

SHARP MEAN VALUE ESTIMATES IN THE ABSENCE OF TRANSLATION-DILATION INVARIANCE

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ABSTRACT. We establish sharp mean value estimates for various exponential sums lacking translation-dilation invariance properties. For example, suppose that $k \geq 1$ and $s > 0$. Then, for each $\varepsilon > 0$, one has

$$\int_{[0,1]^k} \left| \sum_{1 \leq h \leq X} \sum_{1 \leq z \leq X} e(hz\alpha_1 + \dots + hz^k\alpha_k) \right|^{2s} d\alpha \ll X^\varepsilon (X^s + X^{2s-k(k+3)/2}),$$

confirming the main conjecture for a new family of exponential sums. Our methods are based on the application of very recent work of the author employing the nested efficient congruencing method.

1. INTRODUCTION

Transformational progress has been achieved in the last decade or so concerning mean value estimates for exponential sums, achieving in some cases the sharp estimates conjectured to hold (see [1, 3, 6, 7], for example). This progress is almost entirely limited to Vinogradov systems of equations, and analogues of multidimensional type also possessing translation-dilation invariance properties. Our goal in this paper is to expand the list of families amenable to such sharp analysis beyond the class of translation-dilation invariant systems. This goal was achieved for a new class of exponential sums in one variable by the author in recent work [9]. We turn in this memoir to a family of bivariate exponential sums, also not of translation-dilation invariant type, for which we achieve similar success.

We begin our account of the new systems amenable to our methods with a consideration of the exponential sum

$$f_k(\alpha; X) = \sum_{1 \leq h \leq X} \sum_{1 \leq z \leq X} e(h\alpha_1 + hz\alpha_2 + \dots + hz^{k-1}\alpha_k). \quad (1.1)$$

Here, as is conventional, we write $e(\theta)$ for $e^{2\pi i\theta}$. We shall be interested in the mean value

$$B_{s,k}(X) = \int_{[0,1]^k} |f_k(\alpha; X)|^s d\alpha. \quad (1.2)$$

Theorem 1.1. *Suppose that $k \geq 2$ and s is a positive real number. Then, for each $\varepsilon > 0$, one has*

$$B_{s,k}(X) \ll X^\varepsilon (X^s + X^{2s-k(k+1)/2}). \quad (1.3)$$

Note that a trivial estimate for the exponential sum $f_k(\alpha; X)$ yields the bound

$$|f_k(\alpha; X)| \leq X^2,$$

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since the sum (1.1) has at most X^2 summands. Thus, the first term in the upper bound (1.3) reflects square-root cancellation on average. Meanwhile, the sum of the degrees of the monomials hz^{j-1} ($1 \leq j \leq k$) occurring in the argument of the exponential sum (1.1) is equal to $\frac{1}{2}k(k+1)$. The set of α with

$$|\alpha_j| \leq \frac{1}{(100k)} X^{-j} \quad (1 \leq j \leq k)$$

makes a contribution to the integral in (1.2) establishing the lower bound

$$B_{s,k}(X) \gg X^{2s-k(k+1)/2}.$$

A consideration of diagonal solutions consequently reveals that the upper bound (1.3) is sharp up to the factor of X^ε .

We emphasise that the parameter s in Theorem 1.1 is not constrained to be an even integer. However, when s is an even integer, say $s = 2t$, then it follows via orthogonality that $B_{s,k}(X)$ counts the number of integral solutions of the system of equations

$$\sum_{i=1}^t h_i z_i^{j-1} = \sum_{i=t+1}^{2t} h_i z_i^{j-1} \quad (1 \leq j \leq k). \quad (1.4)$$

While the system of equations (1.4) is translation-invariant with respect to the shift $z_i \mapsto z_i + a$ ($1 \leq i \leq 2t$), there is no translation invariance property for the variables h_i . Thus, the system (1.4) does not belong to the class of translation-dilation invariant systems amenable to methods based on efficient congruencing and decoupling. Earlier methods based on the use of polynomial identities and divisor function estimates are available. These recover the conclusion of Theorem 1.1 in the range $s = 2t \leq k+1$ (see [2, Theorem 2.1] in the case $r = 1$ and $t = k$, and the earlier work of the author [5, Lemma 2.1] devoted to the case $k = 3$ and $s = 4$).

