

THE MAIN CONJECTURE FOR A VINOGRADOV SUBSYSTEM

TREVOR D. WOOLEY

ABSTRACT. When $k \geq 2$ and $s > 0$, we establish that for each $\varepsilon > 0$, one has

$$\int_{[0,1)^{k-1}} \left| \sum_{1 \leq x \leq X} e(\alpha_2 x^2 + \dots + \alpha_k x^k) \right|^{2s} d\alpha \ll X^{s+\varepsilon} + X^{2s-(k^2+k-2)/2},$$

confirming the main conjecture for a new family of exponential sums beyond those of Vinogradov (translation-dilation invariant) type. Our methods are based on very recent work of the author employing the nested efficient congruencing method. Consequently, analogues of our results hold also in the setting of number fields and function fields.

1. INTRODUCTION

Our goal in this memoir is an optimal mean value estimate for exponential sums constituting the main conjecture for a Vinogradov subsystem. Let k be a natural number with $k \geq 2$. When $\beta = (\beta_2, \dots, \beta_k) \in \mathbb{R}^{k-1}$ and X is a large positive integer, define

$$f_k(\beta; X) = \sum_{1 \leq x \leq X} e(\beta_2 x^2 + \dots + \beta_k x^k), \quad (1.1)$$

where, as usual, we write $e(z) = e^{2\pi iz}$. For the time being, we abbreviate $(\beta_2, \dots, \beta_k)$ to β and adopt the convention that the differential $d\beta$ denotes $d\beta_2 \cdots d\beta_k$. The main conclusion of this memoir is the mean value estimate described in the following theorem.

Theorem 1.1. *Suppose that $k \geq 2$ and $s > 0$. Then for each $\varepsilon > 0$, one has*

$$\int_{[0,1)^{k-1}} |f_k(\beta; X)|^{2s} d\beta \ll X^{s+\varepsilon} + X^{2s-(k^2+k-2)/2}. \quad (1.2)$$

We pause to assess the strength of Theorem 1.1. When $s \in \mathbb{N}$, we find via orthogonality that the left hand side of (1.2) counts the number of integral solutions of the system

$$x_1^j + \dots + x_s^j = y_1^j + \dots + y_s^j \quad (2 \leq j \leq k), \quad (1.3)$$

with $1 \leq x_i, y_i \leq X$ ($1 \leq i \leq s$). The solutions of (1.3) with $\{x_1, \dots, x_s\} = \{y_1, \dots, y_s\}$ make a contribution of size $s!X^s + O(X^{s-1})$ to the mean value on the left hand side of (1.2). Meanwhile, the set of $(\beta_2, \dots, \beta_k) \in [0,1)^{k-1}$, with $0 \leq \beta_j \leq (6k)^{-1}X^{-j}$, makes a contribution at least as large as $(6k)^{1-k}X^{-(k^2+k-2)/2}(X/2)^{2s}$. Thus, we find that

$$\int_{[0,1)^{k-1}} |f_k(\beta; X)|^{2s} d\beta \gg X^s + X^{2s-(k^2+k-2)/2},$$

and it is apparent that Theorem 1.1 is essentially optimal.

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For comparison, work [12, Corollary 1.2] utilising efficient congruencing to address relatives of Vinogradov's mean value theorem implies that the upper bound (1.2) holds for $s \leq k(k-1)/2$. In subsequent investigations of the author joint with Brandes (see [3, Theorem 1.1]), this range was extended to the interval $s \leq (k^2-1)/2$. The conclusion of Theorem 1.1 further extends this range to the entire critical interval $s \leq (k^2+k-2)/2$, from which the main conjecture (1.2) follows in full via the trivial estimate $|f_k(\boldsymbol{\beta}; X)| = O(X)$.

The conclusion of Theorem 1.1 provides a family of mean values, corresponding to systems of equations (1.3) of additive type, for which we are able to establish sharp upper bounds matching the main conjectures for the system. Classically, such sharp mean value estimates were available only for the special case of Theorem 1.1 in which $k = 2$, a situation that concerns mean values of quadratic exponential sums. An addition to this elementary classical example was made only in the previous decade, when the bound

$$\int_{[0,1]^k} \left| \sum_{1 \leq x \leq X} e(\alpha_1 x + \dots + \alpha_k x^k) \right|^{2s} d\boldsymbol{\alpha} \ll X^{s+\varepsilon} + X^{2s-k(k+1)/2} \quad (1.4)$$

was established. This, the main conjecture in Vinogradov's mean value theorem, was established in work of Bourgain, Demeter and Guth [1], and the author [10, 12]. We should note that the special case of this estimate with $k = 2$ was also classical.

