ON WARING'S PROBLEM FOR LARGER POWERS

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Abstract. Let $G(k)$ denote the least number s having the property that every sufficiently large natural number is the sum of at most s positive integral k-th powers. Then for all $k \in \mathbb{N}$, one has

$$G(k) \leqslant \lceil k(\log k + 4.20032) \rceil.$$

Our new methods improve on all bounds available hitherto when $k \geqslant 14$.

1. Introduction

Since the introduction by Hardy and Littlewood of their circle method a century ago (see [3]), it has been possible to surmise progress associated with this technology from corresponding advances in the theory of Waring's problem. As is usual, we denote by $G(k)$ the least number s having the property that every sufficiently large natural number is the sum of at most s positive integral k-th powers. The initial bound $G(k) \leqslant (k-2)2^{k-1} + 5$ of Hardy and Littlewood [4] was improved rapidly over the next four decades, culminating in 1959 with Vinogradov's bound

$$G(k) \leqslant k(2\log k + 4\log\log k + 2\log\log\log k + 13) \quad (k \geqslant 170,000)$$

(see [18]). The latter bound was subsequently improved by Karatsuba [7], and shortly thereafter by Vaughan [12], showing that

$$G(k) \leqslant 2k(\log k + \log\log k + 1 + \log 2 + O(\log\log k/\log k)).$$

A little over three decades after the work of Vinogradov, the second author obtained a bound roughly half that of this earlier work, establishing the bound

$$G(k) \leqslant k(\log k + \log\log k + 2 + O(\log\log k/\log k))$$

(see [19, 20] and [22, Theorem 1.4]). Our primary goal in this memoir is the removal of the secondary term of size $k\log\log k$.

Theorem 1.1. *For all $k \in \mathbb{N}$, one has $G(k) \leqslant \lceil k(\log k + 4.20032) \rceil$.*

The conclusion of this theorem constitutes the largest improvement in available bounds for $G(k)$, when k is large, since the progress achieved thirty years ago by the second author [19, 20]. The upper bound presented in Theorem 1.1 is in fact an approximation to one asymptotically very slightly stronger. In order to describe this result, we introduce some

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auxiliary constants. Let ω be the unique real solution, with $\omega \geqslant 1$, of the transcendental equation

$$\omega - 2 - 1/\omega = \log \omega. \tag{1.1}$$

We then put

$$C_1 = 2 + \log(\omega^2 - 3 - 2/\omega) \quad \text{and} \quad C_2 = \frac{\omega^2 + 3\omega - 2}{\omega^2 - \omega - 2}. \tag{1.2}$$

A modest computation reveals that

$$\omega = 3.548292\ldots, \quad C_1 = 4.200189\ldots \quad \text{and} \quad C_2 = 3.015478\ldots.$$

Theorem 1.2. *For all $k \in \mathbb{N}$, one has $G(k) < k(\log k + C_1) + C_2$.*

It transpires that the new ideas underlying the progress exhibited in Theorems 1.1 and 1.2 apply not only for very large values of k, but also for exponents of moderate size.

Theorem 1.3. *When $14 \leqslant k \leqslant 20$, one has $G(k) \leqslant H(k)$, where $H(k)$ is defined by means of Table 1.*

k	14	15	16	17	18	19	20
$H(k)$	89	97	105	113	121	129	137

Table 1. Upper bounds for $G(k)$ when $14 \leqslant k \leqslant 20$.

For comparison, recent work of the second author [24] delivers the bounds $G(14) \leqslant 90$, $G(15) \leqslant 99$, $G(16) \leqslant 108$, while rather earlier investigations of Vaughan and Wooley [17] obtained $G(17) \leqslant 117$, $G(18) \leqslant 125$, $G(19) \leqslant 134$, $G(20) \leqslant 142$. For values of k smaller than 14, although superior to the bounds of [17], our new methods do not improve on those obtained in [24].

Two ideas underlie our approach to the theorems above, one old and one new. A novel mean value estimate for moments of smooth Weyl sums over sets of minor arcs of intermediate and large height is essential for our findings. This new tool is of utility in bounding mean values restricted to sets of arcs excluding those of classical major arc type, and hence is applicable in pruning problems. A simple but crude version of this idea occurs as [2, Lemma 2.3], where mean values over sets of major arcs of large height are estimated in terms of complete mean values over shortened exponential sums. This idea, in turn, has [9, Lemma 5.6] as a less flexible and more restricted precursor. While a version of [2, Lemma 2.3] is obtained in Theorem 4.2 which applies to lower moments than were accessible hitherto, the treatment of the present memoir also delivers analogous bounds for moments restricted to minor arcs. Crucial to our applications is the observation that the latter estimates are at their most powerful when the associated set of minor arcs is of maximal height relative to the length of the shortened exponential sums occurring within our argument. Readers seeking clarity beyond these rough and murky remarks would do well to inspect the account in §5 of the ideas delivering Theorem 5.3.

This brings us to the second, much older, idea that we exploit. Minor arc estimates of conventional type for smooth Weyl sums over k-th powers can be substantially improved when their argument lies on an extreme set of minor arcs, rather than on a conventional such set. This idea has been utilized previously in work of Heath-Brown [6] and Karatsuba [8] on fractional parts of αn^k. A flexible analysis using sets of smooth numbers of utility in applications of the circle method can be found in [22]. These improved minor arc estimates can be applied through the novel mean value estimates to which we alluded in the previous paragraph, surmounting difficulties associated with intermediate sets of arcs that previously obstructed their use. The details associated with this plan of attack are described in §5.

We begin the main discourse of this memoir in §2 by introducing the infrastructure required for a discussion of mean values associated with smooth Weyl sums. This section already introduces ideas that relate intermediate sets of arcs of differing heights. The delicate analysis involved in considering mean values restricted to sets of intermediate arcs requires a careful decomposition of smooth Weyl sums, and this we discuss in §3. Thus prepared, we establish our first mean value estimate in §4, completing the proof of Theorem 4.2. In order to exploit the mean value estimate provided in this theorem, we revisit estimates of Weyl type for smooth Weyl sums in §5, providing in Theorem 5.3 an estimate of minor arc type that should be flexible enough for future application beyond the present memoir. In §6 we turn to the application central to this paper, namely Waring's problem, and we describe a general analysis. Explicit bounds for $G(k)$ are then derived for larger k in §7, establishing Theorems 1.1 and 1.2. In §8, we consider intermediate values of k using the tables of exponents made available in [17], and thereby we complete the proof of Theorem 1.3. Finally, in §9, we briefly outline the consequences of our new bounds for problems concerning the representation of almost all positive integers as sums of positive integral k-th powers.

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2. Infrastructure

We initiate the proof of the mean value estimates provided in Theorems 4.2 and 5.3 by introducing infrastructure necessary for the ensuing discussion. A central role is played by the set of R-smooth integers not exceeding P, namely

$$\mathscr{A}(P,R)=\{n\in[1,P]\cap\mathbb{Z}:p|n \text{ implies } p\leqslant R\}.$$

Here, and throughout this memoir, the letter p is used to denote a prime number. Recall the usual convention of writing $e(z)$ for $e^{2\pi iz}$. Then, associated with this set $\mathscr{A}(P,R)$ are the smooth Weyl sum

$$f(\alpha;P,R)=\sum_{x\in\mathscr{A}(P,R)}e(\alpha x^k),$$

and, for each positive real number s, the mean value

$$U_s(P,R)=\int_0^1|f(\alpha;P,R)|^s\,\mathrm{d}\alpha.$$

A real number Δ_s is referred to as an *admissible exponent* (for k) if it has the property that, whenever $\varepsilon>0$ and η is a positive number sufficiently small in terms of ε, k and s, then whenever $1\leqslant R\leqslant P^\eta$ and P is sufficiently large, one has

$$U_s(P,R)\ll P^{s-k+\Delta_s+\varepsilon}.$$

Here and throughout, with P the underlying parameter, the constant implicit in Vinogradov's notation may depend on ε, η, k and s. It is easily verified that for all positive numbers s, one has $\Delta_s \geqslant 0$. It is a simple exercise in interpolation, moreover, to confirm that for each $\eta > 0$ one has $U_s(P, P^\eta) \gg P^{s/2}$. Thus, for all $s > 0$ one has

$$\Delta_s \geqslant \max\{0, k - s/2\}.$$

In the opposite direction, one has the trivial upper bound $U_s(P, R) \ll P^s$. Hence $\Delta_s = k$ is an admissible exponent. We may therefore suppose that $\Delta_s \leqslant k$, and we shall do so whenever this is convenient.

We draw a trivial consequence from the definition of an admissible exponent important enough that we summarise the conclusion in the form of a lemma.

Lemma 2.1. *Suppose that Δ_s is an admissible exponent for k and that ε is a positive number. Then there exists a positive number η, depending at most on ε, k and s, with the following property. Suppose that P is sufficiently large in terms of ε, η, k and s, and further that $1 \leqslant R \leqslant P^\eta$. Then, uniformly in $1 \leqslant Y \leqslant P$, one has the bound*

$$U_s(Y, R) \ll P^\varepsilon Y^{s-k+\Delta_s}.$$

Proof. Fix ε, k and s, so that in our use of Vinogradov's notation we may suppress any mention of quantities depending on these numbers, and write $\mu_s = s - k + \Delta_s$. If we assume that Δ_s is admissible for k, there exists a positive number η_1, depending at most on ε, k and s, and satisfying $\eta_1 < \varepsilon$ and the following property. Whenever X is sufficiently large in terms of η_1, say $X \geqslant X_0(\eta_1)$, and $1 \leqslant R \leqslant X^{\eta_1}$, one has $U_s(X, R) \ll X^{\mu_s+\varepsilon}$. Now consider a real number P sufficiently large in terms of η_1, and suppose that $1 \leqslant Y \leqslant P$. We put $\eta = \eta_1^2/s$ and take R to be a real number with $1 \leqslant R \leqslant P^\eta$. There are three different regimes for Y that we must consider. First, if $Y \leqslant X_0(\eta_1)$, then a trivial estimate yields the bound

$$U_s(Y, R) \leqslant Y^s \leqslant X_0(\eta_1)^s \ll 1.$$

Next, when $X_0(\eta_1) < Y \leqslant R^{1/\eta_1}$, the same trivial estimate now reveals that

$$U_s(Y, R) \leqslant Y^s \leqslant R^{s/\eta_1} \leqslant P^{\eta s/\eta_1} = P^{\eta_1} \leqslant P^\varepsilon.$$

Finally, when $Y \geqslant X_0(\eta_1)$ and $R^{1/\eta_1} < Y \leqslant P$, we have $R < Y^{\eta_1}$, and then it follows from the above discussion that we have

$$U_s(Y, R) \ll Y^{\mu_s+\varepsilon} \ll P^\varepsilon Y^{\mu_s}.$$

By collecting together these estimates, we conclude that the last bound holds uniformly in $1 \leqslant Y \leqslant P$. This completes the proof of the lemma. □

In order to facilitate concision, from this point onwards we adopt the extended ε, R notation routinely employed by scholars working with smooth Weyl sums while applying the Hardy-Littlewood method. Thus, whenever a statement involves the letter ε, then it is asserted that the statement holds for any positive real number assigned to ε. Implicit constants stemming from Vinogradov or Landau symbols may depend on ε, as well as ambient parameters implicitly fixed such as k and s. If a statement also involves the letter R, either implicitly or explicitly, then it is asserted that for any $\varepsilon > 0$ there is a number $\eta > 0$ such that the statement holds uniformly for $2 \leqslant R \leqslant P^\eta$. Our arguments will involve only a finite number of statements, and consequently we may pass to the smallest of the numbers η that arise in this way, and then have all estimates in force with the same positive number η. Notice that η may be assumed sufficiently small in terms of k, s and ε.

