqslant 2$. Then $R(k,k) \leqslant (6k+6)^{2k}$.*

For comparison, the aforementioned work of Scourfield [12] would deliver a much weaker bound of the general shape $R(k,k) \ll \exp(ck^2)$ for a suitable $c > 0$. It is worth remarking, however, that the heuristic arguments noted in the discussion following the statement of Corollary 1.2 suggest that one should have bounds of the shape $R(k,1) \ll k$ and $R(k,k) \ll e^k$.

Our proofs of Theorems 1.1 and 1.3 are based on applications of the Hardy-Littlewood method, and the basic infrastructure associated with this treatment is outlined in §2. Then, in §3 we prepare a novel Weyl-type estimate for exponential sums over smooth numbers. This eases our path in subsequent discussions and will likely be of independent interest. We combine this estimate with an upper bound for mean values of smooth Weyl sums in §4, making use of our recent work [3] concerning Waring's problem. Thereby, we obtain an acceptable upper bound for appropriate sets of minor arcs relevant to Theorem 1.1 and the second conclusion of Theorem 1.3. A refinement of this approach in §5 applies for the minor arc contribution needed for the proof of the first conclusion of Theorem 1.3. The corresponding major arc contributions are discussed in §6, the positivity of the singular series requiring some additional discussion in §7.

In this paper the letter p is reserved to denote a prime number. We use the standard notation $p^h \| n$ to indicate that $p^h | n$ and $p^{h+1} \nmid n$. Also, we write $\|\theta\|$ for $\min\{|\theta - n| : n \in \mathbb{Z}\}$ and $e(z)$ for $e^{2\pi i z}$.

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2. Preliminary infrastructure

The proofs of Theorems 1.1 and 1.3 make use of the Hardy-Littlewood method, with smooth Weyl sums playing a pivotal role. We denote the set of R-smooth integers not exceeding P by $\mathscr{A}(P,R)$, so that

$$\mathscr{A}(P,R)=\{n\in[1,P]\cap\mathbb{Z}:p|n \text{ implies } p\leqslant R\}.$$

We note that the standard theory of smooth numbers shows that whenever $\eta\in(0,1)$, then there is a positive number c_η with the property that $\text{card}(\mathscr{A}(P,R))\sim c_\eta P$ as $P\to\infty$ (see for example [19, Lemma 12.1]).

Fix $k_i\in\mathbb{N}$ $(i\geqslant 1)$ with $2\leqslant k_1\leqslant k_2\leqslant\ldots$. Let s be a natural number, and define $\theta=\theta_s(\mathbf{k})$ by putting

$$\theta_s(\mathbf{k})=\sum_{i=1}^s\frac{1}{k_i}. \tag{2.1}$$

For now, it suffices to remark that we have in mind imposing the condition $\theta>2$, although we shall later impose more onerous conditions on s. We consider a natural number n sufficiently large in terms of s and $k_1,\ldots,k_s$, and we seek a representation of n in the form

$$x_1^{k_1}+x_2^{k_2}+\ldots+x_s^{k_s}=n. \tag{2.2}$$

When $1\leqslant i\leqslant s$, we put

$$P_i=n^{1/k_i} \tag{2.3}$$

and observe that all positive integral solutions of the Diophantine equation (2.2) satisfy the bound $x_i\leqslant P_i$ $(1\leqslant i\leqslant s)$. Fix η to be a positive number sufficiently small in terms of s and $k_1,\ldots,k_s$, in a manner that will become clear in due course. Our goal is to establish a lower bound for the number $T(n;\eta)$ of solutions of the equation (2.2) with $x_i\in\mathscr{A}(P_i,P_i^\eta)$ $(1\leqslant i\leqslant s)$.

The smooth Weyl sums $f_i(\alpha)$ that are key to our arguments are defined by

$$f_i(\alpha)=\sum_{x\in\mathscr{A}(P_i,P_i^\eta)}e(\alpha x^{k_i}). \tag{2.4}$$

Writing

$$\mathscr{F}(\alpha)=f_1(\alpha)f_2(\alpha)\cdots f_s(\alpha), \tag{2.5}$$

it follows via orthogonality that

$$T(n;\eta)=\int_0^1\mathscr{F}(\alpha)e(-n\alpha)\,\mathrm{d}\alpha. \tag{2.6}$$

We derive an asymptotic formula for $T(n;\eta)$ by means of the circle method, the successful application of which requires the introduction of a Hardy-Littlewood dissection. Write $L=\log n$. We take the set of major arcs $\mathfrak{K}$ to be the union of the intervals

$$\mathfrak{K}(q,a)=\{\alpha\in[0,1):|\alpha-a/q|\leqslant L^{1/15}n^{-1}\},$$

with $0\leqslant a\leqslant q\leqslant L^{1/15}$ and $(a,q)=1$. The set of minor arcs complementary to $\mathfrak{K}$ is then $\mathfrak{k}=[0,1)\setminus\mathfrak{K}$. Our first objective, which we complete in §§4 and 5, is to establish that for a suitable positive number δ, provided that s is suitably large in terms of $\mathbf{k}$, one has an upper bound of the shape

$$\int_{\mathfrak{k}} |\mathscr{F}(\alpha)|\,\mathrm{d}\alpha \ll n^{\theta-1}L^{-\delta}. \tag{2.7}$$

The major arc asymptotics is then the central theme of §§6 and 7, where we confirm the lower bound

$$\int_{\mathfrak{K}} \mathscr{F}(\alpha)e(-n\alpha)\,\mathrm{d}\alpha \gg n^{\theta-1}, \tag{2.8}$$

again for suitably large values of s, and with n sufficiently large in terms of s, $\mathbf{k}$ and η. By combining the bounds (2.7) and (2.8) within (2.6), we conclude that $T(n;\eta) \gg n^{\theta-1}$, so that all large enough integers n possess a representation in the shape (2.2). This confirms the respective conclusions of Theorems 1.1 and 1.3, the only problem remaining being that of determining how large s must be so that the estimates (2.7) and (2.8) hold true. Appropriate bounds on s will be determined in §§4, 5 and 7.

3. An estimate of Weyl-type

This section concerns estimates for the exponential sums $f_i(\alpha)$ of use on sets of minor arcs more general than the arcs $\mathfrak{k}$ introduced in the previous section. Consider a natural number $k \geqslant 2$ and a large positive number P. We take Q to be a parameter with $1 \leqslant Q \leqslant P^{k/2}$. The major arcs $\mathfrak{M}(Q)$ are then defined to be the union of the sets

$$\mathfrak{M}(q,a;Q) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant QP^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q)=1$. The complementary set of minor arcs is then defined by putting $\mathfrak{m}(Q) = [0,1) \setminus \mathfrak{M}(Q)$. Finally, we make use of the dyadically truncated set of arcs $\mathfrak{N}(Q) = \mathfrak{M}(Q) \setminus \mathfrak{M}(Q/2)$. Notice that, as a consequence of Dirichlet's approximation theorem, one has $[0,1) = \mathfrak{M}(P^{k/2})$.

Our interest lies in estimates for the exponential sum

$$f(\alpha;P,R) = \sum_{x \in \mathscr{A}(P,R)} e(\alpha x^k),$$

valid when R is a positive number with $R \leqslant P^\eta$ for a suitably small positive number η, and $\alpha \in \mathfrak{N}(Q)$. In order to describe these estimates, we recall the concept of an admissible exponent from the theory of smooth Weyl sums. A real number Δ_s is referred to as an *admissible exponent* (for k) if it has the property that, whenever $\varepsilon > 0$ and η is a positive number sufficiently small in terms of ε, k and s, then whenever $2 \leqslant R \leqslant P^\eta$ and P is sufficiently large, one has

$$\int_0^1 |f(\alpha;P,R)|^s\,\mathrm{d}\alpha \ll P^{s-k+\Delta_s+\varepsilon}.$$

Here, the underlying parameter is P and the constant implicit in Vinogradov's notation may depend on ε, η, k and s. One may confirm that for all positive numbers s, there is no loss of generality in supposing that one has $\max\{0, k-s/2\} \leqslant \Delta_s \leqslant k$.