The reader may wonder whether the methods of this memoir extend to handle mean values of exponential sums of the shape (1.1) in which the missing linear term is replaced by a missing term of higher degree. Thus, one might consider the exponential sum

$$f_{k,d}(\boldsymbol{\beta}; X) = \sum_{1 \leq x \leq X} e\left(\sum_{\substack{1 \leq l \leq k \\ l \neq d}} \beta_l x^l\right),$$

and seek the conjectured mean value estimate

$$\int_{[0,1]^{k-1}} |f_{k,d}(\boldsymbol{\beta}; X)|^{2s} d\boldsymbol{\beta} \ll X^{s+\varepsilon} + X^{2s-(k^2+k-2d)/2}.$$

This scenario is explored in [3, Theorem 1.2], where one finds that the strategy of this memoir does not achieve optimal estimates for $d > 1$. The key difficulty is evident in our proof of Theorem 1.1. By orthogonality, the missing monomial x^d in the exponential sum $f_{k,d}(\boldsymbol{\beta}; X)$ is associated to an omitted diagonal equation in a system analogous to (1.3). This equation may be recovered by introducing an auxiliary variable h having size of order X^d . The reader may inspect the definition (2.1) of the exponential sum $g_k(\boldsymbol{\beta}; X)$ below to surmise how it is that this auxiliary variable makes its presence known in our proof. In order to obtain sharp mean value estimates for the exponential sum $f_{k,d}(\boldsymbol{\beta}; X)$, one would need to obtain $(d+1)$ -th power cancellation in mean values involving the analogue of $g_k(\boldsymbol{\alpha}; X)$, a prospect inaccessible in pure mean value estimates of this sum when $d > 1$.

We organise this paper as follows. First, in §2, we derive a mean value estimate for an auxiliary bivariate exponential sum by applying our recent work [13]. Here, in the rational integer case of our main theorem, earlier work of Guo and Zorin-Kranich [4] would suffice as input. However, we seek results applicable also in number fields and function fields, and for this reason the appeal to our work [13] is critical. A detail worth noting is that care must be taken to provide mean value estimates valid for odd as well as even order moments, so that some additional attention is required en route. In §3 we

apply this auxiliary mean value to prove Theorem 1.1 for $s \leq (k^2 + k - 2)/2$, deferring to §4 an account of the sharp ε -free estimate available for $s > (k^2 + k - 2)/2$. In §§5 and 6, respectively, we explain how estimates analogous to that of Theorem 1.1 may be derived in the setting of number fields and function fields. Finally, in §7, we make additional comments concerning the cubic case of our main theorem, a topic with an extensive history that provides a canvas to explore discrete restriction variants of our results.

Some notational conventions assist in our exposition. We make extensive use of vector notation. Thus, for example, we write $1 \leq \mathbf{x} \leq X$ to denote that every coordinate x_i of $\mathbf{x}$ satisfies $1 \leq x_i \leq X$. Also, given a situation in which the parameter ε has not already been fixed, in any statement involving ε , we implicitly assert that the statement holds for each $\varepsilon > 0$. Implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on ε .

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2. AN AUXILIARY MEAN VALUE THEOREM

The proof of Theorem 1.1 makes essential use of a mean value estimate for a certain auxiliary exponential sum not of translation-dilation invariant type. In this section we relate this mean value to a more complicated enveloping mean value that is fully translation-dilation invariant. We begin with some notation. When $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^k$, define the polynomials

$$\varphi_k(z; \boldsymbol{\alpha}) = \sum_{i=1}^k \alpha_i z^i \quad \text{and} \quad \psi_k(z, h; \boldsymbol{\beta}) = \sum_{i=1}^k \beta_i h z^{i-1}.$$

We consider the exponential sums

$$g_k(\boldsymbol{\beta}; X) = \sum_{1 \leq z \leq X} \left| \sum_{1 \leq h \leq X} e(\psi_k(z, h; \boldsymbol{\beta})) \right|^2, \quad (2.1)$$

$$b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X) = \sum_{1 \leq z \leq X} \sum_{1 \leq h \leq X} e(\varphi_k(z; \boldsymbol{\alpha}) + \psi_k(z, h; \boldsymbol{\beta})), \quad (2.2)$$

and the respective associated mean values

$$A_{s,k}(X) = \int_{[0,1]^k} |g_k(\boldsymbol{\beta}; X)|^s d\boldsymbol{\beta}, \quad (2.3)$$

$$B_{s,k}(X) = \int_{[0,1]^k} \int_{[0,1]^k} |b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)|^{2s} d\boldsymbol{\alpha} d\boldsymbol{\beta}. \quad (2.4)$$

The goal of this section is the mean value estimate contained in the following lemma.

Lemma 2.1. *When $k \geq 2$ and $s > 0$, one has*

$$A_{s,k}(X) \ll X^{s+\varepsilon} (X^s + X^{2s-k(k+1)/2}).$$

Proof. Write $t = k(k+1)/2$. We begin by considering the special case of the mean value (2.4) in which $s = t$. We begin by rewriting $g_k(\boldsymbol{\beta}; X)$ in terms of $b_k(\boldsymbol{\alpha}, \boldsymbol{\beta}; X)$. Thus, on

recalling (2.1) and (2.2), it follows via orthogonality that

$$g_k(\beta; X) = \sum_{1 \leq z \leq X} \left| \sum_{1 \leq h \leq X} e(\psi_k(z, h; \beta)) \right|^2 = \int_{[0,1]^k} |b_k(\alpha, \beta; X)|^2 d\alpha.$$

