RATIONAL LINES ON DIAGONAL HYPERSURFACES AND SUBCONVEXITY VIA THE CIRCLE METHOD

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Abstract. Fix $k, s, n \in \mathbb{N}$, and consider non-zero integers $c_1, \ldots, c_s$, not all of the same sign. Provided that $s \geqslant k(k+1)$, we establish a Hasse principle for the existence of lines having integral coordinates lying on the affine diagonal hypersurface defined by the equation $c_1x_1^k + \ldots + c_sx_s^k = n$. This conclusion surmounts the conventional convexity barrier tantamount to the square-root cancellation limit for this problem.

1. Introduction

The investigation of rational linear spaces on algebraic varieties was pursued by Brauer [6] and Birch [2] as a key step in their inductive strategies for establishing the existence of rational points on complete intersections. This initial work in the middle of the last century has more recently evolved, in contributions of Parsell [11, 12] and Brandes [4], to encompass quantitative considerations. In this paper, we also investigate the abundance of rational lines, but now on affine diagonal hypersurfaces. By applying the Hardy-Littlewood (circle) method, we derive a certain Hasse principle for the existence of lines having integral coordinates lying on the hypersurface. A notable feature of our application is that it goes beyond the convexity limit of the circle method, by which we mean the square-root barrier that ordinarily restricts the method to problems in which the number of available variables exceeds twice the inherent degree.

In order to describe our conclusions more precisely, we must introduce some notation. We fix natural numbers $k \geqslant 2$ and s, and we consider the affine hypersurface defined by the diagonal equation

$$c_1x_1^k + \ldots + c_sx_s^k = n, \tag{1.1}$$

in which $c_i \in \mathbb{Z} \setminus \{0\}$ $(1 \leqslant i \leqslant s)$ and $n \in \mathbb{Z} \setminus \{0\}$ are fixed. We assume, in particular, that the coefficients c_i are neither all positive nor all negative. For each exponent k, a conventional application of the circle method confirms the existence of a positive number $s_0(k)$ having the property that the solutions of the equation (1.1) satisfy the weak approximation property provided only that $s \geqslant s_0(k)$. Indeed, it follows from the work and methods of earlier scholars that when $2 \leqslant k \leqslant 15$ one has $s_0(k) \leqslant t_0(k)$, where $t_0(k)$ is defined according to Table 1 below (see [9, 10, 13, 14, 16, 17, 22] for the necessary ideas). Meanwhile, recent work of the author with Brüdern [9] may be routinely applied to confirm that $s_0(k) \leqslant \lceil k(\log k + 4.20032)\rceil$. Moreover, subject to real and p-adic solubility hypotheses, it follows under the same conditions that the

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equation (1.1) possesses an abundance of integral solutions in which, for each $r \geqslant 1$, there is no r-tuple $(i_1, i_2, \ldots, i_r)$ of indices with $1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant s$ for which

$$c_{i_1} x_{i_1}^k + c_{i_2} x_{i_2}^k + \ldots + c_{i_r} x_{i_r}^k = 0.$$

Henceforth, we refer to the latter as the condition that there be *no vanishing sub-sums*, and we note in particular that it implies that no variable x_i is equal to 0.

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t_0(k)$	4	7	12	17	24	31	39	47	55	63	72	81	89	97

TABLE 1. Upper bounds for $s_0(k)$ when $2 \leqslant k \leqslant 15$.

Our interest in this paper lies with the existence of linear solution spaces of the equation (1.1) of the shape $\mathbf{x} = \mathbf{y} + t\mathbf{z}$, with $\mathbf{y} \in (\mathbb{Z} \setminus \{0\})^s$ and $\mathbf{z} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$. Subject to local solubility conditions and the hypothesis $s \geqslant s_0(k)$, it follows from the above discussion that there exists an s-tuple $\mathbf{y} \in (\mathbb{Z} \setminus \{0\})^s$ satisfying the equation

$$c_1 y_1^k + \ldots + c_s y_s^k = n. \tag{1.2}$$

With this solution fixed, we denote by $N_{s,k}(B; \mathbf{y})$ the number of integral s-tuples $\mathbf{z} \in \mathbb{Z}^s \cap [-B, B]^s$ for which the equation

$$c_1 (y_1 + t z_1)^k + \ldots + c_s (y_s + t z_s)^k = n \tag{1.3}$$

holds identically as a polynomial in t. By expanding the powers in (1.3) via the binomial theorem, and recalling (1.2), one sees that the condition on these s-tuples is equivalent to insisting that $\mathbf{z}$ satisfy the system of equations

$$\sum_{i=1}^s c_i y_i^{k-j} z_i^j = 0 \quad (1 \leqslant j \leqslant k). \tag{1.4}$$

We choose in this paper to focus on the situation with $k \geqslant 3$. The situation with $k = 1$ is a matter for linear algebra, while that with $k = 2$ is accessible to the theory of quadratic forms. Indeed, by eliminating a variable between the linear and quadratic equations in (1.4), one sees that the problem of determining $N_{s,2}(B; \mathbf{y})$ is equivalent to the classical problem of counting integral solutions of a homogeneous quadratic equation in $s - 1$ variables subject to a congruence condition, and this is well-understood for all s.

Theorem 1.1. *Let s and k be natural numbers with $k \geqslant 3$ and $s \geqslant k(k+1)$. Also, let $c_1, \ldots, c_s$ and n be fixed non-zero integers, with $c_1, \ldots, c_s$ neither all positive nor all negative. Suppose that $y_1, \ldots, y_s$ are non-zero integers satisfying the equation* (1.2)*. Then, provided that the system* (1.4) *has non-singular real and p-adic solutions for every prime number p, there is a positive number $\mathscr{C}_{s,k}(\mathbf{y})$ for which*

$$N_{s,k}(B; \mathbf{y}) = \mathscr{C}_{s,k}(\mathbf{y}) B^{s-k(k+1)/2} + o(B^{s-k(k+1)/2}). \tag{1.5}$$

Some remarks are in order concerning the nature of the conclusion provided by Theorem 1.1. First, since $s_0(k) \leqslant k(k+1)$ for all natural numbers k, the discussion above ensures that there are plenty of solutions $\mathbf{y} \in (\mathbb{Z} \setminus \{0\})^s$ satisfying the equation (1.2) whenever local solubility conditions permit such a conclusion. Here, it is apparent that obstructions to p-adic solubility may be present when the bulk of the coefficients c_i are divisible by p, and yet n is not. However, one may regard the first important hypothesis of this theorem as being essentially harmless. Next, as we demonstrate in §7, the existence of non-singular real and p-adic solutions of the system (1.4) follows in two simple circumstances occurring generically. First, should the solution $\mathbf{y} \in (\mathbb{Z} \setminus \{0\})^s$ of the equation (1.2) satisfy the condition that there be no vanishing subsums, then any solution $\mathbf{z} \neq \mathbf{0}$ of the system (1.4) over $\mathbb{R}$ or $\mathbb{Q}_p$ is automatically non-singular. Secondly, subject only to the condition that $y_1, \ldots, y_s$ are non-zero integers satisying the equation (1.2), any solution $\mathbf{z}$ of the system (1.4) over $\mathbb{R}$ or $\mathbb{Q}_p$ is non-singular whenever $z_i \neq 0$ for $1 \leqslant i \leqslant s$.

Finally, as the reader will have anticipated, one may interpret the coefficient $\mathscr{C}_{s,k}(\mathbf{y})$ appearing in the asymptotic formula (1.5) as a product of local densities, the description of which requires some preparation. When p is a prime number and $h \in \mathbb{N}$, write $M_p(h)$ for the number of solutions $\mathbf{z}$ of the system (1.4) with $\mathbf{z} \in (\mathbb{Z}/p^h\mathbb{Z})^s$. Also, when $\eta > 0$, denote by $M_\infty(\eta)$ the volume of the subset of $[-1,1]^s$ defined by the inequalities

$$\left| \sum_{i=1}^s c_i y_i^{k-j} z_i^j \right| < \eta \quad (1 \leqslant j \leqslant k).$$

Then the limits

$$\sigma_\infty = \lim_{\eta \to 0+} (2\eta)^{-k} M_\infty(\eta) \quad \text{and} \quad \sigma_p = \lim_{h \to \infty} p^{h(k-s)} M_p(h),$$

when they exist, respectively define the real and p-adic densities of solutions of the system (1.4). We show in §5 that, under the hypotheses of the statement of Theorem 1.1, both limits exist, and one has $\mathscr{C}_{s,k}(\mathbf{y}) = \sigma_\infty \prod_p \sigma_p$, where the product is taken over all prime numbers p. Furthermore, one has $1 \ll \mathscr{C}_{s,k}(\mathbf{y}) \ll 1$.