The conclusion of Theorem 1.1 is a consequence of a more general conclusion, as we now explain. We consider polynomials $\varphi_j \in \mathbb{Z}[t]$ ($1 \leq j \leq k$), and the Wronskian

$$W_0(t; \varphi) = \det(\varphi_j^{(i-1)}(t))_{1 \leq i, j \leq k},$$

in which we write $\varphi_j^{(r)}(t)$ for the r -th derivative $d^r \varphi_j(t)/dt^r$. We introduce the exponential sum $F_k(\alpha; X) = F_k(\alpha; X; \varphi)$, defined by

$$F_k(\alpha; X; \varphi) = \sum_{1 \leq h \leq X} \sum_{1 \leq z \leq X} e(h\varphi_1(z)\alpha_1 + \dots + h\varphi_k(z)\alpha_k). \quad (1.5)$$

Finally, when $s > 0$, we define the mean value

$$B_{s,k}(X; \varphi) = \int_{[0,1]^k} |F_k(\alpha; X; \varphi)|^s d\alpha. \quad (1.6)$$

Theorem 1.2. *Suppose that $k \geq 1$ and $\varphi_j \in \mathbb{Z}[t]$ ($1 \leq j \leq k$) is a system of polynomials with $W_0(t; \varphi) \neq 0$. Let s be a positive number with $s \leq k(k+1)/2$. Then for each $\varepsilon > 0$, one has*

$$B_{s,k}(X; \varphi) \ll X^{s+\varepsilon}.$$

The conclusion of Theorem 1.2 shows that when $s \leq k(k+1)/2$, the mean value $B_{s,k}(X; \boldsymbol{\varphi})$ exhibits square-root cancellation relative to the trivial estimate

$$|F_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi})| \leq X^2.$$

This new mean value estimate is notably sharp insofar as it applies to exponential sums not of translation-dilation invariant type.

Next, define

$$g_k(\boldsymbol{\alpha}; X) = \sum_{1 \leq h \leq X} \sum_{1 \leq z \leq X} e(hz\alpha_1 + \dots + hz^k\alpha_k), \quad (1.7)$$

and, when $s > 0$, put

$$C_{s,k}(X) = \int_{[0,1]^k} |g_k(\boldsymbol{\alpha}; X)|^s d\boldsymbol{\alpha}. \quad (1.8)$$

Then, as a consequence of Theorem 1.2, one has

$$C_{s,k}(X) \ll X^{s+\varepsilon} \quad (1.9)$$

whenever $s \leq \frac{1}{2}k(k+1)$. By applying ideas aligned with our previous work [9] relating to relatives of Vinogradov's system, we are able to obtain sharp estimates for all $s > 0$.

Theorem 1.3. *Suppose that $k \geq 1$ and $s > 0$. Then for each $\varepsilon > 0$, one has*

$$C_{s,k}(X) \ll X^\varepsilon (X^s + X^{2s-k(k+3)/2}).$$

Thus, one sees that the bound (1.9) holds whenever $s \leq \frac{1}{2}k(k+3)$. Note that since the sum of the degrees of the monomials $hz, \dots, hz^k$ is equal to

$$\sum_{j=2}^{k+1} j = \frac{1}{2}k(k+3),$$

the upper bound for $C_{s,k}(X)$ recorded in Theorem 1.3 achieves that predicted by the standard heuristics of the subject. Indeed, the lower bound

$$C_{s,k}(X) \gg X^s + X^{s-k(k+3)/2}$$

follows via a consideration of diagonal solutions and the contribution from a single major arc centred at $\mathbf{0}$.

The proofs of Theorems 1.1, 1.2 and 1.3 all depend, in one way or another, on estimates for mean values of a weighted analogue of the exponential sum $F_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi})$. Here, we adapt the methods introduced in [9, §§2 and 3] to the present setting. Thus, on defining

$$G_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi}) = \sum_{1 \leq z \leq X} \left| \sum_{1 \leq h \leq X} e(h\varphi_1(z)\alpha_1 + \dots + h\varphi_k(z)\alpha_k) \right|^2, \quad (1.10)$$

we find that the mean value

$$A_{s,k}^*(X; \boldsymbol{\varphi}) = \int_{[0,1]^k} |G_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi})|^s d\boldsymbol{\alpha} \quad (1.11)$$

may be related to the mean value $B_{s,k}(X; \boldsymbol{\varphi})$ defined via (1.6). When s is even, this is more or less immediate via orthogonality, but in general this relationship is far more subtle to establish, since Diophantine interpretations are then absent. It transpires that

the mean value $A_{s,k}^*(X; \boldsymbol{\varphi})$ is readily amenable to estimates available from our earlier work [8] based on efficient congruencing in the ring $\mathbb{Z}[t]$ modulo t^2 .

We establish the relationship between our work [8] and the mean value $A_{s,k}^*(X; \boldsymbol{\varphi})$ in §2, thus laying the groundwork for what is to come. Then, in §3, we show how the mean value $A_{s,k}^*(X; \boldsymbol{\varphi})$ may be related to the corresponding unweighted mean value $B_{s,k}(X; \boldsymbol{\varphi})$. This work establishes Theorem 1.2. Little additional work is required to extract the conclusion of Theorem 1.1, and this we also complete in §3. The proof of Theorem 1.3 is discussed in §4 via the application of a kind of translation-dilation invariance idea previously utilised in earlier work of the author (see [4, 9]).