We shall have cause to consider sets of integers, all of whose prime divisors divide a fixed integer. In this context, we make use of transparent though disturbing notation, writing $u|q^\infty$ to denote that whenever p is a prime and $p|u$, then $p|q$. Then, when $q \in \mathbb{N}$, we define the set

$$\mathscr{C}_q(P,R) = \{n \in \mathscr{A}(P,R) : n|q^\infty\},$$

consisting of R-smooth natural numbers not exceeding P, each having squarefree kernel dividing q. We recall that, while $\mathrm{card}(\mathscr{A}(P,R)) \gg_\eta P$ when $R \geqslant P^\eta$, the set $\mathscr{C}_q(P,R)$ is very thin provided that q is not too large.

Lemma 2.2. *Suppose that C is a positive number. Then, uniformly for positive integers q with $q \leqslant P^C$, one has* $\mathrm{card}(\mathscr{C}_q(P,R)) \ll P^\varepsilon$.

Proof. The desired conclusion is immediate from [20, Lemma 2.1]. □

Our interest lies in mean values of $f(\alpha,P,R)$ analogous to $U_s(P,R)$, though with domains of integration given by intermediate sets of arcs from a Hardy-Littlewood dissection. Let Q be a parameter with $1 \leqslant Q \leqslant P^{k/2}$. When q is a natural number with $1 \leqslant q \leqslant Q$, we define the set of arcs $\mathfrak{M}_q(Q,P)$ to be the union of the sets

$$\mathfrak{M}_{q,a}(Q,P) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant QP^{-k}\},$$

with $0 \leqslant a \leqslant q$ and $(a,q)=1$, and then put

$$\mathfrak{M}(Q,P) = \bigcup_{1\leqslant q\leqslant Q} \mathfrak{M}_q(Q,P).$$

It is convenient to extend these definitions so that $\mathfrak{M}_q(Q,P) = \emptyset$ when $q > Q$. The related dyadically truncated set of arcs $\mathfrak{N}(Q,P)$ may then be defined by

$$\mathfrak{N}(Q,P) = \mathfrak{M}(Q,P) \setminus \mathfrak{M}(Q/2,P).$$

Associated with this set are the collections of arcs

$$\mathfrak{N}_q(Q,P) = \mathfrak{M}_q(Q,P) \setminus \mathfrak{M}_q(Q/2,P).$$

By Dirichlet's approximation theorem, given $\alpha \in [0,1)$, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \leqslant a \leqslant q \leqslant P^{k/2}$, $(a,q)=1$ and $|q\alpha - a| \leqslant P^{-k/2}$. Thus we see that $\alpha \in \mathfrak{M}(P^{k/2},P)$. Hence, in particular, we have

$$[0,1) = \bigcup_{j=0}^{L} \mathfrak{N}(2^{-j}P^{k/2},P),$$

in which

$$L = \left\lfloor \frac{k\log P}{2\log 2} \right\rfloor. \tag{2.1}$$

It therefore follows that

$$\begin{aligned}U_s(P,R)&=\sum_{j=0}^{L}\int_{\mathfrak{N}(2^{-j}P^{k/2},P)}|f(\alpha;P,R)|^s\,\mathrm{d}\alpha\\&\ll(\log P)\max_{1\leqslant Q\leqslant P^{k/2}}\int_{\mathfrak{N}(Q,P)}|f(\alpha;P,R)|^s\,\mathrm{d}\alpha.\end{aligned}$$

An important feature of the mean value on the right hand side here is a certain scaling property of the associated set $\mathfrak{N}(Q,P)$. We summarise this property in the form of a lemma.

Lemma 2.3. *Let $F:\mathbb{R}\to\mathbb{C}$ be a 1-periodic integrable function. Suppose that $w\in\mathbb{N}$ satisfies the property that $1\leqslant Q\leqslant\frac{1}{2}(P/w)^{k/2}$. Then whenever $q\in\mathbb{N}$ satisfies $(q,w)=1$, one has*

$$\int_{\mathfrak{M}_q(Q,P)}F(\alpha w^k)\,\mathrm{d}\alpha=w^{-k}\int_{\mathfrak{M}_q(Q,P/w)}F(\beta)\,\mathrm{d}\beta.$$

Proof. Let

$$I=[-q^{-1}QP^{-k},q^{-1}QP^{-k}]\quad\text{and}\quad J=[-q^{-1}Qw^kP^{-k},q^{-1}Qw^kP^{-k}].$$

The hypothesis $Q\leqslant\frac{1}{2}(P/w)^{k/2}$ ensures that the arcs comprising $\mathfrak{M}_q(Q,P/w)$ are disjoint. Since F has period 1, we infer that

$$\int_{\mathfrak{M}_q(Q,P/w)}F(\beta)\,\mathrm{d}\beta=\sum_{\substack{b=1\\(b,q)=1}}^{q}\int_J F\Bigl(\frac{b}{q}+\gamma\Bigr)\,\mathrm{d}\gamma.\tag{2.2}$$

Likewise, we find that

$$\begin{aligned}\int_{\mathfrak{M}_q(Q,P)}F(\alpha w^k)\,\mathrm{d}\alpha&=\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_I F\Bigl(\Bigl(\frac{a}{q}+\beta\Bigr)w^k\Bigr)\,\mathrm{d}\beta\\&=w^{-k}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_J F\Bigl(\frac{aw^k}{q}+\gamma\Bigr)\,\mathrm{d}\gamma.\end{aligned}\tag{2.3}$$

By hypothesis $(q,w)=1$, whence the mapping $a\mapsto aw^k$ induces a bijection on the reduced residue classes modulo q. Once again using the hypothesis that F has period one, it now follows that the sums on the right hand sides of (2.2) and (2.3) are equal. This proves the lemma. □

3. A decomposition of the smooth Weyl sum

We are unable to apply Lemma 2.3 directly with $F(\beta)=|f(\beta;P,R)|^s$. However, following a decomposition of the smooth Weyl sum $f(\beta;P,R)$, we are able to achieve a conclusion tantamount to such an application. Here, the coprimality condition $(q,w)=1$ of Lemma 2.3 figures prominently in the analysis. We begin by isolating a part of the smooth Weyl sum $f(\alpha;P,R)$ in which a large factor w of the argument is available coprime to an auxiliary variable q. With this objective in mind, we introduce the auxiliary exponential sums

$$f_q^*(\alpha;P,M,R)=\sum_{\substack{v\in\mathscr{A}(P,R)\\ v>M\\ (v,q)=1}}\sum_{u\in\mathscr{C}_q(P/v,R)} e(\alpha(uv)^k) \tag{3.1}$$

and

$$f_q^\dagger(\alpha;P,M,R)=\sum_{\substack{v\in\mathscr{A}(M,R)\\ (v,q)=1}}\sum_{u\in\mathscr{C}_q(P/v,R)} e(\alpha(uv)^k). \tag{3.2}$$

Lemma 3.1. *Let $q\in\mathbb{N}$. Then*

$$f(\alpha;P,R)=f_q^*(\alpha;P,M,R)+f_q^\dagger(\alpha;P,M,R).$$

Proof. Consider an integer $x\in\mathscr{A}(P,R)$, and let u denote the largest divisor of x with $u|q^\infty$. Put $v=x/u$. Then either $v\leqslant M$, in which case $v\in\mathscr{A}(M,R)$, or else $v>M$ and $v\in\mathscr{A}(P,R)$. In both cases, one has $x=uv$ with $u\in\mathscr{C}_q(P/v,R)$ and $(v,q)=1$. The conclusion of the lemma follows at once. □

It transpires that the contribution of the exponential sum $f_q^\dagger(\alpha;P,M,R)$ is easily handled via a trivial estimate.

Lemma 3.2. *Let Q be a parameter with $1\leqslant Q\leqslant P^{k/2}$. Then, whenever $1\leqslant q\leqslant Q$, one has*

$$\int_{\mathfrak{M}_q(Q,P)}|f_q^\dagger(\alpha;P,M,R)|^s\,\mathrm{d}\alpha\ll QM^sP^{\varepsilon-k}.$$

Proof. By applying Lemma 2.2 together with a trivial estimate for the sum over v in (3.2), we see that

$$|f_q^\dagger(\alpha;P,M,R)|\leqslant\sum_{v\leqslant M}\sum_{u\in\mathscr{C}_q(P/v,R)}1\ll P^\varepsilon M.$$

Thus, since $\mathrm{mes}(\mathfrak{M}_q(Q,P))\ll QP^{-k}$, we deduce that

$$\int_{\mathfrak{M}_q(Q,P)}|f_q^\dagger(\alpha;P,M,R)|^s\,\mathrm{d}\alpha\ll QP^{-k}(P^\varepsilon M)^s,$$

and the conclusion of the lemma follows. □

In order to analyse the exponential sum $f_q^*(\alpha;P,M,R)$ further, we recall a decomposition of the smooth numbers utilised in work of Vaughan [12]. In this context, we introduce a subset of the smooth numbers $\mathscr{A}(P,R)$ given by

$$\mathscr{B}(M,\pi,R)=\{v\in\mathscr{A}(M\pi,R):v>M,\ \pi|v,\text{ and }\pi'|v\text{ implies }\pi'\geqslant\pi\}.$$

Both here and in the remainder of this memoir, we reserve the symbols π and π' to denote prime numbers. We also require the exponential sum

$$g_{q,\pi}^*(\alpha;P,m,R)=\sum_{\substack{w\in\mathscr{A}(P/m,\pi)\\ (w,q)=1}}\sum_{u\in\mathscr{C}_q(P/(mw),R)} e(\alpha(wu)^k). \tag{3.3}$$

Lemma 3.3. *Let $q \in \mathbb{N}$. Then whenever $M \geqslant R$, one has*

$$f_q^*(\alpha; P, M, R) = \sum_{\pi \leqslant R} \sum_{\substack{m \in \mathscr{B}(M,\pi,R) \\ (m,q)=1}} g_{q,\pi}^*(\alpha m^k; P, m, R).$$

Proof. It follows from [12, Lemma 10.1] that for each $v \in \mathscr{A}(P, R)$ satisfying $v > M \geqslant R$, there is a unique triple (π, m, w) with $v = mw$, $w \in \mathscr{A}(P/m, \pi)$ and $m \in \mathscr{B}(M, \pi, R)$. On noting that the coprimality conditions $(m, q) = (w, q) = 1$ are inherited from the constraint $(v, q) = 1$, the conclusion of the lemma follows from the definition (3.1) of $f_q^*(\alpha; P, M, R)$. □

We complete this section by combining the conclusions of Lemmata 3.1, 3.2 and 3.3 so as to obtain a mean value estimate of considerable utility. In order to abbreviate notation at this point, we introduce the mean value $I_q(M; \mathfrak{B})$ defined for $\mathfrak{B}$ equal to either $\mathfrak{M}$ or $\mathfrak{N}$ by

$$I_q(M; \mathfrak{B}) = \sum_{\pi \leqslant R} \sum_{\substack{m \in \mathscr{B}(M,\pi,R) \\ (m,q)=1}} \int_{\mathfrak{B}_q(Q,P)} |g_{q,\pi}^*(\alpha m^k; P, m, R)|^s \, \mathrm{d}\alpha. \tag{3.4}$$

Lemma 3.4. *Let Q be a real number with $1 \leqslant Q \leqslant P^{k/2}$, and suppose that s is a real number with $s > 1$. Then whenever $M \geqslant R$ and $1 \leqslant q \leqslant Q$, one has*

$$\int_{\mathfrak{N}_q(Q,P)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha \ll (MR)^{s-1} I_q(M; \mathfrak{N}) + QM^s P^{\varepsilon - k}.$$

The same conclusion also holds when $\mathfrak{M}$ replaces $\mathfrak{N}$ throughout.

