EQUIDISTRIBUTION OF POLYNOMIAL SEQUENCES IN FUNCTION FIELDS: RESOLUTION OF A CONJECTURE

J. CHAMPAGNE, Z. GE, T. H. LÊ, Y.-R. LIU, AND T. D. WOOLEY

Abstract. Let $\mathbb{F}_q$ be the finite field of q elements having characteristic p, and denote by $\mathbb{K}_\infty = \mathbb{F}_q((1/t))$ the field of formal Laurent series in $1/t$. We consider the equidistribution in $\mathbb{T} = \mathbb{K}_\infty/\mathbb{F}_q[t]$ of the values of polynomials $f(u) \in \mathbb{K}_\infty[u]$ as u varies over $\mathbb{F}_q[t]$. Let $\mathcal{K}$ be a finite set of positive integers, and suppose that $\alpha_r \in \mathbb{K}_\infty$ for $r \in \mathcal{K} \cup \{0\}$. We show that the polynomial $\sum_{r\in\mathcal{K}\cup\{0\}} \alpha_r u^r$ is equidistributed in $\mathbb{T}$ whenever α_k is irrational for some $k \in \mathcal{K}$ satisfying $p \nmid k$, and also $p^v k \notin \mathcal{K}$ for any positive integer v. This conclusion resolves in full a conjecture made jointly by the third, fourth and fifth authors.

1. Introduction

Our focus in this paper lies on an analogue for function fields of the equidistribution theory initiated by Weyl [8]. Write $\mathbb{Z}^+$ for the set of positive integers and $\{a\}$ for the fractional part of a real number a, by which we mean $a - \lfloor a \rfloor$, where $\lfloor a \rfloor$ denotes the largest integer not exceeding a. We recall that in the classical setting, a sequence $(s_n)_{n=1}^\infty$ of real numbers is said to be *equidistributed modulo* 1 if, for any interval $[\alpha, \beta] \subset [0, 1)$, we have

$$\lim_{N\to\infty} N^{-1}\text{card}\{n \in [1, N] \cap \mathbb{Z}^+ : \{s_n\} \in [\alpha, \beta]\} = \beta - \alpha.$$

The celebrated observation of Weyl is that whenever $f(u) = \sum_{r=0}^k \alpha_r u^r$ is a polynomial with real coefficients in which one, at least, of the coefficients $\alpha_1, \ldots, \alpha_r$ is irrational, then the sequence $(f(n))_{n=1}^\infty$ is equidistributed modulo 1. Although an analogue of this conclusion in the function field setting fails in general, in this paper we are able to derive a conclusion that is the best possible analogue involving an irrationality hypothesis on only one coefficient.

In order to describe our new conclusions, we require some notational infrastructure. Let $\mathbb{F}_q$ denote the finite field of q elements whose characteristic is p, and suppose that $q = p^m$ throughout. Let $\mathbb{K} = \mathbb{F}_q(t)$ be the field of fractions of the associated polynomial ring $\mathbb{F}_q[t]$. Given $f/g \in \mathbb{K}$, with $f, g \in \mathbb{F}_q[t]$ and $g \neq 0$, we define the norm $|f/g| = q^{\deg f - \deg g}$, with the convention that $\deg 0 = -\infty$. The completion of $\mathbb{K}$ with respect to this norm is $\mathbb{K}_\infty = \mathbb{F}_q((1/t))$, the field of formal Laurent series in $1/t$. Thus, the elements α of $\mathbb{K}_\infty$

2020 *Mathematics Subject Classification.* 11J71, 11T55.

Key words and phrases. Equidistribution, function field analogues.

The third author is supported by NSF grant DMS-2246921 and a travel award from the Simons Foundation, the fourth author is supported by an NSERC discovery grant, while the fifth author is supported by NSF grant DMS-2502625 and Simons Fellowship in Mathematics SFM-00011955. The fifth author is grateful to the Institute for Advanced Study, Princeton, for hosting his sabbatical, during which period this paper was completed.

can be written in the form $\alpha = \sum_{i=-\infty}^{n} a_i t^i$ for some $n \in \mathbb{Z}^+$ and $a_i \in \mathbb{F}_q$ $(i \leqslant n)$. We may define $\operatorname{ord} \alpha$ for $\alpha \in \mathbb{K}_\infty$ by putting $\operatorname{ord} 0 = -\infty$, and otherwise

$$\operatorname{ord}\left(\sum_{i=-\infty}^{n} a_i t^i\right) = \max\{i \in \mathbb{Z} : a_i \neq 0\}.$$

One then has the norm on $\mathbb{K}_\infty$, given by $|\alpha| = q^{\operatorname{ord} \alpha}$, that coincides for $\alpha \in \mathbb{K}$ with the norm that we defined earlier. We note in passing that the coefficient a_{-1} here is called the *residue* of α, denoted by $\operatorname{res}(\alpha)$. Thus, when $n \in \mathbb{Z}^+$, we have

$$\operatorname{res}\left(\sum_{i=-\infty}^{n} a_i t^i\right) = a_{-1}. \tag{1.1}$$

As will be familiar to conversant readers, the function field analogues of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are respectively $\mathbb{F}_q[t]$, $\mathbb{K}$ and $\mathbb{K}_\infty$. Let

$$\mathbb{T} = \{\alpha \in \mathbb{K}_\infty : \operatorname{ord} \alpha < 0\}.$$

This compact subgroup of $\mathbb{K}_\infty$ is the analogue of the unit interval $[0, 1)$ in the classical setting. When $\alpha = \sum_{i=-\infty}^{n} a_i t^i \in \mathbb{K}_\infty$ with $n \in \mathbb{Z}^+$, we define

$$\{\alpha\} = \sum_{i \leqslant -1} a_i t^i \in \mathbb{T}$$

to be the *fractional part* of α. Let μ be a normalized Haar measure on $\mathbb{T}$ such that $\mu(\mathbb{T}) = 1$. A function field analogue of equidistribution modulo 1 was introduced by Carlitz [3, §4].

Definition 1.1. Let f be a $\mathbb{K}_\infty$-valued function defined over $\mathbb{F}_q[t]$. We say that $(f(u))_{u \in \mathbb{F}_q[t]}$ is *equidistributed in* $\mathbb{T}$ if, for every open ball $\mathcal{B} \subset \mathbb{T}$, one has

$$\lim_{N \to \infty} q^{-N} \operatorname{card}\{u \in \mathbb{F}_q[t] : \operatorname{ord} u < N \text{ and } \{f(u)\} \in \mathcal{B}\} = \mu(\mathcal{B}).$$

The reader will find in [6, Definition 1.1] an alternative definition involving *cylinder sets*. A moment of reflection will reveal the latter definition to be equivalent to that of Definition 1.1 above.

