The asymptotic formulae in Waring's problem for cubes

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Abstract. Write $R_s(n)$ for the number of representations of a natural number as the sum of s positive integral cubes. We investigate the higher moments of $R_s(n)$, establishing asymptotic formulae for $\sum_{n\le N} R_s(n)^h$ when $s=5$ and $h=3$, when $s=6$ and $h<28/5$, and when $s=7$ and $h\le 12$. Hitherto, such an asymptotic formula was available only when $s\ge 4$ and $h=2$.

1. Introduction

Waring's problem for cubes has been a guiding theme in the additive theory of numbers since the early use of analytic methods in the subject. It has been conjectured that the number $R_s(n)$ of representations of the natural number n as the sum of s positive integral cubes satisfies the asymptotic formula

$$R_s(n)=\frac{\Gamma(4/3)^s}{\Gamma(s/3)}\mathfrak{S}_s(n)n^{s/3-1}(1+o(1))\quad (n\to\infty),\tag{1.1}$$

provided only that $s\ge 4$. Here $\mathfrak{S}_s(n)$ is the singular series associated with sums of s cubes, namely

$$\mathfrak{S}_s(n)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\Big(q^{-1}\sum_{r=1}^{q}e(ar^3/q)\Big)^s e(-an/q),\tag{1.2}$$

in which we have written $e(z)$ for $e^{2\pi iz}$. Evidence for (1.1) may be provided by statistical considerations. As a showcase for the analytic method that now bears their name, Hardy and Littlewood established (1.1) for $s\ge 9$ (see [7] and Theorem 1 of [8]). A more formal application of their method supports this asymptotic relation already when $s\ge 4$. When $s=5$, Hardy and Littlewood confirmed their formula (1.1) for most values of n. They proved that

$$\sum_{n\le N}\Big|R_5(n)-\frac{\Gamma(4/3)^5}{\Gamma(5/3)}\mathfrak{S}_5(n)n^{2/3}\Big|^2\ll N^{13/6+\varepsilon}\tag{1.3}$$

(see Theorem 1 and §6.1 of [9]). Further progress was made by Vaughan [13], about sixty years later. He established the formula for $R_8(n)$ in the form

$$R_8(n)=\frac{\Gamma(4/3)^8}{\Gamma(8/3)}\mathfrak{S}_8(n)n^{5/3}\big(1+O\big((\log n)^{-\vartheta}\big)\big),$$

for some $\vartheta>0$, and also showed the formula for $R_4(n)$ to be correct in mean square, as a consequence of the bound

$$\sum_{n\le N}\Big|R_4(n)-\Gamma(4/3)^3\mathfrak{S}_4(n)n^{1/3}\Big|^2\ll N^{5/3}(\log N)^{-\vartheta}.$$

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More recently, Boklan [1] demonstrated that any value of ϑ smaller than 3 is admissible in these estimates.

In this paper we study higher moments of the error terms that arise in these asymptotic formulae. To illustrate ideas, we temporarily concentrate on the case of sums of five cubes. No results of significance were obtained hitherto that go beyond (1.3). Our first result provides a cubic moment analogue of this formula.

THEOREM 1.1. *One has*

$$\sum_{n\le N}\left|R_5(n)-\frac{\Gamma(4/3)^5}{\Gamma(5/3)}\mathfrak{S}_5(n)n^{2/3}\right|^3 \ll N^3(\log N)^{\varepsilon-4}.$$

One may apply this result to establish the asymptotic formula

$$\sum_{n\le N} R_5(n)^3 = CN^3 + O(N^3(\log N)^{\varepsilon-4}), \tag{1.4}$$

where C is a suitable positive constant. The left hand side of (1.4) is equal to the number of solutions of the diophantine system

$$x_1^3+\ldots+x_5^3 = y_1^3+\ldots+y_5^3 = z_1^3+\ldots+z_5^3 \le N, \tag{1.5}$$

in positive integers x_j, y_j, z_j $(1\le j\le 5)$. As experts in the field will readily recognise, systems of two additive cubic equations as in (1.5) require at least 16 variables, or otherwise deny treatment by traditional use of the circle method. The latter comment also applies to the conclusion of Theorem 1.1. However, recent work of the authors [5] suggests that the solutions of (1.5) may be counted with greater precision than in (1.4).