By applying Hölder's inequality, we therefore deduce from (2.3) and (2.4) that

$$\begin{aligned} A_{t,k}(X) &= \int_{[0,1]^k} \left(\int_{[0,1]^k} |b_k(\alpha, \beta; X)|^2 d\alpha \right)^t d\beta \\ &\leq \int_{[0,1]^k} \int_{[0,1]^k} |b_k(\alpha, \beta; X)|^{2t} d\alpha d\beta = B_{t,k}(X). \end{aligned} \quad (2.5)$$

By orthogonality, the conclusion of [13, Corollary 1.4] shows that

$$B_{t,k}(X) \ll X^\varepsilon (X^{2t} + X^{4t-k(k+1)}) \ll X^{2t+\varepsilon}. \quad (2.6)$$

The same conclusion may also be extracted from [4, Theorem 1] with careful attention to notation. Since we have in mind generalisations of our main conclusions to the setting of number fields and function fields, we emphasise the appeal to [13] over [4]. In this way, we deduce from (2.5) that $A_{t,k}(X) \ll X^{2t+\varepsilon}$. This estimate suffices to obtain the upper bound claimed by the lemma in full. Indeed, by applying Hölder's inequality to (2.3), one sees that for $0 < s \leq t$, one has

$$A_{s,k}(X) \leq (A_{t,k}(X))^{s/t} \ll (X^{2t+\varepsilon})^{s/t} \ll X^{2s+\varepsilon}.$$

Meanwhile, by employing the trivial estimate $|g_k(\beta; X)| \leq X^3$, one sees for $s > t$ that

$$A_{s,k}(X) \leq (X^3)^{s-t} A_{t,k}(X) \ll (X^3)^{s-t} X^{2t+\varepsilon} \ll X^{3s-t+\varepsilon}.$$

Thus, whenever $s > 0$, one has the upper bound $A_{s,k}(X) \ll X^\varepsilon (X^{2s} + X^{3s-t})$, and the desired conclusion follows on recalling that $t = k(k+1)/2$. $\square$

3. SHIFTING TO VINOGRADOV'S MEAN VALUE THEOREM

Our strategy for proving Theorem 1.1 is based on the proof of the special case in which $k = 3$ and $s \leq 4$ established in [9, Theorem 1.3], and extended in [3, Theorem 1.1] to handle those situations in which $k \geq 3$ and $s \leq (k^2 - 1)/2$. We adapt this approach so as to avoid odd moments of auxiliary exponential sums lacking a Diophantine interpretation. We begin by introducing additional notation. First, we introduce the exponential sum

$$F_k(\alpha; X) = \sum_{1 \leq x \leq X} e(\alpha_1 x + \dots + \alpha_k x^k), \quad (3.1)$$

that extends the exponential sum $f_k(\beta; X)$ defined in (1.1). Note that $F_k(\alpha; X)$ includes a linear term $\alpha_1 x$ in the argument. We consider both Vinogradov's mean value

$$J_{s,k}(X) = \int_{[0,1]^k} |F_k(\alpha; X)|^{2s} d\alpha, \quad (3.2)$$

and the closely associated mean value

$$I_{s,k}(X) = \int_{[0,1]^{k-1}} |f_k(\beta; X)|^{2s} d\beta. \quad (3.3)$$

We also have need of the exponential sum $g_k(\beta; X)$ defined in (2.1), as well as

$$g_{k,s}^*(\alpha; z; X) = \sum_{1 \leq h \leq 2sX} e(h\alpha_1 + 2hz\alpha_2 + \dots + khz^{k-1}\alpha_k). \quad (3.4)$$

Lemma 3.1. *When s is a natural number, one has*

$$I_{s,k}(X) \ll X^{-2} \int_{[0,1]^k} |F_k(\alpha; 2X)|^{2s} g_k(\alpha; 2sX) d\alpha.$$

Proof. Define δ_j to be 1 when $j = 1$, and 0 otherwise. Then by interpreting the mean value $I_{s,k}(X)$ in terms of the system (1.3) via orthogonality, we see that $I_{s,k}(X)$ counts the number of integral solutions of the system of equations

$$\sum_{i=1}^s (x_i^j - y_i^j) = \delta_j h \quad (1 \leq j \leq k), \quad (3.5)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq X$ and $|h| \leq sX$. Here, the range for h has been chosen sufficiently large that the equation with $j = 1$ in (3.5) is automatically satisfied. Denote by $I_{s,k}^*(X)$ the number of solutions of the system of equations

$$\sum_{i=1}^s (x_i^j - y_i^j) = \delta_j (h_1 - h_2) \quad (1 \leq j \leq k), \quad (3.6)$$

with $1 \leq \mathbf{x}, \mathbf{y} \leq X$ and $1 \leq \mathbf{h} \leq 2sX$. Observe that, when $|h| \leq sX$, there are at least X choices for h_1 and h_2 with $1 \leq \mathbf{h} \leq 2sX$ satisfying the equation $h = h_1 - h_2$. Thus,

$$I_{s,k}(X) \leq X^{-1} I_{s,k}^*(X). \quad (3.7)$$

We next consider a shift of each variable x_i, y_i in (3.6) by an integer z with $1 \leq z \leq X$. By the binomial theorem, for each shift z , one finds that $(\mathbf{x}, \mathbf{y})$ is a solution of (3.6) if and only if it is also a solution of the system

$$\sum_{i=1}^s ((x_i + z)^j - (y_i + z)^j) = j(h_1 - h_2)z^{j-1} \quad (1 \leq j \leq k).$$