The problem of finding a proper function field analogue of Weyl's equidistribution theorem for polynomials is the main subject of this paper, and is a problem that was first considered by Carlitz [3] in 1952. We recall that an element $\alpha \in \mathbb{K}_\infty$ is called *irrational* when $\alpha \notin \mathbb{K}$. In [3, Theorem 12], Carlitz shows that when $f \in \mathbb{K}_\infty[x]$ has degree less than p, and $f(x) - f(0)$ has an irrational coefficient, then $(f(u))_{u \in \mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$. This conclusion was subsequently recovered by Dijksma [4] with a stronger notion of equidistribution. As remarked by Carlitz himself, the condition $\deg f < p$ cannot be removed, as there exist uncountably many irrational elements $\alpha \in \mathbb{K}_\infty$ having the property that $(\alpha u^p)_{u \in \mathbb{F}_q[t]}$ is not equidistributed in $\mathbb{T}$ (see also the discussion of [6, Example 1.2]). However, the third, fourth and fifth authors have obtained equidistribution results that address many situations in which $\deg f \geqslant p$ (see [6, Theorem 1.4 and Proposition 5.2]). These conclusions make progress towards a conjecture made by these authors (see [6, Conjecture 1.3]) that, if true, would be the best possible result that could hold in which an irrationality hypothesis is imposed on a single coefficient. Our main goal in this paper is to resolve this conjecture in full.

Theorem 1.2. *Let $\mathcal{K}$ be a finite set of positive integers, suppose that $\alpha_r \in \mathbb{K}_\infty$ for $r \in \mathcal{K} \cup \{0\}$, and define*

$$f(x) = \sum_{r \in \mathcal{K} \cup \{0\}} \alpha_r x^r.$$

Suppose that α_k is irrational for some $k \in \mathcal{K}$ satisfying $p \nmid k$ and furthermore $p^v k \notin \mathcal{K}$ for any $v \in \mathbb{Z}^+$. Then the sequence $(f(u))_{u \in \mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$.

A related though less definitive conclusion is obtained in [6, Theorem 1.4]. In order to describe this result, given a set of positive integers $\mathcal{K}$, we introduce the *shadow* of $\mathcal{K}$, defined to be the set

$$\mathcal{S}(\mathcal{K}) = \left\{ j \in \mathbb{Z}^+ : p \nmid \binom{r}{j} \text{ for some } r \in \mathcal{K} \right\}.$$

Notice that $\mathcal{K} \subseteq \mathcal{S}(\mathcal{K})$ for all sets $\mathcal{K}$ of positive integers. The conclusion of [6, Theorem 1.4] is similar to that of Theorem 1.2 above, save that the condition $p^v k \notin \mathcal{K}$ for any $v \in \mathbb{Z}^+$, in the latter, is replaced by the more stringent hypothesis $p^v k \notin \mathcal{S}(\mathcal{K})$ for any $v \in \mathbb{Z}^+$.

A conclusion slightly more general than that of [6, Theorem 1.4] is obtained in [6, Proposition 5.2]. This too falls short of establishing [6, Conjecture 1.3], now established in Theorem 1.2 above. Consider again a set $\mathcal{K}$ of positive integers. In order to describe [6, Proposition 5.2], we begin by defining the set

$$\mathcal{K}^* = \{k \in \mathcal{K} : p \nmid k \text{ and } p^v k \notin \mathcal{S}(\mathcal{K}) \text{ for any } v \in \mathbb{Z}^+\}.$$

We now put $\mathcal{K}_0 = \mathcal{K}$, and inductively define for each $n \geqslant 1$ the set

$$\mathcal{K}_n = \mathcal{K}_{n-1} \setminus \mathcal{K}^*_{n-1}.$$

We then define the set of indices

$$\widetilde{\mathcal{K}} = \bigcup_{n=0}^{\infty} \mathcal{K}^*_n. \tag{1.2}$$

With this notation, the conclusion of [6, Proposition 5.2] shows that the conclusion of Theorem 1.2 holds whenever α_k is irrational for some $k \in \widetilde{\mathcal{K}}$. As we have noted already, this condition is again more stringent than that imposed in Theorem 1.2.

We remark that, with little additional effort, the improvement of [6, Theorem 1.4] embodied in Theorem 1.2 can be imported also into the conclusions of [6, Theorems 6.3 and 7.3]. We leave to the reader the pedestrian details of confirming such enhancements.

The conclusion of Theorem 1.2 has been claimed in recent work of Ackelsberg [1, Theorem 1.2], based on earlier work of Bergelson and Leibman [2, Theorem 0.3]. The latter work is based on a variant of van der Corput differencing. When working in rings of positive characteristic p, van der Corput differencing shares the same defect as Weyl differencing, in that the number of differencing steps taken is limited by this characteristic, potentially preventing the exponential sum under discussion from being bounded non-trivially when it contains monomials of degree larger than p. Although Bergelson and Leibman take care to work around obstacles that this well-known observation imposes, there is apparently an infelicity in some aspect of their argument. Indeed, we are able to provide a counter-example to the corollary of [2, Theorem 0.3] presented in [1, Theorem 1.2] that provides the basis for Ackelsberg's claimed proof of Theorem 1.2. It is possible that some fix is available for this implied infelicity. Our own proof avoids certain difficulties associated with Weyl and van der Corput differencing by instead making use of ideas based on Vinogradov's mean value theorem and the large sieve inequality, as described in [6]. We stress that, even if this infelicity were to be resolved, the method presented here has the advantage of permitting quantitative bounds to be obtained, as is made apparent in Theorem 4.3.

Our paper is organised as follows. Having recalled some basic features of the harmonic analysis of function fields in §2, paying close attention to the role of the Frobenius map, we explore implications for additive polynomials in §3. This provides the groundwork for our proof of Theorem 1.2 in §4. We conclude in §5 by providing a counter-example to a claim in the conclusion of [1, Theorem 1.2] that we mentioned in the previous paragraph.

The work presented in this paper was the result of a zoom collaboration initiated in mid-2023. Although our investigations were independent of those of Bergelson and Leibman [2] and Ackelsberg [1], we were prompted to accelerate our publication of this paper on learning of the submission of the latter preprint to the arXiv.

2. Preliminaries

We begin our deliberations with an account of certain features of the analysis of function fields playing a role in our discussion of Theorem 1.2. First, in order to describe an analogue of Weyl's criterion for equidistribution in the function field setting, we introduce some additional notation. Recall that we assume $q = p^m$. Let $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denote the familiar trace map, given explicitly by

$$\mathrm{tr}(a) = a + a^p + a^{p^2} + \ldots + a^{p^{m-1}}.$$

We note that this map is additive, in the sense that $\mathrm{tr}(a+b) = \mathrm{tr}(a) + \mathrm{tr}(b)$ whenever $a, b \in \mathbb{F}_q$. We next define the additive character $e_q : \mathbb{F}_q \to \mathbb{C}^\times$ by taking $e_q(a) = e^{2\pi \mathrm{i}\,\mathrm{tr}(a)/p}$. This character induces a map, which we denote by $e(\cdot)$, from $\mathbb{K}_\infty$ to $\mathbb{C}^\times$, defined for each $\alpha \in \mathbb{K}_\infty$ by putting

$$e(\alpha) = e_q(\mathrm{res}\,\alpha),$$

where $\mathrm{res}\,\alpha$ is defined via (1.1). One can check that $e : \mathbb{K}_\infty \to \mathbb{C}^\times$ is a non-trivial continuous additive character on $\mathbb{K}_\infty$. The following analogue of Weyl's criterion for equidistribution was introduced by Carlitz.

