

STRUCTURE AND PAUCITY IN AFFINE DIAGONAL SYSTEMS, I

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ABSTRACT. Let $\varepsilon > 0$ and $\mathbf{h} \in \mathbb{Z}^3$. We show that whenever P is large and the system

$$x_1^j + x_2^j - y_1^j - y_2^j = h_j \quad (j = 1, 2, 3)$$

has more than P^ε integral solutions with $1 \leq x_i, y_i \leq P$, then there exist natural numbers a and b with $h_j = a^j - b^j$ ($j = 1, 2, 3$). This example illustrates the theme that, either the Diophantine system has a paucity of integral solutions, or else the coefficient tuple \mathbf{h} is highly structured. We examine related paucity problems as well as some consequences for problems involving more variables.

1. INTRODUCTION

This paper concerns the influence of coefficient structure on the paucity of integral solutions in certain systems of affine diagonal equations. What we have in mind is best illustrated by introducing our first example of interest. Let P be a large natural number, and consider a coefficient triple $(h_1, h_2, h_3) \in \mathbb{Z}^3$. We denote by $S_2(P; \mathbf{h})$ the number of integral solutions of the system

$$\left. \begin{aligned} x_1^3 + x_2^3 - y_1^3 - y_2^3 &= h_3 \\ x_1^2 + x_2^2 - y_1^2 - y_2^2 &= h_2 \\ x_1 + x_2 - y_1 - y_2 &= h_1 \end{aligned} \right\}, \quad (1.1)$$

with $1 \leq x_i, y_i \leq P$ ($i = 1, 2$). It is apparent that for certain triples $\mathbf{h} = (h_1, h_2, h_3)$, there are many integral solutions to the system (1.1) counted by $S_2(P; \mathbf{h})$. When $\mathbf{h} = (0, 0, 0)$, for example, there are $2P^2 - P$ diagonal solutions with $\{x_1, x_2\} = \{y_1, y_2\}$. Furthermore, when a and b are distinct natural numbers with $1 \leq a, b \leq P$ and

$$h_j = a^j - b^j \quad (j = 1, 2, 3), \quad (1.2)$$

there are P solutions with $x_1 = y_1$, $x_2 = a$, $y_2 = b$. Our goal is to prove that there is a paucity of integral solutions not belonging to one of these structured sets.

Theorem 1.1. *Let $\eta \in (0, 1)$ be fixed, and let P be sufficiently large in terms of η . Suppose that $\mathbf{h} \in \mathbb{Z}^3$ is a coefficient triple with the property that $S_2(P; \mathbf{h}) > P^\eta$. Then, the tuple \mathbf{h} satisfies one of the following two conditions:*

- (a) *for some integers a and b with $a \neq b$ and $1 \leq a, b \leq P$, one has $h_j = a^j - b^j$ ($j = 1, 2, 3$), in which case $S_2(P; \mathbf{h}) = 4P$;*
- (b) *one has $h_j = 0$ ($j = 1, 2, 3$), in which case $S_2(P; \mathbf{0}) = 2P^2 - P$.*

In particular, if \mathbf{h} satisfies neither condition (a) nor condition (b), then $S_2(P; \mathbf{h}) = O(P^\varepsilon)$.

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We note that in scenario (a), the solutions of (1.1) take the form \mathbf{x}, \mathbf{y} with

$$\{x_1, x_2\} = \{a, z\} \quad \text{and} \quad \{y_1, y_2\} = \{b, z\},$$

wherein z is any integer with $1 \leq z \leq P$. Solutions in scenario (b), meanwhile, are the diagonal ones with $\{x_1, x_2\} = \{y_1, y_2\}$.

The wider theme that we explore in this paper is illustrated well by Theorem 1.1. Either the system (1.1) has very few integral solutions, or else the coefficient tuple \mathbf{h} is highly structured, with an explicit structure aligned with the equations comprising (1.1). These ideas extend to systems in a larger number of variables. When $t \in \mathbb{N}$, denote by $S_t(P; \mathbf{h})$ the number of integral solutions of the system

$$\sum_{i=1}^t (x_i^j - y_i^j) = h_j \quad (j = 1, 2, 3),$$

with $1 \leq x_i, y_i \leq P$ ($1 \leq i \leq t$). Improving on earlier work of Brandes and Hughes [1], it was shown in Wooley [7, Theorem 1.1] that $S_t(P; \mathbf{h}) \ll P^{t-1/2+\varepsilon}$ whenever $h_1 \neq 0$ and $1 \leq t \leq 5$, and also when $h_2 \neq 0$ and $1 \leq t \leq 4$. We note also that an asymptotic formula for $S_6(P; \mathbf{h})$ is established in [6, Theorems 1.1 and 1.2], provided that either $h_1 \neq 0$ or $h_2 \neq 0$. In addition, when $h_1 = 0$ and $h_2 \neq 0$, it follows from [7, Theorem 1.2] that one has $S_3(P; \mathbf{h}) \ll P^{2+\varepsilon}$. This, in common with all of the previous results cited, offers estimates beyond square-root cancellation, with the upper bound $S_3(P; \mathbf{h}) \ll P^{2+\varepsilon}$ saving a factor $P^{1-\varepsilon}$ over this square-root barrier when $h_1 = 0$ and $h_2 \neq 0$. An immediate consequence of Theorem 1.1 is that this last upper bound now extends to all $\mathbf{h} \neq \mathbf{0}$.

Corollary 1.2. *Suppose that $\mathbf{h} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ and $\varepsilon > 0$. Then one has $S_3(P; \mathbf{h}) \ll P^{2+\varepsilon}$.*

We remark that when \mathbf{h} has the shape determined by (1.2), with $1 \leq a, b \leq P$ and $a \neq b$, then $S_3(P; \mathbf{h}) \gg P^2$. This conclusion follows via an argument that the reader will readily discern from the proof of the corollary. Thus, the conclusion of the corollary may be regarded as essentially sharp.

It will be evident from our discussion of the proof of Theorem 1.1 in §2 that the system (1.1) has particularly convenient features associated with its status as an affine Vinogradov system. In other systems, additional difficulties may be encountered. We illustrate these challenges with a consideration of a system closely related to (1.1). Consider then a coefficient triple $(h_1, h_2, h_4) \in \mathbb{Z}^3$. We denote by $T_2(P; \mathbf{h})$ the number of integral solutions of the system

$$\left. \begin{aligned} x_1^4 + x_2^4 - y_1^4 - y_2^4 &= h_4 \\ x_1^2 + x_2^2 - y_1^2 - y_2^2 &= h_2 \\ x_1 + x_2 - y_1 - y_2 &= h_1 \end{aligned} \right\}, \quad (1.3)$$

with $1 \leq x_i, y_i \leq P$ ($i = 1, 2$). In §3 we establish the following relative of Theorem 1.1.

Theorem 1.3. *Let $\eta \in (0, 1)$ be fixed, and let P be sufficiently large in terms of η . Suppose that $\mathbf{h} \in \mathbb{Z}^3$ is a coefficient triple with the property that $T_2(P; \mathbf{h}) > P^\eta$. Then, the tuple \mathbf{h} satisfies one of the following two conditions:*

- (a) *for some integers a and b with $a \neq b$ and $1 \leq a, b \leq P$, one has $h_j = a^j - b^j$ ($j = 1, 2, 4$), in which case $T_2(P; \mathbf{h}) = 4P$;*
- (b) *one has $h_j = 0$ ($j = 1, 2, 4$), in which case $T_2(P; \mathbf{0}) = 2P^2 - P$.*

In particular, if \mathbf{h} satisfies neither condition (a) nor condition (b), then $T_2(P; \mathbf{h}) = O(P^\varepsilon)$.

