PARTITIO NUMERORUM: SUMS OF SQUARES AND HIGHER POWERS

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1. Introduction

Ever since the arrival of the circle method, its application to Waring's problem is considered the litmus test for the performance of newly introduced refinements. Until recently, the most efficient techniques were dependent on complete moment estimates for smooth Weyl sums [30, 37, 39]. In 2022, developing ideas of Liu and Zhao [24], we manufactured new moment estimates restricted to major arcs [7, 9]. In our bounds, the excess factor over the conjectured size shrinks with the height of the underlying Farey dissection. This is a considerable advantage in situations where estimates of Weyl's type on minor arcs of large height outperform those that one has at hand for classical minor arcs, the latter being defined as the complement of the range where Weyl sums can be evaluated by Poisson summation. This phenomenon is observed, in the current state of knowledge, for smooth Weyl sums, as we now explain.

We require some notation to provide a description in quantitative form. When $1 \leqslant R \leqslant P$, let $\mathscr{A}(P,R)$ denote the set of integers $n \in [1,P]$, all of whose prime divisors are at most R. Given an integer $k \geqslant 2$, let

$$f(\alpha;P,R) = \sum_{x\in\mathscr{A}(P,R)} e(\alpha x^k), \tag{1.1}$$

where $e(z)$ denotes $\mathrm{e}^{2\pi \mathrm{i} z}$. Slightly oversimplifying the situation, when k is large and R is a small power of P, one has the bound

$$f(a/q;P,R) \ll P^{1-1/(10k)} \tag{1.2}$$

whenever $(a,q)=1$ and q is of rough size $P^{k/2}$, corresponding to the slimmest possible and sensible choice of minor arcs. Such a conclusion is essentially contained in the proof of [38, Corollary 2 to Theorem 1.1]. In contrast, for classically defined minor arcs, a bound of the type $f(a/q;P,R) \ll P^{1-\sigma}$ that holds uniformly for $(a,q)=1$ and $P < q \leqslant P^{k/2}$, is available only when σ is approximately $1/(k\log k)$ (see [39, Theorem 1.1]). Our new major arc moments machinery carries the savings offered by the superior estimate (1.2) through a circle method approach to Waring's problem. For all $k \geqslant 14$, this led to new bounds for the least number $G(k)$ with the property that all large

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natural numbers are the sum of at most $G(k)$ positive integral k-th powers. In particular, by virtue of [7, Theorem 1.1], we now have

$$G(k) \leqslant \lceil k(\log k + 4.20032) \rceil.$$

This should be compared with the nearly thirty year old record

$$G(k) \leqslant k(\log k + \log\log k + 2 + o(1))$$

due to the second author (see [39, Theorem 1.4]).

In additive representation problems other than that of Waring, the extra savings that arise from restriction to extreme minor arcs can be more substantial. As an example, consider representations of natural numbers as the sum of a prime and s non-negative integral k-th powers, and let $\mathrm{P}(k)$ be the smallest s such that all large natural numbers admit a representation in the proposed manner. Here our new major arc mean value estimates show that $\mathrm{P}(k)/k$ remains bounded, and with more care [9, Theorem 1.1], one obtains the inequality

$$\mathrm{P}(k) \leqslant ck + 4 \quad (k \geqslant 3), \tag{1.3}$$

where c is the unique real number with $c > 1$ that satisfies the equation

$$2c = 2 + \log(5c - 1).$$

The decimal representation is $c = 2.134693\ldots$. As detailed in [9], the best previous estimate for $\mathrm{P}(k)$ implicit in the literature was the inequality

$$\mathrm{P}(k) \leqslant \tfrac{1}{2}k\left(\log k + \log\log k + 2 + o(1)\right). \tag{1.4}$$

In this problem, therefore, our new devices impact the order of magnitude of the number of k-th powers consumed by the method. In light of the preceding discussion, the reasons for this are transparent. If n is the number to be represented, the implementation of the circle method calls for $P = n^{1/k}$ in (1.1), and the prime appears via the exponential sum

$$g(\alpha) = \sum_{p \leqslant n} e(\alpha p) \log p. \tag{1.5}$$

Here and later, the letter p is reserved to denote a prime number. The extremal minor arc bound is $g(\alpha) \ll n^{4/5}(\log n)^4$ (see [31, Theorem 3.1]) while the uniform bound for $(a, q) = 1$ and $P < q \leqslant n^{1/2} = P^{k/2}$ merely gives

$$g(a/q) \ll nP^{-1/2}(\log n)^4 = n^{1-1/(2k)}(\log n)^4.$$

This last bound is so weak that proofs of (1.4) had to avoid Weyl type bounds for $g(\alpha)$ entirely. Equipped with our new major arc moments, however, one may exploit the full force of Weyl type bounds for $g(\alpha)$. In fact, Vinogradov's bound for exponential sums over primes combines with the simplest major arc moments to deliver a straightforward proof that $\mathrm{P}(k)/k$ is bounded (see [9, Section 5]). The more precise inequality (1.3) requires further ideas that will be discussed in Section 5 below.

In this survey, our goal is to illustrate the potential of major arc moment estimates for additive number theory. As we shall see, there are competing strategies to combine our moment estimates with other ideas, some of which also developed in our recent works [7, 9]. It is hoped that the article serves as a manual for the working number theorist wishing to apply the tool kit introduced in the latter sources. However, as already indicated, a complex interplay of several estimates calls for a new optimisation procedure in each concrete application. It seems hopeless to produce a blueprint for any conceivable future use of the ideas described below. We therefore concentrate on a single Diophantine equation, with some digressions concerning related questions. The topic chosen is that of representations by sums of one square and a number of k-th powers. This problem has a long history already, paralleling the developments with Waring's problem. Thus, for given natural numbers $k \geqslant 3$ and $s \geqslant 1$, let $r_{k,s}(n)$ denote the number of solutions of the Diophantine equation

$$x^2 + y_1^k + y_2^k + \ldots + y_s^k = n, \tag{1.6}$$

in non-negative integers x and y_j $(1 \leqslant j \leqslant s)$. In analogy with the function $G(k)$, we define $G_2(k)$ to be the smallest s with the property that for all large natural numbers n one has $r_{k,s}(n) \geqslant 1$.

The earliest noteworthy contributions to this circle of ideas are due to Stanley [27, Theorems 10, 11 and 12]. She followed the refined strategies of Hardy and Littlewood [15] for Waring's problem, and obtained a complicated upper bound for $G_2(k)$ that is asymptotically equivalent to the simpler inequality $G_2(k) \leqslant k2^{k-3}$, implied by her work. Inter alia, the proof of her Theorem 12 confirms the asymptotic formula for $r_{3,7}(n)$ that a more formal application of the circle method would predict. She subsequently also showed that $G_2(3) \leqslant 6$ (see [28, Theorem II]).

Prominent scholars have taken up this theme. The bound $G_2(3) \leqslant 5$ is due to Watson [36]. Sinnadurai [26] and Hooley [18] established the anticipated asymptotic formula for $r_{3,6}(n)$. Hooley [19, Theorem 4] deduced an asymptotic formula for $r_{3,5}(n)$ from the unproven hypothesis that the Riemann hypothesis is true for certain Hasse-Weil L-functions. At about the same time, Vaughan [29] obtained the lower bound $r_{3,5}(n) \gg n^{7/6}$ for large n. His result coincides with the conditional asymptotic formula in the order of magnitude. For results on $r_{k,s}(n)$ when k is larger we refer to Brüdern and Kawada [3, Theorems 1 and 2], but note that improvements are now routinely available via the mean value estimates of [45, Section 14]. These relatively recent results would combine with a mean value along the lines of Lemma 6.1 below to enable a competent worker to establish an asymptotic formula for $r_{k,s}(n)$ when $s \geqslant t_0(k)$, where in general

$$t_0(k) \leqslant \left\lceil \tfrac{1}{8}(5k^2 - 2k + 1) \right\rceil + \lfloor \sqrt{2k+2} \rfloor.$$

For smaller values of k, these recent improvements permit $t_0(k)$ to be taken as described in the table below. In this context we note that the entry $t_0(4) = 10$ corresponds to an older conclusion recorded in [3].

k	4	5	6	7	8	9	10	11	12
$t_0(k)$	10	15	21	30	39	51	64	77	91

Upper bounds on $G_2(k)$ are not that well documented in the literature, but it was part of the folklore that Vinogradov's work on Waring's problem yields the bound $G_2(k) \leqslant \big(1+o(1)\big)k\log k$. Incorporating more modern smooth number technology [39], the current benchmark should be considered to be the bound

$$G_2(k) \leqslant \tfrac{1}{2}k\big(\log k + \log\log k + O(1)\big).$$

Our first result reduces the order of magnitude of the upper bound to linear dependence on k. We are able to bound the number $s_0(k)$, defined as the smallest integer s with the property that the lower bound

$$r_{k,s}(n) \gg n^{\frac{s}{k}-\frac{1}{2}}$$

holds for all large n. An estimate for $G_2(k)$ is then available via the immediate relation

$$G_2(k) \leqslant s_0(k). \tag{1.7}$$

Theorem 1.1. *Let $k \geqslant 3$. Then*

$$s_0(k) \leqslant \lfloor c_0 k\rfloor + 2,$$

where $c_0 = \frac{3}{4} + 2\log 2 = 2.136294\ldots$. Moreover, one has

$$s_0(k) \leqslant 2k-1 \quad (3 \leqslant k \leqslant 6) \qquad \textit{and} \qquad s_0(k) \leqslant 2k \quad (7 \leqslant k \leqslant 11).$$

For $k=3$ this conclusion recovers the result of Vaughan [29], but our argument is rather different and yields the new result that for any $\eta > 0$, all large n have a representation in the form

$$n = x^2 + y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3$$

in positive integers x, y_j, with $y_j \in \mathscr{A}(P, P^\eta)$. For comparison, Vaughan imposes asymmetric multiplicative constraints on the cubes, and none of the cubes is smooth in his approach.

For $k \geqslant 4$ the results are new. The inequality $s_0(4) \leqslant 7$ deserves special attention because it cannot be reduced further. The following simple observation is the key step to realise this.

Lemma 1.2. *The isomorphism $T:\mathbb{R}^7 \to \mathbb{R}^7$, defined by putting*

$$T(x, y_1, \ldots, y_6) = (X, Y_1, \ldots, Y_6),$$

with $X = 4x$ and $Y_j = 2y_j$ $(1 \leqslant j \leqslant 6)$, restricts to a bijection between the integer solutions of the equations

$$x^2 + y_1^4 + \cdots + y_6^4 = n \tag{1.8}$$

and

$$X^2 + Y_1^4 + \cdots + Y_6^4 = 16n. \tag{1.9}$$

The proof is so simple that we present it at once. It is clear that T maps integer solutions of (1.8) to integer solutions of (1.9). Conversely, let $X, Y_1, \ldots, Y_6$ be a solution of (1.9). Interpreting this equation as a congruence modulo 16, it suffices to observe that the range of X^2 (mod 16) is $\{0,1,4,9\}$, and the range of Y_j^4 (mod 16) is $\{0,1\}$. It now transpires that for any solution of

$$X^2 + Y_1^4 + \cdots + Y_6^4 \equiv 0 \pmod{16}$$

we must have $2 \mid Y_j$ for $1 \leqslant j \leqslant 6$, and hence also $16 \mid X^2$. Thus, on putting $x = \frac{1}{4}X$, $y_j = \frac{1}{2}Y_j$ $(1 \leqslant j \leqslant 6)$, we obtain a solution of (1.8) that T maps to the solution of (1.9) with which we started. This proves Lemma 1.2.

Note that T respects the sign of all coordinates, whence $r_{4,6}(16n) = r_{4,6}(n)$. Repeated application of this relation shows that for all $l, n \in \mathbb{N}$ one has

$$r_{4,6}(16^l \cdot n) = r_{4,6}(n). \tag{1.10}$$

In particular, we see that $r_{4,6}(n)$ remains small on certain geometric progressions. The only solution of (1.8) with $n = 15$ is $x = 3$, $y_j = 1$ $(1 \leqslant j \leqslant 6)$, and so $r_{4,6}(16^l \cdot 15) = 1$ for all l. There are also sporadic natural numbers n_0 with $r_{4,6}(n_0) = 0$. The numbers below 200 with this property are 47, 62, 63, 77, 78, 79, 143, 158 and 159, as the reader may care to check. It would be very interesting to decide whether the list of such numbers n_0 that are not divisible by 16 is finite. From (1.10) we see that one has $r_{4,6}(16^l \cdot 47) = 0$ for all l, and hence $G_2(4) \geqslant 7$. This observation was also known to Stanley (see the remarks following the statement of [27, Theorem 3] in §4.3 of the latter source). In view of (1.7), therefore, Theorem 1.1 has the following corollary.

Theorem 1.3. *One has* $s_0(4) = G_2(4) = 7$.

It is a very rare event that, for a problem of Waring's type, the exact number of variables required to represent all large numbers is determined. Indeed, besides the new result in Theorem 1.3, the only other instances that are documented in the literature are $G(2) = 4$ (established by Lagrange in 1770) and $G(4) = 16$ (confirmed by Davenport [12] in 1939).

A simple variant of the Diophantine equation (1.6) is obtained by restricting the variable x to be a prime number. In this context, let $\widetilde{r}_{k,s}(n)$ be the number of representations of n in the form (1.6) with x prime and y_j natural numbers. Our result on this counting problem features the real number $\theta > 1$ that is the sole solution in this range of the equation $\theta - \log\theta = \frac{11}{8} + \log 4$, and more prominently the number $\widetilde{c} = \frac{1}{2}\theta + \frac{9}{16} + \log 2$. The decimal representation is $\widetilde{c} = 3.3532\ldots$.

Theorem 1.4. *Let* $k \geqslant 5$ *and* $s \geqslant \widetilde{c}k + 3$. *Then* $\widetilde{r}_{k,s}(n) \gg n^{\frac{s}{k} - \frac{1}{2}}(\log n)^{-1}$. *The same conclusion holds when* $8 \leqslant k \leqslant 12$ *and* $s \geqslant \widetilde{s}_0(k)$, *where* $\widetilde{s}_0(k)$ *is defined in the table below.*

k	8	9	10	11	12
$\widetilde{s}_0(k)$	24	27	31	35	38

The second clause of Theorem 1.4 is also valid for $3 \leqslant k \leqslant 7$ with $\widetilde{s}_0(3) = 6$, $\widetilde{s}_0(4) = 8$, $\widetilde{s}_0(5) = 12$, $\widetilde{s}_0(6) = 16$ and $\widetilde{s}_0(7) = 20$. However, these results follow routinely from familiar mean value estimates that have long been known (see Section 6 for comments on this matter).

Further results will be announced as the discourse progresses. The opening in Section 2 introduces a certain additive convolution where one factor is a fairly general arithmetic function, and the other factor is the counting function for the representations by sums of s smooth k-th powers. If the arithmetic function is the indicator function of the squares, or the squares of primes, then the convolution is a lower bound for $r_{k,s}(n)$ and $\widetilde{r}_{k,s}(n)$. In Section 3 we set the scene for the application of the circle method to the convolution sum, and we evaluate the contribution from the major arcs. The more original work starts in Section 4, where a pruning device is installed through the recently found major arcs moments [7, 9, 24]. This yields a lower bound for the convolution, subject to mild and natural conditions. We refer to Theorem 4.3 for a precise statement. Theorem 1.1 turns out to be an immediate corollary of this far more general result. In Sections 5 to 7 we discuss various refinements of the general approach that can be made if the first factor of the convolution carries arithmetic information. With our interest focussed on the equation (1.6), we concentrate mainly on arithmetic functions that are supported on the set of squares. Amongst other results, we give a proof of Theorem 1.4 in Section 6. In the penultimate section, we shall briefly point to further strategies one can try, but limitations of space prevent us from discussing details. The paper finishes with a short appendix concerned with an upper bound for a certain auxiliary exponential sum.

Notation is standard or otherwise explained in the course of the argument. We apply the familiar convention that whenever the letter ε occurs in a statement, then it is asserted that the statement is true for any positive number assigned to ε. Constants implicit in Vinogradov's or Landau's familiar symbols will depend on ε. From Section 4 onwards we apply the extended ε-R-η convention for smooth numbers. This is introduced in the initial segment of Section 4. Several results in the later sections depend on a certain hypothesis, referred to as Hypothesis H and specified in the introductory paragraph of Section 6.