Theorem 2.1. *The sequence $(a_u)_{u \in \mathbb{F}_q[t]} \subset \mathbb{K}_\infty$ is equidistributed in $\mathbb{T}$ if and only if for any $h \in \mathbb{F}_q[t] \setminus \{0\}$, we have*

$$\lim_{N \to \infty} q^{-N} \Bigg| \sum_{\substack{u \in \mathbb{F}_q[t] \\ \mathrm{ord}\, u < N}} e(h a_u) \Bigg| = 0.$$

Proof. This is Carlitz [3, Theorem 4]. □

We end this section by recording a fundamental property of continuous additive characters on $\mathbb{K}_\infty$. This is, in fact, a consequence of a more general result concerning the self-duality of local fields as topological groups.

Theorem 2.2. *Let $\chi : \mathbb{K}_\infty \to \mathbb{C}^\times$ be a continuous additive character on $\mathbb{K}_\infty$. Then there exists a unique $\eta \in \mathbb{K}_\infty$ such that, for all $\xi \in \mathbb{K}_\infty$, one has $\chi(\xi) = e(\eta\xi)$.*

Proof. This is an immediate consequence of the Corollary to Theorem 3 of Weil [7, Section 5 of Chapter 2]. □

In [3] and [6, Example 1.2], it is explained that for certain irrational $\alpha \in \mathbb{K}_\infty$, the sequence $(\alpha u^p)_{u\in\mathbb{F}_q[t]}$ is not equidistributed in $\mathbb{T}$. This is achieved via an explicit computation. One may also explain this failure of equidistribution by explicitly computing the element $\eta \in \mathbb{K}_\infty$, whose existence is assured by Theorem 2.2, having the property that for all $\xi \in \mathbb{K}_\infty$, one has $e(\alpha\xi^p) = e(\eta\xi)$. We record the relevant explicit construction in the form of a lemma. In this context, we note that since we assume that $q = p^m$, the Frobenius map $a \mapsto a^p$ on $\mathbb{F}_q$ possesses an inverse map $b \mapsto b^{1/p}$ given by putting $b^{1/p} = b^{p^{m-1}}$ for each $b \in \mathbb{F}_q$. Equipped with this inverse to Frobenius, we define the mapping $\psi : \mathbb{K}_\infty \to \mathbb{K}_\infty$ by putting

$$\psi\Big(\sum_{i\leqslant n} a_i t^i\Big) = \sum_{pj+p-1\leqslant n} a_{pj+p-1}^{1/p} t^j. \tag{2.1}$$

Lemma 2.3. *For all $\alpha, \xi \in \mathbb{K}_\infty$, one has $e(\alpha\xi^p) = e(\psi(\alpha)\xi)$.*

Proof. Given $\alpha, \xi \in \mathbb{K}_\infty$, we may write

$$\alpha = \sum_{i\in\mathbb{Z}} a_i t^i \quad \text{and} \quad \xi = \sum_{i\in\mathbb{Z}} b_i t^i,$$

where $a_i, b_i \in \mathbb{F}_q$ $(i \in \mathbb{Z})$, and one has $a_i = b_i = 0$ for i sufficiently large. Equipped with this notation, one sees that

$$\begin{aligned} \alpha\xi^p &= \Big(\sum_{i\in\mathbb{Z}} a_i t^i\Big)\Big(\sum_{j\in\mathbb{Z}} b_j t^j\Big)^p \\ &= \Big(\sum_{i\in\mathbb{Z}} a_i t^i\Big)\Big(\sum_{j\in\mathbb{Z}} b_j^p t^{jp}\Big) \\ &= \sum_{l\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} a_{l-jp} b_j^p\Big) t^l. \end{aligned} \tag{2.2}$$

Meanwhile, similarly, one has

$$\begin{aligned} \psi(\alpha)\xi &= \Big(\sum_{i\in\mathbb{Z}} a_{pi+p-1}^{1/p} t^i\Big)\Big(\sum_{j\in\mathbb{Z}} b_j t^j\Big) \\ &= \sum_{l\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} a_{p(l-j)+p-1}^{1/p} b_j\Big) t^l. \end{aligned} \tag{2.3}$$

A comparison of the coefficients of t^{-1} between the formulae (2.2) and (2.3) reveals that

$$(\mathrm{res}(\psi(\alpha)\xi))^p = \Big(\sum_{j\in\mathbb{Z}} a_{-jp-1}^{1/p} b_j\Big)^p = \sum_{j\in\mathbb{Z}} a_{-jp-1} b_j^p = \mathrm{res}(\alpha\xi^{\mathrm{p}}).$$

Since $\mathrm{tr}(c^p) = \mathrm{tr}(c)$ for all $c \in \mathbb{F}_q$, we deduce that $\mathrm{tr}(\mathrm{res}(\psi(\alpha)\xi)) = \mathrm{tr}(\mathrm{res}(\alpha\xi^p))$, whence $e(\psi(\alpha)\xi) = e(\alpha\xi^p)$. This completes the proof of the lemma. □

We now return briefly to discuss the relevance of this lemma for [6, Example 1.2]. In that example, one considers an irrational element $\alpha \in \mathbb{K}_\infty$ of the shape $\alpha = \sum_{i\leqslant n} a_i t^i$, with $a_{-1} = a_{-p-1} = \ldots = 0$. One sees that the map ψ defined in (2.1) has the property that

$$\psi(\alpha) = \sum_{pj+p-1\leqslant n} a_{pj+p-1}^{1/p} t^j \in \mathbb{F}_q[t].$$

Thus, one sees that whenever $u \in \mathbb{F}_q[t]$, one has

$$e(\alpha u^p) = e(\psi(\alpha)u) = 1.$$

In particular, it is evident from this viewpoint that $(\alpha u^p)_{u\in\mathbb{F}_q[t]}$ cannot be equidistributed in $\mathbb{T}$. This example is readily generalized. Let $k \in \mathbb{N}$. Then, in a similar manner, one sees that whenever α and β are elements of $\mathbb{K}_\infty$ having the property that $\psi(\alpha) + \beta \in \mathbb{F}_q[t]$, then

$$e(\alpha u^{kp} + \beta u^k) = e\big((\psi(\alpha) + \beta)u^k\big) = 1.$$

Thus, there exist irrational elements $\alpha, \beta \in \mathbb{K}_\infty$ having the property that

$$\big(\alpha u^{kp} + \beta u^k\big)_{u\in\mathbb{F}_q[t]}$$

is not equidistributed in $\mathbb{T}$. In a certain sense, the monomials u^{kp} and u^k interfere with each other in characteristic p.

3. Additive polynomials

A polynomial $A \in \mathbb{K}_\infty[x]$ is said to be *additive* if the identity $A(x + y) = A(x) + A(y)$ holds in the polynomial ring $\mathbb{K}_\infty[x, y]$. One may verify that every additive polynomial assumes the form

$$A(x) = \sum_{\nu=0}^{n} \alpha_\nu x^{p^\nu}$$

for some non-negative integer n and $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{K}_\infty$. We denote by $\mathcal{A}$ the set of all additive polynomials lying in $\mathbb{K}_\infty[x]$. The relevance of the set of additive polynomials for our present purposes is that every polynomial in $\mathbb{K}_\infty[x]$ may be written canonically in terms of additive polynomials.