Similar comments apply concerning the structure of solutions in scenarios (a) and (b) as discussed following the statement of Theorem 1.1. Moreover, just as in the analogous discussion relating to $S_t(P; \mathbf{h})$, one can examine the situation with additional variables. Denote by $T_t(P; \mathbf{h})$ the number of integral solutions of the system

$$\sum_{i=1}^t (x_i^j - y_i^j) = h_j \quad (j = 1, 2, 4), \quad (1.4)$$

with $1 \leq x_i, y_i \leq P$ ($1 \leq i \leq t$). We present an analogue of Corollary 1.2 relating to the system (1.4).

Corollary 1.4. *Suppose that $\mathbf{h} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ and $\varepsilon > 0$. Then one has $T_3(P; \mathbf{h}) \ll P^{2+\varepsilon}$.*

Just as in our remark following the statement of Corollary 1.2, this conclusion is essentially sharp in the case that $h_j = a^j - b^j$ ($j = 1, 2, 4$), with $1 \leq a, b \leq P$ and $a \neq b$.

We complete this initial foray into the interplay between structure and paucity with another illustrative example. When s and k are natural numbers, and $\mathbf{h} \in \mathbb{Z}^k$, denote by $U_{s,k}(P; \mathbf{h})$ the number of solutions of the Diophantine system

$$\sum_{i=1}^s x_i^{2j-1} = h_j \quad (1 \leq j \leq k), \quad (1.5)$$

with $|x_i| \leq P$ ($1 \leq i \leq s$). This system is an affine version of that considered in work of Brüdern and Robert [2]. We first offer a theorem analogous to Theorems 1.1 and 1.3 showing that limitations to the strength of paucity results imply structure in the tuple \mathbf{h} .

Theorem 1.5. *Let $\eta \in (0, 1)$ be fixed, and let k and r be non-negative integers with $k \geq 2$. Let P be sufficiently large in terms of η , k and r . Suppose that $\mathbf{h} \in \mathbb{Z}^k$ is a coefficient k -tuple having the property that*

$$U_{k+1,k}(P; \mathbf{h}) > P^{r+\eta}. \quad (1.6)$$

Then one has $0 \leq r \leq (k-1)/2$, and for some integers a_i with $|a_i| \leq P$ ($1 \leq i \leq k-1-2r$), one has

$$h_j = a_1^{2j-1} + \dots + a_{k-1-2r}^{2j-1} \quad (1 \leq j \leq k). \quad (1.7)$$

In particular, when r is the largest integer for which the lower bound (1.6) holds for some $\eta \in (0, 1)$, then for every $\varepsilon > 0$, we have

$$P^{r+1} \ll U_{k+1,k}(P; \mathbf{h}) \ll P^{r+1+\varepsilon}.$$

We present a conclusion essentially equivalent to Theorem 1.5 having a converse flavour, demonstrating that structure in the coefficient tuple \mathbf{h} implies precise conclusions concerning paucity estimates.

Theorem 1.6. *Let P be large enough in terms of k , and suppose that $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$ is a coefficient k -tuple. Take $r_0(\mathbf{h})$ to be the smallest natural number r with the property that there exist integers a_i with $|a_i| \leq P$ ($1 \leq i \leq r$) satisfying the equations*

$$h_j = a_1^{2j-1} + \dots + a_r^{2j-1} \quad (1 \leq j \leq k), \quad (1.8)$$

and write

$$\tau(k; \mathbf{h}) = \left\lfloor \frac{k+1 - r_0(\mathbf{h})}{2} \right\rfloor.$$

Then for every $\varepsilon > 0$, we have

$$P^{\tau(k; \mathbf{h})} \leq U_{k+1, k}(P; \mathbf{h}) \ll P^{\tau(k; \mathbf{h}) + \varepsilon}.$$

These theorems show that large values of the counting function $U_{k+1, k}(P; \mathbf{h})$ are associated with a high degree of structure in the coefficient k -tuple \mathbf{h} . Here, the fewer the number of summands in (1.7) required to define \mathbf{h} , the greater the implied structural constraints on \mathbf{h} . Similar conclusions may be derived for the quantity $U_{s, k}(P; \mathbf{h})$ when $s > k + 1$, though we choose not to elaborate on this matter herein. Instead, we present a non-trivial paucity result that follows from Theorem 1.5 in a manner analogous to the corollaries presented above. We emphasise that this conclusion goes beyond the so-called square-root barrier, in the sense that the upper bound established for $U_{2t, k}(P; \mathbf{h})$ grows more slowly than P^t , the square-root of the underlying available supply of variables.

Corollary 1.7. *Suppose that $k \geq 2$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Then, for each $\varepsilon > 0$, one has*

$$U_{2t, k}(P; \mathbf{h}) \ll P^{t-1+\varepsilon} \quad (1 \leq t \leq k).$$

In common with previous work on paucity and quasi-paucity problems (see, for example [3, 5]), our approach to proving Theorems 1.1, 1.3, 1.5 and 1.6 rests on the application of polynomial identities of multiplicative type. In fortunate circumstances, such identities permit divisor function estimates to be deployed showing that there are few integral solutions of the Diophantine systems of interest for typical tuples \mathbf{h} . For atypical choices of \mathbf{h} , however, one must contend with the possibility that these polynomial identities show only that the underlying variables are constrained in terms of divisors of 0, thereby offering no useful information. One therefore requires a careful analysis of these atypical situations and their relation to diagonal structures within the associated Diophantine systems. The reader will gather from the discussion of §2 the ideas necessary for the consideration of related problems.

Our basic parameter is P , a sufficiently large positive integer. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. We make frequent use of vector notation in the form $\mathbf{x} = (x_1, \dots, x_r)$. Here, the dimension r depends on the course of the argument.

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2. PAUCITY IN THE AFFINE CUBIC VINogradov SYSTEM

Key to the proof of Theorem 1.1 is a polynomial identity having a multiplicative flavour. In order to describe this identity, we define the polynomials $s_j = s_j(x, y)$ by putting

$$s_j(x, y) = x^j - y^j \quad (j \geq 1). \quad (2.1)$$

One can check on the back of a small envelope that one has the identity

$$s_1^4 + 3s_2^2 - 4s_1s_3 = 0. \quad (2.2)$$

As a consequence of the relation (2.2), one has an identity relating the 4-variable polynomials $\sigma_j = \sigma_j(\mathbf{x}, \mathbf{y})$, defined by

$$\sigma_j(\mathbf{x}, \mathbf{y}) = x_1^j + x_2^j - y_1^j - y_2^j \quad (j \geq 1). \quad (2.3)$$

Lemma 2.1. *One has*

$$\sigma_1(\mathbf{x}, \mathbf{y})^4 + 3\sigma_2(\mathbf{x}, \mathbf{y})^2 - 4\sigma_1(\mathbf{x}, \mathbf{y})\sigma_3(\mathbf{x}, \mathbf{y}) = 12(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_2 - y_2). \quad (2.4)$$

Proof. If we specialise by setting $x_2 = y_2$, then we find that

$$\sigma_j(\mathbf{x}, \mathbf{y}) = s_j(x_1, y_1) \quad (j = 1, 2, 3).$$

Under such circumstances, the left hand side of (2.4) is equal to $s_1^4 + 3s_2^2 - 4s_1s_3$, which is 0. It therefore follows that the left hand side of (2.4) is divisible by $x_2 - y_2$, and symmetrical arguments reveal that $x_1 - y_1$, $x_1 - y_2$ and $x_2 - y_1$ are also factors. The identity (2.4) then follows on observing that the left and right hand sides of (2.4) have the same degree 4, with the constant factor 12 being fixed by the coefficient of $x_1^2x_2^2$ on the left hand side of the equation. \square

We are now equipped to confirm the conclusion of Theorem 1.1.