The conclusion of Theorem 1.1 shows, subject to natural local solubility conditions and the constraint $s \geqslant k(k+1)$, that there is an abundance of affine lines having integral coefficients passing through each eligible integral point of the hypersurface determined by the equation (1.1). In the situation wherein $s = k(k+1)$, the conclusion of Theorem 1.1 surmounts the convexity barrier in the circle method, since the number of variables is precisely twice the sum of the degrees of the polynomials defining the system of equations (1.4). This subconvexity conclusion is made more apparent by a consideration of the associated exponential sums. When $k \geqslant 3$ and X is a large real number, define $f(\boldsymbol{\alpha}) = f_k(\boldsymbol{\alpha}; X)$ by putting

$$f_k(\boldsymbol{\alpha}; X) = \sum_{|x| \leqslant X} e(\alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_k x^k), \tag{1.6}$$

where, as usual, we write $e(z) = e^{2\pi i z}$. We introduce a Hardy-Littlewood dissection to facilitate discussion. Write $L = X^{1/(8k^2)}$. Then, when

$$1 \leqslant q \leqslant L, \quad 0 \leqslant \mathbf{a} \leqslant q \quad \text{and} \quad (q, \mathbf{a}) = 1,$$

we define the major arc $\mathfrak{P}(q, \mathbf{a})$ by

$$\mathfrak{P}(q, \mathbf{a}) = \{\boldsymbol{\alpha} \in [0,1)^k : |\alpha_j - a_j/q| \leqslant LX^{-j} \ (1 \leqslant j \leqslant k)\}.$$

Here and throughout this paper, we facilitate concision by adopting the use of extended vector notation. Thus, we write $0 \leqslant \mathbf{a} \leqslant q$ to denote that $0 \leqslant a_j \leqslant q$ for $1 \leqslant j \leqslant k$, and we write $(q, \mathbf{a})$ for the greatest common divisor $(q, a_1, \ldots, a_k)$ of q and $a_1, \ldots, a_k$. The arcs $\mathfrak{P}(q, \mathbf{a})$ are disjoint, as is easily verified. Let $\mathfrak{P}$ denote their union, and put $\mathfrak{p} = [0,1)^k \setminus \mathfrak{P}$.

We illustrate the subconvexity estimates available through the approach underlying the proof of Theorem 1.1 with the following conclusion.

Theorem 1.2. *Let k and s be natural numbers with $k \geqslant 3$. Suppose that $c_1, \ldots, c_s$ are non-zero integers satisfying the property that*

$$c_1 + \ldots + c_s \neq 0. \tag{1.7}$$

Then, whenever $1 \leqslant s < k(k+1)$, one has

$$\int_{[0,1)^k} f_k(c_1\boldsymbol{\alpha}; X) \cdots f_k(c_s\boldsymbol{\alpha}; X)\, \mathrm{d}\boldsymbol{\alpha} \ll X^{(s-1)/2+\varepsilon}. \tag{1.8}$$

When $s = k(k+1)$, meanwhile, one has

$$\int_{\mathfrak{p}} f_k(c_1\boldsymbol{\alpha}; X) \cdots f_k(c_s\boldsymbol{\alpha}; X)\, \mathrm{d}\boldsymbol{\alpha} \ll X^{(s-\delta)/2+\varepsilon}, \tag{1.9}$$

where $\delta = 1/(4k^3)$, and when $s > k(k+1)$ one has

$$\int_{\mathfrak{p}} f_k(c_1\boldsymbol{\alpha}; X) \cdots f_k(c_s\boldsymbol{\alpha}; X)\, \mathrm{d}\boldsymbol{\alpha} \ll X^{s-\frac{1}{2}k(k+1)-\frac{1}{2}\delta+\varepsilon}. \tag{1.10}$$

Given the trivial estimate $|f_k(\boldsymbol{\alpha}; X)| \leqslant 2X+1$, the bounds (1.8) and (1.9) plainly go beyond those that would result from square-root cancellation, and consequently constitute subconvexity estimates in the sense described in our work joint with Brüdern [7]. We remark in this context that, with greater effort, it would be possible to establish the estimate (1.9) with a larger value of δ. In this paper we have elected to opt for a more concise account yielding reasonable qualitative results, rather than seek the strongest quantitative results that might be accessible.

We briefly offer a sketch of the strategy underlying the proof of Theorem 1.1, restricting attention to the simpler situation that is the focus of Theorem 1.2. Here, by orthogonality, the mean value

$$\Upsilon = \int_{[0,1)^k} \prod_{i=1}^{s} f_k(c_i\boldsymbol{\alpha}; X)\, \mathrm{d}\boldsymbol{\alpha}$$

on the left hand side of (1.8) counts the number of integral solutions of the system of equations

$$\sum_{i=1}^{s} c_i x_i^j = 0 \quad (1 \leqslant j \leqslant k), \tag{1.11}$$

with $|x_i| \leqslant X$ $(1 \leqslant i \leqslant s)$. For each such solution $\mathbf{x}$, and for every integer y with $1 \leqslant y \leqslant X$, it follows from the binomial theorem that

$$\sum_{i=1}^{s} c_i(x_i + y)^j = -c_0 y^j \quad (1 \leqslant j \leqslant k),$$

where we write $c_0 = -(c_1 + \ldots + c_s)$. We note that our hypothesis (1.7) concerning the coefficients c_i ensures that one has $c_0 \neq 0$. We therefore see that for each integer y with $|y| \leqslant X$, the number of integral solutions of the system (1.11) counted by Υ is bounded above by the number of integral solutions of

$$c_0 y^j + \sum_{i=1}^{s} c_i z_i^j = 0 \quad (1 \leqslant j \leqslant k),$$

with $|z_i| \leqslant 2X$ $(1 \leqslant i \leqslant s)$. By averaging over these values of y and invoking orthogonality, we thus deduce that

$$\int_{[0,1)^k} \prod_{i=1}^s f_k(c_i \boldsymbol{\alpha}; X)\,\mathrm{d}\boldsymbol{\alpha} \leqslant X^{-1} \int_{[0,1)^k} \prod_{j=0}^s f_k(c_j \boldsymbol{\alpha}; 2X)\,\mathrm{d}\boldsymbol{\alpha}. \tag{1.12}$$

By comparison with the mean value (1.8), we now have an additional variable over which to average in (1.12), and it is this which permits us to achieve subconvexity. We note that, in order to analyse the mean value (1.9), which is restricted to minor arcs only, we employ some ideas from harmonic analysis previously deployed in our work [19] devoted to the asymptotic formula in Waring's problem.

This paper is organised as follows. We derive the fundamental lemma, based on the strategy just described, in §2. This work already permits a swift proof of the first subconvex estimate (1.8) recorded in Theorem 1.2. In §3 we begin the proof of a more general variant of the minor arc estimate (1.9) recorded in Theorem 1.2. This lays the foundation of the proof of Theorem 1.1. This preliminary minor arc estimate is converted in §4 into one more accessible to conventional applications of the Hardy-Littlewood method. The major arc analysis required to complete the proof of Theorem 1.1 is then tackled in §5. We complete the proofs of Theorems 1.1 and 1.2 in §6. Finally, in §7, we discuss the non-singularity condition implicit in Theorem 1.1, showing that the existence of non-singular solutions of the system (1.4) is implied by the conditions that we have already noted.

Throughout, the letter ε will denote a positive number. We adopt the convention that whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Our basic parameter will be either X or B, a sufficiently large positive number. In addition, we use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on k, s and ε, as well as other ambient parameters apparent from the context. Finally, we define $\|\theta\|$ for $\theta \in \mathbb{R}$ by putting $\|\theta\| = \min\{|\theta - n| : n \in \mathbb{Z}\}$.

Historical note: The first version of this paper dates from 2014, motivated by the author's proof in January 2014 of the main conjecture in the cubic case of Vinogradov's mean value theorem (see [21], which first appeared as arXiv:1401.3150). The author is grateful to Julia Brandes, Simon Rydin Myerson, Per Salberger and others for their comments on talks on this topic delivered at Warwick, King's College London, Oxford and Göteborg in the period 2014 to 2016 as the associated ideas evolved. These ideas subsequently delivered subconvex conclusions in the Hilbert-Kamke problem (see [26]) and affine variants of Vinogradov's mean value theorem (see [24, 25], and note also [5]).

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2. An averaged mean value

We begin by interpreting the strategy outlined at the end of the introduction as it applies to a mean value not necessarily open to a Diophantine interpretation. This supplies a fairly general conclusion useful in our subsequent deliberations. We suppose throughout that s, k, $\mathbf{y}$ and n are fixed as in the preamble to the statement of Theorem 1.1. When $y \in \mathbb{Z}$, we define $\beta_j(y) = \beta_j(\boldsymbol{\alpha}; y)$ by putting

$$\beta_j(\boldsymbol{\alpha}; y) = y^{k-j}\alpha_j \quad (1 \leqslant j \leqslant k). \tag{2.1}$$

In the proof of the next lemma as well as in its preamble, when $j \in \{k-1, k\}$, we promote concision by abbreviating the differential $\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_j$ to $\mathrm{d}\boldsymbol{\alpha}_j$. Then, when $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, we introduce the mean value

$$I_s(\mathfrak{B}; X) = \int_{\mathfrak{B}} \int_{[0,1)^{k-1}} \prod_{i=1}^{s} f(c_i \boldsymbol{\beta}(y_i)) \, \mathrm{d}\boldsymbol{\alpha}_k, \tag{2.2}$$

in which $f(\boldsymbol{\theta}) = f_k(\boldsymbol{\theta}; X)$ is defined via (1.6). Notice that, by orthogonality, one has $N_{s,k}(B; \mathbf{y}) = I_s([0,1); B)$. We make use of technology associated with Vinogradov's mean value theorem. With this in mind, when $t, k \in \mathbb{N}$, the parameter X is positive, and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, we define

$$\mathfrak{J}_{t,k}(\mathfrak{B}; X) = \int_{\mathfrak{B}} \int_{[0,1)^{k-1}} |f_k(\boldsymbol{\alpha}; X)|^t \, \mathrm{d}\boldsymbol{\alpha}_k. \tag{2.3}$$

Lemma 2.1. *Let $\mathbf{c}, \mathbf{y} \in (\mathbb{Z} \setminus \{0\})^s$, and define*

$$c_0 = -(c_1 y_1^k + \ldots + c_s y_s^k). \tag{2.4}$$

Suppose that $c_0 \neq 0$. Then, whenever $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, one has

$$I_s(\mathfrak{B}; X) \ll X^{-1} (\log X)^{s+1} \prod_{i=0}^{s} \mathfrak{J}_{s+1,k}(c_i \mathfrak{B}; X)^{1/(s+1)}.$$

Here, the constant implicit in Vinogradov's notation may depend on $\mathbf{y}$.