Proof. It follows from Lemma 3.1 that when $\alpha \in [0, 1)$, one has

$$|f(\alpha; P, R)|^s \ll |f_q^*(\alpha; P, M, R)|^s + |f_q^\dagger(\alpha; P, M, R)|^s.$$

Moreover, by applying Hölder's inequality in combination with Lemma 3.3, one obtains the bound

$$\begin{aligned} |f_q^*(\alpha; P, M, R)|^s &= \Big| \sum_{\pi \leqslant R} \sum_{\substack{m \in \mathscr{B}(M,\pi,R) \\ (m,q)=1}} g_{q,\pi}^*(\alpha m^k; P, m, R) \Big|^s \\ &\ll (MR)^{s-1} \sum_{\pi \leqslant R} \sum_{\substack{m \in \mathscr{B}(M,\pi,R) \\ (m,q)=1}} |g_{q,\pi}^*(\alpha m^k; P, m, R)|^s. \end{aligned}$$

Note that $\mathfrak{N}_q(Q, P) \subseteq \mathfrak{M}_q(Q, P)$. Hence, on integrating over $\alpha \in \mathfrak{N}_q(Q, P)$ or $\alpha \in \mathfrak{M}_q(Q, P)$, the lemma now follows from Lemma 3.2. □

4. Mean value estimates over intermediate arcs

The upper bound provided by Lemma 3.4 bounds $f(\alpha; P, R)$ in mean, over a set of intermediate arcs, in terms of an auxiliary mean value. The latter is susceptible to Lemma 2.3, but the presence of factors in the argument lying in $\mathscr{C}_q(P/(mw), R)$ creates difficulties to which we now attend. In this section, we prepare a preliminary mean value using a method that in certain circumstances may be enhanced. These enhancements we defer to the next section.

We begin with a discussion of the exponential sum $g^*_{q,\pi}(\alpha;P,m,R)$. Here, we shall find it useful to introduce a modification of the set $\mathscr{C}_q(P,R)$, namely

$$\mathscr{C}_{q,\pi}(P,R)=\{n\in\mathscr{C}_q(P,R): p|n \text{ implies } p>\pi\}.$$

Lemma 4.1. *One has*

$$g^*_{q,\pi}(\alpha;P,m,R)=\sum_{z\in\mathscr{C}_{q,\pi}(P/m,R)}\ \sum_{x\in\mathscr{A}(P/(mz),\pi)} e(\alpha(xz)^k).$$

Proof. On recalling the definition (3.3) of $g^*_{q,\pi}(\alpha;P,m,R)$, we may interchange the order of summation to obtain

$$g^*_{q,\pi}(\alpha;P,m,R)=\sum_{u\in\mathscr{C}_q(P/m,R)}\ \sum_{\substack{w\in\mathscr{A}(P/(mu),\pi)\\(w,q)=1}} e(\alpha(wu)^k).$$

For each integer $u\in\mathscr{C}_q(P/m,R)$, there is a unique pair of integers (y,z) satisfying $u=yz$, where y has all of its prime divisors no larger than π, and z has no prime divisors less than or equal to π. Thus, we have $y\in\mathscr{C}_q(P/m,\pi)$ and $z\in\mathscr{C}_{q,\pi}(P/m,R)$. Making use of this decomposition, we see that

$$g^*_{q,\pi}(\alpha;P,m,R)=\sum_{z\in\mathscr{C}_{q,\pi}(P/m,R)}\ \sum_{y\in\mathscr{C}_q(P/(mz),\pi)}\ \sum_{\substack{w\in\mathscr{A}(P/(myz),\pi)\\(w,q)=1}} e(\alpha(wyz)^k). \tag{4.1}$$

Notice here that, given any integer $n\in\mathscr{A}(P/(mz),\pi)$, there are unique integers y and w with $n=yw$, and satisfying the condition that y has all of its prime divisors amongst those of q, and w is coprime with q. With such decompositions in mind, we recognise that

$$\sum_{y\in\mathscr{C}_q(P/(mz),\pi)}\ \sum_{\substack{w\in\mathscr{A}(P/(myz),\pi)\\(w,q)=1}} e(\gamma(wy)^k)=\sum_{x\in\mathscr{A}(P/(mz),\pi)} e(\gamma x^k).$$

The conclusion of the lemma follows on substituting this relation into (4.1). □

We now investigate the mean value $I_q(M;\mathfrak{B})$ defined in (3.4) as a prelude to the highlight of this section, a mean value estimate for moments of $f(\alpha;P,R)$ restricted to the set $\mathfrak{M}(Q,P)$. Fix $\mathfrak{B}$ to be either $\mathfrak{M}$ or $\mathfrak{N}$, and fix a real number Q with $1\leqslant Q\leqslant \frac{1}{2}P^{k/2}R^{-k}$. At this point, we put

$$M=P(2Q)^{-2/k}R^{-1}, \tag{4.2}$$

and we observe that our hypothesis on Q ensures that $M\geqslant R$. Then, when $\pi\leqslant R$ and $m\in\mathscr{B}(M,\pi,R)$, one has $m\leqslant M\pi\leqslant P(2Q)^{-2/k}$, and thus $Q\leqslant\frac{1}{2}(P/m)^{k/2}$. The latter condition ensures that the arcs $\mathfrak{M}_{q,a}(Q,P/m)$ are disjoint for $0\leqslant a\leqslant q\leqslant Q$ with $(a,q)=1$. Under these hypotheses on Q and m, therefore, we deduce from (3.4) via Lemma 2.3 that

$$I_q(M;\mathfrak{B})=\sum_{\pi\leqslant R}\ \sum_{\substack{m\in\mathscr{B}(M,\pi,R)\\(m,q)=1}} m^{-k}\int_{\mathfrak{B}_q(Q,P/m)}|g^*_{q,\pi}(\alpha;P,m,R)|^s\,\mathrm{d}\alpha. \tag{4.3}$$

Observe next that, since $\mathscr{C}_{q,\pi}(P/m,R)\subseteq \mathscr{C}_q(P/m,R)$, it follows from Lemma 4.1 together with Lemma 2.2 and Hölder's inequality that when $s>1$, one has

$$\begin{aligned}|g^*_{q,\pi}(\alpha;P,m,R)|^s &\ll P^\varepsilon \sum_{z\in \mathscr{C}_{q,\pi}(P/m,R)} \biggl| \sum_{x\in \mathscr{A}(P/(mz),\pi)} e(\alpha(xz)^k)\biggr|^s\\ &\ll P^\varepsilon \sum_{z\in \mathscr{A}(P/m,R)} |f(\alpha z^k;P/(mz),\pi)|^s.\end{aligned}$$

Write

$$V_s(\pi,m,z;\mathfrak{B})=\int_{\mathfrak{B}(Q,P/m)} |f(\alpha z^k;P/(mz),\pi)|^s\,\mathrm{d}\alpha.$$

Then we deduce via (4.3) that

$$\begin{aligned}\sum_{1\leqslant q\leqslant Q} I_q(M;\mathfrak{B}) &\leqslant \sum_{\pi\leqslant R}\sum_{m\in\mathscr{B}(M,\pi,R)} m^{-k}\sum_{1\leqslant q\leqslant Q}\int_{\mathfrak{B}_q(Q,P/m)} |g^*_{q,\pi}(\alpha;P,m,R)|^s\,\mathrm{d}\alpha\\ &\ll P^\varepsilon \sum_{\pi\leqslant R}\sum_{m\in\mathscr{B}(M,\pi,R)} m^{-k}\sum_{z\in\mathscr{A}(P/m,R)} V_s(\pi,m,z;\mathfrak{B}).\end{aligned}\tag{4.4}$$

The special case of (4.4) with $\mathfrak{B}=\mathfrak{M}$ combines with Lemma 3.4 to deliver the main conclusion of this section. We emphasise that in this statement just as elsewhere, we are making use of the extended ε, R convention.