Lemma 3.1. *Suppose that $f \in \mathbb{K}_\infty[x]$. Then there exists a unique finite set $\mathcal{R}$ of positive integers, each coprime to p, and a unique collection $(A_r)_{r\in\mathcal{R}}$ of non-zero additive polynomials, such that*

$$f(x) = f(0) + \sum_{r\in\mathcal{R}} A_r(x^r). \tag{3.1}$$

Proof. There exists a finite set $\mathcal{I}$ of positive integers, and a collection $(\alpha_i)_{i\in\mathcal{I}}$ of non-zero elements of $\mathbb{K}_\infty$, with the property that

$$f(x) = f(0) + \sum_{i\in\mathcal{I}} \alpha_i x^i.$$

As usual, we write $p^r \| m$ when $p^r | m$ and $p^{r+1} \nmid m$. Then, we define

$$\mathcal{R} = \{p^{-\nu} i : i \in \mathcal{I} \text{ and } p^\nu \| i\}.$$

Thus, we have

$$\begin{aligned} f(x) &= f(0) + \sum_{r\in\mathcal{R}} \sum_{\substack{\nu\in\mathbb{Z}^+\cup\{0\}\\ p^\nu r\in\mathcal{I}}} \alpha_{p^\nu r}(x^r)^{p^\nu} \\ &= f(0) + \sum_{r\in\mathcal{R}} A_r(x^r), \end{aligned} \tag{3.2}$$

where

$$A_r(x) = \sum_{\substack{\nu\in\mathbb{Z}^+\cup\{0\}\\ p^\nu r\in\mathcal{I}}} \alpha_{p^\nu r} x^{p^\nu}.$$

The conclusion of the lemma is immediate from the decomposition (3.2) on observing that $A_r(x)$ is an additive polynomial for each $r \in \mathcal{R}$. □

We next define a map $\tau : \mathcal{A} \to \mathbb{K}_\infty$ given explicitly by means of the formula

$$\tau\left(\sum_{\nu=0}^{n} \alpha_\nu x^{p^\nu}\right) = \sum_{\nu=0}^{n} \psi^\nu(\alpha_\nu), \tag{3.3}$$

in which ψ is the function defined in (2.1), and ψ^ν denotes the ν-fold composition of ψ with itself. Given an additive polynomial

$$A(x) = \sum_{\nu=0}^{n} \alpha_\nu x^{p^\nu},$$

we see from repeated application of Lemma 2.3 that this map has the property that, whenever $\xi \in \mathbb{K}_\infty$, one has

$$e(A(\xi)) = e\left(\sum_{\nu=0}^{n} \alpha_\nu \xi^{p^\nu}\right) = e\left(\sum_{\nu=0}^{n} \psi^\nu(\alpha_\nu)\xi\right) = e(\tau(A)\xi). \tag{3.4}$$

This observation confirms explicitly the conclusion of Theorem 2.2 for the continuous additive character $\xi \mapsto e(A(\xi))$ on $\mathbb{K}_\infty$. We note in particular that (3.3) gives the relation $\tau(\alpha x) = \alpha$, for $\alpha \in \mathbb{K}_\infty$.

The map τ that we have introduced in (3.3) combines with the decomposition given in Lemma 3.1 to replace a given polynomial $f \in \mathbb{K}_\infty[x]$ by a substitute free of monomials of the shape x^{pm} ($m \in \mathbb{Z}^+$). This distillation provides a route by which one may avoid difficulties in applying the work of [6] in investigations concerning equidistribution in $\mathbb{T}$.

Lemma 3.2. *Suppose that $f \in \mathbb{K}_\infty[x]$. Write $f(x)$ in the shape (3.1) for a suitable set of positive integers $\mathcal{R}$, each coprime to p, and a collection $(A_r)_{r\in\mathcal{R}}$ of non-zero additive polynomials. Define*

$$g(x) = f(0) + \sum_{r\in\mathcal{R}} \tau(A_r)x^r. \tag{3.5}$$

Then, for all $\xi \in \mathbb{K}_\infty$, one has $e(f(\xi)) = e(g(\xi))$.

Proof. On making use of (3.1), (3.4) and (3.5), one sees that

$$e(f(\xi)) = e(f(0)) \prod_{r\in\mathcal{R}} e(A_r(\xi^r)) = e(f(0)) \prod_{r\in\mathcal{R}} e(\tau(A_r)\xi^r) = e(g(\xi)).$$

This completes the proof of the lemma. □

4. Equidistribution in $\mathbb{T}$

Our goal in this section is to make use of the substitution principle embodied in Lemma 3.2 so as to apply the conclusions of [6] in order to confirm Theorem 1.2. We begin by recalling a consequence of [6, Proposition 5.2].

Theorem 4.1. *Let $\mathcal{R}$ be a finite set of positive integers coprime to p, and let $f \in \mathbb{K}_\infty[x]$ be a polynomial of the form*

$$f(x) = \alpha_0 + \sum_{r\in\mathcal{R}} \alpha_r x^r,$$

with $\alpha_i \in \mathbb{K}_\infty$ $(i \in \mathcal{R} \cup \{0\})$. Suppose that α_r is irrational for some index $r \in \mathcal{R}$. Then the sequence $(f(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$.

Proof. It follows from the definition (1.2) that when $(r,p) = 1$ for all $r \in \mathcal{R}$, one has $\widetilde{\mathcal{R}} = \mathcal{R}$. In order to confirm this observation, note first that $\mathcal{R}_n \subset \mathcal{R}$ for each $n \geqslant 0$. Then, whenever $\mathcal{R}_n$ is non-empty, its largest element is coprime to p, and hence is also an element of $\mathcal{R}_n^*$. Since each non-empty set $\mathcal{R}_n^*$ contains at least one element not contained in $\mathcal{R}_{n+1}$, it follows that

$$\bigcup_{n=0}^{\mathrm{card}(\mathcal{R})} \mathcal{R}_n^* = \mathcal{R},$$

whence $\widetilde{\mathcal{R}} = \mathcal{R}$, as claimed. The conclusion of the theorem is therefore immediate from [6, Proposition 5.2]. □

We note that a simplified proof of Theorem 4.1 is available in which the set $\widetilde{\mathcal{R}}$, as well as the application of [6, Proposition 5.2], is avoided. Indeed, since $\mathcal{R}$ contains only positive integers coprime to p, the conclusion of [6, Theorem 4.1] remains valid for $\mathcal{R}$ if, in the proof, we replace the shadow partial ordering on $\mathcal{R}^*$ with the standard integer ordering on $\mathcal{R}$. Consequently, the conclusion of [6, Lemma 5.1] also holds when f has its coefficients α_r supported on $r \in \mathcal{R} \cup \{0\}$. One now obtains a proof of Theorem 4.1 by following the corresponding proof of [6, Theorem 1.4].