Throughout, we use X to denote an integer sufficiently large in terms of all ambient parameters, such as ε , k and s . We make use of standard notational conventions applied in this area of mathematics. Thus, throughout, we make liberal use of vector notation in settings not always conventional in nature. For example, we may write $1 \leq \mathbf{x} \leq X$ to denote that every coordinate x_i of $\mathbf{x}$ satisfies $1 \leq x_i \leq X$. Also, given a situation in which the parameter ε has not already been fixed, in any statement involving the letter ε , it is implicitly asserted that the statement holds for each $\varepsilon > 0$. In such circumstances, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε .

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2. AN AUXILIARY MEAN VALUE THEOREM

We make use of estimates associated with a fully translation-dilation invariant system so as to derive upper bounds for the mean value $A_{s,k}^*(X; \boldsymbol{\varphi})$. We achieve a relationship between these quantities by exploiting an idea introduced in [9, §2].

Lemma 2.1. *Suppose that $k \geq 1$ and $\varphi_j \in \mathbb{Z}[t]$ ($1 \leq j \leq k$) is a system of polynomials with $W_0(t; \boldsymbol{\varphi}) \neq 0$. Let s be a positive number with $s \leq k(k+1)/2$. Then for each $\varepsilon > 0$, one has*

$$A_{s,k}^*(X; \boldsymbol{\varphi}) \ll X^{2s+\varepsilon}.$$

Proof. We relate the mean value $A_{s,k}^*(X; \boldsymbol{\varphi})$ to one that envelopes the underlying Diophantine system, in a suitable sense. First, for $1 \leq j \leq k$, we define the polynomial $\psi_j(t)$ by means of the relation $\psi_j'(t) = \varphi_j(t)$, subject to the condition $\psi_j(0) = 0$. Thus, the polynomial ψ_j is an antiderivative of φ_j . By change of variable in the mean value $A_{s,k}^*(X; \boldsymbol{\varphi})$, there is no loss of generality if we replace $\varphi_j(t)$ by $\lambda\varphi_j(t)$, for any non-zero integer λ . By making a choice for λ having the property that $\lambda\psi_j(t) \in \mathbb{Z}[t]$ ($1 \leq j \leq k$), we see that there is likewise no loss of generality in supposing, *ab initio*, that $\varphi_j(t) \in \mathbb{Z}[t]$ ($1 \leq j \leq k$) is a system of polynomials having the property that $\psi_j(t) \in \mathbb{Z}[t]$ ($1 \leq j \leq k$). The purpose of this manoeuvre is to ensure that

$$\psi_j(u + vT) \equiv \psi_j(u) + v\varphi_j(u)T \pmod{T^2},$$

thereby opening the possibility of making use of [8, Theorem 1.1 and Corollary 1.2].

Next, when $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^k$, we define the polynomials

$$\Psi_k(z; \boldsymbol{\alpha}) = \alpha_1\psi_1(z) + \dots + \alpha_k\psi_k(z),$$

and

$$\Phi_k(z; \boldsymbol{\beta}) = \beta_1 \varphi_1(z) + \dots + \beta_k \varphi_k(z). \quad (2.1)$$

We then define the exponential sum

$$b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X) = \sum_{1 \leq z \leq X} \sum_{1 \leq h \leq X} e(\Psi_k(z; \boldsymbol{\alpha}) + h \Phi_k(z; \boldsymbol{\beta})),$$

and the associated mean value

$$B_{s,k}(X) = \int_{[0,1]^k} \int_{[0,1]^k} |b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)|^{2s} d\boldsymbol{\alpha} d\boldsymbol{\beta}. \quad (2.2)$$

Given an integral k -tuple $\mathbf{u}$, denote by $\mathcal{B}(\mathbf{u})$ the set of integers z with $1 \leq z \leq X$ having the property that

$$\psi_j(z) = u_j \quad (1 \leq j \leq k).$$

Note that $\text{card}(\mathcal{B}(\mathbf{u}))$ is bounded above by the maximum of the degrees of ψ_j for $1 \leq j \leq k$. Equipped with this notation, it follows via orthogonality that

$$\int_{[0,1]^k} |b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)|^2 d\boldsymbol{\alpha} = \sum_{\mathbf{u} \in \mathbb{Z}^k} \left| \sum_{\substack{1 \leq z \leq X \\ z \in \mathcal{B}(\mathbf{u})}} \sum_{1 \leq h \leq X} e(h \Phi_k(z; \boldsymbol{\beta})) \right|^2. \quad (2.3)$$

We may suppose that X is sufficiently large in terms of the coefficients of $\boldsymbol{\varphi}$, and hence also $\boldsymbol{\psi}$, and consequently the real zeros of $\varphi_j(t)$ may be considered bounded in terms of $\boldsymbol{\varphi}$. In particular, we have $|\varphi_j(\xi)| \gg 1$ whenever ξ is sufficiently large. Hence, by the mean value theorem, whenever z_1 and z_2 are sufficiently large with $z_2 > z_1$, there exists $\xi \in (z_1, z_2)$ with

$$\left| \frac{\psi_j(z_1) - \psi_j(z_2)}{z_1 - z_2} \right| = |\psi'_j(\xi)| = |\varphi_j(\xi)| \gg 1.$$