Proof of Theorem 1.1. Let $\eta \in (0, 1)$ and $\mathbf{h} \in \mathbb{Z}^3$. Whenever \mathbf{x}, \mathbf{y} is an integral solution of (1.1) counted by $S_2(P; \mathbf{h})$, it follows from (2.3) that $\sigma_j(\mathbf{x}, \mathbf{y}) = h_j$ ($j = 1, 2, 3$). Hence, from Lemma 2.1 we have

$$12(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_2 - y_2) = \Psi(\mathbf{h}), \quad (2.5)$$

in which

$$\Psi(\mathbf{h}) = h_1^4 + 3h_2^2 - 4h_1h_3.$$

Since $1 \leq x_i, y_i \leq P$ ($i = 1, 2$), we have $|\Psi(\mathbf{h})| < 12P^4$. We divide into cases.

Suppose first that $\Psi(\mathbf{h}) \neq 0$. Then a standard divisor function estimate shows that there are $O(P^\varepsilon)$ possible choices for integers d_i ($1 \leq i \leq 4$) having the property that

$$x_1 - y_1 = d_1, \quad x_1 - y_2 = d_2, \quad x_2 - y_1 = d_3, \quad x_2 - y_2 = d_4. \quad (2.6)$$

Fixing any one such choice of \mathbf{d} , we may substitute

$$y_1 = x_1 - d_1, \quad y_2 = x_1 - d_2, \quad x_2 = x_1 - d_1 + d_3 \quad (2.7)$$

into the system (1.1). Thus, we find that

$$x_1^2 + (x_1 - d_1 + d_3)^2 - (x_1 - d_1)^2 - (x_1 - d_2)^2 = h_2, \quad (2.8)$$

whence

$$2x_1(d_2 + d_3) + (d_1 - d_3)^2 - d_1^2 - d_2^2 = h_2. \quad (2.9)$$

This equation fixes the value of x_1 unless $d_3 = -d_2$, which we henceforth assume to be the case. Likewise, one has

$$x_1^3 + (x_1 - d_1 + d_3)^3 - (x_1 - d_1)^3 - (x_1 - d_2)^3 = h_3,$$

whence

$$3x_1((d_1 + d_2)^2 - d_1^2 - d_2^2) - (d_1 + d_2)^3 + d_1^3 + d_2^3 = h_3.$$

This equation fixes the value of x_1 unless

$$(d_1 + d_2)^2 - d_1^2 - d_2^2 = 0. \quad (2.10)$$

We are therefore forced to conclude that the value of x_1 is fixed, unless one has $2d_1d_2 = 0$, which is to say that one at least of d_1 and d_2 is equal to 0. The latter eventuality implies that $x_1 = y_1$ or $x_1 = y_2$. Either such circumstance implies, via (2.5), that $\Psi(\mathbf{h}) = 0$, contradicting our earlier assumption in this case. Thus x_1 is indeed determined via \mathbf{h} and \mathbf{d} , so that we may infer from (2.7) that y_1 , y_2 and x_2 are also determined. In this first scenario in which $\Psi(\mathbf{h}) \neq 0$, we therefore conclude that $S_2(P; \mathbf{h}) \ll P^\varepsilon$.

If it is the case that $S_2(P; \mathbf{h}) > P^\eta$, then whenever P is large enough in terms of ε and η , we cannot be in this first scenario. Thus, we have $\Psi(\mathbf{h}) = 0$, and it follows from (2.5) that

$$x_1 = y_1, \quad \text{or} \quad x_1 = y_2, \quad \text{or} \quad x_2 = y_1, \quad \text{or} \quad x_2 = y_2. \quad (2.11)$$

By relabelling variables, there is no loss of generality in supposing that $x_2 = y_2$. We then deduce from (1.1) that

$$x_1^j - y_1^j = h_j \quad (j = 1, 2, 3). \quad (2.12)$$

In particular, there exist integers a and b with $1 \leq a, b \leq P$ for which one has the relation (1.2), namely $h_j = a^j - b^j$ ($j = 1, 2, 3$). If one has $a = b$, and hence $h_j = 0$ ($j = 1, 2, 3$), then plainly $x_1 = y_1$. Keeping in mind our relabelling of variables, we then have

$$S_2(P; \mathbf{0}) = 2P^2 - P.$$

When $a \neq b$, meanwhile, we find from (2.12) that

$$x_1 - y_1 = a - b \quad \text{and} \quad x_1 + y_1 = \frac{x_1^2 - y_1^2}{x_1 - y_1} = \frac{a^2 - b^2}{a - b} = a + b.$$

Thus $x_1 = a$ and $y_1 = b$, and the solutions of (1.1) of this kind counted by $S_2(P; \mathbf{h})$ satisfy

$$\{x_1, x_2\} = \{a, t\} \quad \text{and} \quad \{y_1, y_2\} = \{b, t\},$$

with $1 \leq t \leq P$. Again noting our earlier relabelling of variables, we find that when $a \neq b$, one has

$$S_2(P; \mathbf{h}) = 4P.$$

This completes the proof of the theorem. \square

The corollary to Theorem 1.1 requires little additional effort to establish.

Proof of Corollary 1.2. We suppose throughout that $\mathbf{h} \neq \mathbf{0}$. One has

$$S_3(P; \mathbf{h}) = \sum_{1 \leq x_3, y_3 \leq P} S_2(P; \mathbf{h}'(x_3, y_3)), \quad (2.13)$$

where $\mathbf{h}' = (h'_1, h'_2, h'_3)$ and

$$h'_j(u, v) = h_j - u^j + v^j \quad (j \geq 1). \quad (2.14)$$

Let $\eta > 0$ be arbitrarily small. From Theorem 1.1, we find that $S_2(P; \mathbf{h}') \leq P^\eta$ unless, for some positive integers a and b with $1 \leq a, b \leq P$, one has $h'_j = a^j - b^j$ ($j = 1, 2, 3$), in which case

$$x_3^j - y_3^j + a^j - b^j = h_j \quad (j = 1, 2, 3). \quad (2.15)$$

When $a \neq b$, one then has $S_2(P; \mathbf{h}') = 4P$, and when $a = b$ one has $S_2(P; \mathbf{h}') = 2P^2 - P$. We therefore conclude from (2.13) that

$$S_3(P; \mathbf{h}) \ll P^{2+\eta} + P\Upsilon_1 + P^2\Upsilon_2, \quad (2.16)$$

where Υ_1 denotes the number of solutions of (2.15) with $1 \leq a, b, x_3, y_3 \leq P$ and $a \neq b$, and Υ_2 denotes the corresponding number of solutions with $a = b = 1$.

A second application of Theorem 1.1 shows that $\Upsilon_1 = O(P)$ whenever $\mathbf{h} \neq \mathbf{0}$. Turning next to the task of estimating Υ_2 , in which case we may assume that $a = b = 1$, we observe that the equations (2.15) yield $h_2 = h_3 = 0$ whenever $h_1 = 0$. Consequently, when $\mathbf{h} \neq \mathbf{0}$ we must have $h_1 \neq 0$, and then we find that

$$x_3 - y_3 = h_1 \quad \text{and} \quad x_3 + y_3 = h_2/h_1.$$

Thus, we infer that x_3 and y_3 are determined uniquely by h_1 and h_2 when $\mathbf{h} \neq \mathbf{0}$. Hence, when $a = b = 1$, one sees that $\Upsilon_2 = O(1)$. Then we conclude from (2.16) that whenever $\mathbf{h} \neq \mathbf{0}$, one has $S_3(P; \mathbf{h}) \ll P^{2+\varepsilon}$. This completes the proof of the corollary. \square

We note that scenario (a) of Theorem 1.1 may be more explicitly described. Thus, if $h_j = a^j - b^j$ ($j = 1, 2, 3$) with $a \neq b$, then

$$h_1 = a - b \quad \text{and} \quad h_2/h_1 = a + b,$$

whence

$$a = \frac{1}{2}(h_2/h_1 + h_1) \quad \text{and} \quad b = \frac{1}{2}(h_2/h_1 - h_1).$$

One then has

$$h_3 = a^3 - b^3 = \frac{1}{8} \left(\left(\frac{h_2}{h_1} + h_1 \right)^3 - \left(\frac{h_2}{h_1} - h_1 \right)^3 \right),$$

so that $4h_1h_3 = h_1^4 + 3h_2^2$, as should have been apparent from the condition $\Psi(\mathbf{h}) = 0$ already encountered in the proof of Theorem 1.1.