In order to facilitate concision, from this point onwards we adopt the extended ε, R notation routinely employed by scholars working with smooth Weyl sums while applying the Hardy-Littlewood method. Thus, whenever a statement involves the letter ε, then it is asserted that the statement holds for any positive real number assigned to ε. Implicit constants stemming from Vinogradov or Landau symbols may depend on ε, as well as ambient parameters implicitly fixed such as k and s. If a statement also involves the letter R, either implicitly or explicitly, then it is asserted that for any $\varepsilon > 0$ there is a number $\eta > 0$ such that the statement holds uniformly for $2 \leqslant R \leqslant P^\eta$. Our arguments will involve only a finite number of statements, and consequently we may pass to the smallest of the numbers η that arise in this way, and then have all estimates in force with the same positive number η. Notice that η may be assumed sufficiently small in terms of k, s and ε.

Associated with a family $(\Delta_s)_{s>0}$ of admissible exponents for k is the number

$$\tau(k) = \max_{w\in\mathbb{N}} \frac{k - 2\Delta_{2w}}{4w^2}, \tag{3.1}$$

an exponent which satisfies the bound $\tau(k) \leqslant 1/(4k)$. For each positive number s, one then has the related number

$$\Delta_s^* = \min_{0\leqslant t\leqslant s-2} \left(\Delta_{s-t} - t\tau(k)\right), \tag{3.2}$$

which we have described elsewhere as an *admissible exponent for minor arcs* (see the preamble to [3, Theorem 5.2] for a discussion of these exponents).

We recall two consequences of our recent work [3] on Waring's problem.

Lemma 3.1. *Suppose that $k \geqslant 2$, $s \geqslant 2$ and Δ_s^* is an admissible exponent for minor arcs satisfying $\Delta_s^* < 0$. Let κ be a positive number with $\kappa \leqslant k/2$. Then, whenever $P^\kappa \leqslant Q \leqslant P^{k/2}$, one has the bound*

$$\int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha \ll_\kappa P^{s-k} Q^{\varepsilon - 2|\Delta_s^*|/k}.$$

Proof. This is immediate from [3, Theorem 5.3]. □

Lemma 3.2. *Suppose that s is a real number with $s \geqslant 4k$ and Δ_s is an admissible exponent. Then whenever Q is a real number with $1 \leqslant Q \leqslant P^{k/2}$, one has the uniform bound*

$$\int_{\mathfrak{M}(Q)} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha \ll P^{s-k} Q^{\varepsilon + 2\Delta_s/k}.$$

Proof. For the sake of concision, write $f(\alpha)$ for $f(\alpha; P, R)$. Suppose first that $P^{1/(2k)} < Q \leqslant P^{k/2}$. Then the conclusion of [3, Theorem 4.2] shows that whenever $s \geqslant 2$, one has

$$\int_{\mathfrak{M}(Q)} |f(\alpha)|^s \,\mathrm{d}\alpha \ll P^{s-k+\varepsilon} Q^{2\Delta_s/k},$$

and the desired conclusion is immediate.

In order to handle the range of Q with $1 \leqslant Q \leqslant P^{1/(2k)}$, we turn to the bounds made available in [21]. We take a pedestrian approach sufficient for our subsequent application, though we note that with greater effort the condition $s \geqslant 4k$ could be relaxed at this point. Suppose first that $\alpha \in \mathfrak{M}(Q)$ for some real number Q satisfying

$$\exp((\log P)^{1/3}) \leqslant Q \leqslant P^{1/(2k)}.$$

When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $0 \leqslant a \leqslant q \leqslant \frac{1}{2}P^{k/2}$, the intervals $\mathfrak{M}(q, a; \frac{1}{2}P^{k/2})$ comprising $\mathfrak{M}(\frac{1}{2}P^{k/2})$ are disjoint. For $\alpha \in \mathfrak{M}(q, a; \frac{1}{2}P^{k/2}) \subseteq \mathfrak{M}(\frac{1}{2}P^{k/2})$, we put

$$\Upsilon(\alpha) = (q + P^k|q\alpha - a|)^{-1}.$$

Meanwhile, for $\alpha \in [0,1) \setminus \mathfrak{M}(\frac{1}{2}P^{k/2})$, we put $\Upsilon(\alpha) = 0$. Note that when $\alpha \in \mathfrak{N}(Q)$ one has $\Upsilon(\alpha) \ll Q^{-1}$. Then as a consequence of [21, Lemma 7.2], much as in the argument leading to [3, equation (6.3)], we find that when $\alpha \in \mathfrak{N}(Q)$, one has

$$f(\alpha) \ll P(\log P)^3 \Upsilon(\alpha)^{-\varepsilon+1/(2k)} + P^{1-1/(2k)} \ll PQ^{2\varepsilon-1/(2k)}.$$

We remark in this context that the constraint $k \geqslant 3$ of [3, equation (6.3)] is unnecessary in present circumstances. When instead

$$1 \leqslant Q \leqslant \exp((\log P)^{1/3}),$$

we appeal to [21, Lemma 8.5], deducing as in the cognate argument associated with [3, Theorem 6.1] that

$$f(\alpha) \ll P\Upsilon(\alpha)^{-\varepsilon+1/k} + P\exp(-(\log P)^{1/3}) \ll PQ^{-1/(2k)}.$$

Again, the constraint $k \geqslant 3$ of [3] is unnecessary in present circumstances. Thus, in view of the hypothesis $s \geqslant 4k$, it follows that when $1 \leqslant Q \leqslant P^{1/(2k)}$ one has

$$\begin{aligned}\int_{\mathfrak{N}(Q)} |f(\alpha)|^s \,\mathrm{d}\alpha &\ll P^s Q^{\varepsilon - s/(2k)} \mathrm{mes}(\mathfrak{M}(Q)) \\ &\ll P^s Q^{\varepsilon - s/(2k)} (Q^2 P^{-k}) \ll P^{s-k} Q^{\varepsilon}.\end{aligned}$$

This again delivers the estimate asserted in the statement of the lemma, since

$$\int_{\mathfrak{M}(Q)} |f(\alpha)|^s \,\mathrm{d}\alpha \leqslant \sum_{\substack{l=0 \\ 2^l \leqslant Q}}^{\infty} \int_{\mathfrak{N}(2^{-l}Q)} |f(\alpha)|^s \,\mathrm{d}\alpha \ll P^{s-k} Q^{\varepsilon}.$$

This completes the proof of the lemma. □

We obtain a pointwise bound for $f(\alpha) = f(\alpha; P, R)$ when $\alpha \in \mathfrak{m}(Q)$ by application of the Sobolev-Gallagher inequality.