This conclusion is applicable to the substitute polynomials generated by Lemma 3.2.

Lemma 4.2. *Let $f \in \mathbb{K}_\infty[x]$, and write $f(x)$ in the shape (3.1) for a suitable set of positive integers $\mathcal{R}$, each coprime to p, and a collection $(A_r)_{r\in\mathcal{R}}$ of non-zero additive polynomials. Suppose that, for all $h \in \mathbb{F}_q[t] \setminus \{0\}$, there exists $r \in \mathcal{R}$ having the property that $\tau(hA_r)$ is irrational. Then the sequence $(f(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$.*

Proof. Fix an arbitrary element $h \in \mathbb{F}_q[t] \setminus \{0\}$. The hypotheses of the lemma ensure that

$$hf(x) = hf(0) + \sum_{r\in\mathcal{R}} hA_r(x^r),$$

where $(hA_r)_{r\in\mathcal{R}}$ is a collection of polynomials in $\mathcal{A} \setminus \{0\}$. By Lemma 3.2, the polynomial

$$g_h(x) = hf(0) + \sum_{r\in\mathcal{R}} \tau(hA_r)x^r$$

satisfies the property that, for all $u \in \mathbb{F}_q[t]$, one has

$$e(hf(u)) = e(g_h(u)).$$

Since the polynomial $g_h(x)$ satisfies the hypotheses of Theorem 4.1, we find that $(g_h(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$. We therefore infer from Theorem 2.1 that

$$\lim_{N\to\infty} q^{-N}\Bigg|\sum_{\substack{u\in\mathbb{F}_q[t]\\ \mathrm{ord}\, u<N}} e(hf(u))\Bigg| = \lim_{N\to\infty} q^{-N}\Bigg|\sum_{\substack{u\in\mathbb{F}_q[t]\\ \mathrm{ord}\, u<N}} e(g_h(u))\Bigg| = 0.$$

However, the element $h\in\mathbb{F}_q[t]\backslash\{0\}$ was chosen arbitrarily. Thus, again applying Theorem 2.1, we infer that $(f(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$. This completes the proof of the lemma. □

The proof of Theorem 1.2 is accomplished by verifying that its hypotheses imply those of Lemma 4.2.

The proof of Theorem 1.2. Let $\mathcal{K}$ be a finite set of positive integers, and suppose that $\alpha_r\in\mathbb{K}_\infty\setminus\{0\}$ for $r\in\mathcal{K}\cup\{0\}$. We suppose in addition that α_k is irrational for some $k\in\mathcal{K}$ satisfying $p\nmid k$ and furthermore $p^v k\notin\mathcal{K}$ for any $v\in\mathbb{Z}^+$. Defining

$$f(x)=\sum_{i\in\mathcal{K}\cup\{0\}}\alpha_i x^i,$$

it follows from Lemma 3.1 (assisted by its proof) that we can write f in the form

$$f(x)=\alpha_0+\sum_{r\in\mathcal{R}}A_r(x^r),$$

where

$$\mathcal{R}=\{p^{-\nu}i : i\in\mathcal{K} \text{ and } p^\nu\|i\}$$

and

$$A_r(x)=\sum_{\substack{\nu\in\mathbb{Z}^+\cup\{0\}\\ p^\nu r\in\mathcal{K}}}\alpha_{p^\nu r}x^{p^\nu}\quad (r\in\mathcal{R}).$$

We note, in particular, that one has $k\in\mathcal{R}$. Since $p^v k\notin\mathcal{K}$ for any $v\in\mathbb{Z}^+$, moreover, we find that $p^\nu k\in\mathcal{K}$ if and only if $\nu=0$, and thus $A_k(x)=\alpha_k x$.

We now observe that the discussion following (3.3) ensures that

$$\tau(hA_k)=\tau(h\alpha_k x)=h\alpha_k.$$

Our hypothesis that α_k is irrational, moreover, implies that $h\alpha_k$ is irrational for all $h\in\mathbb{F}_q[t]\setminus\{0\}$, whence $\tau(hA_k)$ is irrational. Then we conclude from Lemma 4.2 that $(f(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$. This completes the proof of Theorem 1.2. □

The argument that we applied in the proof of Theorem 1.2 can be incorporated into the method of proof of [6, Theorem 4.1] so as to deliver the following refinement of [6, Proposition 4.2]. This constitutes the quantitative bound to which we alluded in the introduction. Since the proof of this result introduces no new ideas beyond those already introduced in our proof of Theorem 1.2 and that of [6, Theorem 4.1], we leave the details as an exercise for the reader.

Theorem 4.3. *Fix q and a finite set $\mathcal{K}\subset\mathbb{Z}^+$. There exist positive constants c and C, depending only on $\mathcal{K}$ and q, such that the following holds. Let $\varepsilon>0$ and let N*

be sufficiently large in terms of $\mathcal{K}$, ε and q. Suppose that $f(x)=\sum_{r\in\mathcal{K}\cup\{0\}}\alpha_r x^r$ is a polynomial with coefficients in $\mathbb{K}_\infty$ satisfying the bound

$$\Biggl|\sum_{\substack{u\in\mathbb{F}_q[t]\\ \operatorname{ord}u<N}} e(f(u))\Biggr|\geq q^{N-\eta},$$

for some positive number η with $\eta\leqslant cN$. Then, for any $k\in\mathcal{K}$ satisfying $p\nmid k$ and furthermore $p^v k\notin\mathcal{K}$ for any $v\in\mathbb{Z}^+$, there exist $a_k\in\mathbb{F}_q[t]$ and monic $g_k\in\mathbb{F}_q[t]$ such that

$$\operatorname{ord}(g_k\alpha_k-a_k)<-kN+\varepsilon N+C\eta\quad\textit{and}\quad\operatorname{ord}g_k\leq\varepsilon N+C\eta.$$

5. An example related to work of Bergelson and Leibman

We complete our discussion of equidistribution in $\mathbb{T}$ by exhibiting an example that provides evidence of an infelicity in the work of Bergelson and Leibman [2] that apparently invalidates the proof of Theorem 1.2 given by Ackelsberg [1]. Since this example illuminates certain aspects of the theory of equidistribution in positive characteristic, we provide a largely self-contained account. We formulate the relevant claimed corollary in the form of a hypothesis.

Hypothesis 5.1. *Suppose that $\mathcal{R}$ is a set of positive integers, each coprime to p, let $(A_r)_{r\in\mathcal{R}}$ be a collection of additive polynomials, and define*

$$f(x)=\sum_{r\in\mathcal{R}}A_r(x^r).$$

If, for some $r\in\mathcal{R}$, the sequence $(A_r(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$, then $(f(u))_{u\in\mathbb{F}_q[t]}$ is also equidistributed in $\mathbb{T}$.

This assertion is an immediate consequence of the final conclusion of Ackelsberg [1, Theorem 2.1], which he attributes to [2, Theorem 0.3]. In the language of Ackelsberg, the notion of equidistribution in $\mathbb{T}$ is replaced by the notion of being well-distributed in $\mathbb{T}$. However, if $(A_r(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$ for an additive polynomial $A_r(x)$, then it is well-distributed in $\mathbb{T}$, and the hypothesis preliminary to the conclusion of Hypothesis 5.1 suffices (according to [1, Theorem 2.1]) to ensure that $(f(u))_{u\in\mathbb{F}_q[t]}$ is well-distributed in $\mathbb{T}$, and hence also equidistributed in $\mathbb{T}$. We shall exhibit an example demonstrating that Hypothesis 5.1 may fail.