2. A certain counting problem

In this section we build up some infrastructure to formulate a general additive counting problem that involves a number of k-th powers and a general sequence. First we fix natural numbers $k \geqslant 3$ and $s \geqslant 1$, once and for all. Further, we fix a real number η with $0 < \eta \leqslant 1$. Our main parameter is n, a large natural number. We then define

$$P = n^{1/k} \quad \text{and} \quad L = \log n. \tag{2.1}$$

For $0 \leqslant m \leqslant n$ let $\varrho(m)$ denote the number of solutions of the equation

$$m = x_1^k + x_2^k + \ldots + x_s^k$$

with $x_j \in \mathscr{A}(P, P^\eta)$. Next, fix an arithmetic function $w : \mathbb{N} \to \mathbb{C}$ that we shall refer to as the *weight*. Our principal object of study is the convolution sum

$$\nu(n) = \sum_{m \leqslant n} w(m)\varrho(n-m). \tag{2.2}$$

In particular, our goal is to derive lower bounds for $\nu(n)$ in the special case where w takes real non-negative values only. We would then like to choose s as small as is possible for the given weight. If we are successful for some $\eta > 0$, then thanks to the monotonicity of ν in η, the lower bound will be available also for larger values of η. Of course ϱ, and hence also ν, depend on k, s and η, and ν also on w, but we consider these parameters as 'frozen', and therefore suppress them in favour of concise notation here and elsewhere in the paper. We refer to the data k, s, η and w as the parameters.

In most but not all applications in this paper, the weight will be supported on the integral squares. For example, if we choose w as the indicator function of $\{l^2 : l \in \mathbb{N}\}$, then in view of (2.1) the convolution sum $\nu(n)$ is equal to the number of solutions of (1.6) in natural numbers x and y_j with $y_j \in \mathscr{A}(P, P^\eta)$ $(1 \leqslant j \leqslant s)$. In this case, therefore, we have

$$r_{k,s}(n) \geqslant \nu(n). \tag{2.3}$$

One certainly has to impose some severe restrictions on the weight w to even expect that $\nu(n)$ behaves somewhat regularly. We approach $\nu(n)$ via the circle method. Following the ideas developed in [9], we aim to explore the interplay between the k-th powers and the weight. In [9], the role of the weight was played by the primes, but it turns out that the arithmetic structure of the primes is of little relevance, since it is Vinogradov's pointwise estimate for the size of the trigonometric sum (1.5) that is important. In fact, it is possible to obtain a surprisingly powerful and fairly general result for weights where the associated exponential sum obeys an estimate that resembles a consequence of Weyl's inequality. We require a considerable amount of further notation to make this precise.

We work with a Farey dissection of the interval $[0,1]$ of order $2\sqrt{n}$. Let

$$0 \leqslant a \leqslant q \leqslant \tfrac{1}{2}\sqrt{n} \quad \text{and} \quad (a,q) = 1.$$

The intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1] : |q\alpha - a| \leqslant \tfrac{1}{2}n^{-1/2}\}$$

are disjoint. We denote their union by $\mathfrak{M}$. The set $\mathfrak{m} = [0,1] \setminus \mathfrak{M}$ is described as the extreme minor arcs in [7]. By Dirichlet's theorem on Diophantine approximation, for each $\alpha \in [0,1]$, one finds coprime numbers a and q with $1 \leqslant q \leqslant 2\sqrt{n}$ and $|q\alpha - a| \leqslant \frac{1}{2}n^{-1/2}$. Note that $\alpha \in \mathfrak{m}$ implies that $q > \frac{1}{2}\sqrt{n}$, whence $\mathfrak{m}$ is the disjoint union of certain subintervals, again denoted by $\mathfrak{M}(q,a)$, of the intervals

$$\{\alpha \in [0,1] : |q\alpha - a| \leqslant \tfrac{1}{2}n^{-1/2}\}$$

as a and q range over $1 \leqslant a \leqslant q$, $(a,q) = 1$ and $\frac{1}{2}\sqrt{n} < q \leqslant 2\sqrt{n}$. Making use of this notation, we define a function $\Upsilon : [0,1] \to [0,1]$ by taking

$$\Upsilon(\alpha) = (q + n|q\alpha - a|)^{-1}$$

when $\alpha \in \mathfrak{M}(q,a)$ and $0 \leqslant a \leqslant q \leqslant 2\sqrt{n}$, with $(a,q) = 1$. Note that one has the bounds

$$n^{-1/2} \ll \Upsilon(\alpha) \ll n^{-1/2}$$

uniformly for $\alpha \in \mathfrak{m}$.

With the weight w we associate the exponential sum

$$W(\alpha)=\sum_{m\leqslant n} w(m)e(\alpha m) \tag{2.4}$$

and the norm

$$\|W\|=\sum_{m\leqslant n}|w(m)|.$$

We record here that for real non-negative weights w one has $\|W\|=W(0)$.

Given a positive number ϕ, the weight w is called a ϕ-weight if the estimate

$$W(\alpha)\ll \|W\|\Upsilon(\alpha)^{\phi-\varepsilon}$$

holds uniformly for $\alpha\in[0,1]$. If the weight w is the indicator function of a set $\mathscr{W}$, then we say that $\mathscr{W}$ is a ϕ-set.

We illustrate this concept with a number of examples. First, fix an integer h and consider the indicator function of the set of h-th powers of natural numbers. In this case, the exponential sum $W(\alpha)$ becomes the familiar Weyl sum

$$g_h(\alpha)=\sum_{x\leqslant n^{1/h}} e(\alpha x^h). \tag{2.5}$$

Our main concern is the set of squares, and here we quote the enhanced version of Weyl's inequality asserting that $g_2(\alpha)\ll n^{1/2}\Upsilon(\alpha)^{1/2}$ (see [32, Theorem 4]). This shows a little more than just that the squares form a $\frac{1}{2}$-set.

For larger h, the situation is more subtle. If $\alpha=\beta+a/q$, with $(a,q)=1$, $q\leqslant n^{1/h}$ and $|\beta|\leqslant q^{-1}n^{1/h-1}$, then it follows from Theorems 4.1 and 4.2, together with Lemma 2.8, of Vaughan [31] that

$$g_h(\beta+a/q)\ll (n/q)^{1/h}(1+n|\beta|)^{-1/h}. \tag{2.6}$$

If $\alpha\in[0,1]$ is not of the form where (2.6) applies, then the simplest version of Weyl's inequality (see [31, Lemma 2.4]) yields the bound

$$g_h(\alpha)\ll (n^{1/h})^{1-2^{1-h}+\varepsilon}. \tag{2.7}$$

Combining the last two estimates we arrive at the uniform upper bound

$$g_h(\alpha)\ll n^{1/h}\Upsilon(\alpha)^{(2^{2-h}-\varepsilon)/h}, \tag{2.8}$$

and see that the h-th powers form a $2^{2-h}/h$-set. Of course, the estimate (2.8) is much weaker than (2.6) in situations where the latter is applicable, but (2.8) coincides with (2.7) on the extreme minor arcs, and this delimits the size of ϕ in this case. It should be noted that Weyl's inequality alone, coupled with a familiar transference principle (see [42, Lemma 14.1]), yields a bound that is essentially (2.8), but inflated by a factor n^ε, and this is of no use here. Indeed, for a weight to be a ϕ-weight, the inequality $W(\alpha)\ll\|W\|\Upsilon(\alpha)^\phi$ may fail by a factor q^ε on the intervals $\mathfrak{M}(q,a)$, but not by a factor n^ε.

Work of Heath-Brown [17, Theorem 1] offers scope for improving the classical form of Weyl's inequality when $h\geqslant 6$. Relatively recent progress with Vinogradov's mean value theorem, meanwhile, also leads to improvements for $h\geqslant 6$, by combining [45, Corollary 1.3] and [31, Theorem 5.2], for example. If one applies more recent variants of the former idea for $h = 6$, and the latter for $h \geqslant 7$, to obtain a refinement of [31, Lemma 2.4], and then makes use of these substitutes for Weyl's inequality in the above argument, then one concludes as follows.

Lemma 2.1. *Let h be a natural number.*
(a) *If $2 \leqslant h \leqslant 5$, then the h-th powers form a $2^{2-h}/h$-set.*
(b) *The 6-th powers form a $\frac{1}{72}$-set.*
(c) *If $h \geqslant 7$, then the h-th powers form a $2/(h^2(h-1))$-set.*

Proof. The conclusions (a) and (c) are easy consequences of the argument outlined in the preamble to the statement of the lemma, and so we concentrate here on the proof of the assertion (b). Suppose then that $h = 6$. If $\alpha \in [0, 1]$ is in the form where (2.6) applies, then the desired conclusion is immediate. When $\alpha = \beta + a/q$, with $(a, q) = 1$, $q \leqslant n^{1/2}$ and $|\beta| \leqslant q^{-1}n^{-1/2}$, one has

$$n^{-1/2} \ll (q + qn|\beta|)^{-1} = \Upsilon(\alpha).$$

Thus, in the situation in which (2.6) does not apply, it follows from [43, Corollary 1.4] that

$$\begin{aligned} g_6(\beta + a/q) &\ll (n^{1/6})^{1+\varepsilon} \left(\Upsilon(\alpha)^{1/64} + (n^{-1/6}\Upsilon(\alpha))^{1/96}\right) \\ &\ll n^{1/6}\Upsilon(\alpha)^{1/72-\varepsilon}. \end{aligned}$$

The desired conclusion therefore holds in all circumstances. □

Our second example concerns the primes. Now taking $w(p) = \log p$ for primes p, and $w(n) = 0$ otherwise, the sum $W(\alpha)$ in (2.4) becomes the sum $g(\alpha)$ defined in (1.5). Here one may apply Vaughan's version of Vinogradov's estimate for exponential sums over primes and then apply the transference principle. This has been detailed in [9, Lemma 4.2] to the effect that

$$g(\alpha) \ll (n\Upsilon(\alpha)^{1/2} + n^{4/5})L^4. \tag{2.9}$$

This suggests the following result.

Lemma 2.2. *The arithmetic function ϖ defined by $\varpi(p) = \log p$ for primes p, and $\varpi(n) = 0$ otherwise, is a $\frac{2}{5}$-weight.*

It is easy to deduce Lemma 2.2 from (2.9). In fact, the bound (2.9) implies that $g(\alpha) \ll n\Upsilon(\alpha)^{2/5}$, except in those situations in which $\alpha = \beta + a/q$ with $(a, q) = 1$ and $q + qn|\beta| \leqslant L^{40}$. In these exceptional situations, one applies [31, Lemma 3.1] to confirm the bound

$$g(\alpha) \ll n\varphi(q)^{-1}(1 + n|\beta|)^{-1}, \tag{2.10}$$

in which $\varphi(q)$ denotes the Euler totient. This more than suffices to establish Lemma 2.2.

This last example can be developed in various directions. We are particularly interested in squares, and here the estimate of Ghosh [13, Theorem 2] for trigonometric sums over squares of primes combines with the major arc bounds of Kumchev [22, Theorem 3] and Hua [20, Lemmata 7.15 and 7.16] to conclude as follows.

Lemma 2.3. *The squares of primes form a* $\frac{1}{8}$*-set.*

Similarly, one may show that the h-th powers of primes are a ϕ-set, for some $\phi > 0$. Admissible values for ϕ can be read off from the work of Zhao [46, Lemmata 2.1 and 2.3] and Kumchev [22, Theorem 3], once again in combination with [20, Lemmata 7.15 and 7.16]. The results one obtains in this way are susceptible to considerable improvement if $h \geqslant 6$ and the implications of our modern understanding of Vinogradov's mean value are brought into play. Such improvements are recorded in [23, Lemma 2.2]. Thus, when $h \geqslant 3$, one may show that the h-th powers of primes are a $\phi(h)$-set, where $\phi(3) = 1/18$, $\phi(4) = 1/48$, $\phi(5) = 1/120$ and $\phi(h) = 2/(3h^2(h-1))$ $(h \geqslant 6)$.

It should also be noted that Vaughan's estimate for the exponential sum over primes holds, mutatis mutandis, for the exponential sum formed with the Möbius function (see [14, Theorem 2.1]). Therefore, by the argument that proves [9, Lemma 4.2], the sum

$$\mathrm{M}(\alpha) = \sum_{m \leqslant n} \mu(m) e(\alpha m)$$

may replace $g(\alpha)$ in (2.9). As a substitute for (2.10), we have recourse to Davenport's bound [11], asserting that for any $A > 1$ one has

$$\mathrm{M}(\alpha) \ll nL^{-A} \tag{2.11}$$

uniformly for $\alpha \in \mathbb{R}$. This demonstrates the following variant of Lemma 2.2.

Lemma 2.4. *The Möbius function is a* $\frac{2}{5}$*-weight.*

Many other weights, and in particular many indicator functions of sequences of polynomial growth, are ϕ-sets, for some $\phi > 0$. This includes the set of values of integer-valued polynomials, and the values of such polynomials at prime argument. It is also the case that classical multiplicative functions such as the divisor functions $\tau(m)$ and $\sigma(m)$, and Euler's totient $\varphi(m)$, are ϕ-weights for certain $\phi > 0$. Equipped with these examples, the reader may care to see a prototype of the results concerning the convolution ν that we have in mind. The following theorem can be presented at this stage, though the proof will be completed in Section 4 only. We refer to a non-negative weight w as *regular* if for all large natural numbers n one has

$$\sum_{m \leqslant n/2} w(m) \gg \sum_{m \leqslant n} w(m).$$

This extra condition is harmless and typically void for natural sequences of polynomial growth. In particular, all the concrete non-negative weights discussed above are regular.

Theorem 2.5. *Fix a set of parameters including a regular* ϕ*-weight, for some* $\phi \in (0, 1]$*, and put*

$$c_1(\phi) = 1 + \log 2 - \tfrac{1}{2}\phi - \log \phi. \tag{2.12}$$

Let $s_1(k)$ be the smallest even integer exceeding $c_1(\phi)k$, and let s be an integer with $s \geqslant s_1(k)$. Then, if k is not a power of 2, one has $\nu(n) \gg \|W\|n^{s/k-1}$. Meanwhile, if $k \geqslant 4$ is a power of 2, then the same conclusion holds subject to the additional hypothesis $s \geqslant 4k$.

The significance of this result is that the condition on s in this theorem is linear in k. Hitherto, results of this type have been within reach only when $\phi \geqslant 1$. In fact, when $\phi = 1$, it is easy to give a proof of Theorem 2.5 based on familiar pruning devices such as [1, Lemma 2]. Our new major arc moment estimates are key to establish such results for all positive ϕ. Note here also that one has $c_1(\phi)k < s_1(k) \leqslant c_1(\phi)k + 2$. Moreover, if $\phi \in (0, 1]$ is rational, then $c_1(\phi)$ is irrational, and so in this case we have $s_1(k) < c_1(\phi)k + 2$, whence $s_1(k) \leqslant \lfloor c_1(\phi)k \rfloor + 2$. Here, we take $\phi = \frac{1}{2}$ and recall that the squares form a $\frac{1}{2}$-set. On recalling the lower bound (2.3) and noting from (2.12) that one has $c_1\left(\frac{1}{2}\right) = \frac{3}{4} + \log 4$, we now see that Theorem 2.5 implies Theorem 1.1 in all cases where k is not a power of 2. It will turn out that the missing cases are covered by a more precise form of Theorem 2.5 to be presented in Section 4.