Theorem 4.2. *Suppose that s is a real number with $s\geqslant 2$ and Δ_s is an admissible exponent. Then whenever Q is a real number with $1\leqslant Q\leqslant P^{k/2}$, one has the uniform bound*

$$\int_{\mathfrak{M}(Q,P)} |f(\alpha;P,R)|^s\,\mathrm{d}\alpha \ll P^{s-k+\varepsilon}Q^{2\Delta_s/k}.$$

Proof. We begin by observing that the conclusion is immediate from the definition of an admissible exponent when $\frac{1}{2}P^{k/2}R^{-k}<Q\leqslant P^{k/2}$, for in such circumstances one has

$$\int_{\mathfrak{M}(Q,P)} |f(\alpha;P,R)|^s\,\mathrm{d}\alpha \leqslant U_s(P,R)\ll P^{s-k+\Delta_s+\varepsilon}\ll P^{s-k+2\varepsilon}Q^{2\Delta_s/k}.$$

We may therefore suppose henceforth that $1\leqslant Q\leqslant \frac{1}{2}P^{k/2}R^{-k}$. In view of (4.2), one then has also $M\geqslant R$. For each summand m in the relation (4.4), one trivially has $\mathfrak{M}(Q,P/m)\subseteq[0,1)$. Thus, by means of a change of variable we deduce that

$$V_s(\pi,m,z;\mathfrak{M})\leqslant \int_0^1 |f(\alpha z^k;P/(mz),\pi)|^s\,\mathrm{d}\alpha = U_s(P/(mz),\pi).$$

We hence infer from Lemma 2.1 and (4.4) that when $s-k+\Delta_s\geqslant 1$, one has

$$\begin{aligned}\sum_{1\leqslant q\leqslant Q} I_q(M;\mathfrak{M}) &\ll P^\varepsilon \sum_{\pi\leqslant R}\sum_{m\in\mathscr{B}(M,\pi,R)} m^{-k}\sum_{z\in\mathscr{A}(P/m,R)} \Bigl(\frac{P}{mz}\Bigr)^{s-k+\Delta_s}\\ &\ll P^{-k+2\varepsilon}\sum_{\pi\leqslant R}\sum_{m\in\mathscr{B}(M,\pi,R)} \Bigl(\frac{P}{m}\Bigr)^{s+\Delta_s}\\ &\ll P^{s-k+3\varepsilon}M^{1-s}\Bigl(\frac{P}{M}\Bigr)^{\Delta_s}.\end{aligned}$$

The condition $s - k + \Delta_s \geqslant 1$ is satisfied so long as $s \geqslant 2$, for as we have already observed, it is always the case that $\Delta_s \geqslant k - s/2$. We therefore conclude from Lemma 3.4 that

$$\begin{aligned}\int_{\mathfrak{M}(Q,P)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha &= \sum_{1 \leqslant q \leqslant Q} \int_{\mathfrak{M}_q(Q,P)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha \\ &\ll (MR)^{s-1} \sum_{1 \leqslant q \leqslant Q} I_q(M; \mathfrak{M}) + Q^2 M^s P^{\varepsilon - k} \\ &\ll P^{s-k+\varepsilon} (P/M)^{\Delta_s} + Q^2 M^s P^{\varepsilon - k}.\end{aligned}$$

Thus, on recalling our choice (4.2) for M, we conclude that

$$\int_{\mathfrak{M}(Q,P)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha \ll P^{s-k+\varepsilon} \left(Q^{2\Delta_s/k} + Q^{2-2s/k} \right).$$

The conclusion of the theorem follows on observing that $\Delta_s \geqslant k - s$, whence the first term on the right hand side majorises the second. □

We remark that a version of Theorem 4.2 appears as [2, Lemma 2.3], though in that version the condition $s \geqslant k + 1$ is imposed. The proof of that lemma is in many ways more straightforward, with the price being a more restrictive constraint on s. As we shall see in the next section, the approach that we have taken in this memoir also offers the option of retaining minor arc information.

5. Mean value estimates restricted to minor arcs

The conclusion of Theorem 4.2 provides a mean value estimate over an intermediate set of major arcs $\mathfrak{M}(Q, P)$. If instead we integrate over the truncated set $\mathfrak{N}(Q, P)$, then we are removing the points from $\mathfrak{M}(Q, P)$ of small height, and the resulting mean value is relevant to the estimation of the minor arc contribution. Suppose that $1 \leqslant Q \leqslant \frac{1}{2} P^{k/2}$ and put $X = Q^{2/k}$. Then in very rough terms, one can interpret the argument leading to Theorem 4.2 as delivering a bound of the flavour

$$\begin{aligned}\int_{\mathfrak{M}(Q,P)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha &\ll (P/X)^{s-k+\varepsilon} \int_{\mathfrak{M}(\frac{1}{2}X^{k/2}, X)} |f(\alpha; X, R)|^s \, \mathrm{d}\alpha \\ &\ll (P/X)^{s-k+\varepsilon} U_s(X, R).\end{aligned}$$

Our goal now is to obtain an analogous bound of the general shape

$$\int_{\mathfrak{N}(Q,P)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha \ll (P/X)^{s-k+\varepsilon} \int_{\mathfrak{N}(\frac{1}{2}X^{k/2}, X)} |f(\alpha; X, R)|^s \, \mathrm{d}\alpha.$$

The set $\mathfrak{N}(\frac{1}{2}X^{k/2}, X)$ is an extreme set of minor arcs. Here, when α lies on $\mathfrak{N}(\frac{1}{2}X^{k/2}, X)$, it is known that the smooth Weyl sum $f(\alpha; X, R)$ is $O(X^{1-c/k})$, for a suitable positive number c. Since this bound is considerably sharper than conventional minor arc bounds for $f(\alpha; X, R)$, which would lose a factor of roughly $\log k$ in the Weyl exponent, one has rather sharper bounds for

$$\int_{\mathfrak{N}(Q,P)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha$$

than were available hitherto, at least when s is fairly large.

We begin by deriving a consequence of [22, Lemma 3.1].

Lemma 5.1. *Let t be an even integer, and suppose that the exponent Δ_t is admissible. Then whenever $b\in\mathbb{Z}$, $r\in\mathbb{N}$ and $(b,r)=1$, one has*

$$f(\alpha;P,R)\ll r^\varepsilon P^{1+\varepsilon}\left(P^{\Delta_t}\left(\Theta^{-1}+P^{-k/2}+P^{-k}\Theta\right)\right)^{2/t^2}+P^{1/2+\varepsilon},$$

in which we write $\Theta=r+P^k|r\alpha-b|$.

Proof. Suppose that $\frac{1}{2}<\lambda<1$, $M=P^\lambda$ and $\alpha\in\mathbb{R}$. Suppose further that $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$ and $|\alpha-a/q|\leqslant 1/q^2$. Then [22, Lemma 3.1] establishes that for all even natural numbers t and w, one has

$$\begin{aligned}f(\alpha;P,R)\ll\, & q^\varepsilon P^{1+\varepsilon}\left(M^{\Delta_w}(P/M)^{\Delta_t}\left(q^{-1}+M^{-k}+(P/M)^{-k}+qP^{-k}\right)\right)^{2/(tw)}\\ &+M.\end{aligned}$$

We take $w=t$ and $\lambda=\frac{1}{2}+\delta$, for a small fixed positive number δ. Thus

$$f(\alpha;P,R)\ll q^\varepsilon P^{1+k\delta}\left(P^{\Delta_t}\left(q^{-1}+P^{-k/2}+qP^{-k}\right)\right)^{2/t^2}+P^{1/2+\delta}.$$

We now apply a standard transference principle (see [23, Lemma 14.1]) to see that the same conclusion holds for all $b\in\mathbb{Z}$ and $r\in\mathbb{N}$ with $(b,r)=1$ when we replace q by $\Theta=r+P^k|r\alpha-b|$ throughout. The conclusion of the lemma therefore follows, since δ may be taken arbitrarily small. □

The most powerful consequences of Lemma 5.1 are made available by applying Dirichlet's approximation theorem to obtain integers b and r with $(b,r)=1$ and $1\leqslant r\leqslant P^{k/2}$ for which $|r\alpha-b|\leqslant P^{-k/2}$. In such circumstances, Lemma 5.1 is most effective when α satisfies the condition that $r>cP^{k/2}$, for some fixed $c>0$. One then has $f(\alpha;P,R)\ll P^{1-\tau(t,k)+\varepsilon}+P^{1/2+\varepsilon}$, where

$$\tau(t,k)=\frac{k-2\Delta_t}{t^2}.$$

Since $\Delta_t\geqslant\max\{k-t/2,0\}$, one sees that

$$\tau(t,k)\leqslant\min\left\{\frac{t-k}{t^2},\frac{k}{t^2}\right\}\leqslant\frac{1}{4k},$$

and thus our estimate for $f(\alpha;P,R)$ simplifies to $f(\alpha;P,R)\ll P^{1-\tau(t,k)+\varepsilon}$. To extract the most from this bound, we introduce the number

$$\tau(k)=\max_{w\in\mathbb{N}}\frac{k-2\Delta_{2w}}{4w^2},\tag{5.1}$$

and then have

$$f(\alpha;P,R)\ll P^{1-\tau(k)+\varepsilon}.\tag{5.2}$$

The number $\tau(k)$ will be of significance in the argument below. It appears also in slightly different guises in work of Karatsuba [8] and Heath-Brown [6].

We now return to the rescaling argument underlying the work of §4. In this context, we introduce an auxiliary exponent. Suppose that s is a real number with $s\geqslant 2$, and that the exponents Δ_u are admissible for $2\leqslant u\leqslant s$. We define

$$\Delta_s^*=\min_{0\leqslant t\leqslant s-2}\left(\Delta_{s-t}-t\tau(k)\right),\tag{5.3}$$

and refer to Δ_s^* as an *admissible exponent for minor arcs.*

Theorem 5.2. *Suppose that $s \geqslant 2$, and that Δ_s^* is an admissible exponent for minor arcs. Then whenever $1 \leqslant Q \leqslant \frac{1}{2}P^{k/2}R^{-k}$, one has the uniform bound*

$$\int_{\mathfrak{N}(Q,P)} |f(\alpha;P,R)|^s \,\mathrm{d}\alpha \ll P^{s-k+\varepsilon}Q^{2\Delta_s^*/k}.$$

Proof. We again fix M according to equation (4.2), and we recall from (4.4) that when $1 \leqslant Q \leqslant \frac{1}{2}P^{k/2}R^{-k}$, one has

$$\sum_{1\leqslant q\leqslant Q} I_q(M;\mathfrak{N}) \ll P^\varepsilon \sum_{\pi\leqslant R}\sum_{m\in\mathscr{B}(M,\pi,R)} m^{-k} \sum_{z\in\mathscr{A}(P/m,R)} V_s(\pi,m,z;\mathfrak{N}), \tag{5.4}$$

where

$$V_s(\pi,m,z;\mathfrak{N}) = \int_{\mathfrak{N}(Q,P/m)} |f(\alpha z^k;P/(mz),\pi)|^s \,\mathrm{d}\alpha. \tag{5.5}$$

We apply Lemma 5.1 to estimate $f(\alpha z^k;P/(mz),\pi)$ when $\alpha \in \mathfrak{N}(Q,P/m)$. In the latter circumstances, one has $\alpha \in \mathfrak{M}(Q,P/m)\setminus\mathfrak{M}(Q/2,P/m)$. Thus, there exist integers a and q with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q)=1$ for which one has $|q\alpha-a| \leqslant Q(P/m)^{-k}$, and either $q > Q/2$ or $|q\alpha - a| > \frac{1}{2}Q(P/m)^{-k}$. Consider a fixed integer $z \in \mathscr{A}(P/m,R)$. Then as a consequence of these relations, if we put

$$r = \frac{q}{(q,z^k)} \quad \text{and} \quad b = \frac{az^k}{(q,z^k)},$$

then we find that $(b,r)=1$ with $r \leqslant Q$ and $|r(\alpha z^k)-b| \leqslant Q(P/(mz))^{-k}$. Moreover, one has either $r > \frac{1}{2}Qz^{-k}$ or $|r(\alpha z^k)-b| > \frac{1}{2}Q(P/m)^{-k}$. Thus, in particular,

$$\tfrac{1}{2}Qz^{-k} < r + \left(\frac{P}{mz}\right)^k |r(\alpha z^k)-b| \leqslant 2Q.$$