Lemma 3.3. *Suppose that $k \geqslant 2$ and $0 < \rho(k) < 2\tau(k)/k$. Then, uniformly in $1 \leqslant Q \leqslant P^{k/2}$, one has the bound*

$$\sup_{\alpha \in \mathfrak{m}(Q)} |f(\alpha; P, R)| \ll PQ^{-\rho(k)}.$$

Proof. We consider in the first instance the situation in which $P^{1/(2k)} \leqslant Q \leqslant P^{k/2}$. Here, we apply Lemma 3.1 with $s = u + tk$, where $u = 4^k$ and t is sufficiently large in terms of k. The value of u here has been chosen large enough that the classical theory of Waring's problem is comfortably applicable. With more care one could work with a choice for u little more than $k \log k$. On considering the underlying Diophantine equation, working with the value of u already chosen, it follows from Hua's lemma and a routine application of the circle method along the lines described in [19, Chapter 2] that

$$\int_0^1 |f(\alpha)|^u \,\mathrm{d}\alpha \leqslant \int_0^1 \Bigg| \sum_{1 \leqslant x \leqslant P} e(\alpha x^k) \Bigg|^u \,\mathrm{d}\alpha \ll P^{u-k}. \tag{3.3}$$

In particular, the exponent $\Delta_{s-tk} = 0$ is admissible for k, and thus it follows from (3.2) that $\Delta_s^* = -tk\tau(k)$ is an admissible exponent for minor arcs. We therefore infer from Lemma 3.1 that

$$\int_{\mathfrak{m}(Q)} |f(\alpha)|^s \,\mathrm{d}\alpha \ll P^{s-k} Q^{\varepsilon - 2t\tau(k)}. \tag{3.4}$$

Consider next a real number α with $\alpha \in \mathfrak{m}(Q)$, and let δ be any real number with $|\delta| \leqslant P^{-k}$. Suppose, if possible, that $\alpha + \delta \in \mathfrak{M}(Q/2)$. In such circumstances, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, $1 \leqslant q \leqslant Q/2$ and $|q(\alpha+\delta) - a| \leqslant \frac{1}{2}QP^{-k}$. Consequently, one has

$$|q\alpha - a| \leqslant q|\delta| + \tfrac{1}{2}QP^{-k} \leqslant QP^{-k},$$

whence $\alpha \in \mathfrak{M}(Q)$. This yields a contradiction, so we are forced to conclude that $\alpha + \delta \in \mathfrak{m}(Q/2)$. This observation allows us to estimate $f(\alpha)$ pointwise on $\mathfrak{m}(Q)$ in terms of mean values for $f(\alpha)$ over $\mathfrak{m}(Q/2)$. Indeed, as a consequence of the Sobolev-Gallagher inequality (see for example Montgomery [10, Lemma 1.1]), we have

$$|f(\alpha)|^s \leqslant (2P^{-k})^{-1} \int_{|\beta-\alpha| \leqslant P^{-k}} |f(\beta)|^s \,\mathrm{d}\beta + s \int_{|\beta-\alpha| \leqslant P^{-k}} |f'(\beta) f(\beta)^{s-1}| \,\mathrm{d}\beta.$$

Hence, whenever $\alpha \in \mathfrak{m}(Q)$, we infer that

$$|f(\alpha)|^s \ll P^k I_1 + I_2, \tag{3.5}$$

where

$$I_1 = \int_{\mathfrak{m}(Q/2)} |f(\beta)|^s \,\mathrm{d}\beta \quad \text{and} \quad I_2 = \int_{\mathfrak{m}(Q/2)} |f'(\beta) f(\beta)^{s-1}| \,\mathrm{d}\beta.$$

The bound

$$I_1 \ll P^{s-k} Q^{\varepsilon - 2t\tau(k)} \tag{3.6}$$

follows from (3.4). Meanwhile, by applying Hölder's inequality, we see that

$$I_2 \leqslant I_3^{1/u} I_4^{1-1/u}, \tag{3.7}$$

where

$$I_3 = \int_0^1 |f'(\beta)|^u \,\mathrm{d}\beta \quad \text{and} \quad I_4 = \int_{\mathfrak{m}(Q/2)} |f(\beta)|^v \,\mathrm{d}\beta, \tag{3.8}$$

in which

$$v = \frac{s-1}{1-1/u}.$$

Recall that $u = 4^k$ is even. Then since

$$f'(\beta) = 2\pi \mathrm{i} \sum_{x \in \mathscr{A}(P,R)} x^k e(\beta x^k),$$

it follows from (3.8) by considering the underlying Diophantine equations that

$$I_3 \leqslant (2\pi P^k)^u \int_0^1 |f(\beta)|^u \,\mathrm{d}\beta.$$

On recalling (3.3), therefore, we deduce that

$$I_3 \ll (P^k)^u P^{u-k}. \tag{3.9}$$

Meanwhile, since $s > u$ we have $v > s$, and so it follows from (3.8) via Lemma 3.1 that

$$I_4 \ll P^{v-k} Q^{\varepsilon - 2|\Delta_v^*|/k}, \tag{3.10}$$

where

$$\Delta_v^* = -(v-u)\tau(k) = -\Big(\frac{s-u}{1-1/u}\Big)\tau(k) = -\frac{tk\tau(k)}{1-1/u}.$$

On substituting (3.9) and (3.10) into (3.7), we find that

$$I_2 \ll P^k (P^{u-k})^{1/u} (P^{v-k})^{1-1/u} Q^{\varepsilon - 2t\tau(k)} \ll P^s Q^{\varepsilon - 2t\tau(k)}. \tag{3.11}$$

On substituting (3.6) and (3.11) into (3.5), we conclude that

$$|f(\alpha)|^s \ll P^s Q^{\varepsilon - 2t\tau(k)}.$$

Thus, whenever $\alpha \in \mathfrak{m}(Q)$, we have $f(\alpha) \ll PQ^{\varepsilon-\kappa}$, where

$$\kappa = \frac{2t\tau(k)}{u+tk}.$$

We now take t sufficiently large in terms of k and obtain the upper bound

$$f(\alpha) \ll PQ^{\varepsilon - 2\tau(k)/k}.$$

This confirms the upper bound that we sought when $P^{1/(2k)} \leqslant Q \leqslant P^{k/2}$.

In order to handle the range of Q with $1 \leqslant Q < P^{1/(2k)}$, just as in the proof of Lemma 3.2 we turn to the bounds made available in [21]. For the sake of concision we adopt the notation of the proof of the latter lemma. Suppose first that one has $(\log P)^{60ks} \leqslant Q \leqslant P^{1/(2k)}$ and $\alpha \in \mathfrak{m}(Q)$. Then [21, Lemma 7.2] delivers the bound

$$\begin{aligned} f(\alpha) &\ll P(\log P)^3 \Upsilon(\alpha)^{-\varepsilon+1/(2k)} + P^{1-\tau(k)+\varepsilon} \\ &\ll PQ^{-1/(3k)} \ll PQ^{\varepsilon-2\tau(k)/k}. \end{aligned}$$

When instead $1 \leqslant Q \leqslant (\log P)^{60ks}$, we appeal to [21, Lemma 8.5], deducing that

$$\begin{aligned} f(\alpha) &\ll P\Upsilon(\alpha)^{-\varepsilon+1/k} + P(\log P)^{-60ks} \\ &\ll PQ^{-1/(2k)} \ll PQ^{\varepsilon-2\tau(k)/k}. \end{aligned}$$

Thus, in all circumstances, we have the estimate asserted in the statement of the lemma, and the proof is complete. □

For the purposes of this paper, we apply a bound for $\tau(k)$ sufficient for our applications, though falling very slightly short of the sharpest bound attainable using current technology. In this context, it is useful to introduce the exponent $\omega = \omega(k)$, defined by

$$\omega(k) = 1/(Dk^2), \quad \text{where} \quad D = 4.5139506. \tag{3.12}$$

Lemma 3.4. *When $k \geqslant 2$ and $1 \leqslant Q \leqslant P^{k/2}$, one has the uniform bound*

$$\sup_{\alpha \in \mathfrak{m}(Q)} |f(\alpha; P, R)| \ll PQ^{-\omega}.$$

Proof. When $k \geqslant 4$, it is shown in [3, Lemma 7.1] that there is a family of admissible exponents satisfying the property that $\tau(k) \geqslant (Ck)^{-1}$, where $C = 9.027901 < 2D$. Thus

$$\frac{2}{k}\tau(k) > \frac{1}{Dk^2},$$

and the desired conclusion follows from Lemma 3.3.