It follows that for each fixed integer z with $1 \leq z \leq X$, the mean value $I_{s,k}^*(X)$ is bounded above by the number of integral solutions of the system

$$\sum_{i=1}^s (u_i^j - v_i^j) = j(h_1 - h_2)z^{j-1} \quad (1 \leq j \leq k),$$

with $1 \leq \mathbf{u}, \mathbf{v} \leq 2X$ and $1 \leq \mathbf{h} \leq 2sX$. By averaging over these choices for z and applying orthogonality, therefore, we deduce from (3.1), (3.4) and (2.1) that

$$\begin{aligned} I_{s,k}^*(X) &\leq X^{-1} \sum_{1 \leq z \leq X} \int_{[0,1]^k} |F_k(\alpha; 2X)|^{2s} |g_{k,s}^*(\alpha; z; X)|^2 d\alpha \\ &\leq X^{-1} \int_{[0,1]^k} |F_k(\alpha; 2X)|^{2s} g_k(\alpha; 2sX) d\alpha. \end{aligned}$$

The desired conclusion is now immediate from (3.7). $\square$

In advance of our proof of Theorem 1.1, we pause to record the mean value

$$J_{s,k}(X) \ll X^{s+\varepsilon} + X^{2s-k(k+1)/2} \quad (3.8)$$

that, in view of the definitions (3.1) and (3.2), has already been reported in (1.4). We begin by establishing a mean value very slightly weaker than that given in Theorem 1.1.

Lemma 3.2. *Suppose that $k \geq 2$ and $s > 0$. Then for each $\varepsilon > 0$, one has*

$$I_{s,k}(X) \ll X^\varepsilon (X^s + X^{2s-(k^2+k-2)/2}).$$

Proof. We again write $t = k(k+1)/2$. Observe that from Lemma 3.1, one has

$$I_{t-1,k}(X) \ll X^{-2} \int_{[0,1]^k} |F_k(\boldsymbol{\alpha}; 2X)|^{2t-2} g_k(\boldsymbol{\alpha}; (2t-2)X) d\boldsymbol{\alpha}.$$

Hence, on recalling (2.3) and (3.2), an application of Hölder's inequality combines with (3.8) and Lemma 2.1 to yield the bound

$$\begin{aligned} I_{t-1,k}(X) &\ll X^{-2} (J_{t,k}(2X))^{1-1/t} (A_{t,k}((2t-2)X))^{1/t} \\ &\ll X^{-2} (X^{t+\varepsilon})^{1-1/t} (X^{2t+\varepsilon})^{1/t}. \end{aligned}$$

We therefore conclude that $I_{t-1,k} \ll X^{t-1+\varepsilon}$, a bound that suffices to deliver the conclusion of the lemma in full, as we now explain. Indeed, when $0 < s \leq t-1$, one applies Hölder's inequality within (3.3) to obtain

$$I_{s,k}(X) \leq (I_{t-1,k}(X))^{s/(t-1)} \ll X^{s+\varepsilon}.$$

Meanwhile, when $s > t-1$, the trivial estimate $|f_k(\boldsymbol{\beta}; X)| \leq X$ yields

$$I_{s,k}(X) \leq X^{2s-(2t-2)} I_{t-1,k}(X) \ll X^{2s-(t-1)+\varepsilon}.$$

Thus, for any $s > 0$, one has $I_{s,k}(X) \ll X^\varepsilon (X^s + X^{2s-(t-1)})$, and the desired conclusion follows on recalling that $t = k(k+1)/2$. $\square$

4. SHARP ESTIMATES BEYOND THE CRITICAL RANGE

The conclusion of Lemma 3.2 suffices to establish Theorem 1.1 when $s \leq (k^2 + k - 2)/2$. However, in the complementary range $s > (k^2 + k - 2)/2$, the desired upper bound is missed by a factor of X^ε . It is the purpose of this section to repair this slight deficiency.

Lemma 4.1. *Suppose that $k \geq 2$ and $s > (k^2 + k - 2)/2$. Then one has*

$$I_{s,k}(X) \ll X^{2s-(k^2+k-2)/2}.$$

Proof. We apply the Hardy-Littlewood method. In order to streamline our discussion, write $u = (k^2 + k - 2)/2$, and put $\delta = (s - u)/k$. Then the hypotheses of the statement of the lemma ensure that $\delta > 0$. Write $L = X^{1/(4k)}$. Then, when

$$1 \leq q \leq L, \quad 0 \leq \mathbf{a} \leq q \quad \text{and} \quad (q, a_2, \dots, a_k) = 1,$$

we define the major arc $\mathfrak{M}(q, \mathbf{a})$ by

$$\mathfrak{M}(q, \mathbf{a}) = \{\boldsymbol{\alpha} \in [0, 1)^{k-1} : |\alpha_j - a_j/q| \leq LX^{-j} \ (2 \leq j \leq k)\}.$$

The arcs $\mathfrak{M}(q, \mathbf{a})$ are disjoint, as one may easily check. Let $\mathfrak{M}$ denote the union of these arcs $\mathfrak{M}(q, \mathbf{a})$, and put $\mathfrak{m} = [0, 1)^{k-1} \setminus \mathfrak{M}$.