Proof. We make use of the translation invariance underlying a blown-up version of the system of Diophantine equations underlying the mean value (2.2). Write

$$\psi(u; \boldsymbol{\theta}) = \theta_1 u + \ldots + \theta_k u^k. \tag{2.5}$$

Observe first that for each index i, and every integral shift z, it follows from (1.6) that one has

$$f(\boldsymbol{\beta}(y_i); X) = \sum_{|x - y_i z| \leqslant X} e(\psi(x - y_i z; \boldsymbol{\beta}(y_i))). \tag{2.6}$$

Write

$$\mathfrak{f}_{i,z}(\boldsymbol{\alpha}; \gamma) = \sum_{|x| \leqslant 2X} e\left(\psi(x - y_i z; \boldsymbol{\beta}(y_i)) + \gamma(x - y_i z)\right). \tag{2.7}$$

In addition, define

$$K(\gamma) = \sum_{|w| \leqslant X} e(-\gamma w), \tag{2.8}$$

and put

$$\Lambda = \min_{1 \leqslant i \leqslant s} |y_i|^{-1}.$$

Then we deduce from (2.6) via orthogonality that when $|z| \leqslant \Lambda X$, one has

$$f(\boldsymbol{\beta}(y_i); X) = \int_0^1 \mathfrak{f}_{i,z}(\boldsymbol{\alpha}; \gamma) K(\gamma) \, \mathrm{d}\gamma. \tag{2.9}$$

Next, define

$$\mathfrak{F}_z(\boldsymbol{\alpha}; \boldsymbol{\gamma}) = \prod_{i=1}^{s} \mathfrak{f}_{i,z}(c_i \boldsymbol{\alpha}; \gamma_i). \tag{2.10}$$

Then, on substituting (2.9) into (2.2), we deduce that for each integer z satisfying $|z|\leqslant \Lambda X$, one has

$$I_s(\mathfrak{B};X)=\int_{[0,1)^s}\mathscr{I}(\boldsymbol{\gamma};z)\widetilde{K}(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma}, \tag{2.11}$$

where

$$\mathscr{I}(\boldsymbol{\gamma};z)=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}\mathfrak{F}_z(\boldsymbol{\alpha};\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\alpha}_k \tag{2.12}$$

and

$$\widetilde{K}(\boldsymbol{\gamma})=\prod_{i=1}^s K(\gamma_i). \tag{2.13}$$

By orthogonality, one finds that

$$\int_{[0,1)^{k-1}}\mathfrak{F}_z(\boldsymbol{\alpha};\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\alpha}_{k-1}=\sum_{|\mathbf{x}|\leqslant 2X}\Delta(\alpha_k,\boldsymbol{\gamma},z), \tag{2.14}$$

where $\Delta(\theta,\boldsymbol{\gamma},z)$ is equal to

$$e\left(\theta\sum_{i=1}^s c_i(x_i-y_iz)^k+\sum_{i=1}^s(x_i-y_iz)\gamma_i\right),$$

when

$$\sum_{i=1}^s c_iy_i^{k-j}(x_i-y_iz)^j=0\quad (1\leqslant j\leqslant k-1), \tag{2.15}$$

and otherwise $\Delta(\theta,\boldsymbol{\gamma},z)$ is equal to 0.

By applying the binomial theorem and recalling (2.4), one discerns that whenever the system (2.15) is satisfied by the s-tuple $\mathbf{x}$, then

$$c_0z^j+\sum_{i=1}^s c_iy_i^{k-j}x_i^j=0\quad (1\leqslant j\leqslant k-1),$$

and hence

$$\sum_{i=1}^s c_i(x_i-y_iz)^k=c_0z^k+\sum_{i=1}^s c_ix_i^k.$$

Then, on recalling (2.5), it follows from (2.14) that

$$\int_{[0,1)^{k-1}}\mathfrak{F}_z(\boldsymbol{\alpha};\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\alpha}_{k-1}=e(-z\boldsymbol{\gamma}\cdot\mathbf{y})\int_{[0,1)^{k-1}}\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})e(c_0\psi(z;\boldsymbol{\alpha}))\,\mathrm{d}\boldsymbol{\alpha}_{k-1}.$$

From here, we are led from the relation (2.12) to the formula

$$\mathscr{I}(\boldsymbol{\gamma};z)=e(-z\boldsymbol{\gamma}\cdot\mathbf{y})\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})e(c_0\psi(z;\boldsymbol{\alpha}))\,\mathrm{d}\boldsymbol{\alpha}_k.$$

Recalling the notation (1.6), we may consequently conclude thus far that

$$\sum_{|z|\leqslant\Lambda X}\mathscr{I}(\boldsymbol{\gamma};z)=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}\mathfrak{F}_0(\boldsymbol{\alpha};\boldsymbol{\gamma})f(c_0\boldsymbol{\alpha}-\widetilde{\boldsymbol{\gamma}};\Lambda X)\,\mathrm{d}\boldsymbol{\alpha}_k, \tag{2.16}$$

where $\widetilde{\boldsymbol{\gamma}}$ is defined by putting $\widetilde{\gamma}_1=\boldsymbol{\gamma}\cdot\mathbf{y}$ and $\widetilde{\gamma}_j=0$ $(2\leqslant j\leqslant k)$.

It is convenient at this point to set $y_0=1$ and to apply orthogonality just as in the argument leading to (2.9). Thus, on recalling (2.7), we see that

$$f(\boldsymbol{\alpha};\Lambda X)=\int_0^1\mathfrak{f}_{0,0}(\boldsymbol{\alpha};\gamma)K_0(\gamma)\,\mathrm{d}\gamma,$$

where

$$K_0(\gamma)=\sum_{|z|\leqslant \Lambda X} e(-\gamma z). \tag{2.17}$$

Thus, by applying Hölder's inequality to (2.16) and recalling (2.10), we see that

$$\Bigl|\sum_{|z|\leqslant \Lambda X}\mathscr{I}(\boldsymbol{\gamma};z)\Bigr|\leqslant \int_0^1\Bigl(\prod_{i=0}^s \Omega_i\Bigr)^{1/(s+1)}|K_0(\gamma_0)|\,\mathrm{d}\gamma_0, \tag{2.18}$$

where

$$\Omega_0=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|\mathfrak{f}_{0,0}(c_0\boldsymbol{\alpha}-\widetilde{\boldsymbol{\gamma}};\gamma_0)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}_k$$

and

$$\Omega_i=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|\mathfrak{f}_{i,0}(c_i\boldsymbol{\alpha};\gamma_i)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}_k\quad (1\leqslant i\leqslant s).$$

By a change of variable and application of periodicity modulo 1, we find from (2.1) and (2.7) that for $0\leqslant i\leqslant s$, one has

$$\Omega_i=\int_{\mathfrak{B}}\int_{[0,1)^{k-1}}|\mathfrak{f}_{i,0}(c_i\boldsymbol{\alpha};0)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}_k=c_i^{-1}\int_{c_i\mathfrak{B}}\int_{[0,1)^{k-1}}|\mathfrak{f}_{i,0}(\boldsymbol{\alpha};0)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}_k.$$

It therefore follows from (2.3) via orthogonality that $\Omega_i=c_i^{-1}\mathfrak{J}_{s+1,k}(c_i\mathfrak{B};X)$. Thus we infer from (2.18) that

$$\Bigl|\sum_{|z|\leqslant \Lambda X}\mathscr{I}(\boldsymbol{\gamma};z)\Bigr|\leqslant \Bigl(\prod_{i=0}^s \mathfrak{J}_{s+1,k}(c_i\mathfrak{B};X)\Bigr)^{1/(s+1)}\int_0^1|K_0(\gamma_0)|\,\mathrm{d}\gamma_0.$$

On substituting this estimate into (2.11) and recalling (2.13), we therefore obtain

$$\begin{aligned}I_s(\mathfrak{B};X)&\leqslant (2\Lambda X)^{-1}\int_{[0,1)^s}\Bigl|\sum_{|z|\leqslant \Lambda X}\mathscr{I}(\boldsymbol{\gamma};z)\widetilde{K}(\boldsymbol{\gamma})\Bigr|\,\mathrm{d}\boldsymbol{\gamma}\\ &\ll X^{-1}\prod_{i=0}^s\Bigl(\mathfrak{J}_{s+1,k}(c_i\mathfrak{B};X)^{1/(s+1)}\int_0^1|K_i(\gamma_i)|\,\mathrm{d}\gamma_i\Bigr),\end{aligned} \tag{2.19}$$

in which we have taken the expedient step of writing $K_i(\gamma_i)$ for $K(\gamma_i)$ when $1\leqslant i\leqslant s$. Recall (2.8) and (2.17). Then the elementary bound $K_i(\gamma)\ll \min\{X,\|\gamma\|^{-1}\}$ shows, as is familiar, that

$$\int_0^1|K_i(\gamma_i)|\,\mathrm{d}\gamma_i\ll \log X\quad (0\leqslant i\leqslant s).$$

Thus, we conclude from (2.19) that

$$I_s(\mathfrak{B};X)\ll X^{-1}(\log X)^{s+1}\prod_{i=0}^s\mathfrak{J}_{s+1,k}(c_i\mathfrak{B};X)^{1/(s+1)}.$$

Here, we stress that the constant implicit in Vinogradov's notation may depend on $\mathbf{y}$. This completes the proof of the lemma. □

An almost immediate consequence of Lemma 2.1 delivers the first conclusion of Theorem 1.2.