We begin by recording a trivial observation that streamlines subsequent aspects of our discussion.

Lemma 5.2. *Suppose that $A(x)\in\mathbb{K}_\infty[x]$ is an additive polynomial. Then the following are equivalent:*

(a) *for all $h\in\mathbb{F}_q[t]\setminus\{0\}$, the character $x\mapsto e(hA(x))$ is non-trivial on $\mathbb{F}_q[t]$;*
(b) *for all $h\in\mathbb{F}_q[t]\setminus\{0\}$, one has $\tau(hA)\notin\mathbb{F}_q[t]$;*
(c) *the sequence $(A(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$;*
(d) *the sequence $(A(u))_{u\in\mathbb{F}_q[t]}$ is well-distributed in $\mathbb{T}$.*

Proof. In view of the relation (3.4), the character $u\mapsto e(hA(u))$ is non-trivial on $\mathbb{F}_q[t]$ if and only if the character $u\mapsto e(\tau(hA)u)$ is likewise non-trivial on $\mathbb{F}_q[t]$. But the latter holds if and only if $\tau(hA)\notin\mathbb{F}_q[t]$. Thus we find that (a) holds if and only if (b) holds.

Next, we observe that from Theorem 2.1 it follows that $(A(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$ if and only if, for any $h\in\mathbb{F}_q[t]\setminus\{0\}$, we have

$$\lim_{N\to\infty} q^{-N}\Bigg|\sum_{\substack{u\in\mathbb{F}_q[t]\\ \operatorname{ord} u<N}} e(hA(u))\Bigg| = 0. \tag{5.1}$$

Since $e(hA(u))=e(\tau(hA)u)$, we see that when $\tau(hA)\in\mathbb{F}_q[t]$, one has $e(hA(u))=1$. Meanwhile, when $\tau(hA)\notin\mathbb{F}_q[t]$, it follows from Kubota [5, Lemma 7] that

$$\sum_{\substack{u\in\mathbb{F}_q[t]\\ \operatorname{ord} u<N}} e(hA(u)) = \begin{cases} q^N, & \text{when } \operatorname{ord}\{\tau(hA)\}<-N,\\ 0, & \text{when } \operatorname{ord}\{\tau(hA)\}\geqslant -N.\end{cases}$$

Thus, provided that $\{\tau(hA)\}\neq 0$, we see that for large enough values of N one has $\operatorname{ord}\{\tau(hA)\}\geqslant -N$, and hence the relation (5.1) must hold. We have therefore shown that $(A(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$ if and only if, for any $h\in\mathbb{F}_q[t]\setminus\{0\}$, one has $\tau(hA)\notin\mathbb{F}_q[t]$. Thus, we conclude that (c) holds if and only if (b) holds.

Finally, it is a trivial consequence of the definition of well-distribution employed in [1] that $(A(u))_{u\in\mathbb{F}_q[t]}$ is well-distributed in $\mathbb{T}$ if and only if it is equidistributed in $\mathbb{T}$. Indeed, assume that the latter holds. Then, since the additive property of A ensures that $A(u+a)=A(u)+A(a)$ for each $a\in\mathbb{F}_q[t]$, the sequence $(A(u+a))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$, uniformly in a. It follows that $A:\mathbb{F}_q[t]\to\mathbb{K}_\infty$ is equidistributed in $\mathbb{T}$ along any Følner sequence, and hence $(A(u))_{u\in\mathbb{F}_q[t]}$ is well-distributed in $\mathbb{T}$. The reverse implication is trivial. Thus, we conclude that (d) holds if and only if (c) holds.

On combining these conclusions, we complete the proof of the lemma. □

Our counter-example to Hypothesis 5.1 rests on the following result concerning additive polynomials.

Theorem 5.3. *Suppose that $\gamma\in\mathbb{K}_\infty\setminus\mathbb{F}_q[t]$. Then there exists an additive polynomial $A(x)\in\mathbb{K}_\infty[x]$ having the properties:*

(a) *one has $\tau(A)=\gamma$;*
(b) *for all $h\in\mathbb{F}_q[t]\setminus\{0,1\}$, one has $\tau(hA)\notin\mathbb{F}_q[t]$.*

We defer the proof of this conclusion for the time-being, and first concentrate on applying this result to obtain a counter-example to Hypothesis 5.1.

Counter-example to Hypothesis 5.1. We begin by observing that the conclusion of Theorem 5.3 shows that there exists an additive polynomial $A(x)$ having the property that $\tau(A)=1/t$, and satisfying the condition that

$$\tau(hA)\notin\mathbb{F}_q[t]\quad\text{for all}\quad h\in\mathbb{F}_q[t]\setminus\{0,1\}.$$

In particular, it then follows from Lemma 5.2 that the sequence $(A(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$. When $\operatorname{char}(\mathbb{F}_q)\neq 2$, our counter-example to Hypothesis 5.1 is given by the polynomial

$$f(x)=A(x^{q+1})-x^2/t. \tag{5.2}$$

In order to confirm that the polynomial f defined in (5.2) does indeed furnish a counter-example to Hypothesis 5.1, put

$$\mathcal{R}=\{q+1,2\},\quad A_{q+1}(x)=A(x),\quad\text{and}\quad A_2(x)=-x/t.$$

We have seen that the sequence $(A_{q+1}(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$, so Hypothesis 5.1 implies that $(f(u))_{u\in\mathbb{F}_q[t]}$ is also equidistributed in $\mathbb{T}$. However, on making use of (3.4), we find that

$$e(f(u)) = e\Big(A(u^{q+1}) - \frac{u^2}{t}\Big) = e\Big(\tau(A)u^{q+1} - \frac{u^2}{t}\Big) = e\Big(\frac{u^{q+1}}{t} - \frac{u^2}{t}\Big).$$

But for all $u \in \mathbb{F}_q[t]$, one has $u^q - u \equiv 0 \pmod{t}$, and hence

$$e(f(u)) = e\Big(\frac{u(u^q - u)}{t}\Big) = 1.$$

Consequently, the sequence $(f(u))_{u\in\mathbb{F}_q[t]}$ cannot be equidistributed in $\mathbb{T}$, since it fails the analogue of Weyl's criterion exhibited in Theorem 2.1. This conclusion contradicts our earlier deduction that Hypothesis 5.1 implies that $(f(u))_{u\in\mathbb{F}_q[t]}$ is equidistributed in $\mathbb{T}$. We are therefore forced to infer that Hypothesis 5.1 fails for the example (5.2). □

The reader may care to check that the polynomial f defined in (5.2) also furnishes a counter-example to Hypothesis 5.1 when $\mathrm{char}(\mathbb{F}_q) = 2$. In this scenario, we put

$$\mathcal{R} = \{q+1, 1\}, \quad A_{q+1}(x) = A(x), \quad \text{and} \quad A_1(x) = x^2/t.$$

The argument applied above in the case $\mathrm{char}(\mathbb{F}_q) \neq 2$ now shows, mutatis mutandis, that $(f(u))_{u\in\mathbb{F}_q[t]}$ is not equidistributed in $\mathbb{T}$, in contradiction with the earlier consequence of Hypothesis 5.1.