When k is equal to 2 or 3, we appeal to the formula (3.1) with the crude bound on admissible exponents available from Hua's lemma (see [19, Lemma 2.5]). Thus, we have the admissible exponent $\Delta_{2^k} = 0$ since

$$\int_0^1 |f(\alpha)|^{2^k} \, \mathrm{d}\alpha \leqslant \int_0^1 \Big| \sum_{1 \leqslant x \leqslant P} e(\alpha x^k) \Big|^{2^k} \mathrm{d}\alpha \ll P^{2^k - k + \varepsilon},$$

and hence we deduce via (3.1) that $\tau(2) \geqslant 1/8$ and $\tau(3) \geqslant 3/64$. Thus

$$\frac{2}{2}\tau(2) \geqslant \frac{1}{8} > \frac{1}{18} > \omega(2) \quad \text{and} \quad \frac{2}{3}\tau(3) \geqslant \frac{1}{32} > \frac{1}{40} > \omega(3).$$

In each of these cases, the desired conclusion again follows from Lemma 3.3. □

We finish this section with a formulation of our new minor arc estimate of sufficient flexibility that further applications may be anticipated.

Theorem 3.5. *Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$. Then one has*

$$f(\alpha; P, R) \ll P\left(\lambda^{-1} + \lambda P^{-k}\right)^{\omega},$$

where $\lambda = q + P^k|q\alpha - a|$.

Proof. We begin by establishing the superficially weaker assertion that, whenever $\alpha \in \mathbb{R}$, $c \in \mathbb{Z}$ and $t \in \mathbb{N}$ satisfy $(c, t) = 1$ and $|\alpha - c/t| \leqslant 1/t^2$, then

$$f(\alpha) \ll P(t^{-1} + tP^{-k})^{\omega}. \tag{3.13}$$

From this assertion, it follows via a standard transference principle (see for example [24, Lemma 14.1]) that the conclusion of the lemma holds.

Suppose then that c and t satisfy the relations $(c, t) = 1$ and $|\alpha - c/t| \leqslant 1/t^2$. We apply Dirichlet's approximation theorem. Thus, there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with $(b, r) = 1$ satisfying $1 \leqslant r \leqslant P^{k/2}$ and $|r\alpha - b| \leqslant P^{-k/2}$. We now put

$$Q = \max\{r, P^k|r\alpha - b|\} \leqslant P^{k/2},$$

so that $r \leqslant Q$ and $|r\alpha - b| \leqslant QP^{-k}$, and either $r \geqslant Q$ or $|r\alpha - b| \geqslant QP^{-k}$. Thus $\alpha \in \mathfrak{M}(Q) \setminus \mathfrak{M}(Q/2) \subseteq \mathfrak{m}(Q/2)$, and it follows from Lemma 3.4 that

$$f(\alpha) \ll PQ^{-\omega}. \tag{3.14}$$

When $c/t \neq b/r$, it follows from the triangle inequality that

$$\frac{1}{tr} \leqslant \left|\frac{c}{t} - \frac{b}{r}\right| \leqslant \left|\alpha - \frac{c}{t}\right| + \left|\alpha - \frac{b}{r}\right| \leqslant \frac{1}{t^2} + \frac{Q}{rP^k},$$

whence

$$1 \leqslant \frac{r}{t} + \frac{tQ}{P^k}.$$

Thus, we have either $r \geqslant \frac{1}{2}t$ or $Q \geqslant \frac{1}{2}P^k/t$. When instead $c/t = b/r$, we have $b = c$ and $r = t$, and the same conclusion holds. In either case, therefore, we find that $Q = \max\{r, P^k|r\alpha - b|\} \geqslant \frac{1}{2}\min\{t, P^k/t\}$. Thus, we infer from (3.14) that

$$f(\alpha) \ll P\left(t^{-\omega} + (P^k/t)^{-\omega}\right) \ll P(t^{-1} + tP^{-k})^{\omega}.$$

Thus the desired conclusion (3.13) follows, and the proof of the theorem is complete. □

4. The minor arc contribution for ascending powers

We now address the representation problem (2.2) and adopt the notation of §2. In situations wherein k_2 may be substantially larger than k_1, we apply a Weyl-type estimate only for the exponential sum $f_1(\alpha)$, estimating the remaining ones in mean. Put

$$F_1(\alpha) = f_1(\alpha), \quad G_1(\alpha) = f_2(\alpha)f_3(\alpha)\cdots f_s(\alpha), \tag{4.1}$$

and note that in view of (2.1), (2.3) and (2.4), one has $F_1(0)G_1(0) \asymp n^{\theta}$. We take Q to be a parameter with $1 \leqslant Q \leqslant n^{1/2}$ and define a Hardy-Littlewood dissection in accordance with that introduced in §3. Thus, the major arcs $\mathfrak{M}(Q)$ are defined to be the union of the sets

$$\mathfrak{M}(q,a;Q)=\{\alpha\in[0,1):|q\alpha-a|\leqslant Qn^{-1}\},$$

with $0\leqslant a\leqslant q\leqslant Q$ and $(a,q)=1$, and the associated set of minor arcs are defined by setting $\mathfrak{m}(Q)=[0,1)\setminus\mathfrak{M}(Q)$. Also, we put $\mathfrak{N}(Q)=\mathfrak{M}(Q)\setminus\mathfrak{M}(Q/2)$. Note that $\mathfrak{N}(Q)\subseteq\mathfrak{m}(Q/2)$. Since $n=P_i^{k_i}$ for each i, these definitions align with those of §3 when considering the smooth Weyl sum $f_i(\alpha)$.

We begin by recording a Weyl-type estimate for $F_1(\alpha)$.

Lemma 4.1. *When $1\leqslant Q\leqslant n^{1/2}$, one has the bound*

$$\sup_{\alpha\in\mathfrak{m}(Q)}|F_1(\alpha)|\ll F_1(0)Q^{-1/(Dk_1^2)}.$$

Proof. In view of (4.1), this estimate is immediate from Lemma 3.4. □

The mean value estimate that we obtain for $G_1(\alpha)$ depends on admissible exponent bounds. Here we note that, whenever v is even, the corollary to [22, Theorem 2.1] shows that the exponent Δ_v is admissible for $k\geqslant 4$, where Δ_v is the unique positive solution of the equation

$$\Delta_v e^{\Delta_v/k}=ke^{1-v/k}. \tag{4.2}$$

When k is equal to 2 or 3, the admissible exponents available from Hua's lemma show that the real numbers Δ_v defined via (4.2) are admissible. Of course, much sharper estimates are known in these cases (see [23] for the sharpest available conclusions when $k=3$). We note that the exponent Δ_s in [22] corresponds to our Δ_v with $v=2s$, owing to the slightly different definition employed therein.

We next provide an upper bound for the mean value of $|G_1(\alpha)|$ over the intermediate set of arcs $\mathfrak{N}(Q)$. In this context, it is convenient to introduce the quantity

$$\Phi_1=\sum_{i=2}^{s}\frac{1}{k_i}. \tag{4.3}$$

Lemma 4.2. *When $1\leqslant Q\leqslant n^{1/2}$, one has the bound*

$$\int_{\mathfrak{N}(Q)}|G_1(\alpha)|\,\mathrm{d}\alpha\ll G_1(0)n^{-1}Q^{2\Theta_1},$$

where $\Theta_1=e^{1-\Phi_1+2/k_2}$.