3. The circle method: initial steps

We continue to fix a set of parameters k, s, η and w. Further, from now on, we abbreviate $f(\alpha; P, P^\eta)$ to $f(\alpha)$. Then, by (2.2), (2.4) and orthogonality, one has

$$\nu(n) = \int_0^1 W(\alpha) f(\alpha)^s e(-\alpha n)\, \mathrm{d}\alpha. \tag{3.1}$$

For measurable sets $\mathfrak{a} \subset [0, 1]$ we write

$$\nu_{\mathfrak{a}}(n) = \int_{\mathfrak{a}} W(\alpha) f(\alpha)^s e(-\alpha n)\, \mathrm{d}\alpha, \tag{3.2}$$

so that $\nu(n) = \nu_{[0,1]}(n)$. It is routine to evaluate the major arc contribution to the integral (3.1). This will be possible with a mild lower bound on s relative to k, and no further condition on the parameters. The result features the singular series for sums of k-th powers, as it appears in the theory of Waring's problem. In this context, we introduce the Gauss sum

$$S(q, a) = \sum_{x=1}^{q} e(ax^k/q),$$

the auxiliary sum

$$A_m(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(q, a)^s e(-am/q),$$

and then define the formal singular series by

$$\mathfrak{S}(m) = \sum_{q=1}^{\infty} q^{-s} A_m(q). \tag{3.3}$$

Recall that for $s \geqslant 4$ and $m \in \mathbb{N}$ this series converges absolutely to a non-negative number (this is [31, Theorem 4.3]). When $m = 0$ the singular series still converges absolutely for $s \geqslant k + 2$. This follows from [31, Lemma 4.9]. Thus, in particular, uniformly for $m \geqslant 0$, one has

$$\mathfrak{S}(m) \ll 1. \tag{3.4}$$

The core major arcs $\mathfrak{K}$ are the union of the disjoint intervals

$$\mathfrak{K}(q, a) = \{\alpha \in [0, 1] : |\alpha - a/q| \leqslant L^{1/15}/n\}$$

with coprime integers a and q running over $0 \leqslant a \leqslant q \leqslant L^{1/15}$. The first step is to compute $\nu_{\mathfrak{K}}(n)$, and in preparation for this task, we evaluate the integral

$$I(n, m) = \int_{\mathfrak{K}} f(\alpha)^s e(-\alpha m) \, \mathrm{d}\alpha. \tag{3.5}$$

Lemma 3.1. *Fix a set of parameters with $s \geqslant k + 2$. Then there is a positive number $C = C_{k,s,\eta}$ with the property that, uniformly for $0 \leqslant m \leqslant n$, one has*

$$I(n, m) = C\mathfrak{S}(m)m^{s/k-1} + O(n^{s/k-1}L^{-1/(16k)}). \tag{3.6}$$

Proof. We write

$$v(\beta) = \frac{1}{k} \sum_{u \leqslant n} u^{1/k-1} e(\beta u).$$

Let $\alpha \in \mathfrak{K}$. Then, by [30, Lemma 5.4], there is a positive number $c = c(\eta)$ with the property that, whenever α is in an interval $\mathfrak{K}(q, a)$ that is part of the union that forms $\mathfrak{K}$, then one has

$$f(\alpha) = cq^{-1}S(q, a)v(\alpha - a/q) + O(PL^{-1/4}). \tag{3.7}$$

The trivial bound $|q^{-1}S(q, a)v(\beta)| \ll P$ suffices to conclude that

$$f(\alpha)^s = c^s q^{-s} S(q, a)^s v(\alpha - a/q)^s + O(P^s L^{-1/4}).$$

We multiply by $e(-\beta m)$ and integrate. Since the measure of $\mathfrak{K}$ is $O(L^{1/5}/n)$, we infer that

$$I(n, m) = c^s \sum_{q \leqslant L^{1/15}} q^{-s} A_m(q) \int_{-L^{1/15}/n}^{L^{1/15}/n} v(\beta)^s e(-\beta m) \, \mathrm{d}\beta + O(P^{s-k} L^{-1/20}). \tag{3.8}$$

Note that the sum and the integral in (3.8) disengage.

The sum in (3.8) is a partial sum of the series (3.3), and by a variant of the argument underlying [31, Lemma 4.9], the difference between these expressions is readily seen to be bounded by $O(L^{-1/(16k)})$. Similarly, the singular integral

$$\mathfrak{J}(n, m) = \int_{-1/2}^{1/2} v(\beta)^s e(-\beta m) \, \mathrm{d}\beta \tag{3.9}$$

differs from the integral on the right hand side of (3.8) by at most

$$2 \int_{L^{1/15}/n}^{1/2} |v(\beta)|^s \, \mathrm{d}\beta \ll P^s \int_{L^{1/15}/n}^{1/2} (1 + n|\beta|)^{-s/k} \, \mathrm{d}\beta \ll P^{s-k} L^{-1/(15k)}.$$

Here we have routinely applied [31, Lemma 2.8]. It now follows that

$$I(n,m) = c^s\big(\mathfrak{S}(m) + O(L^{-1/(16k)})\big)\big(\mathfrak{J}(n,m) + O(P^{s-k}L^{-1/(15k)})\big).$$

By (3.9) and orthogonality, recalling that $0 \leqslant m \leqslant n$, we find that

$$\mathfrak{J}(n,m) = \sum_{\substack{u_1+u_2+\cdots+u_s=m \\ 1\leqslant u_j \leqslant n}} (u_1u_2\cdots u_s)^{1/k-1} = \mathfrak{J}(m,m).$$

This shows that $\mathfrak{J}(n,0) = 0$. When $m > 0$, our expression $\mathfrak{J}(m,m)$ is the quantity $J_s(m) = J(m)$ evaluated in [31, Theorem 2.3]. With that result and the uniform upper bound (3.4), we arrive at the asymptotic relation (3.6) with $C = c(\eta)^s\Gamma(1+1/k)^s/\Gamma(s/k)$. □

By substituting (2.4) into (3.2) and recalling (3.5), we deduce from Lemma 3.1 that

$$\begin{aligned}\nu_{\mathfrak{K}}(n) &= \sum_{m\leqslant n} w(m)I(n,n-m) \\ &= C\sum_{m\leqslant n} w(m)\mathfrak{S}(n-m)(n-m)^{s/k-1} + O\left(\|W\|n^{s/k-1}L^{-1/(16k)}\right).\end{aligned} \tag{3.10}$$

In the important special case where the weight is non-negative, it is easy to extract a lower bound for $\nu_{\mathfrak{K}}(n)$ from the asymptotic relation (3.10). The following result suffices for most Diophantine applications.

Lemma 3.2. *Fix a set of parameters with $s \geqslant \frac{3}{2}k$ and a non-negative weight. If k is not a power of 2, then*

$$\nu_{\mathfrak{K}}(n) \gg n^{s/k-1}\bigg(\sum_{m\leqslant n/2} w(m) - O\big(W(0)L^{-1/(16k)}\big)\bigg). \tag{3.11}$$

If k is a power of 2, let $\mathscr{R}$ denote the union of residue classes $j \pmod{4k}$ with $1 \leqslant j \leqslant s$. Then

$$\nu_{\mathfrak{K}}(n) \gg n^{s/k-1}\bigg(\sum_{\substack{m\leqslant n/2 \\ n-m\in\mathscr{R}}} w(m) - O\big(W(0)L^{-1/(16k)}\big)\bigg). \tag{3.12}$$

Note here that for $s \geqslant 4k$ one has $\mathscr{R} = \mathbb{Z}$, and (3.12) reduces to (3.11). Also, note that for most values of k, the condition $s \geqslant \frac{3}{2}k$ can be relaxed (see the discussion of the function $\Gamma(k)$ by Hardy and Littlewood in [16]).

Proof. Omitting terms with $m > n/2$ from (3.10) neglects a non-negative contribution. For $m \leqslant n/2$ one has $(n-m)^{s/k-1} \gg n^{s/k-1}$. If k is not a power of 2, then the hypothesis $s \geqslant \frac{3}{2}k$ ensures that the lower bound $\mathfrak{S}(n-m) \gg 1$ holds uniformly for $m < n$ (see [31, Theorem 4.6]), and (3.11) follows. If k is a power of 2, then for $m < n$ one still has $\mathfrak{S}(n-m) \gg 1$ uniformly for $n - m \in \mathscr{R}$. This follows from [31, Lemma 2.13 and Theorem 4.5], observing that in this scenario the congruence

$$x_1^k + x_2^k + \cdots + x_s^k \equiv n - m \pmod{4k}$$

has a solution with $x_1 = 1$ and $x_j \in \{0, 1\}$ for $2 \leqslant j \leqslant s$. We now infer (3.12) in the same way as we arrived at (3.11). □

Although the preceding lemma identifies the expected lower bound for $\nu(n)$ correctly, the choice of core major arcs is extremely slim. The most flexible pruning devices based on our recent major arc moments lose a generic factor n^ε, and it is therefore desirable to work with major arcs of height a small power of n. We proceed to show that it is possible to extend the major arcs appropriately at low cost.

We begin by introducing some additional notation to augment the Hardy-Littlewood dissection of the unit interval $[0, 1]$ into major and minor arcs $\mathfrak{M}$ and $\mathfrak{m}$, with the associated individual arcs $\mathfrak{M}(q, a)$, as introduced in Section 2. Let Q be a parameter with $1 \leqslant Q \leqslant 2\sqrt{n}$. When $1 \leqslant Q \leqslant \frac{1}{2}\sqrt{n}$, the major arcs $\mathfrak{M}(Q)$ are defined to be the union of the sets

$$\mathfrak{M}(q, a; Q) = \{\alpha \in [0, 1] : |q\alpha - a| \leqslant QP^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a, q) = 1$. When instead $\frac{1}{2}\sqrt{n} < Q \leqslant 2\sqrt{n}$, we define $\mathfrak{M}(Q)$ to be the union of $\mathfrak{M}(\frac{1}{2}\sqrt{n})$ and the arcs $\mathfrak{M}(q, a)$ with $1 \leqslant a \leqslant q$, $(a, q) = 1$ and $\frac{1}{2}\sqrt{n} < q \leqslant Q$. Frequently, we make use of the truncated set of arcs $\mathfrak{N}(Q) = \mathfrak{M}(Q) \setminus \mathfrak{M}(Q/4)$. In this context, we note that $\mathfrak{N}(2\sqrt{n}) = \mathfrak{m}$. In this notation, we then see directly from the definition of Υ that for all Q under consideration and $\alpha \in \mathfrak{N}(Q)$, one has

$$\Upsilon(\alpha) \ll Q^{-1}. \tag{3.13}$$

We now take

$$\mathfrak{L} = \mathfrak{M}(P^{1/2})$$

as the extended set of major arcs. Note that $\mathfrak{M}(L^{1/15}) \subset \mathfrak{K}$, so that $\mathfrak{L} \setminus \mathfrak{K}$ is contained in the union of the sets $\mathfrak{N}(Q)$ as Q varies over numbers $Q = 4^{-j}P^{1/2}$ with $j \geqslant 0$ and

$$L^{1/15} < Q \leqslant P^{1/2}. \tag{3.14}$$

At this stage, we invoke our recent estimate [10, Corollary 1.4], yielding

$$\int_{\mathfrak{N}(Q)} |f(\alpha)|^t \,\mathrm{d}\alpha \ll P^{t-k}Q^{-\omega} \tag{3.15}$$

that is valid for $1 \leqslant Q \leqslant P^{1/2}$ and real numbers t and ω with

$$t > 2\lfloor k/2 \rfloor + 4 \quad \text{and} \quad \omega < \frac{t - 2\lfloor k/2 \rfloor - 4}{2k}.$$

It would now be possible to sum (3.15) over Q as in (3.14). For $s \geqslant 2\lfloor k/2 \rfloor + 5$ this would give the bounds

$$\int_{\mathfrak{L}\setminus\mathfrak{K}} |f(\alpha)|^s \,\mathrm{d}\alpha \ll P^{s-k}L^{-1/(31k)} \quad \text{and} \quad \int_{\mathfrak{L}} |f(\alpha)|^s \,\mathrm{d}\alpha \ll P^{s-k}. \tag{3.16}$$

However, the condition $s \geqslant 2\lfloor k/2 \rfloor + 5$ forces us to suppose that $s \geqslant 9$ when $k = 4$, for example, and this is not good enough to cover the case $k = 4$ of

Theorem 1.1. We therefore seek help from another pruning device. From [25, Lemma 11.1] we conclude that for $Q \leqslant P$ and $\theta > 1$ one has

$$\int_{\mathfrak{M}(Q)} \Upsilon(\alpha)^{\theta} |f(\alpha)|^2 \,\mathrm{d}\alpha \ll P^{2-k}. \tag{3.17}$$

Equipped with (3.15) and (3.17) we are able to establish the following estimates. Here, the set $\mathscr{R}$ is the same as that introduced in the statement of Lemma 3.2.

Theorem 3.3. *Fix a set of parameters involving a non-negative ϕ-weight w, for some $\phi \in (0,1]$. Suppose that $s \geqslant \frac{3}{2}k$, and that*

$$s > (1-\phi)(2\lfloor k/2 \rfloor + 4) + 2\phi. \tag{3.18}$$

If k is not a power of 2, then

$$\nu_{\mathfrak{L}}(n) \gg n^{s/k-1} \Big(\sum_{m \leqslant n/2} w(m) - o(W(0)) \Big).$$

Meanwhile, if k is a power of 2, then

$$\nu_{\mathfrak{L}}(n) \gg n^{s/k-1} \Big(\sum_{\substack{m \leqslant n/2 \\ n-m \in \mathscr{R}}} w(m) - o(W(0)) \Big).$$

Proof. Throughout, let Q be in the range specified in (3.14). Let δ be a real number with $0 < \delta < \phi$, and define the real number t through the equation

$$2\,\frac{\phi-\delta}{1+\delta} + t\left(1 - \frac{\phi-\delta}{1+\delta}\right) = s. \tag{3.19}$$

Note that t increases to $(s-2\phi)/(1-\phi)$ as δ shrinks to 0. Thus, in view of (3.18), we may choose δ to ensure that t is larger than $2\lfloor k/2 \rfloor + 4$. By (3.19) and Hölder's inequality,

$$\begin{aligned} \int_{\mathfrak{N}(Q)} |W(\alpha) f(\alpha)^s| \,\mathrm{d}\alpha &\ll W(0) \int_{\mathfrak{N}(Q)} \Upsilon(\alpha)^{\phi-\delta} |f(\alpha)|^s \,\mathrm{d}\alpha \\ &\leqslant W(0) I_1^{\frac{\phi-\delta}{1+\delta}} I_2^{1-\frac{\phi-\delta}{1+\delta}}, \end{aligned}$$

where

$$I_1 = \int_{\mathfrak{M}(Q)} \Upsilon(\alpha)^{1+\delta} |f(\alpha)|^2 \,\mathrm{d}\alpha \quad \text{and} \quad I_2 = \int_{\mathfrak{N}(Q)} |f(\alpha)|^t \,\mathrm{d}\alpha.$$

We apply (3.15) and (3.17) to conclude that there is a positive number σ with

$$\int_{\mathfrak{N}(Q)} |W(\alpha) f(\alpha)^s| \,\mathrm{d}\alpha \ll W(0) P^{s-k} Q^{-\sigma}.$$

We sum over Q as in (3.14) to infer that

$$\nu_{\mathfrak{L} \setminus \mathfrak{K}}(n) = o(W(0) n^{s/k-1}).$$

Reference to Lemma 3.2 completes the proof. □

The proof of this theorem shows that the error terms in the conclusions of Theorem 3.3 can be reduced in size to $O(W(0)L^{-\tau})$, for some potentially tiny positive number τ. For the Diophantine applications that we have in mind, this is irrelevant. However, in some situations, it is desirable to do better. We illustrate this in the particular case where the weight is the Möbius function. The sum $W(\alpha)$ then becomes $\mathrm{M}(\alpha)$. In the following result, the condition on s can be relaxed at the cost of a more involved argument.

Lemma 3.4. *Let $A \geqslant 1$ be a real number, and suppose that $s \geqslant 2\lfloor k/2\rfloor + 5$. Then*

$$\int_{\mathfrak{L}} |\mathrm{M}(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^s L^{-A}.$$

Proof. Combine the second inequality of (3.16) with (2.11). □

4. Minor arcs: pruning by height

With bounds for the contribution from the major arcs $\mathfrak{L}$ in hand, we turn to the minor arcs $\mathfrak{l} = [0,1] \setminus \mathfrak{L}$ and describe a first and simple pruning argument that satisfactorily estimates $\nu_{\mathfrak{l}}(n)$ for all ϕ-weights with $\phi > 0$. As a first step, we note that $\mathfrak{l}$ is a subset of the union of the slices $\mathfrak{N}(Q)$ with $Q = 2^{1-j}\sqrt{n}$, $j \geqslant 0$ and $Q \geqslant P^{1/2}$. It follows that for some such Q we have

$$\nu_{\mathfrak{l}}(n) \ll L \int_{\mathfrak{N}(Q)} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha. \tag{4.1}$$

Here we have sliced the minor arcs $\mathfrak{l}$ according to the height Q of the underlying major arcs $\mathfrak{M}(Q)$. This is the technique of *pruning by height*.

We now bring in our major arc moment estimates from [7, 9]. This requires the concept of admissible exponents. To introduce this, fix $k \geqslant 3$. The number Δ_t is an *admissible exponent* for the positive number t (and exponent k) if, for any fixed positive number ε there exists a positive number η such that, whenever $1 \leqslant R \leqslant P^\eta$, one has

$$\int_0^1 |f(\alpha; P, R)|^t\,\mathrm{d}\alpha \ll P^{t-k+\Delta_t+\varepsilon}. \tag{4.2}$$

In our arguments only finitely many admissible exponents occur. Since one may replace an allowed positive value η by a smaller one without affecting the definition of an admissible exponent, it is possible to work with the same value of η, for all the admissible exponents in play. In particular, we may use the same function $f(\alpha) = f(\alpha; P, P^\eta)$ in the moments defining the finitely many admissible exponents that are in use. Therefore, from this point onwards, we apply the extended ε-R-η convention. Thus, if a statement involves ε and the letter R, then it is asserted that there is a number $\eta > 0$ such that the statement holds uniformly for $2 \leqslant R \leqslant P^\eta$. Again, if one calls upon finitely many such statements, one may pass to a situation where the same value of η occurs in all these statements, and we may then take $R = P^\eta$.