We recall the definition (1.1) of $f_k(\boldsymbol{\alpha}; X)$, and when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a}) \subseteq \mathfrak{M}$, we put

$$V(\boldsymbol{\alpha}; q, \mathbf{a}) = q^{-1} S(q, \mathbf{a}) I(\boldsymbol{\alpha} - \mathbf{a}/q; X),$$

where

$$S(q, \mathbf{a}) = \sum_{r=1}^q e((a_2 r^2 + \dots + a_k r^k)/q)$$

and

$$I(\boldsymbol{\beta}; X) = \int_0^X e(\beta_2 \gamma^2 + \dots + \beta_k \gamma^k) d\gamma.$$

We then define $V(\boldsymbol{\alpha})$ to be $V(\boldsymbol{\alpha}; q; \mathbf{a})$ when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a}) \subseteq \mathfrak{M}$, and to be 0 otherwise.

The minor arc contribution is estimated by an argument akin to that associated with [11, equation (7.1)]. Thus, for each $\varepsilon > 0$, one has

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{m}} |f_k(\boldsymbol{\alpha}; X)| \ll X^{1-\tau+\varepsilon},$$

where $\tau^{-1} = 8k^2$. Since $s - u = k\delta > 0$, one finds via Lemma 3.2 that

$$\begin{aligned} \int_{\mathfrak{m}} |f_k(\boldsymbol{\alpha}; X)|^{2s} d\boldsymbol{\alpha} &\ll \left(\sup_{\boldsymbol{\alpha} \in \mathfrak{m}} |f_k(\boldsymbol{\alpha}; X)| \right)^{2s-2u} \int_{[0,1]^{k-1}} |f_k(\boldsymbol{\alpha}; X)|^{2u} d\boldsymbol{\alpha} \\ &\ll (X^{1-\tau})^{2k\delta} X^{u+\varepsilon}, \end{aligned}$$

whence

$$\int_{\mathfrak{m}} |f_k(\boldsymbol{\alpha}; X)|^{2s} d\boldsymbol{\alpha} = o(X^{2s-u}). \quad (4.1)$$

Meanwhile, as in the proof of [11, Lemma 7.1], one finds that when $\boldsymbol{\alpha} \in \mathfrak{M}(q, \mathbf{a}) \subseteq \mathfrak{M}$, one has $f_k(\boldsymbol{\alpha}; X) - V(\boldsymbol{\alpha}; q, \mathbf{a}) \ll L^2$, whence

$$|f_k(\boldsymbol{\alpha}; X)|^{2s} - |V(\boldsymbol{\alpha}; q, \mathbf{a})|^{2s} \ll L^2 X^{2s-1}.$$

Since $\text{mes}(\mathfrak{M}) \ll L^{2k-1} X^{-u}$, we deduce that

$$\int_{\mathfrak{M}} |f_k(\boldsymbol{\alpha}; X)|^{2s} d\boldsymbol{\alpha} - \int_{\mathfrak{M}} |V(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} \ll (L^{2k+1}/X) X^{2s-u} = o(X^{2s-u}). \quad (4.2)$$

Next, on recalling the definitions of $V(\boldsymbol{\alpha})$ and $\mathfrak{M}$, we find that

$$\int_{\mathfrak{M}} |V(\boldsymbol{\alpha})|^{2s} d\boldsymbol{\alpha} \leq \mathfrak{S} J, \quad (4.3)$$

where

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a_2=1 \\ (q, a_2, \dots, a_k)=1}}^q \dots \sum_{a_k=1}^q |q^{-1} S(q, \mathbf{a})|^{2s} \quad \text{and} \quad J = \int_{\mathbb{R}^{k-1}} |I(\boldsymbol{\beta}; X)|^{2s} d\boldsymbol{\beta}.$$

When $(q, a_2, \dots, a_k) = 1$, we find from [5, Theorem 7.1] that $S(q, \mathbf{a}) \ll q^{1-1/k+\varepsilon}$. Then, since $2u/k = k+1-2/k$, we deduce that

$$\mathfrak{S} \ll \sum_{q=1}^{\infty} q^{k-1+\varepsilon} (q^{-1/k})^{2u+2k\delta} \ll \sum_{q=1}^{\infty} q^{-1-\delta} \ll 1. \quad (4.4)$$

Likewise, from [5, Theorem 7.3], one has

$$I(\beta; X) \ll X \left(1 + |\beta_2|X^2 + \dots + |\beta_k|X^k\right)^{-1/k} \leq X \prod_{j=2}^k (1 + |\beta_j|X^j)^{-1/(k^2-k)}.$$