THEOREM 1.2. *There exists a positive constant C such that*

$$\sum_{n\le N} R_5(n)^3 = CN^3 + O(N^{35/12+\varepsilon}).$$

Moreover,

$$\sum_{n\le N}\left(R_5(n)-\frac{\Gamma(4/3)^5}{\Gamma(5/3)}\mathfrak{S}_5(n)n^{2/3}\right)^3 \ll N^{35/12+\varepsilon}.$$

Note that a result of comparable strength for the mean of $R_5(n)^2$ follows from (1.3).

Our methods also apply when more cubes are present in the representation problem. We record here the analogues of Theorem 1.1 when $s=6$ or 7.

THEOREM 1.3. *For any positive number h smaller than $28/5$, there is a positive number δ such that*

$$\sum_{n\le N}\left|R_6(n)-\Gamma(4/3)^6\mathfrak{S}_6(n)n\right|^h \ll N^{h+1-\delta}.$$

Moreover, one has

$$\sum_{n\le N}\left|R_7(n)-\tfrac{3}{4}\Gamma(4/3)^6\mathfrak{S}_7(n)n^{4/3}\right|^{12} \ll N^{17}(\log N)^{\varepsilon-25}.$$

From these bounds, one may obtain asymptotic formulae for the moments

$$\sum_{n\le N} R_s(n)^h,$$

when $s = 6$ and $h < 28/5$, and when $s = 7$ and $h \le 12$, in the same way as one deduces (1.4) from Theorem 1.1.

Perhaps not unexpectedly for workers in the field, we derive Theorems 1.1 and 1.3 from related estimates for certain minor arc integrals. Some notation is required before we can describe this in detail. Let

$$f(\alpha) = \sum_{x\le N^{1/3}} e(\alpha x^3) \tag{1.6}$$

denote a cubic Weyl sum, and define the minor arcs $\mathfrak{m}$ as the set of all $\alpha \in [0,1]$ such that whenever $q \in \mathbb{N}$, and $q\alpha$ differs from an integer by at most $N^{-3/4}$, then $q > N^{1/4}$. By orthogonality, when $n \le N$, one has

$$R_s(n) = \int_0^1 f(\alpha)^s e(-\alpha n)\, d\alpha, \tag{1.7}$$

and the contribution of the minor arcs $\mathfrak{m}$ to this integral is usually the significant contribution to the difference

$$E_s(n) = R_s(n) - \frac{\Gamma(4/3)^s}{\Gamma(s/3)} \mathfrak{S}_s(n) n^{s/3-1}. \tag{1.8}$$

Thus, quadratic means such as (1.3) are related to the mean values

$$\sum_{n\le N} \Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha \Big|^2.$$

By Bessel's inequality, the latter does not exceed

$$\int_{\mathfrak{m}} |f(\alpha)|^{2s}\, d\alpha.$$

Such integrals are very familiar in the Hardy-Littlewood circle method. In the absence of a suitable analogue of Bessel's inequality, higher moments of minor arc integrals do not seem to have occurred in the literature so far. Here we introduce a method that allows one to count the number of integers n for which the Fourier coefficient

$$\int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha$$

is abnormally large. This has the potential to bound higher moments of this integral, and we shall conclude as follows.