Proof. Define the exponents

$$t_i=k_i\Phi_1\quad(2\leqslant i\leqslant s). \tag{4.4}$$

Then it follows from (4.3) that we have

$$\sum_{i=2}^{s}\frac{1}{t_i}=\frac{1}{\Phi_1}\sum_{i=2}^{s}\frac{1}{k_i}=1,$$

and hence an application of Hölder's inequality leads us from (4.1) to the bound

$$\int_{\mathfrak{N}(Q)}|G_1(\alpha)|\,\mathrm{d}\alpha\leqslant\prod_{i=2}^{s}I_i^{1/t_i}, \tag{4.5}$$

where

$$I_i=\int_{\mathfrak{N}(Q)}|f_i(\alpha)|^{t_i}\,\mathrm{d}\alpha. \tag{4.6}$$

For each index i, the largest even integer not exceeding t_i is larger than t_i-2, and hence it follows from (4.2) that there is an exponent Δ_{t_i} admissible for k_i with

$$\Delta_{t_i}<k_ie^{1-(t_i-2)/k_i}.$$

Since $\mathfrak{N}(Q)\subseteq\mathfrak{M}(Q)$, we find from (4.6) via Lemma 3.2 that

$$I_i\ll P_i^{t_i-k_i}Q^{\varepsilon+2\Delta_{t_i}/k_i}.$$

Thus, in view of (2.3) and (4.4), we have

$$I_i\ll P_i^{t_i}n^{-1}Q^{2\delta_i}\quad(2\leqslant i\leqslant s), \tag{4.7}$$

where

$$\delta_i=e^{1-\Phi_1+2/k_i}.$$

On substituting (4.7) into (4.5), we conclude that

$$\int_{\mathfrak{N}(Q)}|G_1(\alpha)|\,\mathrm{d}\alpha\ll G_1(0)n^{-1}Q^{2\Omega_1}, \tag{4.8}$$

where

$$\Omega_1=\sum_{i=2}^s\frac{\delta_i}{t_i}\leqslant\delta_2\sum_{i=2}^s\frac{1}{t_i}=e^{1-\Phi_1+2/k_2}=\Theta_1.$$

The conclusion of the lemma is therefore immediate from (4.8). □

By combining the conclusions of Lemmata 4.1 and 4.2, we obtain a minor arc estimate sufficient for our proof of Theorem 1.1. Here and henceforth, we fix η to be a positive number sufficiently small in terms of $k_1,k_2,\dots,k_s$, and ε, in the context of the estimates of this and the previous section relevant for the various admissible exponents encountered. Also, we recall the notation of writing L for $\log n$.

Lemma 4.3. *Suppose that*

$$\sum_{i=2}^s\frac{1}{k_i}>2\log k_1+\frac{2}{k_2}+1+\log(2D). \tag{4.9}$$

Then there is a positive number δ having the property that

$$\int_{\mathfrak{k}}|\mathscr{F}(\alpha)|\,\mathrm{d}\alpha\ll n^{\theta-1}L^{-\delta}.$$

Proof. By referring to the definition of $\mathfrak{k}$ in §2, we see that $\mathfrak{k}\subset\mathfrak{m}(L^{1/15})$. When $L^{1/15}\leqslant Q\leqslant n^{1/2}$, it follows from Lemmata 4.1 and 4.2 that

$$\begin{aligned}\int_{\mathfrak{N}(Q)}|F_1(\alpha)G_1(\alpha)|\,\mathrm{d}\alpha&\leqslant\Big(\sup_{\alpha\in\mathfrak{N}(Q)}|F_1(\alpha)|\Big)\int_{\mathfrak{N}(Q)}|G_1(\alpha)|\,\mathrm{d}\alpha\\&\ll F_1(0)Q^{-1/(Dk_1^2)}G_1(0)n^{-1}Q^{2\Theta_1}.\end{aligned}$$

Provided that the hypothesis (4.9) holds, it follows from (4.3) that

$$e^{\Phi_1}>2e^{1+2/k_2}Dk_1^2,$$

whence $2\Theta_1<1/(Dk_1^2)$. Put

$$\delta=\frac{1}{15}\Big(\frac{1}{Dk_1^2}-2\Theta_1\Big).$$

Then, on recalling (2.5), we may conclude thus far that

$$\int_{\mathfrak{N}(Q)} |\mathscr{F}(\alpha)|\,\mathrm{d}\alpha \ll n^{\theta-1}Q^{-15\delta}.$$

But $\mathfrak{k}$ is covered by the sets $\mathfrak{N}(Q)$ via a dyadic dissection, and we see that

$$\begin{aligned}\int_{\mathfrak{k}} |\mathscr{F}(\alpha)|\,\mathrm{d}\alpha &\leqslant \sum_{\substack{l=0\\ 2^l\leqslant n^{1/2}L^{-1/15}}}^{\infty} \int_{\mathfrak{N}(2^{-l}n^{1/2})} |\mathscr{F}(\alpha)|\,\mathrm{d}\alpha\\ &\ll n^{\theta-1} \sum_{\substack{l=0\\ 2^l\leqslant n^{1/2}L^{-1/15}}}^{\infty} (2^{-l}n^{1/2})^{-15\delta}\\ &\ll n^{\theta-1}L^{-\delta}.\end{aligned}$$

□

We complete this section by addressing the particular situation relevant to the second conclusion of Theorem 1.3.

Corollary 4.4. *Suppose that k and r are natural numbers with $r \geqslant k \geqslant 2$, and put $k_i = k + r(i-1)$ $(1 \leqslant i \leqslant s)$. Then, provided that $s \geqslant (6k+6)^{2r}$, there is a positive number δ having the property that*

$$\int_{\mathfrak{k}} |\mathscr{F}(\alpha)|\,\mathrm{d}\alpha \ll n^{\theta-1}L^{-\delta}.$$

Proof. We apply Lemma 4.3, observing that by a familiar argument one has

$$\sum_{i=2}^{s} \frac{1}{k+r(i-1)} > \int_1^s \frac{\mathrm{d}t}{k+rt} = \frac{1}{r}\log\left(\frac{k+rs}{k+r}\right).$$

Thus, provided that one has

$$\frac{1}{r}\log\left(\frac{k+rs}{k+r}\right) \geqslant 2\log k + \frac{2}{k+r} + 1 + \log(2D), \tag{4.10}$$

then the conclusion of the corollary follows from Lemma 4.3. We observe that the hypothesis $r \geqslant k$ ensures that

$$\frac{1}{k+r} \leqslant \frac{1}{2k} < \frac{1}{k} - \frac{1}{2k^2} < \log\left(1 + \frac{1}{k}\right),$$

and hence $e^{1/(k+r)} < 1 + 1/k$. It follows that the lower bound (4.10) is satisfied provided that

$$k + rs \geqslant (2eD(k+1)^2)^r(k+r).$$

Thus, on recalling that $k \leqslant r$, we conclude that the lower bound (4.10) holds whenever $s \geqslant 2(2eD)^r(k+1)^{2r}$. In view of (3.12), however, a modicum of computation confirms the bound $2^{1+1/r}eD < 35$ for $r \geqslant 2$, and hence the upper bound asserted in the corollary holds whenever $s \geqslant (6k+6)^{2r}$, completing the proof. □

5. An enhanced minor arc estimate

Given exponents k_i ($i \geqslant 1$) having the property that k_i is small for numerous small indices i, one may sharpen the analysis of the minor arcs presented in the previous section. We illustrate the underlying strategy in this section with a consideration of the situation in which

$$k_i = k + r(i-1) \quad (1 \leqslant i \leqslant s),$$

with r small. We now put

$$F_2(\alpha) = f_1(\alpha)f_2(\alpha)\cdots f_k(\alpha) \quad \text{and} \quad G_2(\alpha) = f_{k+1}(\alpha)f_{k+2}(\alpha)\cdots f_s(\alpha). \tag{5.1}$$

In accord with the discussion of the previous section, we have $F_2(0)G_2(0) \asymp n^\theta$. In all other respects, we adopt the notation of the previous section, with an analysis so similar that we are able to economise on detail.