Since

$$\frac{2s}{k^2 - k} > \frac{k^2 + k - 2}{k^2 - k} = 1 + \frac{2}{k},$$

we therefore find that

$$J \ll X^{2s} \prod_{j=2}^k \left(\int_{-\infty}^{\infty} (1 + |\beta_j|X^j)^{-1-2/k} d\beta_j \right) \ll X^{2s-u}. \quad (4.5)$$

On substituting (4.4) and (4.5) into (4.3), and thence into (4.2), we infer that

$$\int_{\mathfrak{M}} |f_k(\alpha; X)|^{2s} d\alpha = \int_{\mathfrak{M}} |V(\alpha)|^{2s} d\alpha + o(X^{2s-u}) \ll X^{2s-u},$$

whence by (3.3) and (4.1), we have

$$I_{s,k}(X) = \int_{\mathfrak{M}} |f_k(\alpha; X)|^{2s} d\alpha + \int_{\mathfrak{m}} |f_k(\alpha; X)|^{2s} d\alpha \ll X^{2s-u}.$$

The conclusion of the lemma follows on recalling that $u = (k^2 + k - 2)/2$. $\square$

It is a triviality now to complete the proof of Theorem 1.1.

The proof of Theorem 1.1. The conclusion of Theorem 1.1 follows for $s \leq (k^2 + k - 2)/2$ from Lemma 3.2, and for $s > (k^2 + k - 2)/2$ from Lemma 4.1. $\square$

We remark that, with a little additional effort, one would obtain an asymptotic formula for $I_{s,k}(X)$ when $s > (k^2 + k - 2)/2$ by adapting the proof of Lemma 4.1.

Theorem 4.2. *There are positive numbers $C = C(s, k)$ and $\delta = \delta(s, k)$ having the property that, whenever $s > (k^2 + k - 2)/2$, one has*

$$I_{s,k}(X) = CX^{2s-(k^2+k-2)/2} + O(X^{2s-(k^2+k-2)/2-\delta}).$$

We leave the proof of this conclusion to the enthusiastic reader, noting that the proof of [5, Theorem 7.4] provides a pedestrian route plan for the required argument.

5. A SHARP MEAN VALUE ESTIMATE IN NUMBER FIELDS

We begin by describing the infrastructure required to describe mean value estimates in number fields. Consider an algebraic extension K of degree n over $\mathbb{Q}$. Associated with K are the real conjugate fields $K^{(l)}$ ($1 \leq l \leq n_1$), and the pairs of complex conjugate fields $K^{(m)}$ and $K^{(m+n_2)}$ ($n_1 + 1 \leq m \leq n_1 + n_2$). Here, one has $n = n_1 + 2n_2$. Write $\mathfrak{O}_K$ for the ring of integers of K , and fix a basis $\Omega = \{\omega_1, \dots, \omega_n\}$ for $\mathfrak{O}_K$ over $\mathbb{Z}$. When X is a large positive number, we define the box $\mathcal{B}(X) \subset \mathfrak{O}_K$, with $\text{card}(\mathcal{B}(X)) \asymp X$, by taking

$$\mathcal{B}(X) = \left\{ r_1\omega_1 + \dots + r_n\omega_n : r_i \in \left[-\frac{1}{2}X^{1/n}, \frac{1}{2}X^{1/n}\right) \cap \mathbb{Z} \ (1 \leq i \leq n) \right\}.$$

When $\gamma \in K$, denote by $\gamma^{(i)}$ the conjugates of γ , with $\gamma^{(i)} \in K^{(i)}$ ($1 \leq i \leq n$), and the associated trace map $\text{Tr}_{K/\mathbb{Q}}(\gamma) = \gamma^{(1)} + \dots + \gamma^{(n)}$. Next, when $\boldsymbol{\gamma} \in K^n$ and $\boldsymbol{\theta} \in \mathbb{R}^n$, and

$$\lambda(\boldsymbol{\theta}) = \theta_1 \gamma_1 + \dots + \theta_n \gamma_n,$$

we define

$$\lambda^{(i)}(\boldsymbol{\theta}) = \theta_1 \gamma_1^{(i)} + \dots + \theta_n \gamma_n^{(i)} \quad (1 \leq i \leq n).$$

The trace map on K then extends to the map defined by

$$\text{Tr}_{K/\mathbb{Q}}(\lambda(\boldsymbol{\theta})) = \lambda^{(1)}(\boldsymbol{\theta}) + \dots + \lambda^{(n)}(\boldsymbol{\theta}),$$

and we define

$$E(\lambda(\boldsymbol{\theta})) = e(\text{Tr}_{K/\mathbb{Q}}(\lambda(\boldsymbol{\theta}))).$$

Next, let $\mathfrak{d}^{-1}$ denote the inverse different, so that

$$\mathfrak{d}^{-1} = \{\eta \in K : \text{Tr}_{K/\mathbb{Q}}(\eta\xi) \in \mathbb{Z} \text{ for all } \xi \in \mathfrak{O}_K\}.$$

There is a basis $P = \{\rho_1, \dots, \rho_n\}$ of $\mathfrak{d}^{-1}$ dual to Ω having the property that

$$\text{Tr}_{K/\mathbb{Q}}(\rho_i \omega_j) = \begin{cases} 1, & \text{when } i = j, \\ 0, & \text{when } i \neq j. \end{cases}$$