Theorem 1.4. *One has*

$$\sum_{n\le N} \Big| \int_{\mathfrak{m}} f(\alpha)^5 e(-\alpha n)\, d\alpha \Big|^3 \ll N^3 (\log N)^{\varepsilon-4}$$

and

$$\sum_{n\le N}\Big|\int_{\mathfrak m} f(\alpha)^7 e(-\alpha n)\,d\alpha\Big|^{12} \ll N^{17}(\log N)^{\varepsilon-25}.$$

Moreover, for any positive number h with $h<28/5$, there is a real number $\delta>0$ such that

$$\sum_{n\le N}\Big|\int_{\mathfrak m} f(\alpha)^6 e(-\alpha n)\,d\alpha\Big|^{h} \ll N^{h+1-\delta}.$$

Our elaborations begin with a brief summary of the theory of cubic Weyl sums in §2. Then we examine abnormally large Fourier integrals, and use the results to derive Theorem 1.4 and a more general estimate, Theorem 3.2 below. Theorems 1.1 and 1.3 follow from Theorem 1.4 via a familiar and easy route. This we demonstrate, again in greater generality, at the end of §3. The later sections of this memoir are devoted to the proof of Theorem 1.2. The cubic moment of $R_5(n)$ is evaluated by a method that may be viewed either as a two- or three-dimensional version of the circle method. The minor arc work, to be performed in §4, depends on a mean value estimate of the authors [3]. The complementary major arc analysis is largely classical (see §5). Finally, in §6, we complete the proof of Theorem 1.2 by relating the second assertion of the theorem to the cubic moment of $R_5(n)$ and (1.3).

Notation. We apply the following convention concerning the letter ε: whenever ε appears in a statement, it is asserted that the statement is true for any positive real number ε. Implicit constants in Landau- or Vinogradov symbols $O(\cdot)$ and $\ll$ may depend on ε. Note that this allows us to conclude from the assertions $U\ll N^\varepsilon$ and $V\ll N^\varepsilon$ that $UV\ll N^\varepsilon$, for example.

2. Cubic Weyl sums.

This section is a summary of estimates for the Weyl sum (1.6), for frequent use later on. Write $P=N^{1/3}$ and $L=\log N$. In addition, let $\mathfrak M$ denote the union of the intervals

$$\mathfrak M(q,a)=\{\alpha\in[0,1]:\ |q\alpha-a|\le P^{3/4}N^{-1}\}, \tag{2.1}$$

with $0\le a\le q\le P^{3/4}$ and $(a,q)=1$. Note that $[0,1]$ is the disjoint union of $\mathfrak M$ and $\mathfrak m$, as defined above.

Lemma 2.1. *Let $s\ge 5$ and $0<\delta<\frac{1}{12}$. Then, for $1\le n\le N$, one has*

$$\int_{\mathfrak M} f(\alpha)^s e(-\alpha n)\,d\alpha=\frac{\Gamma(4/3)^s}{\Gamma(s/3)}\mathfrak S_s(n)n^{s/3-1}+O\big(N^{s/3-1-\delta}\big).$$

Proof. The method underlying the proof of Theorem 4.4 of Vaughan [14] confirms the asserted asymptotic formula for some $\delta>0$, and an inspection of Vaughan's argument shows that any positive number δ with $\delta<\frac{1}{12}$ is admissible.

In the later sections, precise upper bounds for $f(\alpha)$ on the major arcs $\mathfrak M$ are required. Let

$$S(q,a)=\sum_{x=1}^{q} e(ax^3/q)\quad\text{and}\quad v(\eta)=\int_0^P e(\eta\gamma^3)\,d\gamma.$$

Then, by Theorem 4.1 of Vaughan [14], for any $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \le q \le \frac{1}{6}P$ and $|q\alpha - a| \le 1/(6P^2)$, one has

$$f(\alpha) = q^{-1}S(q,a)v(\alpha - a/q) + O(q^{1/2+\varepsilon}). \tag{2.2}$$

In particular, the formula (2.2) is applicable when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$.