We remark that our proof of Theorem 5.3 shows, in fact, that $A(x)$ may be chosen to have the shape $\alpha x + \beta x^p$. Moreover, without much difficulty, it would be possible to construct such an example with both α and β irrational. This example then leads to a violation of Hypothesis 5.1 for the polynomial

$$f(x) = \alpha x^{q+1} + \beta x^{p(q+1)} - x^2/t,$$

as we have shown. Moreover, as must be the case, this polynomial $f(x)$ fails to be accessible to our Theorem 1.2, as the reader may care to verify.

We devote the remainder of this section to the proof of Theorem 5.3. The difficulty that we must address revolves around the need to control the value of $\tau(hA)$, while at the same time fixing $\tau(A) = \gamma$, for a specified $\gamma \in \mathbb{K}_\infty \setminus \mathbb{F}_q[t]$. In order to assist in achieving this control, our argument will employ the mappings $\psi_l : \mathbb{K}_\infty \to \mathbb{K}_\infty$, defined for $l \in \mathbb{Z}$ by putting

$$\psi_l\Big(\sum_{i\leqslant n} a_i t^i\Big) = \sum_{pj+l\leqslant n} a_{pj+l}^{1/p} t^j, \tag{5.3}$$

in which the map $b \mapsto b^{1/p}$ is again that described in the preamble to the statement of Lemma 2.3. Note that, in this notation, our earlier mapping $\psi : \mathbb{K}_\infty \to \mathbb{K}_\infty$ is simply ψ_{p-1}. We now introduce the map $\Psi : \mathbb{K}_\infty \to \mathbb{K}_\infty^p$ by putting

$$\Psi(\alpha) = \big(\psi_{p-1}(\alpha), \psi_{p-2}(\alpha), \ldots, \psi_0(\alpha)\big). \tag{5.4}$$

This map is in fact equivalent to the *splitting isomorphism* introduced by Bergelson and Leibman [2, page 934], labelled as ψ_1 therein. It is evident from the construction of Ψ that it defines a bijection from $\mathbb{K}_\infty$ to $\mathbb{K}_\infty^p$. The components of Ψ satisfy the following multiplicative property.

Lemma 5.4. *Suppose that $\alpha, \beta \in \mathbb{K}_\infty$ and $l \in \{0, 1, \dots, p-1\}$. Then one has*

$$\psi_l(\alpha\beta) = \sum_{k=0}^{p-1} \psi_k(\alpha)\psi_{l-k}(\beta). \tag{5.5}$$

Proof. Given $\alpha, \beta \in \mathbb{K}_\infty$, we may write

$$\alpha = \sum_{i\in\mathbb{Z}} a_i t^i \quad \text{and} \quad \beta = \sum_{i\in\mathbb{Z}} b_i t^i,$$

where $a_i, b_i \in \mathbb{F}_q$ $(i \in \mathbb{Z})$, and one has $a_i = b_i = 0$ for i sufficiently large. In this notation, one sees that

$$\alpha\beta = \sum_{i\in\mathbb{Z}} \Bigl(\sum_{r\in\mathbb{Z}} a_r b_{i-r} \Bigr) t^i,$$

whence

$$\begin{aligned} \psi_l(\alpha\beta) &= \sum_{j\in\mathbb{Z}} \Bigl(\sum_{r\in\mathbb{Z}} a_r b_{pj+l-r} \Bigr)^{1/p} t^j \\ &= \sum_{j\in\mathbb{Z}} \Bigl(\sum_{k=0}^{p-1} \sum_{m\in\mathbb{Z}} a_{pm+k} b_{p(j-m)+l-k} \Bigr)^{1/p} t^j. \end{aligned} \tag{5.6}$$

Observe next that for each $a, b \in \mathbb{F}_q$, one has

$$(a+b)^{1/p} = (a+b)^{p^{m-1}} = a^{p^{m-1}} + b^{p^{m-1}} = a^{1/p} + b^{1/p}. \tag{5.7}$$

Consequently, for each $k \in \{0, 1, \dots, p-1\}$, one has

$$\begin{aligned} \psi_k(\alpha)\psi_{l-k}(\beta) &= \Bigl(\sum_{m\in\mathbb{Z}} a_{pm+k}^{1/p} t^m \Bigr) \Bigl(\sum_{n\in\mathbb{Z}} b_{pn+l-k}^{1/p} t^n \Bigr) \\ &= \sum_{j\in\mathbb{Z}} \Bigl(\sum_{m\in\mathbb{Z}} a_{pm+k} b_{p(j-m)+l-k} \Bigr)^{1/p} t^j. \end{aligned}$$

We therefore conclude that

$$\sum_{k=0}^{p-1} \psi_k(\alpha)\psi_{l-k}(\beta) = \sum_{j\in\mathbb{Z}} \sum_{k=0}^{p-1} \Bigl(\sum_{m\in\mathbb{Z}} a_{pm+k} b_{p(j-m)+l-k} \Bigr)^{1/p} t^j.$$

An additional appeal to (5.7) therefore leads via (5.6) to the relation (5.5). This completes the proof of the lemma. □

We apply this conclusion to the specific additive polynomial $A(x) = \alpha x + \beta x^p$, with $\alpha, \beta \in \mathbb{K}_\infty$. We observe that in these circumstances, one finds from (2.1) and (3.3) that

$$\tau(hA) = \tau(h\alpha x + h\beta x^p) = \psi^0(h\alpha) + \psi^1(h\beta) = h\alpha + \psi_{p-1}(h\beta).$$

Thus we infer that

$$\tau(hA) = h\alpha + \psi_0(h)\psi_{p-1}(\beta) + \dots + \psi_{p-1}(h)\psi_0(\beta). \tag{5.8}$$

Our goal is to ensure that $\tau(hA) = \gamma$ when $h = 1$, and when $h \in \mathbb{F}_q[t] \setminus \{0, 1\}$, that $\tau(hA)$ avoids lying in $\mathbb{F}_q[t]$. Provided that the coefficients $\alpha, \psi_0(\beta), \ldots, \psi_{p-1}(\beta)$ are sufficiently independent, this goal can be achieved with an investigation of the values of $h, \psi_0(h), \ldots, \psi_{p-1}(h)$. Our discussion is eased by the introduction of the linear form

$$\lambda(h; \alpha, \boldsymbol{\xi}) = h\alpha + \psi_0(h)\xi_0 + \ldots + \psi_{p-1}(h)\xi_{p-1}.$$

Lemma 5.5. *Suppose that $\gamma \in \mathbb{K}_\infty \setminus \mathbb{F}_q[t]$. Then there exist $\alpha, \xi_0, \ldots, \xi_{p-1} \in \mathbb{K}_\infty$ having the properties:*

(a) *one has $\lambda(1; \alpha, \boldsymbol{\xi}) = \gamma$;*
(b) *whenever $h \in \mathbb{F}_q[t] \setminus \{0, 1\}$, one has $\lambda(h; \alpha, \boldsymbol{\xi}) \notin \mathbb{F}_q[t]$.*

Proof. Fix $\gamma \in \mathbb{K}_\infty \setminus \mathbb{F}_q[t]$. We are at liberty to choose $\xi_0, \ldots, \xi_{p-1} \in \mathbb{K}_\infty$ in such a manner that $1, \xi_0, \ldots, \xi_{p-1}$ are linearly independent over $\mathbb{F}_q(t)$. If $\gamma \notin \mathbb{F}_q(t)$, moreover, we may insist on the stronger condition that $1, \gamma, \xi_0, \ldots, \xi_{p-1}$ are linearly independent over $\mathbb{F}_q(t)$. In either case, we put $\alpha = \gamma - \xi_0$.