Lemma 2.2. *Let s and k be natural numbers with $1 \leqslant s < k(k+1)$. Suppose that $\mathbf{c}, \mathbf{y} \in (\mathbb{Z} \setminus \{0\})^s$ and $c_1 y_1^k + \ldots + c_s y_s^k \neq 0$. Then one has*

$$I_s([0,1);X) \ll X^{(s-1)/2+\varepsilon}.$$

Proof. Put $c_0 = -(c_1 y_1^k + \ldots + c_s y_s^k)$. We apply Lemma 2.1 to obtain the bound

$$I_s([0,1);X) \ll X^{\varepsilon-1} \prod_{i=0}^{s} \mathfrak{J}_{s+1,k}(c_i[0,1);X)^{1/(s+1)}. \tag{2.20}$$

Here, in view of the definition (2.3), we have

$$\mathfrak{J}_{s+1,k}(c_i[0,1);X) = |c_i| \int_{[0,1)^k} |f_k(\boldsymbol{\alpha};X)|^{s+1} \,\mathrm{d}\boldsymbol{\alpha}.$$

Since our hypothesis on s ensures that $s+1 \leqslant k(k+1)$, we deduce from the (now confirmed) main conjecture in Vinogradov's mean value theorem (for which see [3, 21, 23]) that

$$\mathfrak{J}_{s+1,k}(c_i[0,1);X) \ll X^{\varepsilon}(X^{(s+1)/2} + X^{s+1-k(k+1)/2}).$$

By substituting this estimate into (2.20), therefore, we conclude that

$$I_s([0,1);X) \ll X^{\varepsilon}(X^{(s-1)/2} + X^{s-k(k+1)/2}).$$

The conclusion of the lemma is now immediate. □

In order to obtain the upper bound (1.8), we have only to set $y_1 = \ldots = y_s = 1$ to conclude from (2.2) and Lemma 2.2 that when $1 \leqslant s < k(k+1)$, one has

$$\int_{[0,1)^k} f_k(c_1\boldsymbol{\alpha};X) \cdots f_k(c_s\boldsymbol{\alpha};X) \,\mathrm{d}\boldsymbol{\alpha} \ll X^{(s-1)/2+\varepsilon}.$$

In the next lemma, and throughout the remainder of the paper, we suppose that $k \geqslant 3$ and $\mathbf{c}, \mathbf{y} \in (\mathbb{Z} \setminus \{0\})^s$. Moreover, putting $c_0 = -(c_1 y_1^k + \ldots + c_s y_s^k)$, we suppose that $c_0 \neq 0$. We next obtain from Lemma 2.1 an estimate of minor arc type. When $1 \leqslant Q \leqslant X$, we define a one-dimensional Hardy-Littlewood dissection as follows. We define the set of major arcs $\mathfrak{M}(Q)$ to be the union of the arcs

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant QX^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$, and then write $\mathfrak{m}(Q) = [0,1) \setminus \mathfrak{M}(Q)$ for the corresponding set of minor arcs.

Next, when k is an integer with $k \geqslant 2$, we define the exponent $\sigma = \sigma(k)$ by taking

$$\sigma(k)^{-1} = \begin{cases} 2^{k-1}, & \text{when } 2 \leqslant k \leqslant 5, \\ k(k-1), & \text{when } k \geqslant 6. \end{cases}$$

Then, when $k \geqslant 2$ and $1 \leqslant Q \leqslant X$, one has

$$\sup_{\alpha_k \in \mathfrak{m}(Q)} \sup_{\boldsymbol{\alpha}_{k-1} \in [0,1)^{k-1}} |f_k(\boldsymbol{\alpha}_k;X)| \ll X^{1+\varepsilon} Q^{-\sigma(k)}. \tag{2.21}$$

The reader may consult [26, Lemma 2.2] for a proof of this conclusion, which makes use of the standard literature.

Lemma 2.3. *When $1 \leqslant Q \leqslant X$ and $s \geqslant k(k+1)$, one has*

$$I_s(\mathfrak{m}(Q);X) \ll X^{s-\frac{1}{2}k(k+1)+\varepsilon} Q^{-\sigma(k)}.$$

Proof. We apply Lemma 2.1 to obtain the bound

$$I_s(\mathfrak{m}(Q);X) \ll X^{\varepsilon-1}\prod_{i=0}^{s}\mathfrak{I}_{s+1,k}(c_i\mathfrak{m}(Q);X)^{1/(s+1)}. \tag{2.22}$$

Here, in view of the definition (2.3), we have

$$\mathfrak{I}_{s+1,k}(c_i\mathfrak{m}(Q);X) = \int_{c_i\mathfrak{m}(Q)}\int_{[0,1)^{k-1}} |f_k(\boldsymbol{\alpha};X)|^{s+1}\,\mathrm{d}\boldsymbol{\alpha}_k.$$

An elementary exercise confirms that $c_i\mathfrak{m}(Q)\subseteq \mathfrak{m}(Q/|c_i|)$ (mod 1), and hence we deduce from (2.21) that

$$\sup_{\alpha_k\in c_i\mathfrak{m}(Q)}\ \sup_{\boldsymbol{\alpha}_{k-1}\in[0,1)^{k-1}} |f_k(\boldsymbol{\alpha}_k;X)| \ll X^{1+\varepsilon}Q^{-\sigma(k)}.$$

Thus, we find that

$$\mathfrak{I}_{s+1,k}(c_i\mathfrak{m}(Q);X) \ll X^{1+\varepsilon}Q^{-\sigma(k)}\int_{[0,1)^k} |f_k(\boldsymbol{\alpha};X)|^{s}\,\mathrm{d}\boldsymbol{\alpha}_k.$$

By applying the (now confirmed) main conjecture in Vinogradov's mean value theorem (see [3, 21, 23]) once again, we therefore conclude that

$$\mathfrak{I}_{s+1,k}(c_i\mathfrak{m}(Q);X) \ll X^{1+\varepsilon}Q^{-\sigma(k)}(X^{s/2}+X^{s-k(k+1)/2}).$$

The conclusion of the lemma follows by substituting this upper bound into (2.22). □

The conclusion of Lemma 2.3 is neither quite sufficient, by itself, to deliver the bound (1.9) of Theorem 1.2, nor the key minor arc input into Theorem 1.1. However, it does provide a bound for the most difficult region of the minor arcs. Our goal in §§3 and 4 is to handle the remaining parts of the minor arcs in the Hardy-Littlewood dissection.

3. A generalised minor arc estimate

Our goal in this section is to lay the foundations for an application of the Hardy-Littlewood method capable of delivering the estimate (1.9) of Theorem 1.2, as well as the conclusion of Theorem 1.1. To this end we introduce a Hardy-Littlewood dissection. First, as a close relative of the mean value $I_s(\mathfrak{B};X)$ introduced in (2.2), we define the mean value $T_s(\mathfrak{A})=T_s(\mathfrak{A};X)$ for measurable sets $\mathfrak{A}\subseteq[0,1)^k$ by writing

$$T_s(\mathfrak{A};X) = \int_{\mathfrak{A}} f(c_1\boldsymbol{\beta}(y_1))\cdots f(c_s\boldsymbol{\beta}(y_s))\,\mathrm{d}\boldsymbol{\alpha}. \tag{3.1}$$

Next, when $1\leqslant Z\leqslant X$, we denote by $\mathfrak{K}(Z)$ the union of the major arcs

$$\mathfrak{K}(q,\mathbf{a};Z)=\{\boldsymbol{\alpha}\in[0,1)^k : |\alpha_j-a_j/q|\leqslant ZX^{-j}\ (1\leqslant j\leqslant k)\},$$

with $1\leqslant q\leqslant Z$, $0\leqslant a_j\leqslant q$ $(1\leqslant j\leqslant k)$ and $(q,\mathbf{a})=1$, and we define the complementary set of minor arcs by putting $\mathfrak{k}(Z)=[0,1)^k\setminus\mathfrak{K}(Z)$. We have already defined the one-dimensional Hardy-Littlewood dissection of $[0,1)$ into sets of arcs $\mathfrak{M}=\mathfrak{M}(Q)$ and $\mathfrak{m}=\mathfrak{m}(Q)$. We now fix $L=X^{1/(8k^2)}$ and $Q=L^k$, and we define k-dimensional sets of arcs by taking $\mathfrak{N}=\mathfrak{K}(Q^2)$ and $\mathfrak{n}=\mathfrak{k}(Q^2)$. We also need the narrow set of major arcs $\mathfrak{P}=\mathfrak{K}(L)$, and the complementary set of minor arcs $\mathfrak{p}=\mathfrak{k}(L)$. In this last dissection, it is convenient to abbreviate $\mathfrak{K}(q,\mathbf{a};L)$ to $\mathfrak{P}(q,\mathbf{a})$.

We note that this last set of major and minor arcs coincide with those defined in the preamble to the statement of Theorem 1.2.

We partition the set of points $(\alpha_1,\ldots,\alpha_k)$ lying in $[0,1)^k$ into four disjoint subsets, namely

$$\begin{aligned}\mathfrak{W}_1&=[0,1)^{k-1}\times\mathfrak{m},\\ \mathfrak{W}_2&=([0,1)^{k-1}\times\mathfrak{M})\cap\mathfrak{n},\\ \mathfrak{W}_3&=([0,1)^{k-1}\times\mathfrak{M})\cap(\mathfrak{N}\setminus\mathfrak{P}),\\ \mathfrak{W}_4&=\mathfrak{P}.\end{aligned}$$

Noting that $\mathfrak{P}\subseteq[0,1)^{k-1}\times\mathfrak{M}$, it follows that $[0,1)^k=\mathfrak{W}_1\cup\ldots\cup\mathfrak{W}_4$. Hence, by orthogonality, we infer that

$$N_{s,k}(X;\mathbf{y})=T_s([0,1)^k)=\sum_{i=1}^4 T_s(\mathfrak{W}_i), \tag{3.2}$$

and further that

$$\int_{\mathfrak{p}} f(c_1\boldsymbol{\beta}(y_1))\cdots f(c_s\boldsymbol{\beta}(y_s))\,\mathrm{d}\boldsymbol{\alpha}=\sum_{i=1}^3 T_s(\mathfrak{W}_i). \tag{3.3}$$

The work of §2 already permits us to announce a satisfactory upper bound for the contribution of the set of arcs $\mathfrak{W}_1$ in (3.2) and (3.3).