Admissible exponents exist. Integrating the trivial estimate $|f(\alpha; P, R)| \leqslant P$ shows that $\Delta_t = k$ is an admissible exponent. Further, it follows easily from

(3.7) that for any fixed choice of $\eta \in (0,1]$ one has $|f(\alpha)| \gg P$ uniformly for $|\alpha| \leqslant 1/(10n)$. Hence

$$\int_0^1 |f(\alpha)|^t \, \mathrm{d}\alpha \gg P^{t-k},$$

irrespective of t, and we see that admissible exponents are non-negative. According to this discussion, when working with admissible exponents, we may suppose that

$$0 \leqslant \Delta_t \leqslant k,$$

and we shall do so whenever this simplifies an argument.

The next lemma is pivotal to all that follows.

Lemma 4.1. *Let $k \geqslant 3$ be given. Suppose that $t \geqslant 2$ is a real number and let Δ_t be an admissible exponent for t. Let Q be a real number with $1 \leqslant Q \leqslant 2\sqrt{n}$. Then*

$$\int_{\mathfrak{M}(Q)} |f(\alpha; P, R)|^t \, \mathrm{d}\alpha \ll P^{t-k+\varepsilon} Q^{2\Delta_t/k}.$$

Proof. For $1 \leqslant Q \leqslant \frac{1}{2}\sqrt{n}$ this is [7, Theorem 4.2]. For $Q = 2\sqrt{n}$ this is a restatement of the definition of an admissible exponent. Meanwhile, when $\frac{1}{2}\sqrt{n} < Q < 2\sqrt{n}$ this follows trivially from the case $Q = 2\sqrt{n}$. □

We now return to (4.1) and suppose that w is a ϕ-weight, for some $\phi > 0$. Then, by (3.13), one has

$$\sup_{\alpha \in \mathfrak{N}(Q)} |W(\alpha)| \ll \|W\| Q^{\varepsilon - \phi}. \tag{4.3}$$

By Lemma 4.1, we see that

$$\int_{\mathfrak{N}(Q)} |W(\alpha) f(\alpha)^s| \, \mathrm{d}\alpha \ll \|W\| Q^{\varepsilon-\phi} P^{s+\varepsilon} n^{-1} Q^{2\Delta_s/k}.$$

For sufficiently small Δ_s the exponent of Q becomes negative. Since we only require $Q \geqslant P^{1/2}$ in (4.1), we may conclude as follows.

Lemma 4.2. *Fix a set of parameters involving a ϕ-weight, for some $\phi > 0$. Suppose that $2\Delta_s < k\phi$. Then there is a number $\delta > 0$ with the property that*

$$\nu_{\mathfrak{l}}(n) \ll \|W\| n^{s/k-1-\delta}.$$

As a first example that illustrates the use of Lemma 4.2, we choose the Möbius function as the weight. By Lemma 2.4, we may take $\phi = \frac{2}{5}$ in Lemma 4.2. Now, since $\nu(n) = \nu_{\mathfrak{L}}(n) + \nu_{\mathfrak{l}}(n)$, we conclude from Lemma 3.4 that whenever s is a natural number with

$$s \geqslant 2\lfloor k/2 \rfloor + 5 \quad \text{and} \quad 5\Delta_s < k, \tag{4.4}$$

then for each $A > 1$ one has

$$\sum_{m \leqslant n} \mu(m) \rho(n-m) \ll n^{s/k} L^{-A}. \tag{4.5}$$

This result should be compared with the analogous result for the prime weight ϖ that is obtained *inter alia* in [9, Section 5]. As we shall see later, the conditions (4.4) can be relaxed by a more elaborate argument.

For Diophantine applications, one combines Lemma 4.2 with Theorem 3.3. The following result is then immediate.

Theorem 4.3. *Fix a set of parameters involving a non-negative ϕ-weight w, for some $\phi \in (0,1]$. Suppose that*

$$s \geqslant \tfrac{3}{2}k, \qquad s > (1-\phi)(2\lfloor k/2\rfloor + 4) + 2\phi, \tag{4.6}$$

and

$$2\Delta_s < k\phi. \tag{4.7}$$

If k is not a power of 2, then

$$\nu(n) \gg n^{s/k-1}\Bigl(\sum_{m\leqslant n/2} w(m) - o(W(0))\Bigr).$$

Meanwhile, if k is a power of 2, then

$$\nu(n) \gg n^{s/k-1}\Bigl(\sum_{\substack{m\leqslant n/2\\ n-m\in\mathscr{R}}} w(m) - o(W(0))\Bigr).$$

For concrete results, with explicit dependence in terms of k and s, one desires admissible exponents that are as small as is possible. When k is large, the smallest known admissible exponents were found by the second author [37, Theorem 3.2]. We use a marginally weaker version of this conclusion. This features the function $\mathrm{H} : (0,\infty) \to (0,1)$, defined by the equation $\mathrm{H}\,\mathrm{e}^{\mathrm{H}} = \mathrm{e}^{1-t}$. It is readily seen that H is smooth, strictly decreasing and bijective. We have

$$\mathrm{H}(t) + \log \mathrm{H}(t) = 1 - t, \tag{4.8}$$

and we may differentiate to infer the relation

$$\mathrm{H}'(t) = -\mathrm{H}(t)/(1+\mathrm{H}(t)). \tag{4.9}$$

The next lemma is [38, Theorem 2.1] when $k \geqslant 4$, and may be verified directly for $k = 3$ using estimates available via Hua's lemma (see [31, Lemma 2.5]).

Lemma 4.4. *Let $k \geqslant 3$ be given. Then, whenever t is an even natural number, the exponent $k\mathrm{H}(t/k)$ is admissible.*

Equation (4.8) makes it easy to compute the inverse function of H. For example, we have $\mathrm{H}(\frac{4}{5}+\log 5) = \frac{1}{5}$. Since H is strictly decreasing, it follows that the upper bound $\mathrm{H}(t) < \frac{1}{5}$ holds for all $t > \frac{4}{5}+\log 5 = 2.4094\ldots$. In particular, whenever s is an even integer with $s > (\frac{4}{5}+\log 5)k$, then there is an admissible exponent $\Delta_s < k/5$, and it follows that the constraint $s \geqslant (\frac{4}{5}+\log 5)k + 2$ implies that both conditions in (4.4) are satisfied for $k \geqslant 3$. In this form, the result in (4.5) compares more directly with the work on prime numbers in [9, Section 5].

The proof of Theorem 2.5. Now recall the function $c_1(\phi)$ introduced in (2.12). By (4.8), we have $\mathrm{H}(c_1(\phi)) = \phi/2$. Since H is decreasing, we see that the condition (4.7) is certainly met for the smallest even integers s satisfying $s/k > c_1(\phi)$. Recalling that for regular weights one has

$$\sum_{m \leqslant n/2} w(m) \gg W(0),$$

we conclude that all cases of Theorem 2.5 where k is not a power of 2 are in fact a corollary of Theorem 4.3. If k is a power of 2, then the comment immediately following the statement of Lemma 3.2 applies, and we may complete the proof of Theorem 2.5 as before. □

Proof of first clause of Theorem 1.1. Recall that the special case of Theorem 1.1 in which k is not a power of 2 has already been deduced from Theorem 2.5 in the discussion following the statement of that conclusion. Next, we consider the case where k is a power of 2 with $k \geqslant 8$, and we temporarily suppose only that $s \geqslant \frac{3}{2}k$. We claim that there is an odd natural number x_0 and a number j with $1 \leqslant j \leqslant s$ for which

$$n - x_0^2 \equiv j \pmod{4k}.$$

To see this, we note that for each $l \in \mathbb{Z}$ the congruence $x_0^2 \equiv 1+8l \pmod{4k}$ has a solution. Of course, the solution x_0 is necessarily odd. Now choose l so that $1 \leqslant n-1-8l \leqslant 8$. Since $s \geqslant \frac{3}{2}k \geqslant 12$, this justifies our claim. In the notation of Lemma 3.2, for all integers x with $1 \leqslant x \leqslant \frac{1}{2}\sqrt{n}$ and $x \equiv x_0 \pmod{4k}$ we have $n - x^2 \in \mathscr{R}$. Observe next that the condition (4.6) holds for $k \geqslant 3$, and in view of (4.8), the condition (4.7) will certainly be satisfied when s is an even integer with $s/k > \frac{3}{4} + \log 4$. Thus, when $s \geqslant s_0(k)$, Theorem 4.3 finally delivers the lower bound $\nu(n) \gg n^{1/2}P^{s-k}$ subject to the constraints implicit in Theorem 1.1 also in the case where k is a power of 2. In the missing case $k = 4$, the second clause of Theorem 1.1 is stronger anyway, so that in view of (2.3) the discussion of the first clause is complete. □

So far we have concentrated on results for all k, with an emphasis on large values of k. For smaller k much better admissible exponents are known. Special attention has been paid to the smallest k, so we begin with these.

Cubes. For the present discussion, we restrict to the situation with $k = 3$. It is a consequence of work of Wooley [40] that $\Delta_5 = 10/17$ is admissible (see the discussion following [40, Lemma 5.1]). Taking $s = 5$ and making use of this admissible exponent, it is readily checked that whenever $\phi > 20/51$, then (4.6) and (4.7) both hold. We formulate the consequences of Theorem 4.3 as our next theorem.

Theorem 4.5. *Suppose that $k = 3$, $s = 5$ and $\phi > 20/51$. Then, whenever w is a non-negative regular ϕ-weight, one has $\nu(n) \gg W(0)n^{2/3}$. In particular, when $0 < \eta \leqslant 1$, the number $\nu_3(n;\eta)$ of solutions of the Diophantine equation*

$$x^2 + y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 = n,$$

in natural numbers x, y_j *with* $y_j \in \mathscr{A}(P, P^\eta)$ $(1 \leqslant j \leqslant 5)$, *satisfies the lower bound* $\nu_3(n;\eta) \gg n^{7/6}$.

Biquadrates. We now restrict to the situation with $k = 4$. The authors [4, Theorem 2] showed that $\Delta_7 = 0.849408$ is admissible. Hence, Theorem 4.3 is applicable whenever $\phi > \frac{1}{2}\Delta_7$, and it delivers the first clause of the following theorem.

Theorem 4.6. *Suppose that* $k = 4$, $s = 7$ *and* $\phi > 0.424704$. *Then, whenever* w *is a non-negative* ϕ*-weight and* n *is large and in a congruence class modulo* $4k$ *where*

$$\sum_{j=1}^{7} \sum_{\substack{m \leqslant n/2 \\ n-m \equiv j \ (\mathrm{mod}\ 16)}} w(m) \gg W(0),$$

then $\nu(n) \gg W(0)n^{3/4}$. *In particular, when* $0 < \eta \leqslant 1$, *the number* $\nu_4(n;\eta)$ *of solutions of the Diophantine equation*

$$x^2 + y_1^4 + y_2^4 + \cdots + y_7^4 = n,$$

in natural numbers x, y_j *with* $y_j \in \mathscr{A}(P, P^\eta)$ $(1 \leqslant j \leqslant 7)$, *satisfies the lower bound* $\nu_4(n;\eta) \gg n^{5/4}$.

The second clause requires a proof. We apply the first clause with w the indicator function of the squares, which is a $\frac{1}{2}$-weight. We are then required to find a lower bound for the quantity

$$\sum_{j=1}^{7} \sum_{\substack{x^2 \leqslant n/2 \\ n-x^2 \equiv j \ (\mathrm{mod}\ 16)}} 1.$$

For each of the congruence classes $n \pmod{16}$ one can find an integer x_0 with $1 \leqslant x_0 \leqslant 4$ for which $n - x_0^2$ lies in one of the classes $j \pmod{16}$, for some integer j with $1 \leqslant j \leqslant 7$. All integers x with $x \equiv x_0 \pmod{16}$ for which $1 \leqslant x \leqslant \frac{1}{2}\sqrt{n}$ will appear in the sum to be bounded, and this sum is therefore $\gg \sqrt{n}$. The second clause of the theorem now follows from the first.

Extensive tables of exponents for moderately sized exponents $k \geqslant 5$ have been provided by Vaughan and Wooley [34, 35]. In interpreting these tables, note that our admissible exponent Δ_{2t} is given by $\lambda_t - 2t + k$ in the notation applied in [34, 35]. Explicit values of λ_t with $t \in \mathbb{N}$ are tabulated in the latter sources. For odd values of $s \in \mathbb{N}$ one applies Schwarz's inequality to (4.2) to see that whenever Δ_{s-1} and Δ_{s+1} are admissible exponents, then so too is

$$\Delta_s = \tfrac{1}{2}(\Delta_{s-1} + \Delta_{s+1}). \tag{4.10}$$

Proof of the second clause of Theorem 1.1. From (4.10) and the table of exponents for $k = 5$ in [34] we see that $\Delta_9 = 1.181868$ is admissible, so there are admissible exponents for $k = 5$, $s = 9$ smaller than $\frac{5}{4}$. By Theorem 4.3 with $\phi = \frac{1}{2}$, this suffices to establish the case $k = 5$ of Theorem 1.1. The reader may care to confirm the cases $6 \leqslant k \leqslant 11$ in the same way. In this context, we point out that the peculiar case where $k = 8$ requires a discussion concerning squares modulo 32, but this is covered by the more general argument given within the proof of the first clause of Theorem 1.1 where we only needed $s \geqslant 8$. Together with the results in Theorems 4.5 and 4.6, we have now covered all cases of Theorem 1.1. □

5. Enhancements: pruning by size

The method described in the preceding section is the most basic strategy to interpret the major arc moment estimates in Lemma 4.1 as a pruning device. It has the advantage that it is applicable whenever a Weyl bound for $W(\alpha)$ is available. If more is known about the weight w, say its density or the arithmetic structure of its support, then one may hope to improve upon Theorem 4.3. There are several approaches to realise this objective, and we begin with a method that was introduced in [9] as *pruning by size*. This innovation has as a precursor a theme explored in [2, Section 3]. The idea is to slice the range of integration $[0,1]$ in (3.1) into pieces of the shape

$$\mathscr{S}(T) = \{\alpha \in [0,1] : \|W\|T^{-1} < |W(\alpha)| \leqslant 2\|W\|T^{-1}\}, \tag{5.1}$$

where T is potentially larger than the savings offered by the Weyl bound for W. It is then no longer possible to conclude that α has a Diophantine approximation with small denominator. Instead, one explores the lower bound for $|W(\alpha)|$ implicit in (5.1) by variants of Chebychev's inequality.

As a simple example, we note that whenever $w \neq 0$, one has

$$\int_{\mathscr{S}(T)} |W(\alpha)|\,\mathrm{d}\alpha \leqslant \frac{T}{\|W\|}\int_0^1 |W(\alpha)|^2\,\mathrm{d}\alpha \ll T\,\frac{\sum_{m\leqslant n} |w(m)|^2}{\sum_{m\leqslant n} |w(m)|}. \tag{5.2}$$

This inequality is particularly effective in the important special case where w approximates the indicator function of a fairly dense set. For the argument to follow, all that is needed are the bounds

$$\sum_{m\leqslant n} |w(m)| \gg n^{1-\varepsilon} \quad \text{and} \quad \sum_{m\leqslant n} |w(m)|^2 \ll n^{1+\varepsilon} \tag{5.3}$$

that show the quotient on the right hand side of (5.2) to be $O(n^\varepsilon)$. The integral on the left hand side of (5.2) is then bounded by $n^\varepsilon T$. We now choose a microscopic $\delta > 0$ and consider the set

$$\mathscr{E} = \{\alpha \in \mathscr{S}(T) : |f(\alpha)|^s \leqslant P^{s-2\delta}T^{-1}\}.$$

By (3.2), (5.2) and (5.3) we conclude that

$$\nu_{\mathscr{E}}(n) \ll n^\varepsilon P^{s-2\delta} \ll \|W\|P^{s-k-\delta}.$$

This is satisfactory, so we may turn our attention to the complementary set $\mathscr{F} = \mathscr{S}(T) \setminus \mathscr{E}$. This set is characterised by the bound $|f(\alpha)|^s > P^{s-2\delta}T^{-1}$.

Applying this lower bound in the same manner as (5.1) was used in deducing (5.2), we now find that for every non-negative number t one has

$$\begin{aligned}\nu_{\mathscr{F}}(n) &\ll \|W\| T^{-1} \int_{\mathscr{F}} |f(\alpha)|^s \,\mathrm{d}\alpha \\ &\ll \|W\| T^{t/s-1} P^{2\delta t/s - t} \int_0^1 |f(\alpha)|^{s+t} \,\mathrm{d}\alpha \\ &\ll \|W\| T^{t/s-1} P^{2\delta t/s + s - k + \Delta_{s+t} + \varepsilon}.\end{aligned}$$

Here one minimises the right hand side relative to t for a given value of T, seeking to obtain a satisfactory upper bound for $\nu_{\mathscr{F}}(n)$ in the full range for T remaining to be considered. In some cases this approach significantly improves upon the conclusions of Theorem 2.5. In [9] we worked out the details for the primes, encoded by the weight ϖ defined in Lemma 2.2. However, as a careful inspection of the argument described in Sections 6–8 of [9] shows, one may apply the method equally well to $\frac{2}{5}$-weights w that satisfy (5.3). In this way one still finds a small positive number δ for which the estimate

$$\nu_{\mathfrak{f}}(n) \ll P^{s-\delta} \tag{5.4}$$

is valid whenever $s \geqslant ck+4$ and $c = 2.134693\ldots$ is the number that occurs in (1.3).

