AN ELEMENTARY DISCRETE INEQUALITY

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Abstract. When $\lambda > 0$, one has

$$\min_{r\in\mathbb{N}} (r + \lambda/r) \leqslant \sqrt{4\lambda+1},$$

with equality if and only if $\lambda = m(m-1)$ for some positive integer m.

School students learn early in life that when $\lambda > 0$, the function $x + \lambda/x$ achieves its minimum value for positive values of x when $x = \sqrt{\lambda}$, in which case the two terms comprising the function are equal. Indeed,

$$x + \lambda/x = 2\sqrt{\lambda} + \left(\sqrt{x} - \frac{\sqrt{\lambda}}{\sqrt{x}} \right)^2 \geqslant 2\sqrt{\lambda},$$

and the conclusion is clear. If instead of minimising over all positive real values of x, one is restricted to work with positive integers, then one can approximate this argument by choosing x to be one of the two positive integers closest to $\sqrt{\lambda}$. A precise form of this conclusion is surely well-known to the cognoscenti, and was apparently known to this author 25 years ago (see [1], and [2] for a more recent application). The purpose of this note is to refresh the author's memory, while also making the conclusion more readily available.

Theorem 1. *When $\lambda > 0$, one has*

$$\min_{r\in\mathbb{N}} (r + \lambda/r) \leqslant \sqrt{4\lambda+1},$$

with equality if and only if $\lambda = m(m-1)$ for some positive integer m.

Proof. Let r be the unique integer satisfying

$$\sqrt{\lambda + \tfrac{1}{4}} - \tfrac{1}{2} < r \leqslant \sqrt{\lambda + \tfrac{1}{4}} + \tfrac{1}{2}.$$

Then we have $r + \lambda/r \leqslant \sqrt{4\lambda+1}$ if and only if

$$r^2 + 2\lambda + \lambda^2/r^2 \leqslant 4\lambda + 1,$$

and this inequality holds if and only if $(r - \lambda/r)^2 \leqslant 1$. Thus we see that the first conclusion of the theorem will be confirmed by verifying that $|r - \lambda/r| \leqslant 1$.

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Put $\delta = r - \sqrt{\lambda + \frac{1}{4}}$, and note that $-\frac{1}{2} < \delta \leqslant \frac{1}{2}$. Then we have

$$\left| r - \frac{\lambda}{r} \right| = \left| \frac{\left(\sqrt{\lambda + \frac{1}{4}} + \delta\right)^2 - \lambda}{\sqrt{\lambda + \frac{1}{4}} + \delta} \right|$$

$$= \left| 2\delta + \frac{\frac{1}{4} - \delta^2}{\sqrt{\lambda + \frac{1}{4}} + \delta} \right|.$$

When $\delta > 0$, we now put $\tau = \frac{1}{2} - \delta$. Then we find that

$$0 < 2\delta + \frac{\frac{1}{4} - \delta^2}{\sqrt{\lambda + \frac{1}{4}} + \delta} \leqslant 1 - 2\tau + \frac{\tau - \tau^2}{1 - \tau} \leqslant 1.$$

When $\delta \leqslant 0$, meanwhile, we put $\tau = \frac{1}{2} + \delta$. A similar argument then yields

$$-\left(2\delta + \frac{\frac{1}{4} - \delta^2}{\sqrt{\lambda + \frac{1}{4}} + \delta} \right) \leqslant 1 - 2\tau - \frac{\tau - \tau^2}{\sqrt{\lambda + \frac{1}{4}}} \leqslant 1$$

and

$$-\left(2\delta + \frac{\frac{1}{4} - \delta^2}{\sqrt{\lambda + \frac{1}{4}} + \delta} \right) \geqslant 1 - 2\tau - \frac{\tau - \tau^2}{\tau} \geqslant -\tfrac{1}{2}.$$

In either case, therefore, we have $|r - \lambda/r| \leqslant 1$. In view of our earlier discussion, this establishes the first conclusion of the theorem.

The upper bound asserted in the first conclusion of the theorem holds with equality if and only if there is an integer r satisfying the equation $r + \lambda/r = \sqrt{4\lambda + 1}$. This relation holds if and only if

$$(r - \tfrac{1}{2}\sqrt{4\lambda + 1})^2 = r^2 - r\sqrt{4\lambda + 1} + \lambda + \tfrac{1}{4} = \tfrac{1}{4},$$

and in turn, this equation holds if and only if

$$r = \pm\tfrac{1}{2} + \tfrac{1}{2}\sqrt{4\lambda + 1}.$$

Thus, one has $(2r \pm 1)^2 = 4\lambda + 1$, so that $\lambda = r^2 \pm r$. The first conclusion of the theorem consequently holds with equality if and only if $\lambda = m(m-1)$ for some positive integer m. □

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References

[1] R. C. Vaughan and T. D. Wooley, *A special case of Vinogradov's mean value theorem*, Acta Arith. **79** (1997), no. 3, 193–204.

[2] T. D. Wooley, *Paucity problems and some relatives of Vinogradov's mean value theorem*, Math. Proc. Cambridge Philos Soc. (in press, to appear), 17pp; arxiv:2107.12238.

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