Lemma 3.1. *When $s\geqslant k(k+1)$, one has*

$$T_s(\mathfrak{W}_1)\ll X^{s-\frac{1}{2}k(k+1)-1/(8k^3)}.$$

Proof. We observe that

$$T_s(\mathfrak{W}_1)=I_s(\mathfrak{m}(Q);X).$$

Thus, on substituting $Q=X^{1/(8k)}$ into Lemma 2.3, noting that $\sigma(k)>1/k^2$ for $k\geqslant 3$, the conclusion of the lemma is immediate. □

4. Further minor arc estimates

We next estimate the contributions arising from the sets of arcs $\mathfrak{W}_2$ and $\mathfrak{W}_3$ within (3.2) and (3.3). We begin with an estimate of Weyl-type for the exponential sum $f(c_i\boldsymbol{\beta}(y_i))$ $(1\leqslant i\leqslant s)$.

Lemma 4.1. *Suppose that $1\leqslant i\leqslant s$ and $c_iy_i\neq 0$. Then*

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{n}}|f(c_i\boldsymbol{\beta}(y_i))|\ll X^{1-1/(6k^2)}\quad\text{and}\quad\sup_{\boldsymbol{\alpha}\in\mathfrak{p}}|f(c_i\boldsymbol{\beta}(y_i))|\ll X^{1-1/(12k^3)}.$$

Proof. We begin by confirming the first bound. Put $\tau=1/(6k^2)$ and $\delta=1/(5k)$. Since $\tau^{-1}>4k(k-1)$ and $\delta>k\tau$, we find from [18, Theorem 1.6] that whenever $|f(c_i\boldsymbol{\beta}(y_i))|\geqslant X^{1-\tau}$, there exist $q\in\mathbb{N}$ and $\mathbf{a}\in\mathbb{Z}^k$ having the property that

$$1\leqslant q\leqslant X^\delta\qquad\text{and}\qquad|c_iq\beta_j(y_i)-a_j|\leqslant X^{\delta-j}\quad(1\leqslant j\leqslant k).$$

Write

$$r=|c_iy_i^{k-1}|q\quad\text{and}\quad b_j=\frac{a_j|c_iy_i^{k-1}|}{c_iy_i^{k-j}}\quad(1\leqslant j\leqslant k).$$

Then, on recalling from (2.1) that we have $\beta_j(y_i)=y_i^{k-j}\alpha_j$ $(1\leqslant j\leqslant k)$, we see that when X is sufficiently large in terms of $\mathbf{y}$, one has

$$|r\alpha_j-b_j|\leqslant |y_i|^{j-1}X^{\delta-j}\leqslant X^{\delta'-j}\quad (1\leqslant j\leqslant k),$$

in which we have written $\delta'=1/(4k)$. In particular, we see that $r\leqslant Q^2$ and $|\alpha_j-b_j/r|\leqslant Q^2X^{-j}$ $(1\leqslant j\leqslant k)$, and hence $\boldsymbol{\alpha}\in\mathfrak{N}$ (mod 1). We therefore infer that whenever X is sufficiently large in terms of k and $\mathbf{y}$, and $\boldsymbol{\alpha}\in\mathfrak{n}$, then one must have $|f(c_i\boldsymbol{\beta}(y_i))|\ll X^{1-\tau}$ $(1\leqslant i\leqslant s)$, and the first conclusion of the lemma follows.

In order to confirm the second bound, we put $\tau=1/(12k^3)$ and $\delta=1/(10k^2)$. We again have $\tau^{-1}>4k(k-1)$ and $\delta>k\tau$, and so the same argument applies mutatis mutandis. Thus, whenever $|f(c_i\boldsymbol{\beta}(y_i))|\geqslant X^{1-\tau}$, one deduces that $\boldsymbol{\alpha}\in\mathfrak{P}$ (mod 1). Consequently, when X is sufficiently large in terms of k and $\mathbf{y}$, and $\boldsymbol{\alpha}\in\mathfrak{p}$, then one must have $|f(c_i\boldsymbol{\beta}(y_i))|\ll X^{1-\tau}$ $(1\leqslant i\leqslant s)$. This delivers the second conclusion and completes the proof of the lemma. □

By combining this Weyl-type estimate with the conclusion of Lemma 2.1, we obtain a satisfactory estimate for $T_s(\mathfrak{W}_2)$ by exploiting the observation that $\mathfrak{W}_2$ has small measure.

Lemma 4.2. *When $c_iy_i\neq 0$ $(1\leqslant i\leqslant s)$ and $s\geqslant k(k+1)$, one has*

$$T_s(\mathfrak{W}_2)\ll X^{s-\frac{1}{2}k(k+1)-1/(16k)}.$$

Proof. When $\alpha_k\in\mathfrak{M}$, define

$$\mathfrak{L}(\alpha_k)=\{(\alpha_1,\ldots,\alpha_{k-1})\in[0,1)^{k-1}:\boldsymbol{\alpha}\in\mathfrak{W}_2\}.$$

Then an application of Hölder's inequality leads from (3.1) to the upper bound

$$T_s(\mathfrak{W}_2)\leqslant\prod_{i=1}^s\biggl(\int_{\mathfrak{M}}I_i(\alpha_k)\,\mathrm{d}\alpha_k\biggr)^{1/s},\tag{4.1}$$

where

$$I_i(\alpha_k)=\int_{\mathfrak{L}(\alpha_k)}|f(c_i\boldsymbol{\beta}(y_i))|^s\,\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{k-1}.$$

Noting that $\mathfrak{W}_2\subseteq\mathfrak{n}$, applying the trivial estimate $|f(c_i\boldsymbol{\beta}(y_i))|\leqslant 2X+1$, and writing $v=k(k-1)/2$, we deduce that

$$I_i(\alpha_k)\ll X^{s-k(k+1)}\Bigl(\sup_{\boldsymbol{\alpha}\in\mathfrak{n}}|f(c_i\boldsymbol{\beta}(y_i))|\Bigr)^{2k}\int_{[0,1)^{k-1}}|f(c_i\boldsymbol{\beta}(y_i))|^{2v}\,\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{k-1}.\tag{4.2}$$

By orthogonality, the mean value here counts the integral solutions of the system of equations

$$c_iy_i^{k-j}\sum_{l=1}^v(x_l^j-z_l^j)=0\quad(1\leqslant j\leqslant k-1),$$

with $1\leqslant\mathbf{x},\mathbf{z}\leqslant X$, each solution being counted with the unimodular weight

$$e(c_i\alpha_k(x_1^k-z_1^k+\ldots+x_v^k-z_v^k)).$$

Thus, applying the (now proven) main conjecture in Vinogradov's mean value theorem (see [3, 21, 23]), we find that one has the bound

$$\int_{[0,1)^{k-1}}|f(c_i\boldsymbol{\beta}(y_i))|^{2v}\,\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_{k-1}\ll\mathfrak{J}_{2v,k-1}([0,1);X)\ll X^{v+\varepsilon},$$

uniformly in α_k.

Making use of the latter bound, we deduce from (4.2) via Lemma 4.1 that

$$I_i(\alpha_k) \ll X^{s-k(k+1)} \left(X^{1-1/(6k^2)}\right)^{2k} X^{v+\varepsilon}.$$

Moreover, we have $\mathrm{mes}(\mathfrak{M}) \ll Q^2 X^{-k}$. Consequently, we infer that

$$\begin{aligned}\int_{\mathfrak{M}} I_i(\alpha_k)\,\mathrm{d}\alpha_k &\ll X^{s-\frac{1}{2}k(k+1)+\varepsilon}(X^{k-1/(3k)})(Q^2X^{-k})\\ &\ll X^{s-\frac{1}{2}k(k+1)+\varepsilon}(Q^2X^{-1/(3k)}).\end{aligned}$$

Since $Q^2 = X^{1/(4k)}$, we conclude that

$$\int_{\mathfrak{M}} I_i(\alpha_k)\,\mathrm{d}\alpha_k \ll X^{s-\frac{1}{2}k(k+1)-1/(16k)}.$$

The conclusion of the lemma follows by substituting this bound into (4.1). □

The analysis of the set of arcs $\mathfrak{W}_3$ requires standard major arc estimates from the theory of Vinogradov's mean value theorem.