As an example of independent interest, we choose the Möbius function. By Lemma 2.4, this is a $\frac{2}{5}$-weight, and (5.3) certainly holds for $\mu(m)$ in the role of the weight. For these reasons, the upper bound (5.4) is true for the Möbius function. The complementary major arc contribution has been worked out in Lemma 3.4. In combination with (5.4), this proves the following result.

Theorem 5.1. *Let $c = 2.134693\ldots$ be the real number that occurs in* (1.3). *Suppose that $k \geqslant 3$, that $s \geqslant ck+4$, and that $A \geqslant 1$. Then, for sufficiently small $\eta > 0$, one has*

$$\sum_{m \leqslant n} \mu(n-m)\rho(m) \ll n^{s/k} (\log n)^{-A}.$$

We remark that the lower bound constraint for s here can be improved for small values of k. For $6 \leqslant k \leqslant 20$, it suffices to suppose that $s \geqslant S_0(k)$ where $S_0(k)$ is defined in [9, Theorem 1.2], for example $S_0(6) = 11$, $S_0(7) = 13$. We encourage readers to challenge themselves with the problem of establishing the conclusion of Theorem 5.1 when $k = 3$ and $s = 4$, and also when $k = 4$ and $s = 6$. For these exercises, the mean values (7.3) and (7.4) are relevant.

Further, the methods of [9] also apply to ϕ-weights for values of ϕ other than $\frac{2}{5}$. In fact, the results in [9] that depend on the Riemann hypothesis for Dirichlet L-functions directly generalise to $\frac{1}{2}$-weights that obey (5.3). Here the Möbius function is again a prominent example. More generally, for ϕ-weights with $\phi > 0$ one may run the arguments of Sections 6–8 in [9] with $2/\phi$ in the role of the parameter θ that occurs in [9, Lemma 5.1]. We refrain from reworking the details here.

Finally, it might be worth pointing out that, by a mild adjustment of our method, one may establish the bounds

$$\sum_{m\leqslant n}\mu(m)\rho(m)\ll n^{s/k}(\log n)^{-A}$$

and

$$\sum_{x_1\in\mathscr{A}(P,P^\eta)}\cdots\sum_{x_s\in\mathscr{A}(P,P^\eta)}\mu(x_1^k+\ldots+x_s^k)\ll P^s(\log P)^{-A},$$

subject to the hypotheses on k,s and A in the statement of Theorem 5.1.

6. Pruning by size for squares

It is time to return to the main theme of this memoir. We proceed to describe pruning by size for ϕ-weights w *supported on the squares*, and for simplicity, we also suppose that w is *non-negative* and satisfies

$$w(m)\ll m^\varepsilon \quad\text{and}\quad W(0)\gg n^{1/2-\varepsilon}. \tag{H}$$

We shall refer to this collection of restrictions for w as Hypothesis H when announcing results.

The estimate (5.2) still applies to the weights now under consideration, but the savings drawn in this way are much weaker. This is because the squares are too sparse. It is more efficient to borrow some of the k-th powers. The method is then implemented via a mixed mean value that is essentially optimal.

Lemma 6.1. *Let r be a natural number, and let* N *denote the number of solutions of the equation*

$$x_1^2-x_2^2=\sum_{j=1}^r(y_j^k-z_j^k) \tag{6.1}$$

in natural numbers x_1,x_2 and y_j,z_j $(1\leqslant j\leqslant r)$ with

$$1\leqslant x_1,x_2\leqslant\sqrt{n},\qquad y_j,z_j\in\mathscr{A}(P,P^\eta).$$

Then

$$\mathrm{N}\ll P^{2r+\varepsilon}+P^{2r-k/2+\Delta_{2r}+\varepsilon}.$$

Proof. We first count solutions of (6.1) where $x_1\neq x_2$. The number of choices for y_j,z_j $(1\leqslant j\leqslant r)$ is at most P^{2r}, and for each such choice, the value of $x_1^2-x_2^2=(x_1-x_2)(x_1+x_2)$ is a fixed non-zero integer no larger than n in absolute value. A familiar divisor function estimate shows that there are no more than $O(n^\varepsilon)$ choices for x_1 and x_2 left. This shows that the solutions with $x_1\neq x_2$ contribute to N an amount no larger than $O(P^{2r+\varepsilon})$.

For the remaining solutions, we have $x_1=x_2$. By orthogonality, the number of such solutions is

$$\lfloor\sqrt{n}\rfloor\int_0^1|f(\alpha)|^{2r}\,\mathrm{d}\alpha\ll n^{1/2}P^{2r-k+\Delta_{2r}+\varepsilon}.$$

This proves the lemma. □

We are ready to embark on the main argument. As usual, we fix a set of parameters, now including a ϕ-weight satisfying Hypothesis H. Suppose that there is a natural number r with

$$2r < s \quad \text{and} \quad 2\Delta_{2r} \leqslant k. \tag{6.2}$$

In addition, we take δ to be a fixed positive number sufficiently small in terms of r, s and k. Then, by (H), we conclude via orthogonality and Lemma 6.1 that

$$\int_0^1 |W(\alpha)^2 f(\alpha)^{2r}|\,\mathrm{d}\alpha \ll P^{2r+\varepsilon}. \tag{6.3}$$

In light of (4.1), our goal is now an estimate for the integral

$$J = \int_{\mathfrak{N}(Q)} |W(\alpha) f(\alpha)^s|\,\mathrm{d}\alpha \tag{6.4}$$

that is uniform for $P^{1/2} \leqslant Q \leqslant 2\sqrt{n}$. With this end in view, let T be a parameter with $T \geqslant 2$ and slice the arcs $\mathfrak{N}(Q)$ into pieces

$$\mathscr{T} = \mathscr{T}(Q,T) = \{\alpha \in \mathfrak{N}(Q) : W(0)/T < |W(\alpha)| \leqslant 2W(0)/T\}. \tag{6.5}$$

Recall here that w is a non-negative ϕ-weight. Hence, by (4.3), one has

$$W(\alpha) \ll W(0) Q^{\varepsilon-\phi}.$$

It follows that $\mathscr{T}$ is empty for $T \leqslant Q^{\phi-\delta}$. We may therefore suppose that

$$T \geqslant Q^{\phi-\delta}.$$

Since the weight satisfies Hypothesis H, we have $W(0) \gg n^{1/2-\varepsilon}$. Proceeding as in the argument producing (5.2), we now deduce from (6.3) that

$$\int_{\mathscr{T}} |W(\alpha) f(\alpha)^{2r}|\,\mathrm{d}\alpha \ll \frac{T}{W(0)} \int_0^1 |W(\alpha)^2 f(\alpha)^{2r}|\,\mathrm{d}\alpha \ll T n^{\varepsilon-1/2} P^{2r}. \tag{6.6}$$

Note that this bound loses a factor $Tn^{2\varepsilon}$ over an acceptable error term. This is similar to the situation in (5.2) with the conditions (5.3) in place, and will be used as a substitute for (5.2) in the discussion to follow.

In the interest of brevity, we now write $u = s - 2r$. Put

$$\mathscr{U} = \{\alpha \in \mathscr{T} : |f(\alpha)|^u \leqslant P^{u-2\delta} T^{-1}\}.$$

Then, by (6.6), we have

$$\int_{\mathscr{U}} |W(\alpha) f(\alpha)^s|\,\mathrm{d}\alpha \ll P^{u-2\delta} T^{-1} \int_{\mathscr{T}} |W(\alpha) f(\alpha)^{2r}|\,\mathrm{d}\alpha \ll n^{-1/2} P^{s-\delta}. \tag{6.7}$$

This will be an acceptable upper bound. We put $\mathscr{V} = \mathscr{T} \setminus \mathscr{U}$. Then, for $\alpha \in \mathscr{V}$, we have

$$|f(\alpha)| > P^{1-2\delta/u} T^{-1/u}.$$

Hence, for every non-negative real number t, one finds that

$$\int_{\mathscr{V}} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll \frac{W(0)}{T}\int_{\mathscr{V}} |f(\alpha)|^s\,\mathrm{d}\alpha$$
$$\ll \frac{W(0)}{T}\,T^{t/u}P^{2\delta t/u-t}\int_{\mathfrak{N}(Q)} |f(\alpha)|^{s+t}\,\mathrm{d}\alpha.$$

Next applying Lemma 4.1, we see that

$$\int_{\mathscr{V}} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll T^{t/u-1}P^{s-k+2\delta t/u}n^{1/2+\varepsilon}Q^{2\Delta_{s+t}/k}.$$

We now suppose that $0 \leqslant t \leqslant u$. Then we may simplify the preceding bound by applying the lower bound $T \geqslant Q^{\phi-\delta}$. This yields the estimate

$$\int_{\mathscr{V}} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^{s+3k\delta}n^{\varepsilon-1/2}Q^{\phi(t/u-1)+2\Delta_{s+t}/k}. \tag{6.8}$$

If t can be so chosen so that the exponent of Q here is negative, then since we have $Q \geqslant P^{1/2}$, one may choose $\delta > 0$ so small that the integral in (6.8) is $O(P^{s-\delta}n^{-1/2})$. In combination with (6.7) we then see that the upper bound

$$\int_{\mathscr{T}} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^{s-\delta}n^{-1/2}$$

holds for all choices of $T \geqslant Q^{\phi-\delta}$. Now, by dyadic slicing, the set of arcs $\mathfrak{N}(Q)$ is the union of $O(L)$ sets $\mathscr{T}(Q,T)$, with $Q^{\phi-\delta} \leqslant T \leqslant n^2$, and the set

$$\mathscr{T}_0 = \{\alpha \in \mathfrak{N}(Q) : |W(\alpha)| \leqslant W(0)n^{-2}\}.$$

Using only the trivial bound $|f(\alpha)| \leqslant P$, we immediately have

$$\int_{\mathscr{T}_0} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll P^sW(0)n^{-2}.$$

We have now proved that, subject to the conditions collected along the way, one has

$$J \ll P^{s-\delta}n^{-1/2} \tag{6.9}$$

uniformly for $P^{1/2} \leqslant Q \leqslant 2\sqrt{n}$. Thus, we may apply (4.1) to conclude that $\nu_{\mathfrak{l}}(n) \ll LP^{s-\delta}n^{-1/2}$. For easier reference, we summarise this result as a lemma.

Lemma 6.2. *Let $\phi \in (0,1]$, and let r be a natural number with $2\Delta_{2r} \leqslant k$. Fix a set of parameters with $s > 2r$, and involving a ϕ-weight that satisfies Hypothesis H. Suppose that there is a real number t with $t \geqslant 0$ and*

$$\frac{2\Delta_{s+t}}{k} < \Big(1 - \frac{t}{s-2r}\Big)\phi. \tag{6.10}$$

Then there is a number $\tau > 0$ such that $\nu_{\mathfrak{l}}(n) \ll n^{s/k-1-\tau}W(0)$.

Proof. Some book-keeping is required to justify this claim. The argument preceding Lemma 6.2 delivers the conclusion of the lemma with $\tau = \delta/(2k)$. Along the way we assumed that (6.2) holds, a condition that is now immediate from the hypotheses of the lemma. In deducing (6.8) we imposed the condition $0 \leqslant t \leqslant u = s - 2r$ on the auxiliary parameter t and then requested that the exponent of Q in (6.8) be negative. We saw that this is so if and only if (6.10) holds. Since admissible exponents are non-negative, the upper bound (6.10) implies that $t \leqslant s - 2r$. The proof is now complete. □

We combine Lemma 6.2 with Theorem 3.3, and then infer the following.

Theorem 6.3. *In addition to the hypotheses of Lemma 6.2, suppose that*

$$s \geqslant \tfrac{3}{2}k \quad \text{and} \quad s > (1-\phi)(2\lfloor k/2 \rfloor + 4) + 2\phi.$$

If k is not a power of 2 and w is a regular weight, then $\nu(n) \gg n^{s/k-1}W(0)$. Meanwhile, if k is a power of 2, then

$$\nu(n) \gg n^{s/k-1}\Bigg(\sum_{\substack{m \leqslant n/2 \\ n-m \in \mathscr{R}}} w(m) - o(W(0)) \Bigg).$$

For an optimal use of this result, note that the right hand side of (6.10) decreases as r increases, so one first determines the smallest natural number r with $2\Delta_{2r} \leqslant k$. For small values of k this can be read off from the tables in [34, 35]. With r now fixed, one may optimise the choice of the real number t. Of course one may choose $t = 0$, but then (6.10) reduces to $2\Delta_s < k\phi$, which is (4.7) in Theorem 4.3. Choosing t larger, it is sometimes possible to improve on Theorem 4.3 considerably. This effect becomes more pronounced if ϕ is smallish. As an introductory example, however, we use Theorem 6.3 to establish Theorem 1.4 for small k.

Proof of Theorem 1.4, part I. We establish the cases $8 \leqslant k \leqslant 12$ of Theorem 1.4 and we prove *en passant* the claim made in its sequel concerning the case $k = 7$. In the table below, we have listed the smallest natural number r with $2\Delta_{2r} \leqslant k$. By Lemma 2.3, for squares of primes, we may take $\phi = \frac{1}{8}$, and then check (6.10) in the form

$$\Delta_{s+t} \leqslant \Delta^*_{s,t}(r),$$

in which we write

$$\Delta^*_{s,t}(r) = \frac{k}{16}\Big(1 - \frac{t}{s-2r}\Big).$$

The table also gives the smallest value of s for which we were able to verify this inequality, the associated value of t, and the values of Δ_{s+t} and $\Delta^*_{s,t}(r)$ corresponding to this choice of parameters. The numerical values for admissible exponents are taken from [35], rounded up in the last digit displayed, and the values of $\Delta^*_{s,t}(r)$ are rounded down in the last digit displayed. With these data, the cases $8 \leqslant k \leqslant 12$ of Theorem 1.4, as well as the bonus sequel case $k = 7$, follow from Theorem 6.3 on observing that in each case, one has $\Delta_{s+t} \leqslant \Delta^*_{s,t}(r)$. □

k	r	Δ_{2r}	s	t	Δ_{s+t}	$\Delta^*_{s,t}(r)$
7	4	3.27	20	6	0.1926	0.2187
8	5	3.50	24	8	0.1892	0.2142
9	5	4.42	27	9	0.2521	0.2647
10	6	4.65	31	11	0.2450	0.2631
11	7	4.89	35	13	0.2414	0.2619
12	7	5.80	38	12	0.3469	0.3750

Data for the proof of Theorem 1.4.

Next, we explore the potential of Theorem 6.3 when k is large. One of our ultimate goals is to complete the proof of Theorem 1.4. We have recourse to the admissible exponents provided by Lemma 4.4, and then the condition $2\Delta_{2r} \leqslant k$ becomes $\mathrm{H}(2r/k) \leqslant \frac{1}{2}$. But from (4.8) we know that $H(\frac{1}{2} + \log 2) = \frac{1}{2}$, so that the smallest possible choice for the integer r is determined by the inequalities

$$(\tfrac{1}{2} + \log 2)k \leqslant 2r < (\tfrac{1}{2} + \log 2)k + 2.$$

We fix this choice of r from now on. Further, we suppose that a set of parameters is given in accordance with the hypotheses of Theorem 6.3.