Consequently, we find that $\text{card}(\mathcal{B}(\mathbf{u})) = 1$ whenever u_j is sufficiently large for any index j with $1 \leq j \leq k$. By making use of this observation within (2.3), we find that for some positive number $c = c(\boldsymbol{\varphi})$, one has

$$\begin{aligned} \int_{[0,1]^k} |b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)|^2 d\boldsymbol{\alpha} &= \sum_{\substack{\mathbf{u} \in \mathbb{Z}^k \\ \text{card}(\mathcal{B}(\mathbf{u}))=1}} \sum_{\substack{1 \leq z \leq X \\ z \in \mathcal{B}(\mathbf{u})}} \left| \sum_{1 \leq h \leq X} e(h \Phi_k(z; \boldsymbol{\beta})) \right|^2 \\ &\quad + O\left(\sum_{\substack{\mathbf{u} \in \mathbb{Z}^k \\ |\mathbf{u}| \leq c(\boldsymbol{\varphi})}} \left| \sum_{\substack{1 \leq z \leq X \\ z \in \mathcal{B}(\mathbf{u})}} \sum_{1 \leq h \leq X} e(h \Phi_k(z; \boldsymbol{\beta})) \right|^2 \right) \\ &= \sum_{1 \leq z \leq X} \left| \sum_{1 \leq h \leq X} e(h \Phi_k(z; \boldsymbol{\beta})) \right|^2 + O(X^2). \end{aligned}$$

Hence, we conclude from (1.10) that

$$\int_{[0,1]^k} |b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)|^2 d\boldsymbol{\alpha} = G_k(\boldsymbol{\beta}; X; \boldsymbol{\varphi}) + O(X^2). \quad (2.4)$$

By applying Hölder's inequality to the relation (2.4) and recalling (2.2), we deduce from (1.11) that

$$\begin{aligned} A_{s,k}^*(X; \boldsymbol{\varphi}) &\ll \int_{[0,1]^k} \left(\int_{[0,1]^k} |b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)|^2 d\boldsymbol{\alpha} \right)^s d\boldsymbol{\beta} + O(X^{2s}) \\ &\ll \int_{[0,1]^k} \int_{[0,1]^k} |b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)|^{2s} d\boldsymbol{\alpha} d\boldsymbol{\beta} + O(X^{2s}) \\ &= B_{s,k}(X) + O(X^{2s}). \end{aligned} \quad (2.5)$$

The mean value $B_{s,k}(X)$ is the subject of [8, Corollary 1.2] in the situation that s is an integer with $s \leq k(k+1)/2$. Indeed, we find via orthogonality that $B_{s,k}(X)$ counts the number of integral solutions of the Diophantine system

$$\left. \begin{aligned} \sum_{i=1}^s (\psi_j(u_i) - \psi_j(u_{s+i})) &= 0, \\ \sum_{i=1}^s (v_i \psi'_j(u_i) - v_{s+i} \psi'_j(u_{s+i})) &= 0 \end{aligned} \right\} \quad (1 \leq j \leq k),$$

with $1 \leq \mathbf{u}, \mathbf{v} \leq X$. Moreover, since

$$\det(\psi_j^{(i)}(t))_{1 \leq i, j \leq k} = \det(\varphi_j^{(i-1)}(t))_{1 \leq i, j \leq k} = W_0(t; \boldsymbol{\varphi}),$$

one sees that the Wronskian condition of [8, Corollary 1.2] is satisfied whenever one has $W_0(t; \boldsymbol{\varphi}) \neq 0$, as we suppose to be the case. On writing $s_0 = k(k+1)/2$, we thus conclude from [8, Corollary 1.2] that $B_{s_0,k}(X) \ll X^{2s_0+\varepsilon}$. An application of Hölder's inequality therefore shows from (2.2) that whenever $0 < s \leq s_0$, one has

$$B_{s,k}(X) \ll (B_{s_0,k}(X))^{s/s_0} \ll X^{2s+\varepsilon}.$$

Thus, we deduce from (2.5) that $A_{s,k}^*(X; \boldsymbol{\varphi}) \ll X^{2s+\varepsilon}$, and the proof of the lemma is complete. $\square$