As a special case of the linear form $\lambda(\boldsymbol{\theta})$ defined above, we define $\alpha = \alpha(\boldsymbol{\theta})$ by

$$\alpha(\boldsymbol{\theta}) = \theta_1 \rho_1 + \dots + \theta_n \rho_n.$$

When such a form occurs in an n -fold integral, we abbreviate the n -fold differential $d\theta_1 \dots d\theta_n$ to $d\alpha$. We also write $\mathbb{T}$ for $[0, 1]^n$. In this notation, we have the orthogonality relation

$$\int_{\mathbb{T}} E(\alpha\gamma) d\alpha = \begin{cases} 1, & \text{when } \gamma = 0, \\ 0, & \text{when } \gamma \in \mathfrak{O}_K \setminus \{0\}. \end{cases}$$

The analogue of Theorem 1.1 for number fields may now be announced.

Theorem 5.1. *Suppose that $k \geq 2$ and $s > 0$. Then for each $\varepsilon > 0$, one has*

$$\int_{\mathbb{T}^{k-1}} \left| \sum_{x \in \mathcal{B}(X)} E(\alpha_2 x^2 + \dots + \alpha_k x^k) \right|^{2s} d\boldsymbol{\alpha} \ll X^\varepsilon (X^s + X^{2s - (k^2 + k - 2)/2}).$$

Proof. In the number field setting, one may proceed just as in the arguments described in §§2 and 3. The conclusion of [13, Theorem 11.2] supplies the analogue of (2.6) needed to prove the number field version of Lemma 2.1. Moreover, the analogue of the bound (3.8) is supplied by [12, Corollary 15.3]. Thus, the argument of §3 applies mutatis mutandis. $\square$

Additional work is needed to obtain the analogue of the §4 arguments capable of delivering an ε -free version of Theorem 5.1 for $s > (k^2 + k - 2)/2$. This we leave as an exercise for the enthusiastic reader.

6. A FUNCTION FIELD VARIANT

We again initiate proceedings in function fields by introducing additional infrastructure. Consider a finite field $\mathbb{F}_q$, with $q = p^l$, of characteristic $p > k$. Associated with the polynomial ring $\mathbb{O} = \mathbb{F}_q[t]$ is the field of fractions $\mathbb{K} = \mathbb{F}_q(t)$. We take as our basic parameter X , a sufficiently large natural number, and we put

$$\mathbb{O}_X = \{\nu \in \mathbb{F}_q[t] : \deg \nu \leq X\}.$$

Write $\mathbb{K}_\infty = \mathbb{F}_q((1/t))$ for the completion of $\mathbb{F}_q(t)$ at ∞ . Each element $\alpha \in \mathbb{K}_\infty \setminus \{0\}$ may be written in the shape $\alpha = \sum_{i \leq n} a_i t^i$, for some $n \in \mathbb{Z}$, with $a_i = a_i(\alpha) \in \mathbb{F}_q$ for $-\infty < i \leq n$ and $a_n \neq 0$. We define $\text{ord } \alpha = n$, adopt the convention that $\text{ord } 0 = -\infty$, and introduce the compact additive subgroup $\mathbb{T} = \{\alpha \in \mathbb{K}_\infty : \text{ord } \alpha < 0\}$ of $\mathbb{K}_\infty$. We may normalise any Haar measure $d\alpha$ on $\mathbb{K}_\infty$ in such a manner that $\int_{\mathbb{T}} d\alpha = 1$.

We next introduce the non-trivial character $e_q : \mathbb{F}_q \rightarrow \mathbb{C}^\times$, defined for $a \in \mathbb{F}_q$ by taking $e_q(a) = e(\text{tr}(a)/p)$, where $\text{tr}(a) = a + a^p + \dots + a^{p^{l-1}}$. Define $\text{res}(\alpha)$ to be $a_{-1}(\alpha)$ when $\text{ord } \alpha \geq -1$, and 0 otherwise. Then, this character $e_q(\cdot)$ induces a map $e : \mathbb{K}_\infty \rightarrow \mathbb{C}^\times$, defined by $e(\alpha) = e_q(\text{res } \alpha)$ for $\alpha \in \mathbb{K}_\infty$. One then has

$$\int_{\mathbb{T}} e(h\alpha) d\alpha = \begin{cases} 1, & \text{when } h = 0, \\ 0, & \text{when } h \in \mathbb{F}_q[t] \setminus \{0\}. \end{cases}$$

We now announce our analogue of Theorem 1.1 for function fields.

Theorem 6.1. *Suppose that $k \geq 2$ and $s > 0$. Then, whenever $\text{ch}(\mathbb{F}_q) > k$ and $\varepsilon > 0$,*

$$\int_{\mathbb{T}^{k-1}} \left| \sum_{x \in \mathbb{O}_X} e(\alpha_2 x^2 + \dots + \alpha_k x^k) \right|^{2s} d\alpha \ll (q^X)^\varepsilon ((q^X)^s + (q^X)^{2s - (k^2 + k - 2)/2}).$$

Proof. We follow §§2 and 3, modified mutatis mutandis, to the function field setting. The conclusion of [13, Theorem 12.1] supplies the analogue of (2.6) needed to prove the analogue of Lemma 2.1, the analogue of (3.8) being supplied by [12, Corollary 17.3]. $\square$

We again leave as an exercise the additional work required to obtain the analogue of the arguments in §4 that deliver an ε -free version of Theorem 6.1 for $s > (k^2 + k - 2)/2$.