3. PAUCITY IN A RELATIVE OF THE AFFINE VINOGRADOV SYSTEM

In initiating the argument employed to establish Theorem 1.1, one rapidly discovers that the system (1.3) central to Theorem 1.3 is more delicate to analyse than that underlying Theorem 1.1. We again define the polynomials $s_j = s_j(x, y)$ as in (2.1). Then, one may check on the back of a somewhat larger envelope that one has the identity

$$s_2^3 + s_1^4 s_2 - 2s_1^2 s_4 = 0. \quad (3.1)$$

As a consequence of this relation, one has an identity relating the 4-variable polynomials $\sigma_j(\mathbf{x}, \mathbf{y})$ defined in (2.3), together with the additional polynomial

$$\tau(\mathbf{x}, \mathbf{y}) = (x_1^2 + x_1x_2 + x_2^2) - (y_1^2 + y_1y_2 + y_2^2). \quad (3.2)$$

Lemma 3.1. *One has*

$$\begin{aligned} \sigma_2(\mathbf{x}, \mathbf{y})^3 + \sigma_1(\mathbf{x}, \mathbf{y})^4 \sigma_2(\mathbf{x}, \mathbf{y}) - 2\sigma_1(\mathbf{x}, \mathbf{y})^2 \sigma_4(\mathbf{x}, \mathbf{y}) \\ = 8(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_2 - y_2)\tau(\mathbf{x}, \mathbf{y}). \end{aligned} \quad (3.3)$$

Proof. We again observe that by specialising $x_2 = y_2$, one has

$$\sigma_j(\mathbf{x}, \mathbf{y}) = s_j(x_1, y_1) \quad (j = 1, 2, 4).$$

Thus, in view of the identity (3.1), it is evident as in the proof of Lemma 2.1 that the left hand side of (3.3) is divisible by $x_2 - y_2$, with symmetrical arguments uncovering the additional factors $x_1 - y_1$, $x_1 - y_2$, and $x_2 - y_1$. One may then divide the left hand side of (3.3) by the product

$$(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_2 - y_2)$$

to reveal the quotient $8\tau(\mathbf{x}, \mathbf{y})$, with $\tau(\mathbf{x}, \mathbf{y})$ defined by (3.2). Readers wishing to conserve energy might choose to employ a computer algebra assistant in this computation. This completes the proof of the lemma. \square

Although we are equipped to establish Theorem 1.3 using Lemma 3.1, the details are more complicated than was the case in the corresponding argument employed in the proof of Theorem 1.1.

Proof of Theorem 1.3. Let $\eta \in (0, 1)$ and $\mathbf{h} \in \mathbb{Z}^3$. Whenever \mathbf{x}, \mathbf{y} is an integral solution of (1.3) counted by $T_2(P; \mathbf{h})$, it follows from (2.3) that $\sigma_j(\mathbf{x}, \mathbf{y}) = h_j$ ($j = 1, 2, 4$). Hence, on applying Lemma 3.1, we perceive that

$$8(x_1 - y_1)(x_1 - y_2)(x_2 - y_1)(x_2 - y_2)\tau(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{h}), \quad (3.4)$$

in which $\Phi(\mathbf{h}) = h_2^3 + h_1^4 h_2 - 2h_1^2 h_4$. On this occasion, since $1 \leq x_i, y_i \leq P$ ($i = 1, 2$), we have $|\Phi(\mathbf{h})| < 24P^6$. We divide into cases.

Suppose first that $\Phi(\mathbf{h}) \neq 0$. Then a standard divisor function estimate shows that there are $O(P^\varepsilon)$ possible choices for integers d_i ($1 \leq i \leq 4$) having the property that the relations (2.6) hold. Fixing any one such choice for \mathbf{d} , we may again substitute (2.7) into the system (1.3). Just as in the proof of Theorem 1.1, we again obtain (2.8) and (2.9). The latter equation fixes the value of x_1 unless $d_3 = -d_2$, which we henceforth assume to be the case. On this occasion, we have

$$x_1^4 + (x_1 - d_1 + d_3)^4 - (x_1 - d_1)^4 - (x_1 - d_2)^4 = h_4,$$

whence

$$6x_1^2((d_1 + d_2)^2 - d_1^2 - d_2^2) - 4x_1((d_1 + d_2)^3 - d_1^3 - d_2^3) + (d_1 + d_2)^4 - d_1^4 - d_2^4 = h_4.$$

This equation fixes the value of x_1 to be one of the two roots of a quadratic equation unless (2.10) holds. We are therefore forced to conclude that the value of x_1 is fixed, unless one at least of d_1 and d_2 is 0, an eventuality that again implies that $x_1 = y_1$ or $x_1 = y_2$. In either circumstance, it follows from (3.4) that $\Phi(\mathbf{h}) = 0$, contradicting our earlier assumption. We thus conclude that x_1 is indeed determined by fixed choices for \mathbf{h} and \mathbf{d} , whence from (2.7) we see that y_1, y_2 and x_2 are also determined. In this first scenario, in which $\Phi(\mathbf{h}) \neq 0$, therefore, we are forced to conclude that

$$T_2(P; \mathbf{h}) \ll P^\varepsilon.$$

If it is the case that $T_2(P; \mathbf{h}) > P^\eta$, then whenever P is large enough in terms of ε and η , we cannot be in the first scenario. Thus, we have $\Phi(\mathbf{h}) = 0$, and it follows from (3.4) that, either one at least of the relations (2.11) must hold, or else

$$\tau(\mathbf{x}, \mathbf{y}) = (x_1^2 + x_1 x_2 + x_2^2) - (y_1^2 + y_1 y_2 + y_2^2) = 0. \quad (3.5)$$

Before embarking on the analysis of this second case, we begin by examining the consequences of the assumption that the relation (3.5) holds. In this situation, it follows from the system of equations (1.3) that

$$\begin{aligned} (y_1 + y_2 + h_1)^2 - 2(y_1^2 + y_1y_2 + y_2^2) + (y_1^2 + y_2^2 + h_2) \\ = (x_1 + x_2)^2 - 2(x_1^2 + x_1x_2 + x_2^2) + (x_1^2 + x_2^2) = 0, \end{aligned}$$

whence

$$2h_1(y_1 + y_2) + h_1^2 + h_2 = 0. \quad (3.6)$$

On substituting this relation into the linear equation of (1.3), it follows that one also has

$$2h_1(x_1 + x_2) - h_1^2 + h_2 = 0. \quad (3.7)$$

We now make use of (3.6) and (3.7), respectively, to deduce the two relations

$$\begin{aligned} 8h_1^2(y_1^2 + y_2^2 + h_2) &= 2(4h_1^2(y_1^2 + h_2) + (h_1^2 + h_2 + 2h_1y_1)^2) \\ &= (4h_1y_1 + (h_1^2 + h_2))^2 + 8h_1^2h_2 + (h_1^2 + h_2)^2 \end{aligned}$$

and

$$\begin{aligned} 8h_1^2(x_1^2 + x_2^2) &= 2(4h_1^2x_1^2 + (h_1^2 - h_2 - 2h_1x_1)^2) \\ &= (4h_1x_1 - (h_1^2 - h_2))^2 + (h_1^2 - h_2)^2. \end{aligned}$$