3. SHARP MOMENT ESTIMATES

In this section we seek to establish Theorems 1.1 and 1.2. It is worth noting that the conclusions of these theorems are immediate corollaries of Lemma 2.1 when s is even and $s \leq \frac{1}{2}k(k+1)$, as we now explain. Indeed, in the latter situation, we may write $s = 2t$ for an integer t , and then it follows via orthogonality that $A_{2t,k}^*(2X; \boldsymbol{\varphi})$ counts the number of integral solutions of the Diophantine system

$$\sum_{i=1}^t (g_i - g'_i) \varphi_j(z_i) = \sum_{i=1}^t (g_{t+i} - g'_{t+i}) \varphi_j(z_{t+i}) \quad (1 \leq j \leq k),$$

with $1 \leq \mathbf{g}, \mathbf{g}', \mathbf{z} \leq 2X$. For a given integer h with $|h| \leq 2X$, the number of solutions of the equation $g - g' = h$, with $1 \leq g, g' \leq 2X$, is precisely $2X - |h|$. Meanwhile, from (1.5) and (1.6), again by orthogonality, we see that $B_{2t,k}(X; \boldsymbol{\varphi})$ counts the number of integral

solutions of the Diophantine system

$$\sum_{i=1}^t h_i \varphi_j(z_i) = \sum_{i=1}^t h_{t+i} \varphi_j(z_{t+i}) \quad (1 \leq j \leq k),$$

with $1 \leq \mathbf{h}, \mathbf{z} \leq X$. Since $2X - |h| \geq X$ when $1 \leq h \leq X$, it follows that

$$A_{2t,k}^*(2X; \boldsymbol{\varphi}) \geq X^{2t} B_{2t,k}(X; \boldsymbol{\varphi}).$$

Consequently, we find from Lemma 2.1 that when $2t \leq k(k+1)/2$, one has

$$B_{2t,k}(X; \boldsymbol{\varphi}) \leq X^{-2t} A_{2t,k}^*(2X; \boldsymbol{\varphi}) \ll X^{2t+\varepsilon}.$$

This establishes Theorem 1.2 for s even and $s \leq \frac{1}{2}k(k+1)$, with Theorem 1.1 following as a corollary under the same hypotheses on s .

The situation with s odd is not amenable to a Diophantine interpretation of the mean value $B_{s,k}(X; \boldsymbol{\varphi})$, and so we must work rather harder to obtain the same conclusion. We begin by establishing a technical lemma. The announcement of this lemma requires that we introduce some additional notation. First, we recall (2.1) and define

$$\Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; X) = \sum_{1 \leq z \leq X} \left| \sum_{1 \leq h \leq X} e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h) \right|^2 e(\gamma z).$$

The reader may care to note, by reference to (1.10), that one has

$$\Gamma_k(\boldsymbol{\alpha}; 0, 0; X) = G_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi}).$$

We also require the auxiliary exponential sum

$$K(\theta; X) = \sum_{1 \leq y \leq X} e(-\theta y). \quad (3.1)$$

Lemma 3.1. *One has*

$$XF_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi}) = \int_0^1 \int_0^1 (\Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; 2X) - \Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; X)) K(\beta; X) K(\gamma; X) d\beta d\gamma.$$

Proof. Observe that the number of solutions of the equation $h_1 - h_2 = h$ with $1 \leq h_i \leq X$ ($i = 1, 2$) is $X - |h|$, when $|h| \leq X$, and otherwise is 0. Thus, we find that

$$\Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; X) = \sum_{1 \leq z \leq X} \sum_{|h| \leq X} (X - |h|) e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h + \gamma z).$$

It follows that

$$\Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; 2X) - \Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; X) = \sum_{i=1}^3 T_i(\boldsymbol{\alpha}; \beta, \gamma; X), \quad (3.2)$$

where

$$\begin{aligned}
T_1(\boldsymbol{\alpha}; \beta, \gamma; X) &= X \sum_{1 \leq z \leq X} \sum_{|h| \leq X} e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h + \gamma z), \\
T_2(\boldsymbol{\alpha}; \beta, \gamma; X) &= \sum_{X < z \leq 2X} \sum_{|h| \leq 2X} (2X - |h|) e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h + \gamma z), \\
T_3(\boldsymbol{\alpha}; \beta, \gamma; X) &= \sum_{1 \leq z \leq X} \sum_{X < |h| \leq 2X} (2X - |h|) e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h + \gamma z).
\end{aligned}$$

On recalling (1.5), we find that as a consequence of orthogonality, one has the relation

$$\begin{aligned}
&\int_0^1 \int_0^1 T_1(\boldsymbol{\alpha}; \beta, \gamma; X) K(\beta; X) K(\gamma; X) d\beta d\gamma \\
&= X \int_0^1 \int_0^1 \sum_{1 \leq z \leq X} \sum_{|h| \leq X} e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h + \gamma z) \sum_{1 \leq y \leq X} e(-\beta y) \sum_{1 \leq w \leq X} e(-\gamma w) d\beta d\gamma \\
&= X F_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi}).
\end{aligned} \tag{3.3}$$

Meanwhile, since the summands in $T_2(\boldsymbol{\alpha}; \beta, \gamma; X)$ satisfy $z > X$, we see in a similar manner that

$$\begin{aligned}
&\int_0^1 T_2(\boldsymbol{\alpha}; \beta, \gamma; X) K(\gamma; X) d\gamma \\
&= \sum_{X < z \leq 2X} \sum_{|h| \leq 2X} (2X - |h|) e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h) \int_0^1 e(\gamma z) \sum_{1 \leq w \leq X} e(-\gamma w) d\gamma = 0.
\end{aligned}$$