Lemma 5.1. *When $1 \leqslant Q \leqslant n^{1/2}$, one has the bound*

$$\sup_{\alpha \in \mathfrak{m}(Q)} |F_2(\alpha)| \ll F_2(0)Q^{-1/(Dk(r+1))}.$$

Proof. As a consequence of Lemma 3.4, it follows from (5.1) that

$$\sup_{\alpha \in \mathfrak{m}(Q)} |F_2(\alpha)| \ll \prod_{i=1}^{k} P_i Q^{-1/(Dk_i^2)} = F_2(0)Q^{-\phi}, \tag{5.2}$$

in which we put

$$\phi = \sum_{i=1}^{k} \frac{1}{D(k+r(i-1))^2}.$$

However, by applying a familiar lower bound, we find that

$$\phi > \frac{1}{D}\int_0^k \frac{\mathrm{d}t}{(k+rt)^2} = \frac{1}{Dr}\Big(\frac{1}{k} - \frac{1}{k(r+1)}\Big) = \frac{1}{Dk(r+1)}.$$

The conclusion of the lemma follows by substituting this estimate into (5.2). $\square$

Our upper bound for the mean value of $|G_2(\alpha)|$ over $\mathfrak{N}(Q)$ is obtained through a modification of the corresponding treatment of $G_1(\alpha)$ in Lemma 4.2. We now put

$$\Phi_2 = \sum_{i=k+1}^{s} \frac{1}{k_i}. \tag{5.3}$$

Lemma 5.2. *When $1 \leqslant Q \leqslant n^{1/2}$, one has the bound*

$$\int_{\mathfrak{N}(Q)} |G_2(\alpha)|\,\mathrm{d}\alpha \ll G_2(0)n^{-1}Q^{2\Theta_2},$$

where $\Theta_2 = e^{1-\Phi_2+2/(k(r+1))}$.

Proof. Define the exponents $t_i = k_i\Phi_2$ ($k+1 \leqslant i \leqslant s$). Then by following the argument of the proof of Lemma 4.2 mutatis mutandis, we obtain the upper bound

$$\int_{\mathfrak{N}(Q)} |G_2(\alpha)|\,\mathrm{d}\alpha \ll \prod_{i=k+1}^{s} \left(P_i^{t_i} n^{-1} Q^{2\delta_i}\right)^{1/t_i},$$

where we now write $\delta_i = e^{1-\Phi_2+2/k_i}$. Thus we infer that

$$\int_{\mathfrak{N}(Q)} |G_2(\alpha)|\,\mathrm{d}\alpha \ll G_2(0)n^{-1}Q^{2\Omega_2},$$

where

$$\Omega_2 = \sum_{i=k+1}^{s} \frac{\delta_i}{t_i} \leqslant \delta_{k+1} \sum_{i=k+1}^{s} \frac{1}{t_i} = e^{1-\Phi_2+2/(k(r+1))} = \Theta_2,$$

and this delivers the conclusion of the lemma. □

We now combine the conclusions of Lemmata 5.1 and 5.2 much as in the proof of Lemma 4.3.

Lemma 5.3. *Suppose that k and r are natural numbers with $r \geqslant 1$ and $k \geqslant 2$, and put $k_i = k + r(i-1)$ $(1 \leqslant i \leqslant s)$. Then, provided that $s \geqslant A(r)(k+1)^{r+1}$, where $A(r) = r^{-1}25^r(r+1)^{r+1}$, there is a positive number δ having the property that*

$$\int_{\mathfrak{k}} |\mathscr{F}(\alpha)|\,\mathrm{d}\alpha \ll n^{\theta-1}L^{-\delta}.$$

Proof. We again have $\mathfrak{k} \subset \mathfrak{m}(L^{1/15})$. When $L^{1/15} \leqslant Q \leqslant n^{1/2}$, we find from Lemmata 5.1 and 5.2 that

$$\int_{\mathfrak{N}(Q)} |F_2(\alpha)G_2(\alpha)|\,\mathrm{d}\alpha \leqslant \Big(\sup_{\alpha\in\mathfrak{N}(Q)} |F_2(\alpha)|\Big) \int_{\mathfrak{N}(Q)} |G_2(\alpha)|\,\mathrm{d}\alpha$$
$$\ll F_2(0)Q^{-1/(Dk(r+1))}G_2(0)n^{-1}Q^{2\Theta_2}. \tag{5.4}$$

On recalling (5.3), we see that

$$\Phi_2 = \sum_{i=k+1}^{s} \frac{1}{k+r(i-1)} > \int_k^s \frac{\mathrm{d}t}{k+rt} = \frac{1}{r}\log\Big(\frac{k+rs}{k(r+1)}\Big),$$

and hence

$$\Theta_2 < e^{1+2/(k(r+1))}\left(\frac{k(r+1)}{k+rs}\right)^{1/r}.$$

Therefore, provided that

$$2e^{1+2/(k(r+1))}\left(\frac{k(r+1)}{k+rs}\right)^{1/r} < \frac{1}{Dk(r+1)}, \tag{5.5}$$

then it follows from (5.4) that there is a positive number δ having the property that

$$\int_{\mathfrak{N}(Q)} |F_2(\alpha)G_2(\alpha)|\,\mathrm{d}\alpha \ll F_2(0)G_2(0)n^{-1}Q^{-15\delta}. \tag{5.6}$$

We now set about establishing the inequality (5.5). Observe first that since $r \geqslant 1$, one has

$$\frac{2r}{k(r+1)^2} \leqslant \frac{1}{2k} < \frac{1}{k} - \frac{1}{2k^2} < \log\Big(1+\frac{1}{k}\Big),$$

and thus

$$e^{2r/(k(r+1))} < (1+1/k)^{r+1}.$$

We consequently infer that (5.5) holds whenever

$$k+rs > k(r+1)(2eDk(r+1))^r(1+1/k)^{r+1}.$$

On recalling (3.12), we find that $2eD < 25$, and thus it follows that (5.5) holds whenever

$$s \geqslant r^{-1}25^r(r+1)^{r+1}(k+1)^{r+1} = A(r)(k+1)^{r+1}.$$

Since this lower bound is one of the hypotheses of the lemma, we may henceforth work under the assumption that the upper bound (5.6) holds.

On recalling (2.5) and (2.1), we find that (5.6) yields the bound

$$\int_{\mathfrak{N}(Q)} |\mathscr{F}(\alpha)|\,\mathrm{d}\alpha \ll n^{\theta-1}Q^{-15\delta}.$$

A comparison with the concluding part of the argument of the proof of Lemma 4.3, using a dyadic dissection of $\mathfrak{k}$ into subsets of the shape $\mathfrak{N}(Q)$, therefore leads us from here to the conclusion of the lemma. □

6. The major arc contribution

The goal of this section is to make progress on establishing the lower bound (2.8) for the contribution of the major arcs to $T(n;\eta)$. Throughout this section and the next, we work under the hypothesis that

$$\theta = \sum_{i=1}^{s} \frac{1}{k_i} > 2. \tag{6.1}$$

The hypotheses available to us in Theorem 1.1 ensure that $\theta > 3$, thereby confirming (6.1) with room to spare. In the first conclusion of Theorem 1.3, meanwhile, we have in particular $s > 25^r(k+1)$, and thus

$$\begin{aligned}\theta = \sum_{i=1}^{s} \frac{1}{k+r(i-1)} &> \int_0^s \frac{\mathrm{d}t}{k+rt} = \frac{1}{r}\log\left(\frac{k+rs}{k}\right)\\ &> \frac{1}{r}\log(25^r) = \log 25 > 2,\end{aligned}$$

and the hypothesis (6.1) again holds. On the other hand, in the second conclusion of Theorem 1.3 one has $r \geqslant k$ and $s \geqslant (6k+6)^{2r}$, whence a similar argument yields

$$\theta > \frac{1}{r}\log\left((6k+6)^r\right) = \log(6k+6) \geqslant \log 18 > 2,$$

and (6.1) holds once again. The upshot of this discussion is that we are cleared in all circumstances to work henceforth under the assumption that (6.1) holds.