In the interest of a compact notation in a rather complex argument to follow, we write

$$s = \sigma k, \quad t = \tau k, \quad 2r = \zeta k, \quad \gamma = \sigma - \zeta.$$

Here the numbers σ, ζ and γ are frozen with the set of parameters while $\tau \geqslant 0$ is at our disposal. Note that the condition $s > 2r$ becomes $\gamma > 0$. Moreover, on noting that $\frac{1}{2}+\log 2$ is an irrational number, we see that the rational number $\zeta = \zeta_k$ satisfies

$$0 < \zeta - (\tfrac{1}{2} + \log 2) < 2/k. \tag{6.11}$$

By hypothesis, we also have $\sigma \geqslant \frac{3}{2}$. For $k \geqslant 5$, it then follows that $\sigma > \zeta$, as is immediate from (6.11) for $k \geqslant 7$, while $\zeta_5 = \frac{6}{5}$ and $\zeta_6 = \frac{4}{3}$. We recall again that we use the admissible exponents provided by Lemma 4.4. With these exponents, the condition (6.10) translates to $2\mathrm{H}(\sigma + \tau) < (1 - \tau/\gamma)\phi$, with the extra constraint that $s + t$ is supposed to be an even integer. This last inequality we recast in the form

$$\frac{\tau}{\gamma} + \frac{2\mathrm{H}(\sigma + \tau)}{\phi} < 1. \tag{6.12}$$

We now wish to decide whether there is a number $\tau \geqslant 0$ such that (6.12) holds. Temporarily we ignore the requirement that $s + t$ should be an even integer and treat the problem within real analysis. As a first step, with $\sigma \geqslant \frac{3}{2}$, $\sigma > \zeta$ and $\phi \in (0, 1]$ as before, we determine

$$E(\sigma, \phi) = \inf_{\tau \geqslant 0} \Big(\frac{\tau}{\gamma} + \frac{2\mathrm{H}(\sigma + \tau)}{\phi}\Big). \tag{6.13}$$

It is important to note that γ and hence also E depends on k. Of course we consider k as fixed, but we shall later take k large. The only appearance of

k in (6.13) is in $\gamma = \sigma - \zeta_k$, though, and γ is perturbed by at most $2/k$. For fixed $k \geqslant 5$, on the domain

$$D = \{(\sigma, \phi) \in \mathbb{R}^2 : 0 < \phi < 2,\ 2\gamma > \phi\}$$

we define the analytic function $F : D \to \mathbb{R}$ by

$$F(\sigma, \phi) = \frac{2}{2\gamma - \phi} + \frac{1}{\gamma}\Big(1 - \sigma - \frac{\phi}{2\gamma - \phi} + \log \frac{2\gamma - \phi}{\phi}\Big). \tag{6.14}$$

Lemma 6.4. *Fix $k \geqslant 5$. Suppose that $\sigma \geqslant \frac{3}{2}$ and $\phi \in (0, 1]$. If $2\gamma > \phi$ and $\mathrm{H}(\sigma) > \phi/(2\gamma - \phi)$, then $E(\sigma, \phi) = F(\sigma, \phi)$. Otherwise*

$$E(\sigma, \phi) = \frac{2\mathrm{H}(\sigma)}{\phi}.$$

Proof. For a given pair (σ, ϕ), the expression on the left hand side of (6.12) defines a smooth function

$$h : [0, \infty) \to \mathbb{R}, \quad h(\tau) = \frac{\tau}{\gamma} + \frac{2\mathrm{H}(\sigma + \tau)}{\phi}.$$

We compute the derivative directly and then apply (4.9) to confirm that

$$h'(\tau) = \frac{1}{\gamma} + \frac{2\mathrm{H}'(\sigma + \tau)}{\phi} = \frac{1}{\gamma} - \frac{2}{\phi} \cdot \frac{\mathrm{H}(\sigma + \tau)}{1 + \mathrm{H}(\sigma + \tau)}. \tag{6.15}$$

The function $\mathrm{H}' : [0, \infty) \to \mathbb{R}$ is strictly increasing and bijects onto $[-\frac{1}{2}, 0)$. In particular, the function h' is strictly increasing too, and its smallest value is

$$h'(0) = \frac{1}{\gamma} - \frac{2}{\phi} \cdot \frac{\mathrm{H}(\sigma)}{1 + \mathrm{H}(\sigma)}.$$

If $h'(0) \geqslant 0$, then h is strictly increasing. Its smallest value is therefore at $\tau = 0$, and we conclude that in this case $E(\sigma, \phi) = h(0) = 2\mathrm{H}(\sigma)/\phi$. By (6.15), the condition $h'(0) \geqslant 0$ is equivalent to $(2\gamma - \phi)\mathrm{H}(\sigma) \leqslant \phi$. If $2\gamma \leqslant \phi$, then this is always true, and if $2\gamma > \phi$, then the condition becomes $\mathrm{H}(\sigma) \leqslant \phi/(2\gamma - \phi)$. Thus far, we have established the second clause of the lemma.

Next, we consider the case where $h'(0) < 0$, a condition that we now know to be equivalent to the two inequalities $2\gamma > \phi$ and $\mathrm{H}(\sigma) > \phi/(2\gamma - \phi)$, as in the first clause of the lemma. Since $\mathrm{H}(\tau)$ decreases to 0 as $\tau \to \infty$, we deduce from (6.15) that

$$\lim_{\tau \to \infty} h'(\tau) = 1/\gamma > 0.$$

Once again because h' is strictly increasing, it first follows that there is a unique number $\tau_0 = \tau_0(\sigma, \phi)$ with $h'(\tau_0) = 0$, and then h will take its minimum at τ_0. This shows that

$$E(\sigma, \phi) = h(\tau_0) = \frac{\tau_0}{\gamma} + \frac{2\mathrm{H}(\sigma + \tau_0)}{\phi}. \tag{6.16}$$

The function τ_0 can be computed by inserting the relation $h'(\tau_0) = 0$ into (6.15), yielding the equation

$$\frac{\phi}{2\gamma} = \frac{\mathrm{H}(\sigma + \tau_0)}{1 + \mathrm{H}(\sigma + \tau_0)}.$$

This we rewrite as

$$\mathrm{H}(\sigma+\tau_0)=\phi/(2\gamma-\phi). \tag{6.17}$$

By (6.17) and (4.8), we find that

$$\tau_0=1-\sigma-\frac{\phi}{2\gamma-\phi}+\log\frac{2\gamma-\phi}{\phi}. \tag{6.18}$$

If we insert (6.17) and (6.18) into (6.16), then we arrive at the expression for E displayed in (6.14). This completes the proof. □

Since the condition (6.10) translated into (6.12), we are now interested in the set of pairs (σ,ϕ) where the upper bound $E(\sigma,\phi)<1$ holds. We avoid undue generality and take a pragmatic perspective. Recall that we work subject to Hypothesis H, so the weight w is supported on the squares, and one has

$$n^{1/2-\varepsilon}\ll W(0)\ll n^{1/2+\varepsilon}.$$

In such circumstances, the situation with $\phi=\frac{1}{2}$ corresponds to square root cancellation on minor arcs, at least nearly so. In all realistic applications, we shall therefore have $\phi\leqslant\frac{1}{2}$, and we assume this now for the rest of this section. Further, we have already assumed that $\sigma\geqslant\frac{3}{2}$, but it will turn out *a posteriori* that our method will not penetrate into the region $\sigma<2$ unless k is very small, and then one would work from the tables in [34, 35] anyway. Thus, we assume that $\sigma\geqslant 2$ as well. Then, for $k\geqslant 4$, one has

$$2\gamma\geqslant 4-2\zeta>3-\log 4-2/k>\tfrac{1}{2}\geqslant\phi.$$

Next, recall from the discussion following Lemma 4.4 that with $c_1(\phi)$ defined via (2.12), we have $\mathrm{H}(c_1(\phi))=\phi/2$. Thus, we find that Theorem 2.5 gives us a lower bound for $\nu(n)$ in all cases where $\mathrm{H}(\sigma)<\phi/2$. In circumstances with $2\gamma-\phi\leqslant 2$, the interval for σ given by the inequality $\mathrm{H}(\sigma)<\phi/(2\gamma-\phi)$ contains the corresponding interval determined by $\mathrm{H}(\sigma)<\phi/2$ (because H is a decreasing function). In this wider range for σ we find that Lemma 6.4 applies and delivers the relation $E(\sigma,\phi)=2\mathrm{H}(\sigma)/\phi$. But then we see that $E(\sigma,\phi)<1$ exactly when $\mathrm{H}(\sigma)<\phi/2$, and as we have already pointed out, this is a situation covered by Theorem 2.5. Hence, an improvement on the conclusion of Theorem 2.5 can only be expected in the case where $2\gamma-\phi>2$. This implies an upper bound on ϕ where improvements can arise, a scenario we now explore.

Recall the function $c_1=c_1(\phi)$ from (2.12) that is characterised by the equation $\mathrm{H}(c_1)=\phi/2$. With $\sigma=c_1$, the condition $2\gamma-\phi>2$ that we currently analyse becomes $\phi<2c_1-2\zeta_k-2$. In view of (4.8), this condition is equivalent to the constraint

$$\begin{aligned}1-c_1=\mathrm{H}(c_1)+\log\mathrm{H}(c_1)&=\tfrac{1}{2}\phi+\log\phi-\log 2\\&>\phi+\log\phi-\log 2-(c_1-\zeta_k-1),\end{aligned}$$

which reduces to

$$\phi+\log\phi<\log 2-\zeta_k.$$

The left hand side here is an increasing function of ϕ. We define ϕ_k to be the unique positive solution of $\phi_k + \log \phi_k = \log 2 - \zeta_k$. Since ζ_k converges to $\frac{1}{2} + \log 2$ as $k \to \infty$, it follows that ϕ_k converges to the unique positive number ϕ^* with

$$\phi^* + \log \phi^* = -\tfrac{1}{2}.$$

Mundane analysis reveals that

$$0.4046 < \phi^* < 0.4047.$$

Note that since $\zeta_k > \frac{1}{2} + \log 2$, then $\phi_k < \phi^*$ for all k. Since the squares are a $\frac{1}{2}$-set and Theorem 1.1 was deduced from Theorem 2.5 with $\phi = \frac{1}{2}$, this tells us that we cannot expect to improve Theorem 1.1 via pruning by size, and definitely not in the way the proof of Theorem 6.3 is designed. We do foresee, however, that Theorem 6.3 will perform much better than Theorem 2.5 in the context of Theorem 1.4 where $\phi = \frac{1}{8}$. From now on we may suppose that $\phi < \phi^*$, an even more stringent request than $\phi \leqslant \frac{1}{2}$. In the scenario where $\phi = \phi^*$, we see that Theorem 2.5 requests that $\sigma > \sigma^*$ where σ^* is defined by means of the equation $\mathrm{H}(\sigma^*) = \phi^*/2$. Recalling (4.8) and the defining equation for ϕ^*, we find that

$$2\sigma^* = \phi^* + 3 + 2\log 2,$$

whence $\sigma^* > 2.3954$. For smaller values of ϕ, moreover, we do not expect positive results from Theorem 6.3 for values of σ smaller than σ^*. This justifies our earlier comment that we may safely suppose that $\sigma \geqslant 2$ in all that follows.

Our next task is to interpret the inequality $E(\sigma, \phi) < 1$. In this context, we define the positive number σ_k through the equation $\mathrm{H}(\sigma_k) = \phi_k/2$. Then, by (4.8), we have

$$\sigma_k = 1 - \tfrac{1}{2}\phi_k + \log(2/\phi_k) = c_1(\phi_k),$$

in the notation (2.12), and the defining equation for ϕ_k then yields the relation

$$2(\sigma_k - \zeta_k) - \phi_k = 2. \tag{6.19}$$

With this identity now given, interpreted in the form $2\gamma - \phi_k = 2$, and working under the assumption that $k \geqslant 5$, it is readily checked from (6.14) that

$$F(\sigma_k, \phi_k) = 1. \tag{6.20}$$

Moreover, the definition of σ_k in conjunction with Lemma 6.4 also gives the relation $E(\sigma_k, \phi_k) = 1$. Further properties of the functions E and F are summarised in the next lemma.

Lemma 6.5. *Fix a natural number k with $k \geqslant 5$. Then one has the following conclusions.*
(a) *For $(\sigma, \phi) \in D$, one has*

$$\gamma\big(F(\sigma, \phi) - 1\big) = 2 - \zeta - 2\gamma + \log \frac{2\gamma - \phi}{\phi}.$$

(b) *For $0 < \phi < \phi_k$ and $\sigma_k \leqslant \sigma \leqslant c_1(\phi)$, one has $E(\sigma, \phi) = F(\sigma, \phi)$ and*

$$\frac{\partial E}{\partial \sigma}(\sigma, \phi) < 0, \qquad \frac{\partial E}{\partial \phi}(\sigma, \phi) < 0.$$

(c) *For* $0 < \phi < \phi_k$*, one has* $F(c_1(\phi), \phi) < 1$.

Proof. Part (a) is a trivial computation, starting with (6.14) and applying the relation $\sigma = \gamma + \zeta$. For the first conclusion of part (b), we have recourse to (6.19) and see that for $\sigma \geqslant \sigma_k$, one has $2\gamma - \phi > 2\gamma - \phi_k \geqslant 2$. But for $\sigma \leqslant c_1(\phi)$, we have $\mathrm{H}(\sigma) \geqslant \phi/2 > \phi/(2\gamma - \phi)$. Lemma 6.4 now confirms that $E(\sigma, \phi) = F(\sigma, \phi)$ in the range currently under consideration.

Next we compute $\partial E/\partial \sigma$. Here we review the proof of Lemma 6.4 and see that in the range where $E(\sigma, \phi)$ is given by $F(\sigma, \phi)$ one may also express $E(\sigma, \phi)$ via (6.16) and (6.17) as

$$E(\sigma, \phi) = \frac{\tau_0(\sigma, \phi)}{\gamma} + \frac{2}{2\gamma - \phi},$$

in which $\tau_0(\sigma, \phi)$ is defined via (6.18). It is immediate that τ_0 and E are analytic functions of (σ, ϕ) on $[\sigma_k, \infty) \times (0, \phi^*)$. Thus, as functions of σ, both are continuously differentiable. On recalling that $\gamma = \sigma - \zeta$ with $\zeta = \zeta_k$ fixed, we find that

$$\frac{\partial E}{\partial \sigma} = \frac{\partial \tau_0/\partial \sigma}{\gamma} - \frac{\tau_0}{\gamma^2} - \frac{4}{(2\gamma - \phi)^2}$$

and

$$\frac{\partial \tau_0}{\partial \sigma} = -1 + \frac{2\phi}{(2\gamma - \phi)^2} + \frac{2}{2\gamma - \phi}. \tag{6.21}$$

We temporarily write $2\gamma - \phi = z$. We attempt to prove that $\partial \tau_0/\partial \sigma < 0$, noting that this inequality can be rewritten as

$$(z - 1)^2 - 1 - 2\phi > 0.$$

This is satisfied when $z > 1 + \sqrt{1 + 2\phi}$, and since $\phi < \phi_k < \phi^*$, we certainly have $\partial \tau_0/\partial \sigma < 0$ when

$$z > 1 + \sqrt{1 + 2\phi^*} = 2.345\ldots.$$

For these values of z we see that $\partial E/\partial \sigma < 0$ because $\tau_0(\sigma, \phi)$ is always positive. As we have noted at the outset of the proof, we may assume that $z = 2\gamma - \phi > 2$, and thus it suffices now to deal with the range $2 < z \leqslant 1 + \sqrt{1 + 2\phi^*}$. By (6.21), in this range for z, we have $\partial \tau_0/\partial \sigma < \phi/2 < \phi^*/2$. Moreover, since $2\gamma \geqslant 2 + \phi$, we have $\gamma \geqslant 1$. It therefore follows that

$$\frac{\partial E}{\partial \sigma} \leqslant \frac{\phi^*}{2} - \frac{4}{(1 + \sqrt{1 + 2\phi^*})^2} < 0.$$

In order to compute $\partial E/\partial \phi$, we use the relation $E(\sigma, \phi) = F(\sigma, \phi)$ and work from (6.14) to see that

$$\frac{\partial E}{\partial \phi} = \frac{2}{z^2} - \frac{1}{\gamma}\Big(\frac{1}{\phi} + \frac{2}{z} + \frac{\phi}{z^2}\Big) = \frac{2}{z^2} - \frac{(z + \phi)^2}{\gamma \phi z^2}.$$

Here again, we have used the abbreviation $z = 2\gamma - \phi$, and we recall that $z \geqslant 2$. Since $\gamma = \frac{1}{2}(z + \phi)$, it therefore follows that

$$\frac{\partial E}{\partial \phi} = \frac{2}{z^2} - \frac{2}{z^2}\Big(1 + \frac{z}{\phi}\Big) < 0.$$

For (c), we apply the identity described in conclusion (a) with $\sigma = c_1(\phi)$. With this choice, the expression in (a) defines a function $Y : (0, \phi_k] \to \mathbb{R}$ given by

$$\begin{aligned} Y(\phi) &= \big(c_1(\phi) - \zeta\big)\big(F(c_1(\phi), \phi) - 1\big) \\ &= 2 + \zeta - 2c_1(\phi) + \log \frac{2c_1(\phi) - 2\zeta - \phi}{\phi}. \end{aligned}$$

By (6.20) we have $Y(\phi_k) = 0$, and for $\phi \in (0, \phi_k)$ we have

$$c_1(\phi) > c_1(\phi_k) = \sigma_k > 2 > \zeta.$$

Hence, it suffices to prove that for the same ϕ one has $Y(\phi) < 0$. We achieve this by showing that Y is increasing on $(0, \phi_k]$, and with this approach in mind, we compute the derivative and find that $Y'(\phi)$ is positive on $(0, \phi_k)$. This then proves (c).