Hence, on substituting these formulæ into the quadratic equation of (1.3), we infer that

$$(4h_1x_1 - (h_1^2 - h_2))^2 - (4h_1y_1 + (h_1^2 + h_2))^2 = 12h_2h_1^2. \quad (3.8)$$

We now begin our investigation of the scenario in which

$$\Phi(\mathbf{h}) = h_2^3 + h_1^4h_2 - 2h_1^2h_4 = 0,$$

but none of the relations (2.11) hold. In that scenario, we must therefore have (3.5), and consequently (3.8). Our goal is to show that these conditions are in fact incompatible with the assumption that $T_2(P; \mathbf{h}) > P^n$. In order to see this, consider first the case in which $h_1 = 0$. Then since $\Phi(\mathbf{h}) = 0$, it follows that $h_2 = 0$, and we have

$$\left. \begin{aligned} x_1^2 + x_2^2 &= y_1^2 + y_2^2 \\ x_1 + x_2 &= y_1 + y_2 \end{aligned} \right\}.$$

This forces us to conclude that $\{x_1, x_2\} = \{y_1, y_2\}$, violating our assumption that none of the relations (2.11) hold. It follows that we must have $h_1 \neq 0$.

Consider next the case where $h_1 \neq 0$ but $h_2 = 0$. This then implies that $h_4 = 0$, so that we must have

$$\left. \begin{aligned} x_1^4 + x_2^4 &= y_1^4 + y_2^4 \\ x_1^2 + x_2^2 &= y_1^2 + y_2^2 \end{aligned} \right\}.$$

These equations imply that $\{x_1^2, x_2^2\} = \{y_1^2, y_2^2\}$, and since $1 \leq x_i, y_i \leq P$, it follows that $\{x_1, x_2\} = \{y_1, y_2\}$. This again contradicts our assumption that none of the relations (2.11) hold.

We may therefore suppose that $12h_2h_1^2 \neq 0$. Under such circumstances, an elementary divisor function estimate leads from (3.8) to the conclusion that there are at most $O(P^\varepsilon)$ possible choices for integers e_1 and e_2 for which

$$\left. \begin{aligned} (4h_1x_1 - (h_1^2 - h_2)) - (4h_1y_1 + (h_1^2 + h_2)) &= e_1 \\ (4h_1x_1 - (h_1^2 - h_2)) + (4h_1y_1 + (h_1^2 + h_2)) &= e_2 \end{aligned} \right\}.$$

Fixing any one choice for e_1 and e_2 , it follows that both x_1 and y_1 are determined. We then find from (3.6) and (3.7) that x_2 and y_2 are also determined. Consequently, the number of solutions of this type is at most $O(P^\varepsilon)$. Since we had assumed that $T_2(P; \mathbf{h}) > P^\eta$, it follows that this scenario is also excluded. This affirms our claim that the assumption that none of the relations (2.11) hold is incompatible with the condition that $T_2(P; \mathbf{h}) > P^\eta$.

It remains to consider the scenario in which one of the relations (2.11) does hold. Here, by relabelling variables, there is no loss of generality in supposing that $x_2 = y_2$. We then deduce from (1.3) that

$$x_1^j - y_1^j = h_j \quad (j = 1, 2, 4). \quad (3.9)$$

In particular, there exist integers a and b with $1 \leq a, b \leq P$ for which one has $h_j = a^j - b^j$ ($j = 1, 2, 4$). If one has $a = b$, and hence $h_j = 0$ ($j = 1, 2, 4$), then plainly $x_1 = y_1$, and we are in case (b) of the theorem. In view of our relabelling of variables, one then has

$$T_2(P; \mathbf{0}) = 2P^2 - P.$$

When $a \neq b$, meanwhile, we find from (3.9), just as in the proof of Theorem 1.1, that $x_1 = a$ and $y_1 = b$, and the solutions of the system (1.3) of this kind counted by $T_2(P; \mathbf{h})$ satisfy

$$\{x_1, x_2\} = \{a, t\}, \quad \{y_1, y_2\} = \{b, t\},$$

with $1 \leq t \leq P$. Again accounting for our earlier relabelling of variables, we see that we are in case (a) of the theorem, and that when $a \neq b$ one has

$$T_2(P; \mathbf{h}) = 4P.$$

We have shown that when P is large enough and $T_2(P; \mathbf{h}) > P^\eta$, then we are either in case (a) or in case (b) of the theorem. In all other circumstances, we have confirmed that $T_2(P; \mathbf{h}) = O(P^\varepsilon)$. The proof of the theorem is therefore complete. \square

We follow the strategy of the proof of Corollary 1.2 in our proof of Corollary 1.4.

Proof of Corollary 1.4. We proceed using an argument almost identical to that establishing Corollary 1.2. We may suppose throughout that $\mathbf{h} \neq \mathbf{0}$. One has

$$T_3(P; \mathbf{h}) = \sum_{1 \leq x_3, y_3 \leq P} T_2(P; \mathbf{h}'(x_3, y_3)), \quad (3.10)$$

where $\mathbf{h}' = (h'_1, h'_2, h'_4)$ is defined as in (2.14). Let $\eta > 0$ be arbitrarily small. From Theorem 1.3, we know that $T_2(P; \mathbf{h}') \leq P^\eta$ unless, for some positive integers a and b with $1 \leq a, b \leq P$, one has $h'_j = a^j - b^j$ ($j = 1, 2, 4$), in which case

$$x_3^j - y_3^j + a^j - b^j = h_j \quad (j = 1, 2, 4). \quad (3.11)$$

When $a \neq b$, one has $T_2(P; \mathbf{h}') = 4P$, and when instead we have $a = b$, then one has $T_2(P; \mathbf{h}') = 2P^2 - P$. We therefore conclude from (3.10) that

$$T_3(P; \mathbf{h}) \ll P^{2+\varepsilon} + P\Upsilon_1 + P^2\Upsilon_2, \quad (3.12)$$

where Υ_1 now denotes the number of solutions of (3.11) with $1 \leq a, b, x_3, y_3 \leq P$ and $a \neq b$, and Υ_2 denotes the corresponding number of solutions with $a = b = 1$.

Again applying Theorem 1.3, we find that $\Upsilon_1 = O(P)$ whenever $\mathbf{h} \neq \mathbf{0}$. In order to bound Υ_2 , we may assume that $a = b = 1$, and observe that the equations (3.11) yield $h_1 = h_2 = h_4 = 0$ whenever $h_1 = 0$. When $h_1 \neq 0$, meanwhile, we find as in the discussion concluding the proof of Corollary 1.2 that

$$x_3 = \frac{1}{2}(h_2/h_1 + h_1) \quad \text{and} \quad y_3 = \frac{1}{2}(h_2/h_1 - h_1).$$

Hence, when $a = b = 1$, one sees that $\Upsilon_2 = O(1)$. Then we conclude from (3.12) that whenever $\mathbf{h} \neq \mathbf{0}$, one has $T_3(P; \mathbf{h}) \ll P^{2+\varepsilon}$. This completes the proof of the corollary. \square

4. PAUCITY IN AFFINE BRÜDERN-ROBERT SYSTEMS

In this section we investigate the Diophantine system (1.5) previously considered, in the case $\mathbf{h} = \mathbf{0}$, by Brüdern and Robert [2]. In order to illustrate our ideas without the distraction of superficial complications, we restrict attention to the situation with $s = k + 1$. Thus, when $\mathbf{h} \in \mathbb{Z}^k$, we consider the number of integral solutions $U_{k+1,k}(P; \mathbf{h})$ of the system of equations

$$\sum_{i=1}^{k+1} x_i^{2j-1} = h_j \quad (1 \leq j \leq k), \quad (4.1)$$

with $|x_i| \leq P$ ($1 \leq i \leq k + 1$).