Suppose next that $\alpha \in \mathfrak{K}(q,a) \subseteq \mathfrak{K}$. The standard theory of smooth Weyl sums (see [20, Lemma 5.4]) shows that there is a positive number $c = c(\eta)$ such that for $1 \leqslant i \leqslant s$, one has

$$f_i(\alpha) = cq^{-1}S_i(q,a)v_i(\alpha - a/q) + O(P_iL^{-1/4}),$$

wherein

$$S_i(q,a) = \sum_{t=1}^{q} e(at^{k_i}/q) \quad \text{and} \quad v_i(\beta) = \frac{1}{k_i}\sum_{m\leqslant n} m^{-1+1/k_i}e(\beta m).$$

Put

$$\mathfrak{J}(n;X) = \int_{-X/n}^{X/n} v_1(\beta)v_2(\beta)\cdots v_s(\beta)e(-\beta n)\,\mathrm{d}\beta. \tag{6.2}$$

Also, write

$$\mathfrak{S}(n;X)=\sum_{1\leqslant q\leqslant X}q^{-s}U_n(q),$$

where

$$U_n(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}S_1(q,a)S_2(q,a)\cdots S_s(q,a)e(-na/q). \tag{6.3}$$

Then since $\mathfrak{K}$ has measure $O(L^{1/5}n^{-1})$, we see that

$$\int_{\mathfrak{K}}\mathscr{F}(\alpha)e(-n\alpha)\,\mathrm{d}\alpha=c^s\mathfrak{J}(n;L^{1/15})\mathfrak{S}(n;L^{1/15})+O(P_1P_2\cdots P_sn^{-1}L^{-1/20}). \tag{6.4}$$

We show in the next section that the singular series

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}q^{-s}U_n(q) \tag{6.5}$$

converges absolutely and uniformly for $n\in\mathbb{N}$, and moreover that $\mathfrak{S}(n)\gg 1$ whenever $s\geqslant 4k_1$ and the condition (6.1) holds. Moreover, under the latter condition we show further that there is a positive number δ such that

$$\mathfrak{S}(n)-\mathfrak{S}(n;X)\ll X^{-\delta}. \tag{6.6}$$

Next, on making use of the bound supplied by [19, Lemma 2.8], one finds that

$$v_i(\beta)\ll P_i(1+n\|\beta\|)^{-1/k_i}\quad(1\leqslant i\leqslant s).$$

Hence, working under the hypothesis (6.1), we deduce from (6.2) that there is a positive number δ such that

$$\mathfrak{J}(n;X)=\mathfrak{J}(n)+O(P_1P_2\cdots P_sn^{-1}X^{-\delta}), \tag{6.7}$$

where

$$\mathfrak{J}(n)=\int_{-1/2}^{1/2}v_1(\beta)v_2(\beta)\cdots v_s(\beta)e(-\beta n)\,\mathrm{d}\beta.$$

A familiar approach paralleling that of [19, Theorem 2.3] shows that

$$\mathfrak{J}(n)=\frac{\Gamma\big(1+\frac{1}{k_1}\big)\Gamma\big(1+\frac{1}{k_2}\big)\cdots\Gamma\big(1+\frac{1}{k_s}\big)}{\Gamma\big(\frac{1}{k_1}+\frac{1}{k_2}+\ldots+\frac{1}{k_s}\big)}n^{\theta-1}\left(1+O(n^{-1/k_s})\right). \tag{6.8}$$

Thus, on combining (6.4) with (6.6), (6.7) and (6.8), we conclude that there is a positive number δ for which

$$\int_{\mathfrak{K}}\mathscr{F}(\alpha)e(-n\alpha)\,\mathrm{d}\alpha=c^s\Gamma(\theta)^{-1}\bigg(\prod_{i=1}^{s}\Gamma\Big(1+\frac{1}{k_i}\Big)\bigg)\mathfrak{S}(n)n^{\theta-1}+O(n^{\theta-1}L^{-\delta}). \tag{6.9}$$

Subject to our verification in the next section that the lower bound $\mathfrak{S}(n)\gg 1$ holds uniformly in n, we conclude from (6.9) that the lower bound (2.8) holds. In combination with the minor arc estimate (2.7), available from Lemma 4.3 under the hypotheses of Theorem 1.1, we conclude that

$$T(n;\eta)=\int_{\mathfrak{K}}\mathscr{F}(\alpha)e(-n\alpha)\,\mathrm{d}\alpha+\int_{\mathfrak{k}}\mathscr{F}(\alpha)e(-n\alpha)\,\mathrm{d}\alpha\gg n^{\theta-1}. \tag{6.10}$$

This completes the proof of Theorem 1.1. In order to establish Theorem 1.3, we observe on the one hand that the upper bound (2.7) follows from Lemma 5.3 when $s\geqslant A(r)(k+1)^{r+1}$. Also, when $r\geqslant k$ and $s\geqslant(6k+6)^{2r}$, the upper bound (2.7) follows from Corollary 4.4. Thus, in either case, we find as before that (6.10) follows in these respective situations, and thus the proof of Theorem 1.3 is now complete.

7. The singular series

In this section we estimate the singular series, confirming (6.6) and the bounds $1 \ll \mathfrak{S}(n) \ll 1$. Our argument parallels the analogous treatment of [12], though we introduce refinements en route. We continue working under the hypothesis (6.1) throughout.

First, from (6.3) and [19, Theorem 4.2], we see that the bound

$$q^{-s}U_n(q) \ll q^{1-1/k_1-1/k_2-\ldots-1/k_s} = q^{1-\theta}$$

holds uniformly in n. Thus, in view of (6.1), there is a positive number δ for which $q^{-s}U_n(q) \ll q^{-1-\delta}$. It follows that the singular series $\mathfrak{S}(n)$ defined in (6.5) converges absolutely and uniformly in n, and moreover one has the bound (6.6). Next, by following the argument underlying the proof of [19, Lemma 2.11], we see that $U_n(q)$ is a multiplicative function of q. In view of (6.5), we may rewrite $\mathfrak{S}(n)$ in the form $\mathfrak{S}(n) = \prod_p \chi_p(n)$, where the product is over all prime numbers p, and

$$\chi_p(n) = \sum_{\nu=0}^{\infty} p^{-s\nu}U_n(p^\nu). \tag{7.1}$$

By orthogonality, this Euler factor is related to the number $M_n(p^\nu)$ of incongruent solutions of the congruence

$$x_1^{k_1} + x_2^{k_2} + \ldots + x_s^{k_s} \equiv n \pmod{p^\nu},$$

via the relation

$$\chi_p(n) = \lim_{\nu\to\infty} p^{(1-s)\nu}M_n(p^\nu). \tag{7.2}$$

A model for the necessary argument, which is standard, may be found in the discussion associated with [19, Lemma 2.12]. The limit (7.2) is seen to exist via the relation (7.1). In particular, the quantity $\chi_p(n)$ is a non-negative number satisfying the relation $\chi_p(n) = 1 + O(p^{-1-\delta})$.

We summarise our deliberations thus far in the form of a lemma.

Lemma 7.1. *Suppose that* (6.1) *holds. Then the series* (6.5) *converges absolutely, and there exists a natural number C with the property that for all integers n, one has*

$$\mathfrak{S}(n) \geqslant \tfrac{1}{2} \prod_{p\leqslant C} \chi_p(n).$$

We have yet to obtain a lower bound for $\chi_p(n)$ when $p \leqslant C$, a matter to which we now attend. Put $D = (k_1, k_2, \ldots, k_s)$, the greatest common divisor of $k_1, k_2, \ldots, k_s$. Define the non-negative integer λ by means of the relation $p^\lambda \| D$. Then we have $p^\lambda | k_i$ for $1 \leqslant i \leqslant s$, and there exists an index j with $1 \leqslant j \leqslant s$ for which $p^\lambda \| k_j$. We show that for each integer n, there is a solution of the congruence

$$x_1^{k_1} + x_2^{k_2} + \ldots + x_s^{k_s} \equiv n \pmod{p^{\lambda+\tau}}, \tag{7.3}$$

with $\tau = 1$ for odd p, and with $\tau = 2$ for $p = 2$, in each case with $(x_j, p) = 1$.