By reference to (2.12), we have $c_1'(\phi) = -\frac{1}{2} - \phi^{-1}$, and so

$$Y'(\phi) = 1 + \frac{1}{\phi} - \frac{2(1 + 1/\phi)}{2c_1(\phi) - 2\zeta - \phi} = \Big(1 + \frac{1}{\phi}\Big)\Big(1 - \frac{2}{2c_1(\phi) - 2\zeta - \phi}\Big).$$

Here, the denominator $2c_1(\phi) - 2\zeta - \phi$ is a specialisation of $2\gamma - \phi$ via the relation $\sigma = c_1(\phi)$, and for $\sigma > \sigma_k$ we know that $2\gamma - \phi > 2$. Hence the second factor in the rightmost expression is positive and we conclude that $Y'(\phi) > 0$. This completes the proof. ☐

Our main argument now comes to a close. By Lemma 6.5(b), we know that $E(\sigma_k, \phi)$ is a decreasing function of $\phi \in (0, \phi_k]$, and we already noted (following (6.20)) that $E(\sigma_k, \phi_k) = 1$. It follows that the lower bound $E(\sigma_k, \phi) > 1$ holds for $0 < \phi < \phi_k$. Now fix a choice for ϕ lying in the latter interval. Then $E(\sigma, \phi)$ is strictly decreasing for $\sigma_k \leqslant \sigma \leqslant c_1(\phi)$, thanks to Lemma 6.5(b) once more. Since the latter also shows that $E(c_1(\phi), \phi) = F(c_1(\phi), \phi)$, we deduce from Lemma 6.5(c) that $E(c_1(\phi), \phi) < 1$. Hence, there is exactly one value of σ with $\sigma_k < \sigma < c_1(\phi)$ having the property that $E(\sigma, \phi) = 1$. We denote this value of σ by $c_2(\phi)$. In conjunction with Lemma 6.5(b), the Implicit Function Theorem shows that $c_2(\phi)$ is an analytic function on the interval $(0, \phi_k)$.

Lemma 6.6. *Suppose that $k \geqslant 5$ and $0 < \phi < \phi_k$. Then*

$$\{\sigma \in [\sigma_k, \infty) : E(\sigma, \phi) < 1\} = (c_2(\phi), \infty).$$

Proof. The argument preceding the statement of the lemma shows that the real numbers $\sigma \in [\sigma_k, c_1(\phi)]$ with $E(\sigma, \phi) < 1$ form the interval $(c_2(\phi), c_1(\phi)]$. For $\sigma > c_1(\phi)$ we have $\mathrm{H}(\sigma) < \phi/2$ (because H is decreasing). By taking $\tau = 0$ in (6.13) we therefore see that $E(\sigma, \phi) \leqslant 2\mathrm{H}(\sigma)/\phi < 1$, as required. ☐

Now suppose that $k \geqslant 5$ and $0 < \phi < \phi_k$. Choose $\sigma_0 \geqslant \sigma_k$ with $E(\sigma_0, \phi) < 1$. Then, by (6.13), there is a number $\tau \geqslant 0$ with

$$\frac{\tau}{\sigma_0 - \zeta} + \frac{2\mathrm{H}(\sigma_0 + \tau)}{\phi} < 1.$$

Typically, the number $(\sigma_0+\tau)k$ will not be an even integer, but we may increase σ_0 to a number σ with $\sigma_0 \leqslant \sigma < \sigma_0 + 2/k$ for which $(\sigma+\tau)k$ is an even integer. In the preceding display, the left hand side is decreasing as a function of σ_0, so we have the upper bound (6.12). In view of the discussion around (6.12), we can now apply Theorem 6.3 with the admissible exponent $\Delta_{s+t} = \Delta_{(\sigma+\tau)k}$ provided by Lemma 4.4. We recall in this context that the latter theorem employs the hypotheses of Lemma 6.2, and in particular (6.10). By Lemma 6.6, we may take any $\sigma_0 > c_2(\phi)$ in this argument. In particular, Theorem 6.3 applies successfully whenever $s \geqslant c_2(\phi)k + 2$. We have thus established the following corollary of Theorem 6.3.

Corollary 6.7. *Fix a set of parameters with*

$$k \geqslant 5, \quad 0 < \phi < \phi_k, \quad s \geqslant c_2(\phi)k + 2$$

and a regular ϕ-weight satisfying Hypothesis H. Then the conclusions of Theorem 6.3 hold.

We found along the way that $c_2(\phi) < c_1(\phi)$ holds for all $\phi < \phi_k$, and thus the corollary is an improvement over Theorem 4.3 in all instances where it applies. We now compute $c_2(\phi)$ numerically for selected values of ϕ. It turns out that the dependence on k is marginal. To carry this out, we apply Lemma 6.5(a) and (b) to present the equation $E(\sigma, \phi) = 1$ in the form

$$2\gamma = 2 - \zeta + \log(2\gamma - \phi) - \log \phi,$$

and interpret this as an equation in $z = 2\gamma - \phi$. This recycles notation already used in the proof of Lemma 6.5, and the equation for z now reads

$$z - \log z = 2 - \zeta - \phi - \log \phi. \tag{6.22}$$

Recall that $\zeta = \zeta_k$, and hence also γ, depends on k. Moreover, the number ϕ_k is defined via the equation $\phi_k + \log \phi_k = \log 2 - \zeta_k$. Then for $\phi \leqslant \phi_k$ the smallest value of the right hand side of (6.22) (with $\zeta = \zeta_k$) is $2 - \log 2$. For $z > 1$ the left hand side is increasing in z, so the solution of (6.22) with $z > 1$ actually satisfies $z \geqslant 2$.

We now work out the impact of the dependence on k in the solution z of (6.22). For a given $k \geqslant 5$ and $\phi < \phi_k$, let $z_k(\phi)$ be the solution in $z > 1$ of (6.22) with $\zeta = \zeta_k$, and let $z^*(\phi)$ be the solution of (6.22) with $\zeta = \frac{1}{2} + \log 2 = \zeta^*$, say, this being the limit of the sequence (ζ_k). Let $\delta_k = \zeta_k - \zeta^*$. Then we have $0 < \delta_k < 2/k$. We let $g(z) = z - \log z$ and then subtract the equations for z_k and z^* defined via (6.22). This yields $g(z^*) - g(z_k) = \delta_k$. We now apply the mean value theorem. This gives us a real number $\alpha \in (z_k, z^*)$ with the property that $\delta_k = g'(\alpha)(z^* - z_k)$. But $g'(\alpha) < 1$ and hence $z^* - z_k > \delta_k$. Moreover, for a given k, the quantities $c_2(\phi)$ and $z_k(\phi)$ are linked via the equation $z_k(\phi) = 2(c_2(\phi) - \zeta_k) - \phi$. Thus, on writing

$$c_2^*(\phi) = \tfrac{1}{2} z^*(\phi) + \zeta^* + \tfrac{1}{2}\phi, \tag{6.23}$$

we deduce that

$$c_2(\phi) = \tfrac{1}{2} z_k(\phi) + \zeta^* + \delta_k + \tfrac{1}{2}\phi < c_2^*(\phi) + \tfrac{1}{2}\delta_k.$$

In particular, one has

$$c_2(\phi) < c_2^*(\phi) + 1/k.$$

We take the opportunity to make an asymptotic comparison of the values of $c_1(\phi)$ and $c_2^*(\phi)$ when ϕ is small.

Lemma 6.8. *As $\kappa \to \infty$, one has*

$$c_1\Big(\frac{1}{\kappa}\Big) = \log \kappa + 1 + \log 2 + O\Big(\frac{1}{\kappa}\Big)$$

and

$$c_2^*\Big(\frac{1}{\kappa}\Big) < \tfrac{1}{2}(\log \kappa + \log\log \kappa) + 1 + \tfrac{1}{2}\zeta^* + O\Big(\frac{\log\log \kappa}{\log \kappa}\Big).$$

Proof. On recalling (2.12), we find that

$$\begin{aligned} c_1\Big(\frac{1}{\kappa}\Big) &= 1 + \log(2\kappa) - \frac{1}{2\kappa} \\ &= \log \kappa + 1 + \log 2 + O\Big(\frac{1}{\kappa}\Big). \end{aligned}$$

At the same time, it follows from equation (6.22) that

$$z^*\Big(\frac{1}{\kappa}\Big) = \log \kappa + \log\log \kappa + 2 - \zeta^* + O\Big(\frac{\log\log \kappa}{\log \kappa}\Big),$$

and hence the bound on $c_2^*(1/\kappa)$ asserted in the statement of the lemma follows from (6.23). □

Suppose that ϕ is a small positive number. Then on writing $\kappa = 1/\phi$, the conclusion of Lemma 6.8 shows that $c_2^*(\phi)$ is no more than about half the size of $c_1(\phi)$.

Since we are primarily interested in upper bounds for $c_2(\phi)$, we have now isolated the dependence on k and are left with the task of solving the equation (6.22) with $\zeta = \zeta^*$. This is an easy job using Newton's iteration. The table below lists some representative values of ϕ, with the associated values of $z^*(\phi)$ and $c_2^*(\phi)$, the latter given by (6.23), rounded up in the last digit displayed. For comparison, the last column records the value of $c_1(\phi) = 1 - \frac{1}{2}\phi - \log(\phi/2)$, again rounded up in the last digit displayed. Along the way we compute $2 - \zeta^* - \phi - \log \phi$, and this value we also list rounded up in the last digit displayed. We recall in this context that $\zeta^* = \frac{1}{2} + \log 2$.

ϕ	$2 - \zeta^* - \phi - \log \phi$	$z^*(\phi)$	$c_2^*(\phi)$	$c_1(\phi)$
3/8	1.41268208	2.2020882	2.481692	2.486477
5/16	1.65750363	2.6210963	2.659946	2.700048
1/4	1.94314719	3.0623200	2.849308	2.954442
3/16	2.29332926	3.5642958	3.069046	3.273374
1/6	2.43194563	3.7550463	3.154004	3.401574
1/8	2.76129437	4.1952465	3.353271	3.710089
1/16	3.51694155	5.1573680	3.803082	4.434486
1/32	4.24133873	6.0396917	4.228619	5.143259
1/64	4.95011091	6.8785135	4.640217	5.844218
1/128	5.65107059	7.6911396	5.042624	6.541272

We give two applications of Corollary 6.7. The first one will complete the proof of Theorem 1.4.

Proof of Theorem 1.4, part II. First observe that the number $\widetilde{c}$ defined in the preamble of Theorem 1.4 is $c_2^*\big(\frac{1}{8}\big)$, and thus $c_2\big(\frac{1}{8}\big) \leqslant \widetilde{c} + 1/k$. Next, recall that the squares of primes form a regular $\frac{1}{8}$-set, and then note that Hypothesis H is satisfied. We may apply Corollary 6.7 and Chebyshev's lower bound to conclude that when $s \geqslant c_2\big(\frac{1}{8}\big)k + 2$ and k is not a power of 2, then one has $\widetilde{r}_{k,s}(n) \gg n^{s/k-1/2}(\log n)^{-1}$ as desired. In particular, the latter asymptotic lower bound holds when $s \geqslant \widetilde{c}k + 3$. If $k \geqslant 8$ is a power of 2 then one still arrives at the same conclusion by an elaboration of the argument presented in the proof of the first clause of Theorem 1.1. □

Our second application is of a somewhat different nature. We consider the exponential sum

$$B(\alpha; M) = \sum_{\substack{p_1 \leqslant M \\ p_1 \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{\substack{p_2 \leqslant M^2 \\ p_2 \equiv 1 \ (\mathrm{mod}\ 3)}} e(\alpha p_1^2 p_2^2). \tag{6.24}$$

This sum behaves like an exponential sum of a regular $\frac{1}{6}$-weight supported on the squares, as we now demonstrate.

Lemma 6.9. *Suppose that $j \in \{1, 2\}$. Then one has*

$$B(j\alpha, n^{1/6}) \ll n^{1/2} L^{-2} \Upsilon(\alpha)^{\varepsilon - 1/6}.$$

This lemma follows by a routine type II sum argument. Because this is hardly the point of the current communication, we postpone a proof to Section 9 and concentrate on its application. We wish to use Corollary 6.7 with $W(\alpha) = B(j\alpha; n^{1/6})$ and $j \in \{1, 2\}$. If $j = 1$, this corresponds to the weight w_B defined by taking $w_B(m) = 1$ when $m = p_1^2 p_2^2$, with primes p_1 and p_2 in the class 1 modulo 3 and $p_1 \leqslant n^{1/6}$, $p_2 \leqslant n^{1/3}$, but with $w_B(m) = 0$ otherwise. This situation does not exactly fit inside the framework described in Section 2 because w_B now depends on n. Fortunately this is not a serious obstacle. The integral

$$\int_0^1 B(\alpha; n^{1/6}) f(\alpha)^s e(-\alpha n)\, \mathrm{d}\alpha \tag{6.25}$$

still provides a lower bound for the number of solutions of (1.6) with $x = p_1 p_2$ and p_1, p_2 constrained as before. All later stages of the argument leading to Theorem 6.3 only involve the values $w(m)$ for $m \leqslant n$, so that we may safely apply the conclusion of Corollary 6.7 to the situation currently under consideration.

Theorem 6.10. *Let $k \geqslant 5$, and let $s \geqslant c_2^*\big(\frac{1}{6}\big)k + 3$. Then, for sufficiently large n, the number $\nu^*(n)$ of solutions of the equation (1.6) in natural numbers, with the variable x restricted to E_2-numbers, satisfies $\nu^*(n) \gg n^{s/k-1/2}(\log n)^{-2}$.*

Here an E_2-number is a natural number with exactly two prime factors. Readers who prefer to have two distinct prime factors may add the condition

$p_1 \neq p_2$ to the definition of B. This introduces an acceptable error of size $n^{1/6}$ in the upper bound presented in Lemma 6.9.

Next we apply Corollary 6.7 to the situation described in (6.25), but now with $B(2\alpha; n^{1/6})$ in place of $B(\alpha; n^{1/6})$. Then we draw a conclusion analogous to that of Theorem 6.10, but now for the analogue $\nu^\dagger(n)$ of $\nu^*(n)$ counting the solutions of the Diophantine problem

$$2(p_1p_2)^2 + y_1^k + y_2^k + \cdots + y_s^k = n,$$

with p_1 and p_2 primes in the class 1 modulo 3 and $p_1 \leqslant n^{1/6}$, $p_2 \leqslant n^{1/3}$. The reader is invited to check that the extra factor 2 does no harm to the congruential constraints modulo $4k$ when k is a power of 2. We now use an idea of Kawada and Wooley [21]. We apply the identity

$$u^4 + v^4 + (u+v)^4 = 2(u^2 + uv + v^2)^2,$$

and observe that for primes p_1 and p_2 in the class 1 modulo 3, there are integers u and v with $p_1p_2 = u^2 + uv + v^2$. This follows from the theory associated with the quadratic number field $\mathbb{Q}(\sqrt{-3})$. This argument shows that the summand $2(p_1p_2)^2$ is a sum of three integral fourth powers, and we conclude as follows.

Theorem 6.11. *Let $k \geqslant 5$, and let $s \geqslant c_2^*\big(\frac{1}{6}\big)k + 3$. Then, for sufficiently large n, the Diophantine equation*

$$x_1^4 + x_2^4 + x_3^4 + y_1^k + y_2^k + \cdots + y_s^k = n \tag{6.26}$$

has solutions in non-negative integers.

There is a more direct approach to the representation problem considered in this theorem. In fact, it would be natural to replace the exponential sum $B(2\alpha; n^{1/6})$, utilised in the treatment above, by $W(\alpha) = g_4(\alpha)^3$, with $g_4(\alpha)$ defined in (2.5). Then the integral (3.1) counts solutions of (6.26). Again, this choice of W does depend on n, but as in the case of the sum B, the method still applies with this choice of $W = g_4^3$. By Lemma 2.1, this corresponds to a $\frac{3}{16}$-weight, and we may apply Theorem 4.3. We then find that the equation (6.26) has solutions when $s > c_1\big(\frac{3}{16}\big)k + 2$. However, the tabulated data show that $c_1\big(\frac{3}{16}\big) = 3.2733\ldots$ and $c_2^*\big(\frac{1}{6}\big) = 3.1540\ldots$, so the exotic approach toward Theorem 6.11 spares about 4 percent of the k-th powers.