Consider a fixed solution (x_1, \dots, x_s) of the system (1.5) counted by $U_{s,k}(P; \mathbf{h})$. It is possible that one may relabel variables in such a manner that, for some integer j with $0 \leq j \leq s/2$, one has

$$x_{2i-1} + x_{2i} = 0 \quad (1 \leq i \leq j), \quad (4.2)$$

and so that for no l and m with $2j < l < m \leq s$ does one have

$$x_l + x_m = 0. \quad (4.3)$$

We shall describe an s -tuple \mathbf{x} having this property as being of *type* j . We write $V_{s,k}^{(j)}(P; \mathbf{h})$ for the number of solutions \mathbf{x} of (1.5) counted by $U_{s,k}(P; \mathbf{h})$ having type j .

Next, for a fixed k -tuple \mathbf{h} , we denote by $I_k(\mathbf{h})$ the largest integer j having the property that there exists some $(k + 1)$ -tuple \mathbf{x} of type j , with $|x_i| \leq P$ ($1 \leq i \leq k + 1$), satisfying (4.1). Thus, we have $I_k(\mathbf{h}) \leq (k + 1)/2$, and

$$U_{k+1,k}(P; \mathbf{h}) = \sum_{0 \leq j \leq I_k(\mathbf{h})} V_{k+1,k}^{(j)}(P; \mathbf{h}). \quad (4.4)$$

Observe that, when $V_{k+1,k}^{(j)}(P; \mathbf{h}) > 0$, the relations (4.2) ensure that $V_{k+1,k}^{(j)}(P; \mathbf{h}) \geq P^j$. It follows, in particular, that

$$U_{k+1,k}(P; \mathbf{h}) \geq P^{I_k(\mathbf{h})}. \quad (4.5)$$

Our proofs of Theorems 1.5 and 1.6 make critical use of a corresponding upper bound which is a consequence of the following auxiliary lemma.

Lemma 4.1. *One has $V_{k+1,k}^{(j)}(P; \mathbf{h}) \ll P^{j+\varepsilon}$.*

From the upper bound supplied by this lemma, we infer via (4.4) that

$$U_{k+1,k}(P; \mathbf{h}) \ll \sum_{0 \leq j \leq I_k(\mathbf{h})} V_{k+1,k}^{(j)}(P; \mathbf{h}) \ll P^{I_k(\mathbf{h})+\varepsilon}. \quad (4.6)$$

Proof of Lemma 4.1. If $j = (k+1)/2$, which is possible only when k is odd, then the relations (4.2) combine with (4.1) to show that $h_j = 0$ ($1 \leq j \leq k$). We relabel variables so as to realise the relations (4.2). Then, given fixed choices for the variables x_{2i-1} ($1 \leq i \leq \frac{1}{2}(k+1)$), the remaining variables are determined uniquely by the relations (4.2). Thus, in this case in which $j = (k+1)/2$, we have $V_{k+1,k}^{((k+1)/2)}(P; \mathbf{h}) \ll P^{(k+1)/2}$, and the conclusion of the lemma follows.

Consider now the situation with $j < (k+1)/2$. Here, by relabelling variables, we find from (4.2) and (4.3) that

$$V_{k+1,k}^{(j)}(P; \mathbf{h}) \ll P^j V_{k+1-2j,k}^{(0)}(P; \mathbf{h}). \quad (4.7)$$

The situation in which $2j \in \{k-1, k\}$ is straightforward to handle. Here, we have $k+1-2j \in \{1, 2\}$. The number of solutions of the equation

$$x_1 = h_1$$

is plainly 1, and so $V_{1,k}^{(0)}(P; \mathbf{h}) \leq V_{1,1}^{(0)}(P; h_1) = 1$. Meanwhile, the number of solutions of the system of equations

$$\left. \begin{aligned} x_1^3 + x_2^3 &= h_2 \\ x_1 + x_2 &= h_1 \end{aligned} \right\}$$

with $x_1 + x_2 \neq 0$ is also $O(1)$. In order to confirm this claim, we observe that since $h_1 = x_1 + x_2 \neq 0$, we have

$$x_1^2 - x_1 x_2 + x_2^2 = h_2/h_1 \quad \text{and} \quad x_1 + x_2 = h_1,$$

whence

$$h_2/h_1 = x_1^2 - x_1(h_1 - x_1) + (h_1 - x_1)^2 = 3x_1^2 - 3x_1 h_1 + h_1^2.$$

This quadratic equation has at most 2 solutions for x_1 , each of which uniquely determines $x_2 = h_1 - x_1$. Thus, in either of these cases, we have $V_{k+1-2j,k}^{(0)}(P; \mathbf{h}) = O(1)$, and the estimate (4.7) then yields the bound $V_{k+1,k}^{(j)}(P; \mathbf{h}) \ll P^{j+\varepsilon}$. The conclusion of the lemma again follows, therefore, when $2j \in \{k-1, k\}$.

When instead $0 \leq 2j \leq k-2$, we proceed in a more sophisticated manner. By discarding information from the highest degree equations in (4.1), we are led from the relation (4.7) to the estimate

$$V_{k+1,k}^{(j)}(P; \mathbf{h}) \ll P^j V_{k+1-2j,k-2j}^{(0)}(P; \mathbf{h}') \quad (0 \leq j \leq \frac{1}{2}(k-2)), \quad (4.8)$$

where we write

$$\mathbf{h}' = (h_1, \dots, h_{k-2j}). \quad (4.9)$$

Put $\kappa = k - 2j$, and examine the expression $V_{\kappa+1,\kappa}^{(0)}(P; \mathbf{h}')$ that occurs in the upper bound (4.8). Define the polynomials $t_j = t_j(\mathbf{x})$ by putting

$$t_j(\mathbf{x}) = \sum_{i=1}^{\kappa-1} x_i^{2j-1} \quad (1 \leq j \leq \kappa).$$

Then it follows from Brüdern and Robert [2, Lemma 1] that there exists a non-zero polynomial $\Upsilon(\mathbf{z}) \in \mathbb{Z}[z_1, \dots, z_\kappa]$ having the property that

$$\Upsilon(t_1(\mathbf{x}), \dots, t_\kappa(\mathbf{x})) = 0. \quad (4.10)$$

As noted in [2, p. 227], this identity also appears under alternate guise in the work of Perron [4, Satz 1]. Furthermore, the discussion of [2, p. 227] shows that the weighted degree of $\Upsilon(\mathbf{z})$ is equal to $\kappa(\kappa + 1)/2$, in the sense that in each monomial of $\Upsilon(\mathbf{z})$ of the shape

$$c_\alpha z_1^{\alpha_1} \cdots z_\kappa^{\alpha_\kappa},$$

with $c_\alpha \in \mathbb{Z}$ and $\alpha_i \geq 0$ ($1 \leq i \leq \kappa$), one has

$$\sum_{i=1}^{\kappa} (2i - 1)\alpha_i = \kappa(\kappa + 1)/2. \quad (4.11)$$

Next, we define the polynomials

$$\tau_j(\mathbf{x}) = \sum_{i=1}^{\kappa+1} x_i^{2j-1} \quad (1 \leq j \leq \kappa).$$