Lemma 4.3. *Suppose that* $u > \frac{1}{2}k(k+1) + 2$ *and* $c_i y_i \neq 0$ *for* $1 \leqslant i \leqslant s$. *Then*

$$\int_{\mathfrak{N}} |f(c_i\boldsymbol{\beta}(y_i))|^u\,\mathrm{d}\boldsymbol{\alpha} \ll X^{u-k(k+1)/2}.$$

Proof. Suppose that $\boldsymbol{\alpha} \in \mathfrak{N}$. Then there exist $q \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^k$ for which $(q, \mathbf{a}) = 1$,

$$1 \leqslant q \leqslant Q^2 \quad \text{and} \quad 0 \leqslant a_j \leqslant q \quad (1 \leqslant j \leqslant k),$$

and such that

$$|\alpha_j - a_j/q| \leqslant Q^2 X^{-j} \quad (1 \leqslant j \leqslant k).$$

In such circumstances, one has

$$|c_i y_i^{k-j}\alpha_j - c_i y_i^{k-j} a_j/q| \leqslant |c_i y_i^{k-j}| Q^2 X^{-j}.$$

Thus, when X is sufficiently large in terms of $\mathbf{y}$, we see that $c_i\boldsymbol{\beta}(y_i) \in \mathfrak{K}(Q^{2+\varepsilon}) \pmod 1$. Hence, applying periodicity modulo 1, we have

$$\int_{\mathfrak{N}} |f(c_i\boldsymbol{\beta}(y_i))|^u\,\mathrm{d}\boldsymbol{\alpha} \ll \int_{\mathfrak{K}(Q^{2+\varepsilon})} |f(\boldsymbol{\beta})|^u\,\mathrm{d}\boldsymbol{\beta}.$$

From here we may apply [20, Lemma 7.1], observing that the set of major arcs $\mathfrak{K}(Q^{2+\varepsilon})$ is a subset of the major arcs employed in the latter source. Thus, in particular, one has

$$\int_{\mathfrak{K}(Q^{2+\varepsilon})} |f(\boldsymbol{\beta})|^u\,\mathrm{d}\boldsymbol{\beta} \ll X^{u-k(k+1)/2},$$

and the conclusion of the lemma follows. □

Lemma 4.4. *When* $c_i y_i \neq 0$ $(1 \leqslant i \leqslant s)$ *and* $s \geqslant k(k+1)$, *one has*

$$T_s(\mathfrak{W}_3) \ll X^{s-\frac{1}{2}k(k+1)-1/(12k^2)}.$$

Proof. An application of Hölder's inequality conveys us from (3.1) to the upper bound

$$T_s(\mathfrak{W}_3) \leqslant \prod_{i=1}^{s} J_i^{1/s}, \tag{4.3}$$

where

$$J_i=\int_{\mathfrak{W}_3}|f(c_i\boldsymbol{\beta}(y_i))|^s\,\mathrm{d}\boldsymbol{\alpha}.$$

Since $\mathfrak{W}_3\subseteq\mathfrak{N}\setminus\mathfrak{P}$, we discern from Lemma 4.1 that

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{W}_3}|f(c_i\boldsymbol{\beta}(y_i))|\leqslant\sup_{\boldsymbol{\alpha}\in\mathfrak{p}}|f(c_i\boldsymbol{\beta}(y_i))|\ll X^{1-1/(12k^3)}.$$

Thus, taking $u=s-k$ and noting that $u\geqslant\frac{1}{2}k(k+1)+3$, it follows from Lemma 4.3 that

$$\begin{aligned}J_i&\leqslant\Bigl(\sup_{\boldsymbol{\alpha}\in\mathfrak{p}}|f(c_i\boldsymbol{\beta}(y_i))|\Bigr)^k\int_{\mathfrak{N}}|f(c_i\boldsymbol{\beta}(y_i))|^u\,\mathrm{d}\boldsymbol{\alpha}\\&\ll\bigl(X^{1-1/(12k^3)}\bigr)^kX^{u-k(k+1)/2}.\end{aligned}$$

The conclusion of the lemma follows on substituting this bound into (4.3). □

5. The analysis of the major arc contribution

By substituting the conclusions of Lemmata 3.1, 4.2 and 4.4 into the relation (3.2), we find that

$$N_{s,k}(X;\mathbf{y})=T_s(\mathfrak{P})+O(X^{s-\frac{1}{2}k(k+1)-1/(8k^3)}).\tag{5.1}$$

The goal of this section is to obtain an asymptotic formula for $T_s(\mathfrak{P})$ that suffices to confirm (1.5), and hence completes the proof of Theorem 1.1.

We begin by introducing the generating functions

$$I(\boldsymbol{\theta};X)=\int_{-X}^{X}e(\theta_1\gamma+\ldots+\theta_k\gamma^k)\,\mathrm{d}\gamma$$

and

$$S(q,\mathbf{a})=\sum_{r=1}^{q}e_q(a_1r+\ldots+a_kr^k),$$

in which $e_q(u)$ denotes $e^{2\pi iu/q}$. Recall the notation (2.1). When $1\leqslant i\leqslant s$, we define

$$I_i(\boldsymbol{\theta};X)=I(c_i\boldsymbol{\beta}(\boldsymbol{\theta};y_i);X)\quad\text{and}\quad S_i(q,\mathbf{a})=S(q,c_i\boldsymbol{\beta}(\mathbf{a};y_i)).$$

Then, when $\boldsymbol{\alpha}\in\mathfrak{P}(q,\mathbf{a})\subseteq\mathfrak{P}$, we write

$$V_i(\boldsymbol{\alpha};q,\mathbf{a})=q^{-1}S_i(q,\mathbf{a})I_i(\boldsymbol{\alpha}-\mathbf{a}/q;X).$$

Define the function $V_i(\boldsymbol{\alpha})$ to be $V_i(\boldsymbol{\alpha};q,\mathbf{a})$ when $\boldsymbol{\alpha}\in\mathfrak{P}(q,\mathbf{a})\subseteq\mathfrak{P}$, and to be 0 otherwise. Then, when $\boldsymbol{\alpha}\in\mathfrak{P}(q,\mathbf{a})\subseteq\mathfrak{P}$, we see from [15, Theorem 7.2] that

$$f(c_i\boldsymbol{\beta}(\boldsymbol{\alpha};y_i))-V_i(\boldsymbol{\alpha};q,\mathbf{a})\ll q+X|q\alpha_1-a_1|+\ldots+X^k|q\alpha_k-a_k|\ll L^2,$$

with the implicit constant in Vinogradov's notation depending at most on c_i, y_i and k. Thus, uniformly for $\boldsymbol{\alpha}\in\mathfrak{P}$, we have the bound

$$\prod_{i=1}^sf(c_i\boldsymbol{\beta}(\boldsymbol{\alpha};y_i))-\prod_{i=1}^sV_i(\boldsymbol{\alpha})\ll X^{s-1+1/(4k^2)}.$$

Write

$$T_s^*(\mathfrak{P})=\int_{\mathfrak{P}}\prod_{i=1}^sV_i(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}.$$

Then since $\mathrm{mes}(\mathfrak{P}) \ll L^{2k+1}X^{-k(k+1)/2}$, we deduce that

$$T_s(\mathfrak{P}) - T_s^*(\mathfrak{P}) \ll X^{s-k(k+1)/2}L^{-1}. \tag{5.2}$$

Next write

$$\Omega(X;D) = [-DX^{-1}, DX^{-1}] \times \cdots \times [-DX^{-k}, DX^{-k}].$$

Then one finds that

$$T_s^*(\mathfrak{P}) = \mathfrak{S}(L)\mathfrak{T}(X;L), \tag{5.3}$$

where

$$\mathfrak{T}(X;D) = \int_{\Omega(X;D)} \prod_{i=1}^{s} I_i(\boldsymbol{\theta};X)\,\mathrm{d}\boldsymbol{\theta} \tag{5.4}$$

and

$$\mathfrak{S}(D) = \sum_{1\leqslant q\leqslant D} \sum_{\substack{1\leqslant \mathbf{a}\leqslant q\\ (q,\mathbf{a})=1}} q^{-s} \prod_{i=1}^{s} S_i(q,\mathbf{a}).$$

We examine the truncated singular integral $\mathfrak{T}(X;D)$ and the truncated singular series $\mathfrak{S}(D)$ in turn. In this context, we recall the definitions of the real density σ_∞ and the p-adic densities σ_p from the sequel to the statement of Theorem 1.1. We begin by examining the integrals

$$I(D) = \int_{\Omega(1;D)} \prod_{i=1}^{s} I_i(\boldsymbol{\theta};1)\,\mathrm{d}\boldsymbol{\theta}$$

and

$$I_\infty = \int_{\mathbb{R}^k} \prod_{i=1}^{s} I_i(\boldsymbol{\theta};1)\,\mathrm{d}\boldsymbol{\theta}.$$

Lemma 5.1. *Suppose that $s \geqslant \frac{1}{2}k(k+1)+3$. Then the limit $I_\infty = \lim_{D\to\infty} I(D)$ exists, and one has*

$$\mathfrak{T}(X;L) = I_\infty X^{s-k(k+1)/2} + O(X^{s-k(k+1)/2}L^{-1/k}).$$

Moreover, one has $I_\infty = \sigma_\infty$, and provided that the system (1.4) *has a non-singular real solution $\mathbf{z}\in\mathbb{R}^s$, one has $\sigma_\infty > 0$.*

Proof. We consider a parameter D with $D \geqslant 1$, and we put

$$\Omega^{\mathrm{c}}(D) = \mathbb{R}^k \setminus \Omega(1;D).$$

We begin by applying Hölder's inequality to the mean value complementary to (5.4), obtaining the bound

$$\int_{\Omega^{\mathrm{c}}(D)} \prod_{i=1}^{s} |I_i(\boldsymbol{\theta};1)|\,\mathrm{d}\boldsymbol{\theta} \leqslant \prod_{i=1}^{s} \biggl(\int_{\Omega^{\mathrm{c}}(D)} |I_i(\boldsymbol{\theta};1)|^s\,\mathrm{d}\boldsymbol{\theta} \biggr)^{1/s}. \tag{5.5}$$

Recall the bound

$$I(\boldsymbol{\theta};1) \ll (1+|\theta_1|+\ldots+|\theta_k|)^{-1/k},$$

available from [15, Theorem 7.3]. When $\boldsymbol{\theta}\in\Omega^{\mathrm{c}}(D)$, one has $|\theta_i| > D$ for some index i with $1\leqslant i\leqslant k$. Then in view of the definition (2.1) of $\boldsymbol{\beta}(\boldsymbol{\theta};y_i)$, we have

$$\sup_{\boldsymbol{\theta}\in\Omega^{\mathrm{c}}(D)} |I_i(\boldsymbol{\theta};1)| \ll_{\mathbf{y}} D^{-1/k}.$$