We therefore deduce from Lemma 5.1 that whenever t is an even integer, then

$$f(\alpha z^k;P/(mz),\pi) \ll Q^\varepsilon \left(\frac{P}{mz}\right)^{1+\varepsilon} \left(\left(\frac{P}{mz}\right)^{\Delta_t} \left(\frac{z^k}{Q} + \left(\frac{mz}{P}\right)^{k/2} + Q\left(\frac{mz}{P}\right)^k\right)\right)^{2/t^2} + \left(\frac{P}{mz}\right)^{1/2+\varepsilon}.$$

We choose $t=2w$ to correspond to the maximum in the definition of $\tau=\tau(k)$ in (5.1), and recall from (4.2) that $Q=\frac{1}{2}(P/(MR))^{k/2}$. Then, when $M < m \leqslant MR$ and $\alpha \in \mathfrak{N}(Q,P/m)$, we conclude that

$$f(\alpha z^k;P/(mz),\pi) \ll \left(\frac{P}{mz}\right)^{1/2+\varepsilon} + P^\varepsilon\left(\frac{P}{m}\right)^{1-\tau} z^{-1+2(k-\Delta_t)/t^2}.$$

Since $\Delta_t \geqslant k-t/2$ and $t \geqslant 2$, we arrive at the upper bound

$$\sup_{\alpha\in\mathfrak{N}(Q,P/m)} |f(\alpha z^k;P/(mz),\pi)| \ll P^\varepsilon (P/m)^{1-\tau} z^{-1/2}. \tag{5.6}$$

We now return to the mean value $V_s(\pi,m,z;\mathfrak{N})$ defined in (5.5). Let t and v be non-negative integers with $s=t+v$. Then it follows from (5.6) that

$$V_s(\pi,m,z;\mathfrak{N})\ll P^\varepsilon (P/m)^{t(1-\tau)}\int_0^1 |f(\alpha z^k;P/(mz),\pi)|^v\,\mathrm{d}\alpha.$$

A change of variable therefore combines with Lemma 2.1 to show that

$$\begin{aligned}V_s(\pi,m,z;\mathfrak{N})&\ll P^\varepsilon (P/m)^{t(1-\tau)}U_v(P/(mz),\pi)\\ &\ll P^{2\varepsilon}(P/m)^{t(1-\tau)}(P/(mz))^{v-k+\Delta_v}.\end{aligned}$$

Since $\Delta_v\geqslant k-v/2$, we see that $v-k+\Delta_v\geqslant 1$ whenever $v\geqslant 2$. On recalling the definition (5.3) of Δ_s^*, therefore, and noting that $v=s-t$, we discern that

$$V_s(\pi,m,z;\mathfrak{N})\ll z^{-1}P^\varepsilon (P/m)^{s-k+\Delta_s^*}.$$

On substituting this upper bound into (5.4), we find that

$$\begin{aligned}\sum_{1\leqslant q\leqslant Q}I_q(M;\mathfrak{N})&\ll P^\varepsilon \sum_{\pi\leqslant R}\sum_{M<m\leqslant MR}m^{-k}(P/m)^{s-k+\Delta_s^*}\sum_{1\leqslant z\leqslant P/m}z^{-1}\\ &\ll P^{s-k+2\varepsilon}M^{1-s}(P/M)^{\Delta_s^*}.\end{aligned}\tag{5.7}$$

We next appeal to Lemma 3.4, proceeding just as in the conclusion of the proof of Theorem 4.2. Thus, making use of the bound (5.7), we obtain

$$\begin{aligned}\int_{\mathfrak{N}(Q,P)}|f(\alpha;P,R)|^s\,\mathrm{d}\alpha&=\sum_{1\leqslant q\leqslant Q}\int_{\mathfrak{N}_q(Q,P)}|f(\alpha;P,R)|^s\,\mathrm{d}\alpha\\ &\ll (MR)^{s-1}\sum_{1\leqslant q\leqslant Q}I_q(M;\mathfrak{N})+Q^2M^sP^{\varepsilon-k}\\ &\ll P^{s-k+\varepsilon}(P/M)^{\Delta_s^*}+Q^2M^sP^{\varepsilon-k}.\end{aligned}$$

Hence, on recalling the choice (4.2) for M, we conclude that

$$\int_{\mathfrak{N}(Q,P)}|f(\alpha;P,R)|^s\,\mathrm{d}\alpha\ll P^{s-k+\varepsilon}(Q^{2-2s/k}+Q^{2\Delta_s^*/k}).\tag{5.8}$$

We have observed already that $\tau(k)\leqslant 1/(4k)$. Thus, since $\Delta_{s-t}\geqslant k-(s-t)$, one sees that for some integer t satisfying $0\leqslant t\leqslant s-2$ (the integer t associated with the definition (5.3) of Δ_s^*), one has

$$\frac{2}{k}\Delta_s^*\geqslant \frac{2}{k}\Bigl(k-(s-t)-\frac{t}{4k}\Bigr)\geqslant 2-\frac{2s}{k}.$$

The desired conclusion is therefore immediate from (5.8). □

This theorem may be exploited to obtain a bound for minor arc contributions of considerable utility in applications of the circle method. In this context, we introduce the set of minor arcs $\mathfrak{m}(Q)=\mathfrak{m}(Q,P)$ given by $\mathfrak{m}(Q)=[0,1]\setminus\mathfrak{M}(Q,P)$. We also abbreviate the major arcs $\mathfrak{M}(Q,P)$ simply to $\mathfrak{M}(Q)$ in circumstances where the implicit second parameter is equal to P and brevity is to be prized above full disclosure.

Theorem 5.3. *Let $s \geqslant 2$ and suppose that Δ_s^* is an admissible exponent for minor arcs satisfying $\Delta_s^* < 0$. Let θ be a positive number with $\theta \leqslant k/2$. Then whenever $P^\theta \leqslant Q \leqslant P^{k/2}$, one has the bound*

$$\int_{\mathfrak{m}(Q)} |f(\alpha;P,R)|^s \,\mathrm{d}\alpha \ll_\theta P^{s-k} Q^{\varepsilon - 2|\Delta_s^*|/k}.$$

Proof. Write

$$J = \left\lceil \frac{\log(P^{k/2}/Q)}{\log 2} \right\rceil \quad \text{and} \quad J_0 = \left\lceil \frac{\log(2R^k)}{\log 2} \right\rceil.$$

We begin by observing that, since $\mathfrak{m}(Q) = [0,1] \setminus \mathfrak{M}(Q,P)$, we have

$$\mathfrak{m}(Q) \subseteq \bigcup_{j=0}^{J} \mathfrak{N}(2^{-j}P^{k/2}, P).$$

When $J_0 < j \leqslant J$, it follows from Theorem 5.2 that

$$\begin{aligned}\int_{\mathfrak{N}(2^{-j}P^{k/2},P)} |f(\alpha;P,R)|^s \,\mathrm{d}\alpha &\ll P^{s-k+\varepsilon} \left(2^{-j}P^{k/2}\right)^{2\Delta_s^*/k} \\ &\ll P^{s-k+\varepsilon} Q^{-2|\Delta_s^*|/k}. \end{aligned} \tag{5.9}$$

Meanwile, when $0 \leqslant j \leqslant J_0$, we may apply the argument underlying the proof of Theorem 5.2. Thus, when $\alpha \in \mathfrak{N}(2^{-j}P^{k/2},P)$, there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with $(b,r) = 1$, $r \leqslant 2^{-j}P^{k/2}$ and $|r\alpha - b| \leqslant 2^{-j}P^{-k/2}$. Since $\alpha \notin \mathfrak{M}(2^{-j-1}P^{k/2},P)$, we have

$$P^{k/2}R^{-k} \ll 2^{-1-j}P^{k/2} \leqslant r + P^k|r\alpha - b| \ll P^{k/2}.$$

By Lemma 5.1 and (5.2), we now have $f(\alpha;P,R) \ll P^{1-\tau(k)+\varepsilon}$. With $s = t + v$, and t and v defined as in the proof of Theorem 5.2, we therefore infer that

$$\begin{aligned}\int_{\mathfrak{N}(2^{-j}P^{k/2},P)} |f(\alpha;P,R)|^s \,\mathrm{d}\alpha &\ll (P^{1-\tau(k)+\varepsilon})^t \int_0^1 |f(\alpha;P,R)|^v \,\mathrm{d}\alpha \\ &\ll P^{s-k+\varepsilon} P^{\Delta_v - t\tau(k)} \\ &\ll P^{s-k+\varepsilon} Q^{-2|\Delta_s^*|/k}.\end{aligned}$$

On combining this estimate with (5.9), we see that

$$\begin{aligned}\int_{\mathfrak{m}(Q)} |f(\alpha;P,R)|^s \,\mathrm{d}\alpha &\ll \sum_{j=0}^{J} \int_{\mathfrak{N}(2^{-j}P^{k/2},P)} |f(\alpha;P,R)|^s \,\mathrm{d}\alpha \\ &\ll P^{s-k+\varepsilon} Q^{-2|\Delta_s^*|/k}.\end{aligned}$$

Since $Q \geqslant P^\theta$ and $\theta > 0$, it suffices to recall the conventions concerning the use of ε and η to complete the proof of the theorem. □

6. The treatment of $G(k)$ in general terms

Our proofs of Theorems 1.1 and 1.2 are largely routine given the flexible nature of Theorem 5.3, so we may be concise in our exposition. We begin with a pruning argument that extends the range of Q in Theorem 5.3 from a power of P to an arbitrarily slowly growing function of P.