The equations

$$\sum_{i=1}^{\kappa+1} x_i^{2j-1} = h_j \quad (1 \leq j \leq \kappa) \quad (4.12)$$

now become $\tau_j(\mathbf{x}) = h_j$ ($1 \leq j \leq \kappa$). In accordance with the discussion surrounding equation (5) of [2], we find via (4.10) that when we make the specialisation $x_\kappa + x_{\kappa+1} = 0$, one has

$$\Upsilon(\tau_1(\mathbf{x}), \dots, \tau_\kappa(\mathbf{x})) = \Upsilon(t_1(\mathbf{x}), \dots, t_\kappa(\mathbf{x})) = 0,$$

and hence the polynomial $\Upsilon(\tau_1(\mathbf{x}), \dots, \tau_\kappa(\mathbf{x}))$ is divisible by $x_\kappa + x_{\kappa+1}$. Symmetrical arguments reveal that $x_i + x_j$ is likewise a factor whenever $1 \leq i < j \leq \kappa + 1$. In view of the relation (4.11), we infer that there exists a non-zero integer $C = C(\kappa)$ having the property that

$$\Upsilon(\tau_1(\mathbf{x}), \dots, \tau_\kappa(\mathbf{x})) = C \prod_{1 \leq i < j \leq \kappa+1} (x_i + x_j).$$

When the variables $x_1, \dots, x_{\kappa+1}$ satisfy (4.12), it therefore follows that one has

$$C \prod_{1 \leq i < j \leq \kappa+1} (x_i + x_j) = \Upsilon(h_1, \dots, h_\kappa). \quad (4.13)$$

Recall from (4.9) our definition of \mathbf{h}' , and suppose that $\Upsilon(\mathbf{h}') = 0$. Then it follows from (4.13) that $x_i + x_j = 0$ for some indices i and j with $1 \leq i < j \leq \kappa + 1$. This conclusion is in conflict with the definition of $V_{\kappa+1,\kappa}^{(0)}(P; \mathbf{h}')$, since any solutions of (4.12) counted by

this quantity are of type 0. This contradiction shows, in fact, that it is not possible that $\Upsilon(\mathbf{h}') = 0$.

We may therefore continue our deliberations under the assumption that $\Upsilon(\mathbf{h}') \neq 0$. Then since $\Upsilon(\mathbf{z})$ has weighted degree $\kappa(\kappa + 1)/2$, we see that $|\Upsilon(\mathbf{h}')| \ll P^{\kappa(\kappa+1)/2}$, and hence we find from an elementary estimate for the divisor function that there are $O(P^\varepsilon)$ possible choices for integers d_{ij} ($1 \leq i < j \leq \kappa + 1$) having the property that

$$x_i + x_j = d_{ij} \quad (1 \leq i < j \leq \kappa + 1).$$

Fix any one such choice for these integers d_{ij} ($1 \leq i < j \leq \kappa + 1$). Since for $1 \leq i \leq \kappa + 1$, one has

$$\sum_{\substack{1 \leq j \leq \kappa+1 \\ j \neq i}} (x_i + x_j) = (\kappa - 1)x_i + \sum_{j=1}^{\kappa+1} x_j = (\kappa - 1)x_i + h_1,$$

we see that

$$(\kappa - 1)x_i = \sum_{\substack{1 \leq j \leq \kappa+1 \\ j \neq i}} d_{ij} - h_1.$$

We are at liberty to suppose that $\kappa \geq 2$. Thus, for $1 \leq i \leq \kappa + 1$, the integer x_i is fixed by \mathbf{h}' and our choice for \mathbf{d} , and one infers that $V_{\kappa+1, \kappa}^{(0)}(P; \mathbf{h}') \ll P^\varepsilon$.

We have shown that, in all circumstances, one has

$$V_{\kappa+1, \kappa}^{(0)}(P; \mathbf{h}') \ll P^\varepsilon.$$

In combination with our earlier analysis of the situation with $2j \in \{k-1, k\}$, the conclusion of the lemma follows upon recalling (4.8). \square

With Lemma 4.1 in hand, we are now equipped to complete the proofs of Theorems 1.5 and 1.6.

Proof of Theorem 1.5. Suppose that $\eta \in (0, 1)$ and r is a non-negative integer, and suppose further that $\mathbf{h} \in \mathbb{Z}^k$ is a coefficient tuple with the property that $U_{k+1, k}(P; \mathbf{h}) > P^{r+\eta}$. Then we see from (4.6) that

$$P^{r+\eta} < U_{k+1, k}(P; \mathbf{h}) \ll P^{I_k(\mathbf{h})+\varepsilon},$$

and thus $I_k(\mathbf{h}) \geq r + 1$. This lower bound ensures that some tuple \mathbf{x} counted by $U_{k+1, k}(P; \mathbf{h})$ has type at least $r + 1$. Then there is a relabelling of variables with the property that

$$x_{2i-1} + x_{2i} = 0 \quad (1 \leq i \leq r + 1).$$

This is, of course, impossible when $2r + 2 > k + 1$. By reference to (4.1), we find that there are integers a_i with $|a_i| \leq P$ ($1 \leq i \leq k - 2r - 1$) having the property that

$$h_j = \sum_{i=1}^{k-2r-1} a_i^{2j-1} \quad (1 \leq j \leq k).$$

Since $I_k(\mathbf{h}) \geq r + 1$, moreover, it follows from (4.5) that $U_{k+1, k}(P; \mathbf{h}) \gg P^{r+1}$. Choosing r to be maximal with the property that $U_{k+1, k}(P; \mathbf{h}) > P^{r+\eta}$ for some $\eta > 0$, we infer that

$$P^{r+1} \ll U_{k+1, k}(P; \mathbf{h}) \leq P^{r+1+\eta}.$$

Since η may be chosen arbitrarily small, we conclude that $P^{r+1} \ll U_{k+1,k}(P; \mathbf{h}) \ll P^{r+1+\varepsilon}$. We have now confirmed all of the conclusions of Theorem 1.5. \square

We turn next to the problem of showing that structure in the coefficient tuple \mathbf{h} implies precise paucity estimates.

Proof of Theorem 1.6. Define $r = r_0(\mathbf{h})$ as in the statement of Theorem 1.6, and suppose that the integers a_1, \dots, a_r satisfy $|a_i| \leq P$ ($1 \leq i \leq r$) and the equations (1.8). Recall the definition of $\tau = \tau(k; \mathbf{h})$ from the statement of the theorem. Then, in light of the estimates (4.5) and (4.6), the proof of the theorem will follow once we confirm that $I_k(\mathbf{h}) = \tau(k; \mathbf{h})$. Note next that, in view of (1.8), any $(k+1)$ -tuple $\mathbf{x} \in \mathbb{Z}^{k+1}$ of the shape $\mathbf{x} = (x_1, \dots, x_{k+1-r}, a_1, \dots, a_r)$, and satisfying the equations

$$\sum_{i=1}^{k+1-r} x_i^{2^j-1} = 0 \quad (1 \leq j \leq k),$$

is a solution of (4.1). It is apparent that any choice of x_1, \dots, x_{k+1-r} , with $x_{2i-1} + x_{2i} = 0$ for $1 \leq i \leq \tau$, complemented with $x_{2\tau+1} = 0$ in the case when $k+1-r$ is odd, provides such a solution. Thus $I_k(\mathbf{h}) \geq \tau$. It is also apparent that no solution of type exceeding τ can exist, since this would imply the existence of a representation of \mathbf{h} as in (1.8) but with $r = k+1 - 2(\tau+1) < r_0(\mathbf{h})$, contradicting the implicit minimality of $r_0(\mathbf{h})$. Thus, we have $I_k(\mathbf{h}) = \tau(k; \mathbf{h})$ as claimed, and the conclusion of the theorem follows. \square

Finally, we turn to the proof of Corollary 1.7.