Thus, leaving the dependence of $\mathbf{y}$ implicit in Vinogradov's notation henceforth, we deduce that

$$\int_{\Omega^c(D)} |I_i(\boldsymbol{\theta};1)|^s \, \mathrm{d}\boldsymbol{\theta} \ll (D^{-1/k})^{s-\frac{1}{2}k(k+1)-2} \int_{\mathbb{R}^k} |I_i(\boldsymbol{\theta};1)|^{\frac{1}{2}k(k+1)+2} \, \mathrm{d}\boldsymbol{\theta}. \tag{5.6}$$

When $t > \frac{1}{2}k(k+1)+1$, the integral

$$\int_{\mathbb{R}^k} |I(\boldsymbol{\theta};1)|^t \, \mathrm{d}\boldsymbol{\theta}$$

converges absolutely (see [1, Theorem 1.3]). Thus, one finds by a change of variable that

$$\int_{\mathbb{R}^k} |I_i(\boldsymbol{\theta};1)|^t \, \mathrm{d}\boldsymbol{\theta} \ll \int_{\mathbb{R}^k} |I(\boldsymbol{\theta};1)|^t \, \mathrm{d}\boldsymbol{\theta} \ll 1.$$

By substituting this bound first into (5.6) and then into (5.5), we obtain the estimate

$$\int_{\Omega^c(D)} \prod_{i=1}^s |I_i(\boldsymbol{\theta};1)| \, \mathrm{d}\boldsymbol{\theta} \ll D^{-1/k}.$$

It therefore follows that the integral I_∞ converges absolutely, and further that one has $I_\infty - I(D) \ll D^{-1/k}$. Moreover, by two changes of variable, we are led from (5.4) to the relation

$$\begin{aligned} \mathfrak{T}(X;L) &= X^{s-k(k+1)/2} \int_{\Omega(1;L)} \prod_{i=1}^s I_i(\boldsymbol{\theta};1) \, \mathrm{d}\boldsymbol{\theta} \\ &= X^{s-k(k+1)/2} \left(I_\infty + O(L^{-1/k}) \right). \end{aligned} \tag{5.7}$$

At this point we recall the definitions of the quantities $M_\infty(\eta)$ and σ_∞, defined in the sequel to the statement of Theorem 1.1. Since the singular integral I_∞ converges absolutely, it follows from the argument of [8, §9] that

$$I_\infty = \lim_{\eta \to 0+} (2\eta)^{-k} M_\infty(\eta) = \sigma_\infty.$$

It is apparent, moreover, that whenever the system of equations (1.4) has a non-singular real solution, then one has $M_\infty(\eta) \gg \eta^k$, and hence $\sigma_\infty > 0$. In view of the conclusion (5.7) already obtained, the proof of the lemma is complete. □

Before discussing the singular series

$$\mathfrak{S} = \lim_{D \to \infty} \mathfrak{S}(D),$$

we introduce the quantity

$$A(q) = \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q,\mathbf{a})=1}} q^{-s} \prod_{i=1}^s S_i(q,\mathbf{a}).$$

Thus, one has

$$\mathfrak{S}(D) = \sum_{1 \leqslant q \leqslant D} A(q),$$

and the singular series is given by the infinite sum

$$\mathfrak{S} = \sum_{q=1}^\infty A(q). \tag{5.8}$$

Lemma 5.2. *Suppose that $s \geqslant k(k+1)$. Then the singular series $\mathfrak{S}$ converges absolutely. Moreover, for each prime number p, the limit σ_p exists, the product over all primes $\prod_p \sigma_p$ converges absolutely, and one has $\mathfrak{S} = \prod_p \sigma_p$. Moreover, one has*

$$\mathfrak{S}(L) = \prod_p \sigma_p + O(L^{-1-1/(3k)}), \tag{5.9}$$

and provided that the system (1.4) *has a non-singular p-adic solution for each prime number p, one has $\prod_p \sigma_p \gg 1$.*

Proof. We may suppose that $s \geqslant k(k+1)$. Put $t = \frac{1}{2}k(k+1) + \frac{5}{2}$. Then, in view of our assumption throughout that $k \geqslant 3$, one sees that

$$s - t \geqslant \tfrac{1}{2}k(k+1) - \tfrac{5}{2} \geqslant k + \tfrac{1}{2},$$

and in particular $\frac{1}{2}k(k+1) + 2 < t < s$. From [15, Theorem 7.1], we find that when $(q, \mathbf{a}) = 1$ one has $S(q, \mathbf{a}) \ll q^{1-1/k+\varepsilon}$. Hence, we deduce that

$$S_i(q, \mathbf{a}) \ll_{\mathbf{y}} q^{1-1/k+\varepsilon}.$$

Again suppressing the implicit dependence on $\mathbf{y}$ in Vinogradov's notation, an application of Hölder's inequality reveals that

$$\begin{aligned} A(q) &\leqslant \prod_{i=1}^{s} \Bigg(\sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q,\mathbf{a})=1}} q^{-s} |S_i(q, \mathbf{a})|^s \Bigg)^{1/s} \\ &\ll q^{\varepsilon - (s-t)/k} \prod_{i=1}^{s} \Bigg(\sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q,\mathbf{a})=1}} q^{-t} |S_i(q, \mathbf{a})|^t \Bigg)^{1/s}. \end{aligned}$$

Since we have arranged parameters so that $s - t \geqslant k + \frac{1}{2}$, we find by means of Hölder's inequality that

$$\sum_{q \geqslant D} |A(q)| \ll D^{-1-1/(3k)} \prod_{i=1}^{s} \Bigg(\sum_{q \geqslant D} \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q,\mathbf{a})=1}} q^{-t} |S_i(q, \mathbf{a})|^t \Bigg)^{1/s}. \tag{5.10}$$

A change of variable supplies the estimate

$$\sum_{q \geqslant D} \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q,\mathbf{a})=1}} q^{-t} |S_i(q, \mathbf{a})|^t \ll \sum_{q=1}^{\infty} \sum_{\substack{1 \leqslant \mathbf{a} \leqslant q \\ (q,\mathbf{a})=1}} q^{-t} |S(q, \mathbf{a})|^t.$$

By reference to [1, Theorem 2.4], the sum on the right hand side here is absolutely convergent for $t > \frac{1}{2}k(k+1) + 2$. We therefore derive from (5.10) the upper bound

$$\sum_{q \geqslant D} |A(q)| \ll D^{-1-1/(3k)}, \tag{5.11}$$

and thus the singular series (5.8) is absolutely convergent, and one has

$$\mathfrak{S} - \mathfrak{S}(L) \ll L^{-1-1/(3k)}. \tag{5.12}$$

The standard theory of singular series shows that the function $A(q)$ is a multiplicative function of q (see [15, §2.6] for the necessary ideas). Moreover, since (5.11) shows that, for each prime number p, one has

$$\sum_{h\geqslant H}|A(p^h)|\ll p^{-H(1+1/(3k))},$$

we see that the limit

$$\lim_{H\to\infty}\sum_{h=0}^{H}A(p^h)$$

exists, and that the infinite sum

$$A_p=\sum_{h=0}^{\infty}A(p^h)$$

is absolutely convergent with $A_p=1+O(p^{-1-1/(3k)})$. Thus the infinite product $\prod_p A_p$ is absolutely convergent and $\mathfrak{S}=\prod_p A_p$. In particular, we deduce from (5.12) that

$$\mathfrak{S}(L)-\prod_p A_p\ll L^{-1-1/(3k)}. \tag{5.13}$$

Once again applying the standard theory of singular series, moreover, one has

$$\sum_{h=0}^{H}A(p^h)=p^{H(k-s)}M_p(H),$$

where $M_p(H)$ denotes the number of solutions of the system

$$\sum_{i=1}^{s}c_iy_i^{k-j}z_i^j\equiv 0 \pmod{p^H}\quad (1\leqslant j\leqslant k),$$

with $1\leqslant \mathbf{z}\leqslant p^H$. Thus we find that A_p is equal to the p-adic density σ_p defined in the sequel to the statement of Theorem 1.1. We are at liberty to assume that the system of equations (1.4) has a non-singular p-adic solution for each prime p. It therefore follows via Hensel's lemma that there is a non-negative integer ν_p satisfying the property that, whenever $H\geqslant \nu_p$, one has

$$M_p(H)\geqslant p^{(H-\nu_p)(s-k)},$$

whence

$$\sigma_p=\lim_{H\to\infty}p^{H(k-s)}M_p(H)\geqslant p^{-(s-k)\nu_p}>0.$$

Then, on recalling that $\sigma_p=1+O(p^{-1-1/(3k)})$, we find that there is a positive integer p_0 with the property that

$$\mathfrak{S}=\prod_p\sigma_p\gg\prod_{p>p_0}(1-p^{-1-1/(4k)})\gg 1,$$

whilst at the same time $\mathfrak{S}\ll 1$. The proof of the lemma is completed on noting that since $A_p=\sigma_p$, the relation (5.13) yields the asymptotic relation (5.9). $\square$

We are now equipped to complete the asymptotic analysis of the major arc contribution $T_s(\mathfrak{P})$.

Lemma 5.3. *Suppose that $s \geqslant k(k+1)$, and the system* (1.4) *has a non-singular real solution, and a non-singular p-adic solution for each prime p. Then one has*

$$T_s(\mathfrak{P}) = \sigma_\infty \Big(\prod_p \sigma_p \Big) X^{s-k(k+1)/2} + o(X^{s-k(k+1)/2}),$$

in which the product over real and p-adic densities is positive.