From the definition (5.3), we have $\psi_0(1) = 1$ and $\psi_1(1) = \ldots = \psi_{p-1}(1) = 0$. Thus, we see that

$$\lambda(1; \alpha, \boldsymbol{\xi}) = \alpha + \xi_0 = \gamma.$$

This confirms the first assertion of the lemma.

We now seek to establish the second assertion of the lemma. By way of seeking a contradiction, suppose, if possible, that there exists $h \in \mathbb{F}_q[t] \setminus \{0, 1\}$ for which $\lambda(h; \alpha, \boldsymbol{\xi}) \in \mathbb{F}_q[t]$. Then there exists $b \in \mathbb{F}_q[t]$ having the property that

$$h\alpha + \psi_0(h)\xi_0 + \ldots + \psi_{p-1}(h)\xi_{p-1} = b. \tag{5.9}$$

Suppose first that $\gamma \in \mathbb{F}_q(t)$. Since $\alpha = \gamma - \xi_0$, we find from (5.9) that

$$(h\gamma - b) \cdot 1 + (\psi_0(h) - h)\xi_0 + \psi_1(h)\xi_1 + \ldots + \psi_{p-1}(h)\xi_{p-1} = 0.$$

Each coefficient of $1, \xi_0, \ldots, \xi_{p-1}$ here is an element of $\mathbb{F}_q(t)$, and hence it follows from the linear independence of $1, \xi_0, \ldots, \xi_{p-1}$ over $\mathbb{F}_q(t)$ that

$$\gamma = b/h, \quad \psi_0(h) = h, \quad \text{and} \quad \psi_1(h) = \ldots = \psi_{p-1}(h) = 0.$$

However, an inspection of the relation (5.3) ensures that when $h \in \mathbb{F}_q[t]$, one has the upper bound $\deg \psi_0(h) \leqslant (\deg h)/p$. Thus, the relation $\psi_0(h) = h$ ensures that $\deg h = 0$, whence $h \in \mathbb{F}_q$. Consequently, we have $\gamma = b/h \in \mathbb{F}_q[t]$, contradicting our hypothesis $\gamma \in \mathbb{K}_\infty \setminus \mathbb{F}_q[t]$.

In the alternative case $\gamma \notin \mathbb{F}_q(t)$, we interpret the relation (5.9) in the form

$$-b \cdot 1 + h\gamma + (\psi_0(h) - h)\xi_0 + \psi_1(h)\xi_1 + \ldots + \psi_{p-1}(h)\xi_{p-1} = 0.$$

In this case, the linear independence of $1, \gamma, \xi_0, \ldots, \xi_{p-1}$ over $\mathbb{F}_q(t)$ forces us to conclude that $b = h = 0$. This again delivers a contradiction, and so the proof of the lemma is now complete. □

We are now equipped to complete the proof of Theorem 5.3.

The proof of Theorem 5.3. Suppose that $\gamma \in \mathbb{K}_\infty \setminus \mathbb{F}_q[t]$. The conclusion of Lemma 5.5 shows that there exist $\alpha, \xi_0, \ldots, \xi_{p-1} \in \mathbb{K}_\infty$ having the property that

$$\lambda(1; \alpha, \boldsymbol{\xi}) = \alpha + \psi_0(1)\xi_0 + \ldots + \psi_{p-1}(1)\xi_{p-1} = \gamma,$$

and further that, when $h \in \mathbb{F}_q[t] \setminus \{0,1\}$, one has

$$h\alpha + \psi_0(h)\xi_0 + \ldots + \psi_{p-1}(h)\xi_{p-1} \notin \mathbb{F}_q[t].$$

Fix any such choice of $\alpha, \xi_0, \ldots, \xi_{p-1}$. Since the map $\Psi : \mathbb{K}_\infty \to \mathbb{K}_\infty^p$ defined via (5.4) is bijective, there exists $\beta \in \mathbb{K}_\infty$ such that

$$\psi_{p-1-l}(\beta) = \xi_l \quad (0 \leqslant l \leqslant p-1).$$

We define as our additive polynomial $A(x) = \alpha x + \beta x^p$. In view of (5.8), we then have

$$\tau(A) = \lambda(1; \alpha, \boldsymbol{\xi}) = \gamma,$$

and, furthermore, when $h \in \mathbb{F}_q[t] \setminus \{0,1\}$, one has

$$\tau(hA) = \lambda(h; \alpha, \boldsymbol{\xi}) \notin \mathbb{F}_q[t].$$

This confirms that the additive polynomial $A(x) \in \mathbb{K}_\infty[x]$ has the properties claimed in the statement of Theorem 5.3, and completes the proof. □

References

[1] E. Ackelsberg, *A Weyl equidistribution theorem over function fields*, preprint available as arXiv:2509.18789.

[2] V. Bergelson and A. Leibman, *A Weyl-type equidistribution theorem in finite characteristic*, Adv. Math. **289** (2016), 928–950.

[3] L. Carlitz, *Diophantine approximation in fields of characteristic p*, Trans. Amer. Math. Soc. **72** (1952), no. 2, 187–208.

[4] A. Dijksma, *Uniform distribution of polynomials over* GF$\{q,x\}$ *in* GF$[q,x]$*. II*, Indag. Math. **32** (1970), 187–195.

[5] R. M. Kubota, *Waring's problem for* $\mathbb{F}_q[x]$, Dissertationes Math. (Rozprawy Mat.) **117** (1974), 60 pp.

[6] T. H. Lê, Y.-R. Liu and T. D. Wooley, *Equidistribution of polynomial sequences in function fields, with applications*, Adv. Math. **479** (2025), article no. 110424, 44pp.

[7] A. Weil, *Basic number theory*, Die Grundlehren der mathematischen Wissenschaften, Band 144, Springer-Verlag New York, New York, 1967.

[8] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), no. 3, 313–352.

JC, ZG and Y-RL: Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, Canada

Email address: jchampag@uwaterloo.ca, z4ge@uwaterloo.ca, yrliu@uwaterloo.ca

THL: Department of Mathematics, University of Mississippi, 305 Hume Hall, University, MS 38677, USA

Email address: leth@olemiss.edu

TDW: Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA

Email address: twooley@purdue.edu