In order to establish this last assertion, suppose temporarily that there is an integer n having the property that (7.3) has no solution with $(x_j, p) = 1$. It then follows that the range of the left hand side of (7.3) modulo $p^{\lambda+\tau}$, with $(x_j, p) = 1$, has at most $p^{\lambda+\tau}-1$ elements. In the first instance we assume that p is odd. Then, the theory of power residues shows that the monomial x^{k_j} takes $(p-1)/(p-1,k_j)$ values modulo $p^{\lambda+1}$ as x varies over $1\leqslant x\leqslant p^{\lambda+1}$ with $(x,p)=1$. Furthermore, for any index i we see that y^{k_i} takes at least $1+(p-1)/(p-1,k_i)$ values modulo $p^{\lambda+1}$ as y varies over $1\leqslant y\leqslant p^{\lambda+1}$. We now repeatedly apply the Cauchy-Davenport theorem (see [19, Lemma 2.14]), beginning with the values of $x_j^{k_j}$, and then adding in the remaining powers step-by-step. On recalling (6.1), we find that with $p\nmid x_j$, the range of the left hand side of (7.3), modulo $p^{\lambda+1}$, contains a number of elements which is at least

$$\sum_{i=1}^{s}\frac{p-1}{(p-1,k_ip^{-\lambda})}\geqslant p^{\lambda}(p-1)\sum_{i=1}^{s}\frac{1}{k_i}>2p^{\lambda}(p-1).$$

This yields a contradiction, since $2p^{\lambda}(p-1)\geqslant p^{\lambda+1}$. Our claim concerning the solubility of the congruence (7.3) is consequently confirmed when p is odd.

We next consider the situation with $p=2$, where $\tau=2$. For some index j with $1\leqslant j\leqslant s$, one has $2^{\lambda}\|k_j$. In (7.3) we take $x_j=1$. We can solve (7.3) with $x_i\in\{0,1\}$ $(1\leqslant i\leqslant s$ and $i\neq j)$ provided that $s\geqslant 2^{\lambda+2}$. However, we have $k_1\geqslant 2^{\lambda}$, and hence the condition that $s\geqslant 4k_1$ suffices to confirm our claim concerning the solubility of the congruence (7.3) in the case that $p=2$.

A routine argument now bounds $M_n(p^{\nu})$ from below. We observe that since $p^{\lambda}\|k_j$, a number coprime to p is a k_j-th power residue modulo $p^{\lambda+\tau}$ if and only if it is a k_j-th power residue modulo p^{ν}, for all $\nu\geqslant\lambda+\tau$. Let $x_1,x_2,\dots,x_s$ be a solution of (7.3), with $(x_j,p)=1$, and let ν be a natural number with $\nu\geqslant\lambda+\tau$. There are $p^{\nu-\lambda-\tau}$ choices for y_i with $y_i\equiv x_i \pmod{p^{\lambda+\tau}}$ and $1\leqslant y_i\leqslant p^{\nu}$. For each such choice with $1\leqslant i\leqslant s$ and $i\neq j$, the integer

$$n-\sum_{\substack{i=1\\ i\neq j}}^{s}x_i^{k_i}$$

is a k_j-th power residue modulo $p^{\lambda+\tau}$, and therefore a k_j-th power residue modulo p^{ν}. Thus, we have $M_n(p^{\nu})\geqslant p^{(s-1)(\nu-\lambda-\tau)}$, so by (7.2) we see that $\chi_p(n)\geqslant p^{-(\lambda+\tau)(s-1)}$. This lower bound holds for all primes p with $p^{\lambda}\|D$ and all $n\in\mathbb{N}$ provided that $s\geqslant 4k_1$ and (6.1) holds.

We summarise these deliberations in the following lemma.

Lemma 7.2. *Suppose that $s\geqslant 4k_1$, and (6.1) holds. Then there is a positive number ω having the property that $\mathfrak{S}(n)\geqslant\omega$ for all $n\in\mathbb{N}$.*

This lemma completes our analysis of the singular series, and thus we have confirmed all of the properties that were needed to complete the analysis of §6. It is worth noting that the condition $s\geqslant 4k_1$ of Lemma 7.2 is automatically satisfied whenever the hypotheses of Theorem 1.1 hold for the exponents $k_1,k_2,\dots,k_s$. In order to verify this claim, observe that

$$\sum_{i=3}^{s}\frac{1}{k_3}\leqslant\frac{s}{k_1}$$

whilst

$$2\log k_1+\frac{1}{k_2}+3.20032>2\log 2+3>4.$$

Thus the hypotheses of Theorem 1.1 can be satisfied only when $s > 4k_1$.

References

[1] J. Brüdern, *Sums of squares and higher powers II*, J. London Math. Soc. (2) **35** (1987), no. 2, 244–250.

[2] J. Brüdern, *A problem in additive number theory*, Math. Proc. Cambridge Philos. Soc. **103** (1988), no. 1, 27–33.

[3] J. Brüdern and T. D. Wooley, *On Waring's problem for larger powers*, submitted, arxiv:2211.10380.

[4] K. B. Ford, *The representation of numbers as sums of unlike powers*, J. London Math. Soc. (2) **51** (1995), no. 1, 14–26.

[5] K. B. Ford, *The representation of numbers as sums of unlike powers. II*, J. Amer. Math. Soc. **9** (1996), no. 4, 919–940.

[6] G. A. Freĭman, *Solution of Waring's problem in a new form*, Uspehi Matem. Nauk (N.S.) **4** (1949), no. 1 (29), 193.

[7] G. A. Freĭman, *On one variant of Waring's problem*, Vestnik Mos. Univ. **3** (1949), 3–16.

[8] C. I. Kuan, D. Lesesvre and X. Xiao, *Sums of even ascending powers*, submitted, arxiv:2001.02429.

[9] J. Liu and L. Zhao, *Representation by sums of unlike powers*, J. Reine Angew. Math. **781** (2021), 19–55.

[10] H. L. Montgomery, *Topics in multiplicative number theory*, Springer-Verlag, Berlin-New York, 1971.

[11] K. F. Roth, *A problem in additive number theory*, Proc. London Math. Soc. (2) **53** (1951), no. 1, 381–395.

[12] E. J. Scourfield, *A generalization of Waring's problem*, J. London Math. Soc. **35** (1960), no. 1, 98–116.

[13] K. Thanigasalam, *On additive number theory*, Acta Arith. **13** (1967/68), no. 3, 237–258.

[14] K. Thanigasalam, *On sums of powers and a related problem*, Acta Arith. **36** (1980), no. 2, 125–141.

[15] K. Thanigasalam, *Addendum and corrigendum to "On sums of powers and a related problem"*, Acta Arith. **42** (1982), no. 4, 425.

[16] K. Thanigasalam, *On certain additive representations of integers*, Portugal. Math. **42** (1983/84), no. 4, 447–465.

[17] R. C. Vaughan, *On the representation of numbers as sums of powers of natural numbers*, Proc. London Math. Soc. (3) **21** (1970), no. 1, 160–180.

[18] R. C. Vaughan, *On sums of mixed powers*, J. London Math. Soc. (2) **3** (1971), no. 4, 677–688.

[19] R. C. Vaughan, *The Hardy-Littlewood method*, 2nd edition, Cambridge University Press, Cambridge, 1997.

[20] R. C. Vaughan, *A new iterative method in Waring's problem*, Acta Math. **162** (1989), no. 1-2, 1–71.

[21] R. C. Vaughan and T. D. Wooley, *On Waring's problem: some refinements*, Proc. London Math. Soc. (3) **63** (1991), no. 1, 35–68.

[22] T. D. Wooley, *The application of a new mean value theorem to the fractional parts of polynomials*, Acta Arith. **65** (1993), no. 2, 163–179.

[23] T. D. Wooley, *Sums of three cubes, II*, Acta Arith. **170** (2015), no. 1, 73–100.

[24] T. D. Wooley, *Rational solutions of pairs of diagonal equations, one cubic and one quadratic*, Proc. London Math. Soc. (3) **110** (2015), no. 2, 325–356.

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