7. The role of Hölder's inequality

The methods described in the last two sections have a competitor that is far more familiar to workers in the field. As experts may have already recognised, the success of our new version of pruning by size for squares depends heavily on the mean value bound (6.3), and the latter is an instance of Lemma 6.1. The mean value is brought into play by slicing the minor arcs according to the size of $|W(\alpha)|$, as in the dissection (6.5). There is another very natural way to make the minor arc estimate depend on the mean value (6.3). Suppose that a set of parameters is given, including a weight that satisfies Hypothesis H. Then the minor arc integral can be bounded via Schwarz's inequality, yielding

$$\int_{\mathfrak{l}} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \leqslant \left(\int_0^1 |W(\alpha)f(\alpha)^r|^2\,\mathrm{d}\alpha\right)^{1/2}\left(\int_{\mathfrak{l}} |f(\alpha)|^{2s-2r}\,\mathrm{d}\alpha\right)^{1/2}. \quad (7.1)$$

As before, we choose r to be the smallest positive integer for which (6.3) applies. Then, if s is so large that there exists a number $\delta > 0$ with

$$\int_{\mathfrak{l}} |f(\alpha)|^{2s-2r}\,\mathrm{d}\alpha \ll P^{2s-2r-k-3\delta}, \quad (7.2)$$

then, by (7.1) and (6.3), one has

$$\int_{\mathfrak{l}} |W(\alpha)f(\alpha)^s|\,\mathrm{d}\alpha \ll n^{1/2}P^{s-k-\delta}.$$

This is a satisfactory minor arcs bound that combines well with the major arc work in Section 3, especially Theorem 3.3. It follows that, subject to (7.2) and the conditions on s imposed in Theorem 3.3, one has the conclusion of Theorem 6.3 for regular weights satisfying Hypothesis H. Note that when $s \geqslant 2k+5$ one does not even need to assume that w is a ϕ-weight here.

The downside of this approach is that when k is large, we require $2s-2r$ to be of size $k\log k + O(k)$ for an estimate of strength sufficient to meet the condition (7.2). The indicator function of the squares of primes satisfies Hypothesis H and is regular, so this argument confirms the estimate $G_2(k) \leqslant (\frac{1}{2}+o(1))k\log k$ that we acknowledged in the introductory part of this memoir to be part of the folklore. The argument also solves (1.6) with x restricted to primes when $s \geqslant (\frac{1}{2}+o(1))k\log k$. These results are not competitive with the work in this paper, but when k is small, the bound (7.1) is surprisingly efficient. It is easy to see that when $k = 3$ or 4, then (6.3) holds with $r = 2$. Further, improving on our earlier work [5, Theorem 2], it was shown in [41, Theorem 1.4] that when $k = 3$ one has

$$\int_0^1 |f(\alpha)|^{38/5}\,\mathrm{d}\alpha \ll P^{23/5}. \quad (7.3)$$

When $k = 4$, meanwhile, we found in [6, Theorem 1.2] that there is a $\theta > 0.044$ with the property that

$$\int_0^1 |f(\alpha)|^{12-\theta}\,\mathrm{d}\alpha \ll P^{8-\theta}. \quad (7.4)$$

One then readily confirms that (7.2) holds with $k = 3$ whenever $2s-2r \geqslant 8$, and with $k = 4$ whenever $2s-2r \geqslant 12$. If we choose $W(\alpha)$ as the exponential sum over squares of primes, for example, then we get the expected lower bound for $\widetilde{r}_{3,6}(n)$ and $\widetilde{r}_{4,8}(n)$. This substantiates the comment that follows the statement of Theorem 1.4, at least for $k = 3$ and 4. For $k = 5, 6$ and 7, one finds from Lemma 6.1 that the bound (6.3) holds with $r = 3, 4$ and 4, respectively. One may then use the same strategy as in the cases $k = 3$ and 4, but unfortunately the required versions of (7.2) are not as easy to cite. In fact, when $k = 5$, the desired bound (7.2) when $s - r = 9$ can be obtained by quoting directly from [34], but adjusting the setup to allow two of the implicit fifth powers to run over the natural numbers. When $k = 6$ we seek a bound analogous to (7.2) when $s - r = 12$, and here one must reach for the more delicate tools made available in [33]. Finally, when $k = 7$, the methods of [44] apply when $s - r = 16$ by again adjusting the setup to allow four of the implicit seventh powers to run over the natural numbers. If one follows this line of thought for $k = 8$, one requires $r \geqslant 5$ and $s - r \geqslant 20$, so it is here where Theorem 1.4 starts to improve upon the simplistic approach.

We return now to the observation that in the direct approach outlined above, we require $2(s-r) \geqslant k \log k + O(k)$. It is possible to modify (7.1), using Hölder's inequality to decrease the weight of the first factor. This makes the dependence between s and k again linear. It turns out that this leads to another proof of Lemma 6.2. In an effort to ease comparison, we apply the same notation as in Section 6. Thus, we fix a set of parameters including a ϕ-weight that satisfies Hypothesis H, and we suppose that s is so large that a natural number r can be chosen with $2r \leqslant s$ and $2\Delta_{2r} \leqslant k$ (this is (6.2)). When $P^{1/2} \leqslant Q \leqslant P^{k/2}$, we then have to estimate the mean value

$$J = \int_{\mathfrak{N}(Q)} |W(\alpha) f(\alpha)^s| \, \mathrm{d}\alpha$$

defined already in (6.4), and the goal is to establish the bound $J \ll n^{-1/2} P^{s-\delta}$, for some $\delta > 0$, uniformly in the indicated range for Q.

We now apply Hölder's inequality to the integral J. Then, for $0 \leqslant v \leqslant \frac{1}{2}$ we infer that

$$J \ll \Big(\sup_{\alpha \in \mathfrak{N}(Q)} |W(\alpha)| \Big)^{1-2v} \Big(\int_0^1 |W(\alpha)^2 f(\alpha)^{2r}| \, \mathrm{d}\alpha \Big)^{v} \Big(\int_{\mathfrak{N}(Q)} |f(\alpha)|^b \, \mathrm{d}\alpha \Big)^{1-v},$$

where $b = b(v)$ is defined by means of the equation

$$s = 2rv + (1 - v)b. \tag{7.5}$$

Note that the special case $v = 0$ is the initial step towards Theorem 4.3, while the situation $v = \frac{1}{2}$ corresponds to the application of Schwarz's inequality in (7.1). Also, we observe that the constraint $s \geqslant 2r$ implies that $b \geqslant s$. We therefore write $b = s + t$ and then have $t \geqslant 0$. In this notation, we recall the upper bound $W(\alpha) \ll n^{1/2+\varepsilon} Q^{-\phi}$, and then apply (6.3) and Lemma 4.1 to conclude that

$$\begin{aligned} J &\ll \big(n^{1/2+\varepsilon} Q^{-\phi}\big)^{1-2v} \big(P^{2r+\varepsilon}\big)^{v} \big(P^{b-k+\varepsilon} Q^{2\Delta_{s+t}/k}\big)^{1-v} \\ &\ll n^{2\varepsilon - 1/2} P^s Q^{-A}, \end{aligned}$$

where

$$A = \phi(1 - 2v) - 2(1 - v)\Delta_{s+t}/k.$$

If there exists some $v \in [0, \frac{1}{2}]$ where $A > 0$, then we have the upper bound (6.9), as required for a successful conclusion. We rewrite the condition $A > 0$ in the form

$$\frac{2\Delta_{s+t}}{k} < \Big(1 - \frac{v}{1 - v}\Big)\phi \tag{7.6}$$

and then check from (7.5) that $t/(s-2r) = v/(1-v)$ to realize that (7.6) is the condition (6.10) that appears in Lemma 6.2. Some care is still required in order to interpret the relationship between the two approaches. Thus, in Lemma 6.2 we see that the allowed range for t is $t \geqslant 0$. However, admissible exponents are non-negative, so whenever (6.10) holds then $t \leqslant s-2r$. In the current situation $b(v)$ is increasing from $b(0) = s$ to $b(\frac{1}{2}) = 2s-2r$, so t varies from 0 to $s-2r$. Consequently, we see that the approach outlined above based on the application of Hölder's inequality does indeed suffice to complete this new proof of Lemma 6.2.

The reader may well wonder whether the two approaches that we have presented, one based on pruning by height, and the alternate based on the application of Hölder's inequality, are identical save for the outfits in which one finds them garbed. Certainly, it seems that the two approaches lead to the same conclusions for the problem that has been our focus herein. We would observe that the availability of two seemingly different approaches often facilitates the first solution to a problem, with one approach more readily accessible to the most natural mode of thinking about the problem. Only in hindsight does one realise that the alternate approach, often involving a less intuitively obvious choice of parameters, nonetheless achieves the desired objective. Thus, we would argue that the availability of two approaches, even if ultimately equivalent for the problem at hand, should propel progress through the flexibility to adopt the most intuitive line of attack. There may also be situations, more complex than those examined in this work, wherein one or other of the two approaches offers definite quantitative advantage.

8. Outlook

So far, we have described general methods to estimate the convolution sum $\nu(n)$ defined in (2.2), and we proposed refinements designed for the Diophantine equation (1.6). Even in the latter narrower environment, potential applications of major arc moments are by no means exhausted. Also, there are further natural questions related to equation (1.6). To mention just a single example, one may restrict the variable x in (1.6) to the squares, or to some other higher power. The resulting Diophantine equations are the special cases of even exponents h in the family of representation problems

$$x^h + y_1^k + y_2^k + \cdots + y_s^k = n, \tag{8.1}$$

where now h and k are given natural numbers. The methods of this paper apply favourably when k is significantly larger than h. Indeed, if $h \geqslant 6$, then Lemma 2.1 shows that the h-th powers form a ϕ-set, with $\phi = 2/(h^2(h-1))$. Observe that in this situation, one has

$$\begin{aligned} c_1(\phi) &= 1 + \log 2 - \tfrac{1}{2}\phi - \log\phi \\ &= 1 + 3\log h + \log\Big(1 - \frac{1}{h}\Big) - \frac{1}{h^2(h-1)}, \end{aligned}$$

so that $c_1(\phi) < 3\log h + 1$. Hence, by Theorem 2.5, whenever $h \geqslant 6$ and n is sufficiently large (in terms of h and k), then whenever

$$s \geqslant 3k\log h + k + 2,$$

the equation (8.1) has solutions in non-negative integers. The essence of this result is that for fixed h, we are able to handle the equation (8.1) when s has dependence on k within the linear regime. The factor $3\log h + 1$ can be significantly improved.

Theorem 8.1. *Let h and k be natural numbers, and suppose that*

$$s > (2\log h + 3.20032)k + 2.$$

Then, for sufficiently large n, the equation (8.1) *has solutions in natural numbers.*

This theorem is the special case $k_1 = h$ and $k_j = k$ $(j \geqslant 2)$ of [8, Theorem 1.1]. Besides the ideas developed in Section 4, the key ingredient is an estimate of Weyl's type for smooth exponential sums that we restate here in a language that fits with the terminology of Section 2.

Lemma 8.2. *Let $D = 4.5139506$, and let $k \geqslant 3$. Then there is a number $\eta > 0$ such that whenever $2 \leqslant R \leqslant P^\eta$, one has*

$$f(\alpha; P, R) \ll P\Upsilon(\alpha)^{1/(Dk^2)}.$$

Proof. This is immediate from [8, Theorem 3.5]. □

Equipped with this lemma, the reader should be able to prove Theorem 8.1 within the philosophical framework of this memoir. We provide a manual for this exercise. By Lemma 8.2, one finds that the set $\{x^h : x \in \mathscr{A}(n^{1/h}, n^{\eta/h})\}$ is a $1/(Dh^2)$-set. This statement has to be taken with a grain of salt, but rectification requires no idea other than the bypass chosen in the proof of Theorem 6.10. Now apply Theorem 2.5 to deduce Theorem 8.1.

The most attractive instances of (8.1) are perhaps the cases where h is small. To stay in tune with our main theme, we briefly discuss the case $h = 4$, corresponding to the restriction of x in (1.6) to squares. The biquadrates form a $\frac{1}{16}$-set, whence Theorem 2.5 shows that whenever $s \geqslant c_1\big(\frac{1}{16}\big)k + 2$ and n is sufficiently large, then the equation

$$x^4 + y_1^k + y_2^k + \cdots + y_s^k = n$$

has solutions in natural numbers. Here, we note that $c_1\big(\frac{1}{16}\big) < 4.4345$.

The work in Sections 6 and 7 suggests that one should be able to improve this very simplistic approach. One would desire a substitute for Lemma 6.1 or (6.3), and this should take the shape

$$\int_0^1 |g_4(\alpha)^2 f(\alpha)^{2r}|\,\mathrm{d}\alpha \ll n^{\varepsilon - 1/2}P^{2r}. \tag{8.2}$$

Here the natural number r should be as small as is possible. The integral in equation (8.2) has a Diophantine interpretation, and as experts in the field would expect, one can extract an "efficient differencing variable" from the

k-th powers to difference the biquadrates. Estimates as strong as (8.2) are within the competence of the circle method when applied to the differenced Diophantine equation. A precursory examination of the matter suggests that one should apply differencing restricted to minor arcs (see [33]) for better performance, and one may then expect a visible improvement of the condition $s \geqslant c_1\left(\frac{1}{16}\right)k+2$. Limitations on space and time force us to postpone a thorough discussion of the matter to another occasion where we intend to illustrate the favourable interplay of our principal new tool, the major arc moment estimates, with differencing processes of various kinds.

9. Appendix: An exponential sum

The sole purpose of this section is to prove Lemma 6.9. We begin with (6.24), taking $M = n^{1/6}$, and apply Cauchy's inequality to the sum over p_2. Thus, we obtain

$$|B(\alpha; M)|^2 \leqslant M^2 \sum_{m \leqslant M^2} \Bigg| \sum_{\substack{p \leqslant M \\ p \equiv 1 \ (\mathrm{mod}\ 3)}} e(\alpha m^2 p^2) \Bigg|^2 .$$

Here we open the square and see that

$$|B(\alpha; M)|^2 \leqslant M^2 \sum_{\substack{p_1, p_2 \leqslant M \\ p_1 \equiv p_2 \equiv 1 \ (\mathrm{mod}\ 3)}} \sum_{m \leqslant M^2} e(\alpha (p_1^2 - p_2^2) m^2).$$

The terms with $p_1 = p_2$ make a total contribution of at most M^5 to the right hand side. For $p_1 \neq p_2$, the factor $p_1^2 - p_2^2$ is non-zero. For a given non-zero integer l with $|l| \leqslant M^2$, a divisor function argument shows that the number of solutions of the equation $p_1^2 - p_2^2 = l$, with $p_1, p_2 \leqslant M$, is at most $O(l^\varepsilon)$. Another application of Cauchy's inequality therefore yields the bound

$$\begin{aligned} |B(\alpha; M)|^4 &\ll \Bigg(M^5 + M^{2+\varepsilon} \sum_{1 \leqslant l \leqslant M^2} \Bigg| \sum_{m \leqslant M^2} e(\alpha l m^2) \Bigg| \Bigg)^2 \\ &\ll M^{10} + M^{6+2\varepsilon} \sum_{1 \leqslant l \leqslant M^2} \sum_{1 \leqslant m_1, m_2 \leqslant M^2} e(\alpha l (m_1^2 - m_2^2)). \end{aligned}$$

Accounting for the diagonal contribution $m_1 = m_2$ in the inner sum, we therefore deduce that

$$|B(\alpha; M)|^4 \ll M^{10+\varepsilon} + M^{6+\varepsilon} \sum_{1 \leqslant m_2 < m_1 \leqslant M^2} \min\left\{ M^2, \|\alpha(m_1^2 - m_2^2)\|^{-1} \right\}.$$

Another divisor function argument paralleling that above consequently delivers the upper bound

$$|B(\alpha; M)|^4 \ll M^{10+\varepsilon} + M^{6+\varepsilon} \sum_{1 \leqslant m \leqslant M^4} \min\left\{ M^6/m, \|\alpha m\|^{-1} \right\}.$$

We next apply a standard reciprocal sums lemma (see [31, Lemma 2.2]). This shows that whenever a and q are coprime with $|q\alpha - a| \leqslant 1/q$, one has

$$|B(\alpha;M)|^4 \ll M^{12+\varepsilon}\Bigl(\frac{1}{q}+\frac{1}{M^2}+\frac{q}{M^6}\Bigr).$$

A familiar transference principle then yields the bound

$$B(\alpha;n^{1/6}) \ll n^{1/2+\varepsilon}\Upsilon(\alpha)^{-1/6}. \tag{9.1}$$

For this argument, we may refer to [31, Exercise 2 of Section 2.8] or [42, Lemma 14.1]. Unfortunately, the estimate (9.1) only proves what is claimed in Lemma 6.9 when $\alpha \in [0,1]\setminus \mathfrak{M}(n^{1/24})$, say. This is because of the presence of the factor n^ε. However, the exponential sum estimates of Kumchev [22, Theorem 3], directly applied to the longer sum over p_2, cover the situation when $\alpha \in \mathfrak{M}(n^{1/24})\setminus \mathfrak{M}(L^A)$, provided that the positive number A is taken large enough. When $\alpha \in \mathfrak{M}(L^A)$, meanwhile, one may refer to the standard literature concerning Weyl sums over prime numbers (see [20, Lemmata 7.15 and 7.16], for example). We may leave this routine part of the argument to the reader.

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