Proof of Corollary 1.7. Suppose that $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$ and $1 \leq t \leq k$. We observe first that it suffices to establish the conclusion of the corollary in the cases with $\lceil (k+1)/2 \rceil \leq t \leq k$ in order to establish it in full generality for $1 \leq t \leq k$. In order to confirm this observation, suppose that t is an integer with $1 \leq t < \lceil (k+1)/2 \rceil$. By considering solutions of the system (1.5) in the case $s = 2t+2$ with $x_{2t+1} = -x_{2t+2}$, we see that

$$PU_{2t,k}(P; \mathbf{h}) \ll U_{2t+2,k}(P; \mathbf{h}). \quad (4.14)$$

By iterating this relation, one finds that whenever it is known that

$$U_{2T,k}(P; \mathbf{h}) \ll P^{T-1+\varepsilon} \quad (\mathbf{h} \in \mathbb{Z}^k \setminus \{0\}), \quad (4.15)$$

with some integer T satisfying $T \geq \lceil (k+1)/2 \rceil$, then it follows that

$$U_{2t,k}(P; \mathbf{h}) \ll P^{t-T} \cdot P^{T-1+\varepsilon} \ll P^{t-1+\varepsilon} \quad (1 \leq t \leq T).$$

This confirms our claim.

We next set about proving the upper bound (4.15) for $\lceil (k+1)/2 \rceil \leq T \leq k$. We observe that

$$U_{2T,k}(P; \mathbf{h}) = \sum_{|x_{k+2}| \leq P} \cdots \sum_{|x_{2T}| \leq P} U_{k+1,k}(P; \mathbf{h}'), \quad (4.16)$$

where we write

$$h'_j = h_j - (x_{k+2}^{2^j-1} + \dots + x_{2T}^{2^j-1}) \quad (1 \leq j \leq k).$$

Let $\eta > 0$ be arbitrarily small, and suppose that $\mathbf{h}' \in \mathbb{Z}^k$ is a coefficient k -tuple with the property that for some non-negative integer r , one has

$$U_{k+1,k}(P; \mathbf{h}') > P^{r+\eta}. \quad (4.17)$$

We take $r = r(\mathbf{h}')$ to be maximal with this property, and note that from Theorem 1.5, one then has $0 \leq r \leq (k-1)/2$. The remaining tuples \mathbf{h}' in which (4.17) does not hold for any such value of r must have the property that

$$U_{k+1,k}(P; \mathbf{h}') \ll P^\varepsilon. \quad (4.18)$$

Note that when (4.17) holds with $0 \leq r \leq (k-1)/2$, it also follows from Theorem 1.5 that there exist integers a_i with $|a_i| \leq P$ ($1 \leq i \leq k-1-2r$) with

$$h_j - (x_{k+2}^{2j-1} + \dots + x_{2T}^{2j-1}) = a_1^{2j-1} + \dots + a_{k-1-2r}^{2j-1} \quad (1 \leq j \leq k). \quad (4.19)$$

In these circumstances, Theorem 1.5 shows that

$$U_{k+1,k}(P; \mathbf{h}') \ll P^{r+1+\varepsilon}.$$

The number of possible choices for \mathbf{h}' of this type is determined by (4.19), and so this number is bounded above by the number of integral solutions of the system

$$\sum_{i=1}^{2T-2r-2} y_i^{2j-1} = h_j \quad (1 \leq j \leq k),$$

with $|y_i| \leq P$ ($1 \leq i \leq 2T-2r-2$). We thus conclude from (4.16) and (4.18) that

$$U_{2T,k}(P; \mathbf{h}) \ll P^{2T-k-1+\varepsilon} + \sum_{0 \leq r \leq (k-1)/2} P^{r+1+\varepsilon} U_{2T-2r-2,k}(P; \mathbf{h}).$$

By employing (4.14), we see that for $0 \leq r \leq (k-1)/2$, one has

$$P^{r+1+\varepsilon} U_{2T-2r-2,k}(P; \mathbf{h}) \ll P^{1+\varepsilon} U_{2T-2,k}(P; \mathbf{h}).$$

Thus, we may conclude at this stage that when $\lceil (k+1)/2 \rceil \leq T \leq k$, one has

$$U_{2T,k}(P; \mathbf{h}) \ll P^{T-1+\varepsilon} + P^{1+\varepsilon} U_{2T-2,k}(P; \mathbf{h}). \quad (4.20)$$

We now divide into cases according to the parity of k . When k is odd, we may iterate the application of (4.20) to infer that

$$U_{2T,k}(P; \mathbf{h}) \ll P^{T-1+\varepsilon} + P^{T-(k+1)/2} U_{k+1,k}(P; \mathbf{h}). \quad (4.21)$$

We now see from Theorem 1.5 that

$$U_{k+1,k}(P; \mathbf{h}) \ll P^{(k-1)/2+\varepsilon}$$

unless $h_j = 0$ ($1 \leq j \leq k$). Since the latter scenario is specifically excluded by the hypotheses of the corollary, it follows from (4.21) that

$$U_{2T,k}(P; \mathbf{h}) \ll P^{T-1+\varepsilon} + P^{T-(k+1)/2} \cdot P^{(k-1)/2+\varepsilon} \ll P^{T-1+\varepsilon}.$$

This confirms (4.15) in the scenario that k is odd.

Consider next the situation in which k is even. Here, we deduce from (4.20) that

$$U_{2T,k}(P; \mathbf{h}) \ll P^{T-1+\varepsilon} + P^{T-(k+2)/2} U_{k+2,k}(P; \mathbf{h}). \quad (4.22)$$

In this situation, we make use of the bound

$$U_{k+2,k}(P; \mathbf{h}) = \sum_{|x| \leq P} U_{k+1,k}(P; \mathbf{h}'),$$

where we now put

$$h'_j = h_j - x^{2j-1} \quad (1 \leq j \leq k). \quad (4.23)$$

In this instance, we see from Theorem 1.5 that

$$U_{k+1,k}(P; \mathbf{h}') \ll P^{(k-2)/2+\varepsilon},$$

unless there exists an integer a with $|a| \leq P$ satisfying the system of equations

$$h'_j = a^{2j-1} \quad (1 \leq j \leq k), \quad (4.24)$$

in which case we have

$$U_{k+1,k}(P; \mathbf{h}') \ll P^{k/2+\varepsilon}.$$

On recalling that $k \geq 2$, we see from the equations (4.23) and (4.24) that $h_1 = x + a$. As in previous arguments we observe that if $h_1 = 0$, we have $h_j = 0$ ($1 \leq j \leq k$), contradicting our assumption that $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Thus $h_1 \neq 0$, and we have

$$h_3/h_1 = x^2 - ax + a^2 = (x + a)^2 - 3a(a + x) + 3a^2 = h_1^2 - 3ah_1 + 3a^2.$$

This determines a up to a possible choice of roots of a quadratic equation, and hence also x via the equation $h_1 = x + a$. Consequently, in instances in which $U_{k+1,k}(P; \mathbf{h}') > P^{(k-2)/2+\eta}$ for some $\eta > 0$, there are $O(1)$ choices for the variables a and x . Then

$$U_{k+2,k}(P; \mathbf{h}) = \sum_{|x| \leq P} U_{k+1,k}(P; \mathbf{h}') \ll P \cdot P^{(k-2)/2+\varepsilon} + 1 \cdot P^{k/2+\varepsilon} \ll P^{k/2+\varepsilon}.$$

On substituting this estimate into (4.22), we conclude that

$$U_{2T,k}(P; \mathbf{h}) \ll P^{T-1+\varepsilon} + P^{T-(k+2)/2+\varepsilon} \cdot P^{k/2+\varepsilon} \ll P^{T-1+2\varepsilon}.$$

This confirms (4.15) in the scenario that k is even.

In view of the preliminary discussion opening this argument, the proof of the corollary is now complete. \square

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