Consequently, we have

$$\int_0^1 \int_0^1 T_2(\boldsymbol{\alpha}; \beta, \gamma; X) K(\beta; X) K(\gamma; X) d\beta d\gamma = 0. \tag{3.4}$$

A similar argument shows that

$$\int_0^1 \int_0^1 T_3(\boldsymbol{\alpha}; \beta, \gamma; X) K(\beta; X) K(\gamma; X) d\beta d\gamma = 0. \tag{3.5}$$

The desired conclusion now follows on combining (3.2)-(3.5). $\square$

The conclusion of Lemma 3.1 permits the estimation of mean values of the exponential sum $F_k(\boldsymbol{\alpha}; X; \boldsymbol{\varphi})$ in terms of $A_{s,k}^*(X; \boldsymbol{\varphi})$.

Lemma 3.2. *One has*

$$B_{s,k}(X; \boldsymbol{\varphi}) \ll X^{-s} (\log(2X))^{2s} (A_{s,k}^*(X; \boldsymbol{\varphi}) + A_{s,k}^*(2X; \boldsymbol{\varphi})).$$

Proof. We find from Lemma 3.1 and (1.6) that $B_{s,k}(X; \boldsymbol{\varphi})$ is equal to

$$X^{-s} \int_{[0,1]^k} \left| \int_0^1 \int_0^1 (\Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; 2X) - \Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; X)) K(\beta; X) K(\gamma; X) d\beta d\gamma \right|^s d\boldsymbol{\alpha}.$$

By Hölder's inequality, one finds that

$$\left| \int_0^1 \int_0^1 (\Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; 2X) - \Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; X)) K(\beta; X) K(\gamma; X) d\beta d\gamma \right|^s \\ \ll \left(\int_0^1 \int_0^1 |K(\beta; X) K(\gamma; X)| d\beta d\gamma \right)^{s-1} (I(\boldsymbol{\alpha}; 2X) + I(\boldsymbol{\alpha}; X)),$$

where

$$I(\boldsymbol{\alpha}; Y) = \int_0^1 \int_0^1 |\Gamma_k(\boldsymbol{\alpha}; \beta, \gamma; Y)|^s |K(\beta; X) K(\gamma; X)| d\beta d\gamma. \quad (3.6)$$

Since

$$\int_0^1 |K(\theta; X)| d\theta \ll \int_0^1 \min\{X, \|\theta\|^{-1}\} d\theta \ll \log(2X), \quad (3.7)$$

we discern thus far that

$$B_{s,k}(X; \boldsymbol{\varphi}) \ll X^{-s} (\log(2X))^{2s-2} \int_{[0,1]^k} (I(\boldsymbol{\alpha}; 2X) + I(\boldsymbol{\alpha}; X)) d\boldsymbol{\alpha}. \quad (3.8)$$

The definition of Γ_k combines with (3.6) and (3.7) to show that

$$I(\boldsymbol{\alpha}; Y) \leq \int_0^1 \int_0^1 \left(\sum_{1 \leq z \leq Y} \left| \sum_{1 \leq h \leq Y} e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h) \right|^2 \right)^s |K(\beta; X) K(\gamma; X)| d\beta d\gamma \\ \ll (\log 2X) \int_0^1 \left(\sum_{1 \leq z \leq Y} \left| \sum_{1 \leq h \leq Y} e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h) \right|^2 \right)^s |K(\beta; X)| d\beta.$$

Hence, we infer that

$$\int_{[0,1]^k} I(\boldsymbol{\alpha}; Y) d\boldsymbol{\alpha} \ll (\log(2X)) \int_0^1 T(\beta; Y) |K(\beta; X)| d\beta, \quad (3.9)$$

where we write

$$T(\beta; Y) = \int_{[0,1]^k} \left(\sum_{1 \leq z \leq Y} \left| \sum_{1 \leq h \leq Y} e(h\Phi_k(z; \boldsymbol{\alpha}) + \beta h) \right|^2 \right)^s d\boldsymbol{\alpha}.$$

By orthogonality, the integral $T(\beta; Y)$ counts the integral solutions of the system of equations

$$\sum_{i=1}^s (h_i - h'_i) \varphi_j(z_i) = 0 \quad (1 \leq j \leq k),$$

with $1 \leq \mathbf{z}, \mathbf{h}, \mathbf{h}' \leq Y$, and with each solution $\mathbf{z}, \mathbf{h}, \mathbf{h}'$ counted with weight

$$e\left(\beta \sum_{i=1}^s (h_i - h'_i)\right).$$

Thus, on recalling (1.11), we discern that

$$|T(\beta; Y)| \leq T(0; Y) = A_{s,k}^*(Y; \boldsymbol{\varphi}).$$

On substituting this upper bound into (3.9), we conclude via (3.7) and (3.8) that

$$\begin{aligned} B_{s,k}(X; \boldsymbol{\varphi}) &\ll X^{-s}(\log(2X))^{2s-1} (A_{s,k}^*(X; \boldsymbol{\varphi}) + A_{s,k}^*(2X; \boldsymbol{\varphi})) \int_0^1 \min\{2X, \|\theta\|^{-1}\} d\theta \\ &\ll X^{-s}(\log(2X))^{2s} (A_{s,k}^*(X; \boldsymbol{\varphi}) + A_{s,k}^*(2X; \boldsymbol{\varphi})). \end{aligned}$$