Proof. By substituting the conclusions of Lemmata 5.1 and 5.2 into (5.3), we find that

$$\begin{aligned} T_s^*(\mathfrak{P}) &= \Big(\mathfrak{S} + O(L^{-1-1/(3k)}) \Big) \Big(\sigma_\infty X^{s-k(k+1)/2} + O(X^{s-k(k+1)/2} L^{-1/k}) \Big) \\ &= \sigma_\infty \Big(\prod_p \sigma_p \Big) X^{s-k(k+1)/2} + O(X^{s-k(k+1)/2} L^{-1/k}). \end{aligned}$$

Moreover, the product over real and p-adic densities here in the leading asymptotic term is positive. We therefore conclude from (5.2) that

$$T_s(\mathfrak{P}) = \sigma_\infty \Big(\prod_p \sigma_p \Big) X^{s-k(k+1)/2} + O(X^{s-k(k+1)/2} L^{-1/k}),$$

and the proof of the lemma is complete. □

6. The proof of Theorems 1.1 and 1.2

The completion of the proofs of our main theorems is now at hand, though we defer to the next section a consideration of the nature of the singularities of the system (1.4).

The proof of Theorem 1.1. On recalling (5.1), we find that when $s \geqslant k(k+1)$ and $c_i y_i \neq 0$ $(1 \leqslant i \leqslant s)$, one has

$$N_{s,k}(X; \mathbf{y}) = T_s(\mathfrak{P}) + o(X^{s-k(k+1)/2}). \tag{6.1}$$

The hypotheses of Theorem 1.1 permit us to assume that the system (1.4) possesses non-singular real and p-adic solutions, for each prime number p. Thus, we deduce from Lemma 5.3 that

$$T_s(\mathfrak{P}) = \mathscr{C}_{s,k}(\mathbf{y}) X^{s-k(k+1)/2} + o(X^{s-k(k+1)/2}),$$

where $\mathscr{C}_{s,k}(\mathbf{y}) = \sigma_\infty \prod_p \sigma_p > 0$. The conclusion of Theorem 1.1 now follows by substituting this asymptotic relation into (6.1). □

The proof of Theorem 1.2. The proof of the upper bound (1.8) has already been accomplished in Lemma 2.2 and the discussion following the latter. Turning now to the proof of the upper bounds (1.9) and (1.10), suppose that $k \geqslant 3$ and $s \geqslant k(k+1)$. We set $y_j = 1$ for $1 \leqslant j \leqslant s$ and put $n = c_1 + \ldots + c_s \neq 0$. In this scenario, we find that (3.3) delivers the estimate

$$\int_{\mathfrak{p}} f_k(c_1 \boldsymbol{\alpha}; X) \cdots f_k(c_s \boldsymbol{\alpha}; X) \, \mathrm{d}\boldsymbol{\alpha} = \sum_{i=1}^{3} T_s(\mathfrak{W}_i), \tag{6.2}$$

where, by virtue of Lemmata 3.1, 4.2 and 4.4,

$$\sum_{i=1}^{3} T_s(\mathfrak{W}_i) \ll X^{s - \frac{1}{2}k(k+1) - 1/(8k^3)}. \tag{6.3}$$

When $s=k(k+1)$, the right hand side here is $O(X^{(s-\delta)/2})$, where $\delta=1/(4k^3)$, and when $s>k(k+1)$, it is instead $O(X^{s-\frac{1}{2}k(k+1)-\frac{1}{2}\delta})$. In either case, therefore, the upper bounds (1.9) and (1.10) follow by substituting (6.3) into (6.2). $\square$

7. The non-singularity of non-zero solutions

Suppose that the system of equations (1.4) has a non-zero solution $\mathbf{z}\ne\mathbf{0}$ lying in either $\mathbb{R}^s$ or $\mathbb{Q}_p^s$, for a given prime p. Our goal in this section is to show that this solution is in fact non-singular under the conditions discussed in the sequel to the statement of Theorem 1.1. We assume throughout that the equation (1.2), with $n\ne 0$ and $c_i\ne 0$ $(1\leqslant i\leqslant s)$, has a solution $\mathbf{y}$ with $y_i\ne 0$ $(1\leqslant i\leqslant s)$. Then, should the system (1.4) have a non-zero solution $\mathbf{z}$ over $\mathbb{R}$, or over $\mathbb{Q}_p$, we find that $\mathbf{z}$ satisfies the system of equations

$$\sum_{i=1}^s c_iy_i^k\,(z_i/y_i)^j=0\quad (1\leqslant j\leqslant k). \tag{7.1}$$

Suppose, by way of deriving a contradiction, that this solution $\mathbf{z}$ is singular. Then, for any k-tuple $(i_1,\dots,i_k)$ of natural numbers satisfying $1\leqslant i_1<i_2<\ldots<i_k\leqslant s$, one must have

$$\det\left(jc_{i_l}y_{i_l}^{k-j}z_{i_l}^{j-1}\right)_{1\leqslant j,l\leqslant k}=0. \tag{7.2}$$

Since $c_iy_i\ne 0$ for $1\leqslant i\leqslant s$, a consideration of Vandermonde determinants reveals that the condition (7.2) is satisfied if and only if

$$0=\det\left(\left(\frac{z_{i_l}}{y_{i_l}}\right)^{j-1}\right)_{1\leqslant j,l\leqslant k}=\prod_{1\leqslant j<l\leqslant k}\left(\frac{z_{i_l}}{y_{i_l}}-\frac{z_{i_j}}{y_{i_j}}\right).$$

This relation implies that

$$\frac{z_{i_l}}{y_{i_l}}=\frac{z_{i_j}}{y_{i_j}},$$

for some indices j and l with $1\leqslant j<l\leqslant k$, and thus we are forced to conclude that the set $\{z_i/y_i:1\leqslant i\leqslant s\}$ contains at most $k-1$ distinct values.

By relabelling indices, we may suppose that, for some integer r with $1\leqslant r\leqslant k-1$, each of the rational numbers

$$\frac{z_1}{y_1},\frac{z_2}{y_2},\dots,\frac{z_r}{y_r},$$

is distinct, and further that, whenever $i>r$, one has

$$\frac{z_i}{y_i}\in\left\{\frac{z_1}{y_1},\dots,\frac{z_r}{y_r}\right\}.$$

We define an equivalence relation on indices by defining $i\sim j$ whenever one has $z_i/y_i=z_j/y_j$. Then, on putting

$$C_i=\sum_{\substack{1\leqslant j\leqslant s\\ j\sim i}}c_j\frac{y_j^k}{y_i^k}\quad (1\leqslant i\leqslant r),$$

we see that the equation (1.2) becomes

$$C_1y_1^k+\ldots+C_ry_r^k=n, \tag{7.3}$$

while the equations (7.1) transform into the new system

$$\sum_{i=1}^{r} C_i y_i^k \Big(\frac{z_i}{y_i}\Big)^j = 0 \quad (1 \leqslant j \leqslant k), \tag{7.4}$$

subject to the condition

$$\frac{z_i}{y_i} \neq \frac{z_l}{y_l} \quad (1 \leqslant i < l \leqslant r). \tag{7.5}$$

Notice here that since $n \neq 0$, it follows from the equation (7.3) that $C_i y_i^k \neq 0$ for some index i with $1 \leqslant i \leqslant r$. Moreover, since $(z_1, \ldots, z_k) \neq \mathbf{0}$, the relation (7.5) ensures that $z_i = 0$ for at most one index i with $1 \leqslant i \leqslant r$, and in such circumstances one must have $r \geqslant 2$.

Should the solution $\mathbf{y}$ of (1.2) satisfy the condition that there be no vanishing subsums, then $C_i \neq 0$ for $1 \leqslant i \leqslant r$. We suppose either that such is the case and $\mathbf{z} \neq \mathbf{0}$, or else that $z_i \neq 0$ for $1 \leqslant i \leqslant s$. In both circumstances we relabel indices in such a manner that $C_i z_i \neq 0$ for $1 \leqslant i \leqslant R$, and $C_i z_i = 0$ for $R < i \leqslant r$. Here, in either scenario, our discussion thus far permits us the assumption that $1 \leqslant R < k$. We now infer from the system of equations (7.4) that

$$\sum_{i=1}^{R} C_i y_i^k \Big(\frac{z_i}{y_i}\Big)^j = 0 \quad (1 \leqslant j \leqslant R).$$

We view these relations as a system of linear equations, with the quantities $C_i y_i^k$ $(1 \leqslant i \leqslant R)$ as variables. Then since in either scenario under consideration, we have $C_i y_i^k \neq 0$ for all indices i with $1 \leqslant i \leqslant R$, we see that

$$\det\bigg(\Big(\frac{z_i}{y_i}\Big)^j\bigg)_{1 \leqslant i,j \leqslant R} = 0.$$

Expanding the Vandermonde determinant, we thus conclude that

$$\bigg(\prod_{l=1}^{R} \frac{z_l}{y_l}\bigg) \prod_{1 \leqslant i < j \leqslant R} \bigg(\frac{z_i}{y_i} - \frac{z_j}{y_j}\bigg) = 0.$$

But the hypothesis (7.5) ensures that the second product on the left hand side is non-zero, and our hypothesis $C_i z_i \neq 0$ for $1 \leqslant i \leqslant R$ ensures that the first product on the left hand side is non-zero. We therefore arrive at a contradiction, so that the solution $\mathbf{z}$ cannot in fact be singular. The conditions in the sequel to the statement of Theorem 1.1 consequently suffice to guarantee the existence of non-singular real and p-adic solutions, as we had claimed.

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