Theorem 6.1. *Suppose that $k \geqslant 3$, $s \geqslant 2k+3$ and Δ_s^* is an admissible exponent for minor arcs with $\Delta_s^* < 0$. Let ν be any positive number with*

$$\nu < \min\left\{ \frac{2|\Delta_s^*|}{k}, \frac{1}{6k} \right\}.$$

Then, when $1 \leqslant Q \leqslant P^{k/2}$, one has the uniform bound

$$\int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha \ll P^{s-k} Q^{-\nu}.$$

Proof. In view of the conclusion of Theorem 5.3, it suffices to consider values of Q with $1 \leqslant Q \leqslant P^{\theta}$, where θ is a fixed positive number small in terms of k and s. We assume in particular that $\theta < 1/k$, whence for $k \geqslant 3$ one has

$$\frac{3}{4} + \frac{\theta}{8} < 1 - \frac{1}{2k}. \tag{6.1}$$

Our starting point is the observation that, as a consequence of Theorem 5.3,

$$\begin{aligned} \int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha &= \int_{\mathfrak{m}(P^{\theta})} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha + \int_{\mathfrak{m}(Q) \setminus \mathfrak{m}(P^{\theta})} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha \\ &\ll P^{s-k} (P^{\theta})^{\varepsilon - 2|\Delta_s^*|/k} + \int_{\mathfrak{m}(Q) \setminus \mathfrak{m}(P^{\theta})} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha \\ &\ll P^{s-k} Q^{-\nu} + \int_{\mathfrak{M}(P^{\theta}) \setminus \mathfrak{M}(Q)} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha. \end{aligned} \tag{6.2}$$

When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $0 \leqslant a \leqslant q \leqslant \frac{1}{2} P^{k/2}$ and $(a, q) = 1$, the intervals $\mathfrak{M}_{q,a}(\frac{1}{2} P^{k/2}, P)$ are disjoint, and for $\alpha \in \mathfrak{M}_{q,a}(\frac{1}{2} P^{k/2}, P)$ we put

$$\Upsilon(\alpha) = (q + P^k |q\alpha - a|)^{-1}.$$

Meanwhile, for $\alpha \in [0,1) \setminus \mathfrak{M}(\frac{1}{2} P^{k/2}, P)$ we put $\Upsilon(\alpha) = 0$. This defines a function $\Upsilon : [0,1) \to [0,1]$. By [14, Lemma 7.2], we find that when

$$2 \leqslant R \leqslant M \leqslant P, \quad |q\alpha - a| \leqslant M/(k(2P)^k R) \quad \text{and} \quad (a, q) = 1,$$

one has

$$f(\alpha; P, R) \ll q^{\varepsilon} L^3 \left(P \Upsilon(\alpha)^{1/(2k)} + (PMR)^{1/2} + q^{1/4} P (R/M)^{1/2} \right)$$

where L is defined by (2.1). But on taking $M = P^{(2+\theta)/4}$ and recalling (6.1), we see that when $q \leqslant P^{\theta}$ one has

$$q^{\varepsilon} L^3 \left((PMR)^{1/2} + q^{1/4} P (R/M)^{1/2} \right) \ll P^{\frac{3}{4} + \frac{\theta}{8} + \varepsilon} R \ll P^{1 - 1/(2k)}.$$

It follows that whenever $\alpha \in \mathfrak{M}(P^{\theta}, P)$, one has the bound

$$f(\alpha; P, R) \ll P L^3 \Upsilon(\alpha)^{-\varepsilon + 1/(2k)} + P^{1 - \tau(k) + \varepsilon}. \tag{6.3}$$

We now put $s = t + v$, where t and v are chosen in accordance with the definition (5.3) of Δ_s^*, just as in the proof of Theorem 5.2. Thus, by substituting (6.3) into (6.2), we obtain the bound

$$\int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \,\mathrm{d}\alpha \ll P^{s-k} Q^{-\nu} + P^{\varepsilon} T_1 + (PL^3)^t T_2, \tag{6.4}$$

where

$$T_1 = \left(P^{1-\tau(k)}\right)^t \int_0^1 |f(\alpha; P, R)|^v \, \mathrm{d}\alpha \tag{6.5}$$

and

$$T_2 = \int_{\mathfrak{M}(P^\theta)\setminus\mathfrak{M}(Q)} \Upsilon(\alpha)^{-\varepsilon+t/(2k)} |f(\alpha; P, R)|^v \, \mathrm{d}\alpha. \tag{6.6}$$

As in the proof of Theorem 5.3, it is apparent from (6.5) that

$$T_1 \ll \left(P^{1-\tau(k)}\right)^t P^{v-k+\Delta_v+\varepsilon} \ll P^{s-k-|\Delta_s^*|+\varepsilon}.$$

Thus we obtain the estimate

$$P^\varepsilon T_1 \ll P^{s-k} Q^{-\nu}. \tag{6.7}$$

Meanwhile, an application of Hölder's inequality to (6.6) reveals that

$$T_2 \leqslant T_3^{(v-2)/(s-2)} T_4^{t/(s-2)}, \tag{6.8}$$

where

$$T_3 = \int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha$$

and

$$T_4 = \int_{\mathfrak{M}(P^\theta)\setminus\mathfrak{M}(Q)} \Upsilon(\alpha)^{-\varepsilon+(s-2)/(2k)} |f(\alpha; P, R)|^2 \, \mathrm{d}\alpha. \tag{6.9}$$

On substituting (6.7) and (6.8) into (6.4), we obtain the estimate

$$T_3 \ll P^{s-k} Q^{-\nu} + (PL^3)^t T_3^{1-t/(s-2)} T_4^{t/(s-2)},$$

whence

$$\int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha \ll P^{s-k} Q^{-\nu} + (PL^3)^{s-2} T_4. \tag{6.10}$$

Thus it remains only to bound the mean value T_4.

When $\alpha \in \mathbb{R}$, it follows from Dirichlet's approximation theorem that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $q \leqslant Q^{-1}P^k$ and $|q\alpha - a| \leqslant QP^{-k}$. When $\alpha \in \mathfrak{M}(P^\theta) \setminus \mathfrak{M}(Q)$, moreover, one has $q + P^k|q\alpha - a| > Q$, and hence $\Upsilon(\alpha) < Q^{-1}$. We therefore deduce from (6.9) that when $s \geqslant 2k+3$, we have the bound

$$T_4 \ll Q^{\varepsilon-1/(4k)} \int_{\mathfrak{M}(P^\theta)} \Upsilon(\alpha)^{1+1/(4k)} |f(\alpha; P, R)|^2 \, \mathrm{d}\alpha.$$

The mean value on the right hand side here is amenable to [10, Lemma 11.1], a pruning lemma that refines earlier work of the first author [1, Lemma 2]. Thus, we obtain the estimate $T_4 \ll Q^{\varepsilon-1/(4k)} P^{2-k}$. After substituting this bound into (6.10), we infer that

$$\int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha \ll P^{s-k} Q^{-\nu} + Q^{\varepsilon-1/(4k)} L^{3s} P^{s-k}.$$

The desired conclusion therefore follows provided that $Q > L^{60ks}$, since then

$$L^{3s} Q^{\varepsilon-1/(4k)} \leqslant Q^{\varepsilon-1/(5k)} \left(L^{60ks} Q^{-1}\right)^{1/(20k)} < Q^{-\nu}.$$

At this point we are reduced to the scenario in which one has $Q \leqslant L^{60ks}$. In this range for Q, we appeal to [14, Lemma 8.5]. Let $A > 0$ be fixed. Then the latter lemma shows that when $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $q \leqslant L^A$, one has the upper bound

$$f(\alpha; P, R) \ll P\Upsilon(\alpha)^{-\varepsilon+1/k} + P\exp\left(-c(\log P)^{1/2}\right)(1 + P^k|\alpha - a/q|),$$

in which $c = c(A) > 0$. When $\alpha \in \mathfrak{M}(L^{60ks}) \setminus \mathfrak{M}(Q)$, one has

$$Q < q + P^k|q\alpha - a| \leqslant 2L^{60ks}.$$

In such circumstances, therefore, we have

$$f(\alpha; P, R) \ll P\Upsilon(\alpha)^{-\varepsilon+1/k} + PL^{-60ks} \ll P\Upsilon(\alpha)^{1/(2k)}Q^{-1/(3k)}.$$

Write

$$T_5 = \int_{\mathfrak{M}(L^{60ks})\setminus\mathfrak{M}(Q)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha.$$

Then we deduce that when $s \geqslant 2k + 3$, one has

$$\begin{aligned} T_5 &\ll (PQ^{-1/(3k)})^{s-2} \int_{\mathfrak{M}(L^{60ks})} \Upsilon(\alpha)^{(s-2)/(2k)}|f(\alpha; P, R)|^2 \, \mathrm{d}\alpha \\ &\ll P^{s-2}Q^{-1/2} \int_{\mathfrak{M}(L^{60ks})} \Upsilon(\alpha)^{1+1/(2k)}|f(\alpha; P, R)|^2 \, \mathrm{d}\alpha. \end{aligned}$$

Observe that $\mathfrak{m}(Q) \setminus \mathfrak{m}(L^{60ks}) = \mathfrak{M}(L^{60ks}) \setminus \mathfrak{M}(Q)$. Then, again employing [10, Lemma 11.1], we conclude that

$$\int_{\mathfrak{m}(Q)\setminus\mathfrak{m}(L^{60ks})} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha = T_5 \ll P^{s-k}Q^{-1/2}.$$

Hence, on applying the conclusion of the theorem already established when $Q \geqslant L^{60ks}$, we obtain

$$\begin{aligned} \int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha &= \int_{\mathfrak{m}(L^{60ks})} |f(\alpha; P, R)|^s \, \mathrm{d}\alpha + T_5 \\ &\ll P^{s-k}(L^{60ks})^{-\nu} + P^{s-k}Q^{-1/2} \\ &\ll P^{s-k}Q^{-\nu}. \end{aligned}$$

The conclusion of the theorem therefore follows also in this last case with $1 \leqslant Q \leqslant L^{60ks}$, and thus the proof of the theorem is complete. □

We are now equipped to bound the quantity $G(k)$ relevant to Waring's problem. We assume that we have available an admissible exponent Δ_u for each positive number u. Then, when $k \geqslant 4$, we define $\tau(k)$ as in (5.1), and we also put

$$G_0(k) = \min_{v \geqslant 2} \left(v + \frac{\Delta_v}{\tau(k)} \right). \tag{6.11}$$

Also, when $s \in \mathbb{N}$, we write $R_{s,k}(n)$ for the number of solutions of the equation

$$x_1^k + \ldots + x_s^k = n, \tag{6.12}$$

with $x_i \in \mathbb{N}$.

Theorem 6.2. *Suppose that $k \geqslant 4$ and $s \geqslant \max\{\lfloor G_0(k)\rfloor + 1, 2k+3\}$. Then provided that the integer n is sufficiently large in terms of k and s, and for each natural number q the congruence*

$$x_1^k + \ldots + x_s^k \equiv n \pmod q$$

possesses a solution with $(x_1, q) = 1$, one has $R_{s,k}(n) \gg n^{s/k-1}$. In particular, when k is not a power of 2 one has $G(k) \leqslant \max\{\lfloor G_0(k)\rfloor + 1, 2k+3\}$, and when k is a power of 2 one has instead $G(k) \leqslant \max\{\lfloor G_0(k)\rfloor + 1, 4k\}$.