This completes the proof of the lemma. $\square$

We are now equipped to complete the proof of Theorem 1.2

The proof of Theorem 1.2. On substituting the upper bound provided by Lemma 2.1 into the conclusion of Lemma 3.2, we find that

$$B_{s,k}(X; \boldsymbol{\varphi}) \ll X^{-s}(\log(2X))^{2s} \cdot (2X)^{2s+\varepsilon} \ll X^{s+2\varepsilon}.$$

This completes the proof of Theorem 1.2. $\square$

Finally, the conclusion of Theorem 1.1 follows as an immediate corollary of Theorem 1.2 on taking $\varphi_j(z) = z^{j-1}$ ($1 \leq j \leq k$), since then $W_0(t; \boldsymbol{\varphi}) \neq 0$.

4. SHARP MOMENT ESTIMATES WITH A MISSING SLICE

We move in this section towards the goal of establishing Theorem 1.3. This conclusion is a mean value estimate related to the upper bound provided in Theorem 1.1 for the mean value of exponential sums $f_k(\boldsymbol{\alpha}; X)$, save that the term $h\alpha_1$ in the argument of the exponential sum is removed in (1.1). In a related setting (see [9, §3]), such a situation was handled by a shift towards a translation-dilation invariant system of equations. We adapt that strategy now to the present setting.

Before announcing our key lemma, we begin by recalling the exponential sums $g_k(\boldsymbol{\alpha}; X)$ and $f_k(\boldsymbol{\alpha}; X)$ defined in (1.7) and (1.1), respectively. We again introduce an auxiliary sum as in the preamble to Lemma 3.1. We define

$$\Lambda_k(\boldsymbol{\alpha}; \beta, \gamma; X) = \sum_{1 \leq z \leq X} \left| \sum_{1 \leq h \leq X} e(hz\alpha_1 + \dots + hz^k\alpha_k + \beta h) \right|^2 e(\gamma z),$$

and we recall the definition of $K(\theta; X)$ from (3.1).

Lemma 4.1. *One has*

$$X g_k(\boldsymbol{\alpha}; X) = \int_0^1 \int_0^1 (\Lambda_k(\boldsymbol{\alpha}; \beta, \gamma; 2X) - \Lambda_k(\boldsymbol{\alpha}; \beta, \gamma; X)) K(\beta; X) K(\gamma; X) d\beta d\gamma.$$

Proof. This is the special case of Lemma 3.1 in which $\varphi_j(z) = z^j$ ($1 \leq j \leq k$). $\square$

We are now equipped to bound the mean value $C_{s,k}(X)$, defined in (1.8), in terms of mean values involving auxiliary exponential sums. We introduce the exponential sum

$$h_k(\boldsymbol{\alpha}; X) = \sum_{1 \leq z \leq X} \left| \sum_{1 \leq h \leq X} e(hz\alpha_1 + \dots + hz^k\alpha_k) \right|^2,$$

and then define a mean value related to $C_{s,k}(X)$, namely

$$C_{s,k}^*(X) = \int_{[0,1]^k} |h_k(\boldsymbol{\alpha}; X)|^s d\boldsymbol{\alpha}.$$

Lemma 4.2. *When s is a natural number, one has*

$$C_{s,k}(X) \ll X^{-s}(\log(2X))^{2s}(C_{s,k}^*(X) + C_{s,k}^*(2X)).$$

Proof. This is the special case of Lemma 3.2 corresponding to the choice of polynomials $\varphi_j(z) = z^j$ ($1 \leq j \leq k$). $\square$

We can now begin the main discussion.

The proof of Theorem 1.3. We follow an argument motivated by that of Lemma 3.1, fixing $s = \frac{1}{2}(k^2 + 3k)$, and focusing on the estimation of $C_{s,k}^*(Y)$. Here, we may take $Y = X$ or $Y = 2X$. Define δ_j to be 1 when $j = 0$, and 0 otherwise. Then, by interpreting the mean value $C_{s,k}^*(Y)$ via orthogonality, one finds that it counts the integral solutions of the system of equations

$$\sum_{i=1}^s (h_i - g_i) z_i^j = \delta_j h \quad (0 \leq j \leq k), \quad (4.1)$$

with $1 \leq \mathbf{h}, \mathbf{g}, \mathbf{z} \leq Y$ and $|h| \leq sY$. Here, we adopt the usual convention that $z_i^0 = 1$. Denote by $I_{s,k}^*(Y)$ the number of solutions of the system of equations