7. SOME COMMENTS ON THE CUBIC CASE OF THEOREM 1.1

We now address the special case of Theorem 1.1 in which $k = 3$. Here, we have

$$\int_0^1 \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^3 + \beta x^2) \right|^{2s} d\alpha d\beta \ll X^{s+\varepsilon} + X^{2s-5},$$

and when $s = 5$, in particular, the upper bound

$$\int_0^1 \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^3 + \beta x^2) \right|^{10} d\alpha d\beta \ll X^{5+\varepsilon}. \quad (7.1)$$

The mean value on the left hand side of (7.1) has been the subject of intensive investigation by the author over the past 35 years. Early work [6] and [7] was restricted to exponential sums over smooth numbers, obtaining analogues of (7.1) in which the exponent 5 is replaced by $35/6$, with subsequent work [8] improving the latter exponent to $17/3$. More

recently, following the author's resolution [10] of the main conjecture in the cubic case of Vinogradov's mean value theorem, the upper bound (7.1) was established with the exponent 5 replaced by $16/3$. The latter work was employed by the author [9] to establish a quantitative Hasse principle for pairs of diagonal Diophantine equations, one cubic and one quadratic, in 11 or more variables. This conclusion was much extended in work of Brandes [2] concerning systems of (many) diagonal Diophantine cubic and quadratic equations. The sharp upper bound (7.1) should foster improvements in the latter results so far as the number of variables is concerned.

One challenge that remains unresolved concerns a discrete restriction analogue of the upper bound (7.1). Let (a_n) denote a sequence of complex numbers. It is plausible that

$$\int_0^1 \int_0^1 \left| \sum_{1 \leq n \leq X} a_n e(\alpha n^3 + \beta n^2) \right|^{10} d\alpha d\beta \ll X^\varepsilon \left(\sum_{1 \leq n \leq X} |a_n|^2 \right)^5. \quad (7.2)$$

However, the methods of this paper establish such a bound only when the a_n are more-or-less 1-bounded and not too thinly supported. The sharpest available unconditional estimate of this flavour would appear to be [11, Corollary 1.3], which gives the bound

$$\int_0^1 \int_0^1 \left| \sum_{1 \leq n \leq X} a_n e(\alpha n^3 + \beta n^2) \right|^{12} d\alpha d\beta \ll X^{1+\varepsilon} \left(\sum_{1 \leq n \leq X} |a_n|^2 \right)^6.$$

Consequently, the proof of (7.2) apparently remains a significant challenge for aficionados.

REFERENCES

- [1] J. Bourgain, C. Demeter and L. Guth, *Proof of the main conjecture in Vinogradov's mean value theorem for degrees higher than three*, Ann. of Math. (2) **184** (2016), no. 2, 633–682.
- [2] J. Brandes, *The Hasse principle for systems of quadratic and cubic diagonal equations*, Q. J. Math. **68** (2017), no. 3, 831–850.
- [3] J. Brandes and T. D. Wooley, *Vinogradov systems with a slice off*, Mathematika **63** (2017), no. 3, 797–817.
- [4] S. Guo and P. Zorin-Kranich, *Decoupling for moment manifolds associated to Arkhipov-Chubarikov-Karatsuba systems*, Adv. Math. **360** (2020), article no. 106889, 56pp.
- [5] R. C. Vaughan, *The Hardy-Littlewood method*, Cambridge University Press, 1997.
- [6] T. D. Wooley, *On simultaneous additive equations and Waring's problem*, Ph.D. thesis, University of London, 1990.
- [7] T. D. Wooley, *On simultaneous additive equations, II*, J. Reine Angew. Math. **419** (1991), 141–198.
- [8] T. D. Wooley, *On simultaneous additive equations, IV*, Mathematika **45** (1998), 319–335.
- [9] T. D. Wooley, *Rational solutions of pairs of diagonal equations, one cubic and one quadratic*, Proc. London Math. Soc. (3) **110** (2015), no. 2, 325–356.
- [10] T. D. Wooley, *The cubic case of the main conjecture in Vinogradov's mean value theorem*, Adv. Math. **294** (2016), 532–561.
- [11] T. D. Wooley, *Discrete Fourier restriction via efficient congruencing*, Internat. Math. Res. Notices **2017** (2017), no. 5, 1342–1389.
- [12] T. D. Wooley, *Nested efficient congruencing and relatives of Vinogradov's mean value theorem*, Proc. London Math. Soc. (3) **118** (2019), no. 4, 942–1016.
- [13] T. D. Wooley, *Efficient congruencing in $\mathbb{Z}[t]$ modulo t^2* , preprint, 55pp.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY STREET, WEST LAFAYETTE, IN 47907-2067, USA

Email address: twooley@purdue.edu