Proof. We first address the claimed asymptotic lower bound $R_{s,k}(n) \gg n^{s/k-1}$, the final conclusions of the theorem following from the standard theory associated with local solubility in Waring's problem (see [13, Theorem 4.6], for example). Consider a natural number n sufficiently large in terms of k and s. Let $P = n^{1/k}$ and $R = P^\eta$, where $\eta > 0$ is sufficiently small, in a manner to be specified in due course. We denote by $r_{s,k}(n)$ the number of representations of n in the form (6.12) with $x_i \in \mathscr{A}(P,R)$ $(1 \leqslant i \leqslant s)$, so that $R_{s,k}(n) \geqslant r_{s,k}(n)$. By orthogonality, one has

$$r_{s,k}(n) = \int_0^1 f(\alpha; P, R)^s e(-n\alpha)\,\mathrm{d}\alpha.$$

We put $Q = L^{1/15}$, and we specify η to be sufficiently small in the context of the (finitely many) admissible exponents that must be discussed in determining $\tau(k)$ and $G_0(k)$. We make use of a simplified Hardy-Littlewood dissection. Thus, we take $\mathfrak{K}$ to be the union of the arcs

$$\mathfrak{K}(q,a) = \{\alpha \in [0,1) : |\alpha - a/q| \leqslant QP^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$, and then put $\mathfrak{k} = [0,1) \setminus \mathfrak{K}$. Thus, by the triangle inequality, we have

$$r_{s,k}(n) = \int_{\mathfrak{K}} f(\alpha; P, R)^s e(-n\alpha)\,\mathrm{d}\alpha + O\biggl(\int_{\mathfrak{k}} |f(\alpha; P, R)|^s\,\mathrm{d}\alpha\biggr). \tag{6.13}$$

We first handle the contribution of the minor arcs $\mathfrak{k}$ within (6.13). Suppose that $s \geqslant \max\{\lfloor G_0(k)\rfloor + 1, 2k+3\}$, and recall (5.3) and (6.11). Then there exists a positive number v with $v \geqslant 2$ and an admissible exponent Δ_v for which the exponent Δ_s^* is admissible for minor arcs, where

$$\Delta_s^* = \Delta_v - (s-v)\tau(k) = -\tau(k)\,(s - G_0(k)) < 0.$$

Put $\nu = \min\{|\Delta_s^*|/k, 1/(18k)\}$. Then we see from Theorem 6.1 that

$$\int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s\,\mathrm{d}\alpha \ll P^{s-k}Q^{-\nu} = P^{s-k}L^{-\nu/15}.$$

Finally, since $\mathfrak{k} \subseteq \mathfrak{m}(Q)$, we may conclude thus far that

$$\int_{\mathfrak{k}} |f(\alpha; P, R)|^s\,\mathrm{d}\alpha \leqslant \int_{\mathfrak{m}(Q)} |f(\alpha; P, R)|^s\,\mathrm{d}\alpha \ll P^{s-k}L^{-\nu/15}. \tag{6.14}$$

Next we attend to the contribution of the major arcs $\mathfrak{K}$. Suppose that $\alpha \in \mathfrak{K}(q,a) \subseteq \mathfrak{K}$. The standard theory of smooth Weyl sums (see [12, Lemma 5.4]) shows that there is a positive number $c = c(\eta)$ such that

$$f(\alpha; P, R) = cq^{-1}S(q,a)v(\alpha - a/q) + O(PL^{-1/4}),$$

wherein

$$S(q,a)=\sum_{r=1}^{q} e(ar^k/q) \quad \text{and} \quad v(\beta)=\frac{1}{k}\sum_{m\leqslant n} m^{-1+1/k}e(\beta m).$$

Since $\mathfrak{K}$ has measure $O(Q^3n^{-1})$, we see that

$$\int_{\mathfrak{K}} f(\alpha;P,R)^s e(-n\alpha)\,\mathrm{d}\alpha = c^s\mathfrak{J}(n,Q)\mathfrak{S}(n,Q)+O(P^{s-k}Q^3L^{-1/4}), \tag{6.15}$$

where

$$\mathfrak{J}(n,X)=\int_{-X/n}^{X/n} v(\beta)^s e(-\beta n)\,\mathrm{d}\beta$$

and

$$\mathfrak{S}(n,X)=\sum_{1\leqslant q\leqslant X}\sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-s}S(q,a)^s e(-na/q).$$

Notice that since $Q=L^{1/15}$, the error term in (6.15) is $O(P^{s-k}L^{-1/20})$. Familiar estimates from the theory of Waring's problem (see [13, Chapters 2 and 4]) show that under the hypotheses on s at hand,

$$\mathfrak{S}(n,X)=\mathfrak{S}(n)+O(X^{-1/k}),$$

where

$$\mathfrak{S}(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-s}S(q,a)^s e(-na/q).$$

Thus, in particular, subject to the hypotheses of the statement of the theorem, one has $\mathfrak{S}(n,X)\gg 1$. Likewise, one finds that

$$\mathfrak{J}(n,X)=\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}n^{s/k-1}+O(n^{s/k-1}X^{-1/k}).$$

Hence, again under the hypotheses of the statement of the theorem, we deduce from (6.15) that

$$\int_{\mathfrak{K}} f(\alpha;P,R)^s e(-n\alpha)\,\mathrm{d}\alpha = c^s\mathfrak{S}(n)\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}n^{s/k-1}+o(n^{s/k-1}). \tag{6.16}$$

On substituting (6.14) and (6.16) into (6.13), we conclude that

$$r_{s,k}(n)=c^s\mathfrak{S}(n)\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}n^{s/k-1}+o(n^{s/k-1}),$$

whence $R_{s,k}(n)\geqslant r_{s,k}(n)\gg n^{s/k-1}$. This completes the proof of the asymptotic lower bound asserted in the statement of the theorem, subject of course to the associated hypotheses on s, and, when $s<4k$ and k is a power of 2, the hypothesis on local solubility. Since we have already confirmed the remaining assertions of the theorem, subject to validity of this asymptotic lower bound, the proof of the theorem is complete. □

7. The proofs of Theorems 1.1 and 1.2

The proof of our main theorems using Theorem 6.2 is relatively routine, involving an optimisation of parameters. We first compute the Weyl-type exponent $\tau(k)$ defined in (5.1). This is essentially the optimisation performed in the proofs of Corollaries 1 and 2 to [21, Theorem 1.1].

We begin by observing that whenever v is even, then the corollary to [21, Theorem 2.1] shows that the exponent Δ_v is admissible for $k \geqslant 4$, where Δ_v is the unique positive solution of the equation

$$\Delta_v e^{\Delta_v/k} = ke^{1-v/k}. \tag{7.1}$$

Notice here that the exponent Δ_s in the statement of this earlier result corresponds to our Δ_v with $v = 2s$, owing to the slightly different definitions employed between [21] and the present memoir. Equipped with these exponents, we now seek to obtain a good approximation to

$$k\tau(k) = \max_{w\in\mathbb{N}} \frac{1 - 2\Delta_{2w}/k}{4w^2/k^2}. \tag{7.2}$$

We explore this quantity by putting $w = \lceil \gamma k \rceil$, where $\gamma > 0$ is a real parameter at our disposal. With the relation (7.1) in mind, we take $\delta = \delta(\gamma)$ to be the positive solution of the equation

$$\delta + \log \delta = 1 - 2\gamma. \tag{7.3}$$

We note that the function $t + \log t$ is increasing for $t > 0$. Then, since the relation (7.1) shows that the exponent Δ_{2w} is admissible, where Δ_{2w} is the unique positive solution of the equation

$$\frac{\Delta_{2w}}{k} + \log \frac{\Delta_{2w}}{k} = 1 - \frac{2w}{k},$$

and $1 - 2w/k \leqslant 1 - 2\gamma$, we infer that $\Delta_{2w} \leqslant k\delta(\gamma)$. We now define $\theta = \theta(\gamma, w)$ by setting $\theta = w - \gamma k$. Thus $0 \leqslant \theta < 1$, and we see that the formula (7.2) delivers the lower bound

$$k\tau(k) \geqslant \max_{\gamma>0} \frac{1 - 2\delta(\gamma)}{4(\gamma + \theta/k)^2}. \tag{7.4}$$

One may now attempt to optimise the choice of γ on the right hand side of (7.4) so as to maximise our lower bound for $\tau(k)$. It transpires that the optimal choice for γ is very close to 1, and so a good approximation to the maximum is found by taking $\gamma = 1$ and hence $\theta = 0$. Solving (7.3) with $\gamma = 1$, it is apparent that δ is constrained to satisfy the equation

$$\delta + \log \delta + 1 = 0.$$

It is not difficult via a Newton iteration to verify that $\delta = 0.2784645\ldots$. With this value of δ, one has

$$k\tau(k) \geqslant \frac{1 - 2\delta}{4} = \frac{1}{9.027900\ldots}. \tag{7.5}$$

Asymptotic information very slightly superior to the lower bound (7.5) is obtained by observing that since $0 \leqslant \theta < 1$, the relation (7.4) yields

$$k\tau(k) \geqslant \max_{\gamma>0} \frac{1 - 2\delta(\gamma)}{4(\gamma + 1/k)^2}.$$

The maximum here corresponds to a value of γ for which

$$\frac{4(\gamma+1/k)^2}{1-2\delta(\gamma)}$$

achieves its minimum. On making use of (7.3) to eliminate γ and substituting ξ for $\delta(\gamma)$, we find that this minimum value is equal to the minimum of the function

$$\kappa(\xi)=\frac{(1-\xi-\log\xi+2/k)^2}{1-2\xi},$$

as ξ varies over the interval $(0,1)$, and that the minimising value of $\delta(\gamma)$ is then equal to the value of ξ corresponding to this minimum. Identifying the value of ξ where $\kappa'(\xi)=0$, we see that ξ satisfies the equation

$$\xi-\frac{1}{\xi}+2+\frac{2}{k}=\log\xi.$$

Thus, if $\omega=3.548292\ldots$ is the positive real number with $\omega\geqslant 1$ satisfying the equation (1.1), namely $\omega-2-1/\omega=\log\omega$, then we find that $\xi=1/\omega+O(1/k)$. We should therefore take δ asymptotically close to $1/\omega$ for large k.

Motivated by this discussion, and recalling the relation (7.3), we put

$$\gamma=\tfrac{1}{2}(1-1/\omega+\log\omega)=0.992320\ldots,$$

and we avoid adjusting this value by the term of size $O(1/k)$ corresponding to the optimal choice. With this very slightly non-optimal choice of γ, we find that

$$k\tau(k)\geqslant\frac{1-2\delta(\gamma)}{4(\gamma+1/k)^2}.$$

Here, in view of (7.3), one has

$$\delta+\log\delta=1-2\gamma=\frac{1}{\omega}+\log\frac{1}{\omega},$$

whence $\delta=1/\omega$. Thus

$$k\tau(k)\geqslant\frac{1-2/\omega}{(1-1/\omega+\log\omega+2/k)^2}=\frac{1}{9.026725\ldots}+O\Big(\frac{1}{k}\Big).$$

We summarise these deliberations in the form of a lemma.