$$\sum_{i=1}^s (h_i - g_i) z_i^j = \delta_j (l_1 - l_2) \quad (0 \leq j \leq k), \quad (4.2)$$

with $1 \leq \mathbf{h}, \mathbf{g}, \mathbf{z} \leq Y$ and $|\mathbf{l}| \leq 2sY$. Then, a comparison of (4.1) and (4.2) reveals that

$$C_{s,k}^*(Y) \leq Y^{-1} I_{s,k}^*(Y). \quad (4.3)$$

We next consider a shift of each variable z_i in (4.2) by any integer w with $1 \leq w \leq Y$. By the binomial theorem, for each shift w , one finds that $\mathbf{g}, \mathbf{h}, \mathbf{z}$ is a solution of (4.2) if and only if it is also a solution of the system

$$\sum_{i=1}^s (h_i - g_i) (z_i + w)^j = (l_1 - l_2) w^j \quad (0 \leq j \leq k).$$

We deduce that for each fixed integer w with $1 \leq w \leq Y$, the mean value $I_{s,k}^*(Y)$ is bounded above by the number of solutions of the system

$$\sum_{i=1}^s (h_i - g_i) w_i^j = (l_1 - l_2) w^j \quad (0 \leq j \leq k), \quad (4.4)$$

with $1 \leq \mathbf{g}, \mathbf{h}, \mathbf{u} \leq 2Y$ and $|\mathbf{l}| \leq 2sY$.

Define the exponential sums

$$r_k(\boldsymbol{\alpha}; Y) = \sum_{1 \leq z \leq Y} \left| \sum_{1 \leq h \leq Y} e(h\alpha_0 + hz\alpha_1 + \dots + hz^k\alpha_k) \right|^2, \quad (4.5)$$

and

$$r_{k,s}^*(\boldsymbol{\alpha}; w; Y) = \sum_{1 \leq h \leq 2sY} e(h\alpha_0 + hw\alpha_1 + \dots + hw^k\alpha_k).$$

Then, by averaging over the above choices for w and applying orthogonality, we deduce by means of (4.4) that

$$\begin{aligned} I_{s,k}^*(Y) &\leq Y^{-1} \sum_{1 \leq w \leq Y} \int_{[0,1]^{k+1}} |r_k(\alpha; 2Y)|^s |r_{k,s}^*(\alpha; w; Y)|^2 d\alpha \\ &\leq Y^{-1} \sum_{1 \leq w \leq 2sY} \int_{[0,1]^{k+1}} |r_k(\alpha; 2Y)|^s |r_{k,s}^*(\alpha; w; Y)|^2 d\alpha \\ &= Y^{-1} \int_{[0,1]^{k+1}} |r_k(\alpha; 2Y)|^s |r_k(\alpha; 2sY)| d\alpha. \end{aligned}$$

Thus, on applying (4.3) and Hölder's inequality, we deduce that

$$C_{s,k}^*(Y) \ll Y^{-2} J_2^{s/(s+1)} J_{2s}^{1/(s+1)}, \quad (4.6)$$

where

$$J_t = \int_{[0,1]^{k+1}} |r_k(\alpha; tY)|^{s+1} d\alpha.$$

Recall the definition (1.11) and put $\varphi_j(z) = z^{j-1}$ ($1 \leq j \leq k+1$). Then, a comparison of (4.5) with (1.10) shows that

$$J_t = A_{s+1,k+1}^*(tY).$$

Note next that since $s = \frac{1}{2}k(k+3)$, one has

$$s+1 = \frac{1}{2}(k^2 + 3k + 2) = \frac{1}{2}(k+1)(k+2).$$

Hence, by Lemma 2.1, one finds from (4.6) that

$$C_{s,k}^*(Y) \ll Y^{-2} \cdot Y^{2s+2+\varepsilon} = Y^{2s+\varepsilon}.$$

On substituting this estimate into Lemma 4.2, we conclude that

$$C_{s,k}(X) \ll X^{-s} \cdot (\log(2X))^{2s} \cdot X^{2s+\varepsilon} \ll X^{s+2\varepsilon}.$$

This estimate has been shown to hold for $s = \frac{1}{2}k(k+3)$. The proof of Theorem 1.3 follows by applying Hölder's inequality to discern that for $0 \leq u \leq \frac{1}{2}k(k+3)$, one has

$$C_{u,k}(X) \leq (C_{s,k}(X))^{u/s} \ll X^{u+\varepsilon},$$

and applying a trivial estimate for $|g_k(\alpha; X)|$ to confirm that for $u > \frac{1}{2}k(k+3)$, one has

$$C_{u,k}(X) \ll (X^2)^{u-s} C_{s,k}(X) \ll X^{2u-\frac{1}{2}k(k+3)+\varepsilon}.$$

This completes the proof of the theorem. $\square$

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