Lemma 7.1. *When $k\geqslant 4$, one has*

$$\tau(k)\geqslant\frac{1}{9.027901k},$$

and also

$$\tau(k)\geqslant\frac{1-2/\omega}{(1-1/\omega+\log\omega+2/k)^2k},$$

where ω is the unique real solution with $\omega\geqslant 1$ of the equation

$$\omega-2-1/\omega=\log\omega.$$

We remark that, following a modest computation, one can confirm that the second lower bound for $\tau(k)$ delivered by this lemma takes the asymptotic form

$$\tau(k) \geqslant \frac{1}{(\omega^2 - 3 - 2/\omega)k} + O\Big(\frac{1}{k^2}\Big).$$

We may now make use of Theorem 6.2, where we must consider the quantity

$$G_0(k) = \min_{v \geqslant 2}\left(v + \frac{\Delta_v}{\tau(k)}\right).$$

Write $\tau(k) = (Dk)^{-1}$, where D may depend on k, but is asymptotic to a constant determined via the conclusion of Lemma 7.1. Given a positive even integer v, we take δ to be the real number with $0 < \delta < 1$ satisfying the equation $\delta + \log\delta = 1 - v/k$. Then, in view of the equation (7.1), one has $\Delta_v = k\delta$, and hence also $\Delta_v = ke^{1-\delta-v/k}$. Consequently,

$$G_0(k) \leqslant v + \frac{\Delta_v}{\tau(k)} \leqslant v + Dk^2e^{1-\delta-v/k}. \tag{7.6}$$

As a corresponding inequality in a real variable v, the right hand side is approximately minimised by taking $v = k(1 + \log(Dk))$. Instead, with v constrained to be an even integer, we take

$$v = 2\left\lfloor \frac{1}{2}k(1 + \log(Dk)) - \frac{1}{2D}\right\rfloor.$$

In this way, one finds that

$$\delta + \log\delta \geqslant 1 - (1 + \log(Dk)) + \frac{1}{Dk} = \frac{1}{Dk} + \log\Big(\frac{1}{Dk}\Big),$$

whence $\delta \geqslant 1/(Dk)$.

Define the real number θ via the relation

$$v = k(1 + \log(Dk)) - \frac{1}{D} - \theta,$$

and note that one then has $0 \leqslant \theta < 2$. In this way, we discern that

$$\begin{aligned} 1 - \delta - \frac{v}{k} &\leqslant 1 - \frac{1}{Dk} - (1 + \log(Dk)) + \frac{1}{Dk} + \frac{\theta}{k} \\ &= -\log(Dk) + \frac{\theta}{k}. \end{aligned}$$

Then we deduce from (7.6) that

$$G_0(k) \leqslant v + \frac{\Delta_v}{\tau(k)} \leqslant k(1 + \log(Dk)) - \frac{1}{D} - \theta + ke^{\theta/k}. \tag{7.7}$$

The function $-\theta + ke^{\theta/k}$ is increasing with θ for $\theta \in [0, 2)$, so is bounded above in this interval by $-2 + ke^{2/k}$. Moreover, the function $k(e^{2/k} - 1)$ is decreasing as a function of k for $k \geqslant 2$. One may check that when $k \geqslant 20$, one has

$$-2 + ke^{2/k} \leqslant k + \frac{1}{9.6694} < k + \frac{1}{D}.$$

In such circumstances, we deduce that

$$-\frac{1}{D} - \theta + ke^{\theta/k} \leqslant -\frac{1}{D} - 2 + ke^{2/k} < k,$$

whence, as a consequence of (7.7), we obtain the bound

$$G_0(k) \leqslant v + \frac{\Delta_v}{\tau(k)} < k(2 + \log(Dk)).$$

In this way, we deduce that for $k \geqslant 20$, one has

$$G_0(k) \leqslant k(\log k + 2 + \log D). \tag{7.8}$$

The proof of Theorem 1.1. By reference to the first bound supplied by Lemma 7.1, one finds that the argument just described may be applied with $D = 9.027901$ whenever $k \geqslant 20$. In such circumstances, one has $2 + \log D \leqslant 4.2003199$, and hence it follows from (7.8) that

$$\lfloor G_0(k) \rfloor \leqslant \lfloor k(\log k + 4.2003199) \rfloor \leqslant \lceil k(\log k + 4.20032) \rceil - 1.$$

The proof of Theorem 1.1 when $k \geqslant 20$ is therefore made complete by reference to Theorem 6.2. For small values of k, one finds that the bounds for $G(k)$ already available in the literature are smaller than $\lceil k(\log k + 4.20032) \rceil$ for $k \leqslant 19$. Indeed, the bound $G(k) \leqslant 2^k + 1$ available via Hua's work (see the corollary to [13, Theorem 2.6], for example) already suffices for $k \leqslant 4$, while for $k \geqslant 14$ one has the bounds already reported in the introduction following the announcement of Theorem 1.3. We can complete this list with the addition of the bounds $G(7) \leqslant 31$, $G(8) \leqslant 39$, $G(9) \leqslant 47$, $G(10) \leqslant 55$, $G(11) \leqslant 63$, $G(12) \leqslant 72$, $G(13) \leqslant 81$, available from [24], together with the bounds $G(5) \leqslant 17$ and $G(6) \leqslant 24$ obtained, respectively, in [16] and [15]. Following this small list of checks, the proof of Theorem 1.1 is complete. □

We note that the bound supplied by Theorem 1.1 is surprisingly competitive even for small values of k. Thus, for example, the bound $G(20) \leqslant 144$ of Theorem 1.1 may be compared with the corresponding bound $G(20) \leqslant 142$ of [17]. Of course, in Theorem 1.3 of the present memoir, we obtain $G(20) \leqslant 137$.

The proof of Theorem 1.2. We now apply the second bound supplied by Lemma 7.1. With this bound in hand, the argument leading to (7.8) may be applied with

$$D = \frac{(\omega - 1 - 2/\omega + 2/k)^2}{1 - 2/\omega},$$

again, whenever $k \geqslant 20$. On recalling the definition (1.2) of C_1 and C_2, we now have

$$\begin{aligned} 2 + \log D &= 2 + \log\Big(\omega^2 - 3 - \frac{2}{\omega}\Big) + 2\log\Big(1 + \frac{2}{k(\omega - 1 - 2/\omega)}\Big) \\ &< C_1 + \frac{4\omega}{k(\omega^2 - \omega - 2)} = C_1 + \frac{C_2 - 1}{k}. \end{aligned}$$

We therefore deduce from (7.8) that

$$G_0(k) + 1 < k(\log k + C_1 + (C_2 - 1)/k) + 1 = k(\log k + C_1) + C_2.$$

The proof of Theorem 1.2 is completed by reference to Theorem 6.2 when $k \geqslant 20$. For the small values of k with $k \leqslant 19$, the bound claimed in the statement of Theorem 1.2 is again confirmed by reference to the previously known upper bounds for $G(k)$ already cited in the proof of Theorem 1.1. □

8. Bounding $G(k)$ for intermediate values of k

Our proof of Theorem 1.3 follows the argument used to establish Theorems 1.1 and 1.2, save that we now make use of the numerical tables of exponents available from [17]. We begin by numerically computing the exponent $\tau(k)$.

Theorem 8.1. *When* $14 \leqslant k \leqslant 20$*, one has* $\tau(k) \leqslant T(k)^{-1}$*, where the exponents* $T(k)$ *are presented in Table 2.*

Proof. We apply the formula

$$T(k) = \Big(\frac{k - 2\Delta_{2w}}{4w^2}\Big)^{-1},$$

available from (5.1), using the values of w and corresponding admissible exponents Δ_{2w} to be found in the tables of [17]. Here, the exponents λ_w of [17] are related to Δ_{2w} via the formula $\Delta_{2w} = \lambda_w - 2w + k$. We record the necessary choice of parameter w, together with the associated admissible exponent Δ_{2w}, rounded up in the final decimal place, in Table 2 below. $\square$

k	$2w$	Δ_{2w}	$T(k)$	v	Δ_v	$G_0(k)$
14	26	4.039939	114.1869	76	0.109356	88.4871
15	28	4.323087	123.3903	82	0.117123	96.4519
16	30	4.606286	132.5981	90	0.108806	104.4275
17	32	4.888677	141.7763	96	0.116203	112.4749
18	34	5.170691	150.9411	104	0.109619	120.5461
19	36	5.451758	160.0695	110	0.116770	128.6914
20	38	5.732224	169.1748	118	0.111388	136.8441

Table 2. Choice of exponents for $13 \leqslant k \leqslant 20$.

We next confirm Theorem 1.3 by utilising the formula $G(k) \leqslant \lfloor G_0(k) \rfloor + 1$ available via Theorem 6.2. Here, we have

$$G_0(k) = v + \frac{\Delta_v}{\tau(k)} = v + T(k)\Delta_v,$$

for a suitably chosen value of v. We present values of v, Δ_v and $G_0(k)$ in Table 2, with the values Δ_v extracted from [17], again all rounded up in the final decimal place presented. The conclusion of Theorem 1.3 follows on noting that $G(k) \leqslant \lfloor G_0(k) \rfloor + 1$ for each value of k in the table. This completes the proof of Theorem 1.3.

9. Remarks on upper bounds for $G^+(k)$

Scholars of the circle method as it applies to Waring's problem will appreciate instantly that the methods of this paper deliver bounds for the number $G^+(k)$, the smallest number s having the property that almost all positive integers (in the sense of natural density) are the sum of at most s positive integral k-th powers. Here, one makes a standard application of Bessel's inequality to estimate the minor arc contribution in mean square, the upshot being the familiar upper bound $G^+(k) \leqslant \frac{1}{2}(H(k)+1)$, whenever $H(k)$ is an upper bound for $G(k)$ obtained by the methods of this paper. The methods here have nothing to contribute to the literature well-known to any worker in the area, so we may record without further delay the following conclusions.

Theorem 9.1. *Suppose that* $k \in \mathbb{N} \setminus \{4, 8, 16, 32\}$. *Then*

$$G^+(k) \leqslant \lceil \tfrac{1}{2} k(\log k + 4.20032) \rceil$$

and

$$G^+(k) < \tfrac{1}{2} k(\log k + C_1) + \tfrac{1}{2}(C_2 + 1).$$

In the exceptional cases $k = 2^j$ *with* $j \in \{2, 3, 4, 5\}$, *one has* $G^+(k) = 4k$. *Moreover, when* $14 \leqslant k \leqslant 20$ *but* $k \neq 16$, *one has* $G^+(k) \leqslant H^+(k)$, *where* $H^+(k)$ *is defined by means of Table 3.*

k	14	15	16	17	18	19	20
$H^+(k)$	45	49	53	57	61	65	69

Table 3. Upper bounds for $G^+(k)$ when $14 \leqslant k \leqslant 20$.

The assertion that $G^+(k) = 4k$ when $k = 2^j$ with $j \in \{2, 3, 4, 5\}$ is not new. This was established by Hardy and Littlewood [5] when $k = 4$, by Vaughan [11] when $k = 8$, and by the second author [20] when $k = 16$ and $k = 32$. It is straightforward, however, to establish the following refinements that more fully reflect the entry $H^+(16) = 53$ from Table 3, and the upper bound implicitly obtained for $k = 32$ in Theorem 9.1.

Theorem 9.2. *Let* k *be either* 16 *or* 32, *and put* $H^+(16) = 53$ *and* $H^+(32) = 123$. *Suppose that* $s \geqslant H^+(k)$ *and that* r *is an integer with* $1 \leqslant r \leqslant s$. *Then almost all positive integers* n *with* $n \equiv r \pmod{4k}$ *are the sum of* s *positive integral* k*-th powers.*

The proof of this conclusion is once again routine for scholars of the circle method, and we refer the reader to earlier literature such as [11] or [20] for the ideas necessary to complete this exercise.

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