Let $\psi(q)$ be the multiplicative function defined on prime powers p^l by

$$\psi(p^{3l}) = p^{-l}, \quad \psi(p^{3l+1}) = 2p^{-l-1/2}, \quad \psi(p^{3l+2}) = p^{-l-1}.$$

Then, by Lemmata 4.3, 4.4 and 4.5 of Vaughan [14], whenever $(a,q)=1$, one has

$$q^{-1}S(q,a) \ll \psi(q) \ll q^{\varepsilon - 1/3}. \tag{2.3}$$

Moreover, integration by parts readily yields the bound

$$v(\eta) \ll P(1 + N|\eta|)^{-1/3}. \tag{2.4}$$

Note that the lower bound $\psi(q) \ge q^{-1/2}$ holds for all $q \in \mathbb{N}$. Hence, by (2.2), (2.3) and (2.4), we infer for $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$ the estimate

$$f(\alpha) \ll \psi(q)P(1 + N|\alpha - a/q|)^{-1/3}. \tag{2.5}$$

The alternative bound

$$f(\alpha) \ll P(q + N|q\alpha - a|)^{-1/3} \tag{2.6}$$

follows by the same argument, but with (2.3) replaced by a reference to Theorem 4.2 of Vaughan [14]. By (2.5) and straightforward estimations, one finds that

$$\int_{\mathfrak{M}} |f(\alpha)|^4 \, d\alpha \ll P \sum_{q \le P} q\psi(q)^4 \ll P^{1+\varepsilon}. \tag{2.7}$$

LEMMA 2.2. *Uniformly for $\alpha \in \mathfrak{m}$, one has $f(\alpha) \ll P^{3/4}L^{1/4+\varepsilon}$.*

This is an enhanced version of Weyl's inequality. A slightly weaker estimate occurs as Lemma 1 of Vaughan [13]. The argument given there yields our Lemma 2.2 when more recent bounds of Hall and Tenenbaum [6] for Hooley's Δ-function are substituted for Vaughan's reference to Hooley [11].

We also require the mean value estimates

$$\int_0^1 |f(\alpha)|^4 \, d\alpha \ll P^2 \quad \text{and} \quad \int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha \ll P^5 L^{\varepsilon - 3} \tag{2.8}$$

that follow, respectively, from Theorem 1 of Hooley [12], and Boklan [1].

LEMMA 2.3. *Let t be a real number with $t \ge 4$. Then*

$$\int_{\mathfrak{m}} |f(\alpha)|^t \, d\alpha \ll P^{\frac{3}{4}t - 1} L^{\varepsilon - \tau}$$

where $\tau = \frac{3}{4}t - 3$ when $4 \le t \le 8$, and where $\tau = 5 - \frac{1}{4}t$ when $t > 8$.

Proof. When $4 \le t \le 8$, we apply Hölder's inequality to interpolate between the two estimates (2.8). When $t > 8$, on the other hand, we use the estimate provided by Lemma 2.2 in conjunction with the second inequality in (2.8).

3. The frequency of abnormal minor arc contributions.

The proof of Theorem 1.4 depends on a method for counting abnormally large Fourier coefficents. Bessel's inequality is the most basic tool in such circumstances, but one can often do significantly better when the Fourier coefficents arise from a diophantine problem. For earlier uses of this observation, see Wooley [15, 16], Brüdern, Kawada and Wooley [2] and Brüdern and Wooley [4]. In the present context, let $\mathcal{Z}_s(T;N)$ denote the set of all natural numbers $n \le N$ for which one has the lower bound

$$\Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha \Big| > T. \tag{3.1}$$

We seek to establish bounds for the number

$$Z = Z_{T,s} = \text{card } \mathcal{Z}_s(T;N) \tag{3.2}$$

of the shape $Z \le P^\beta T^{-2}$, with $\beta = \beta(s)$ as small as is possible. These may be converted in an obvious manner into moment estimates for the quantity on the left hand side of (3.1).

The principal idea for estimating $Z_{T,s}$ is readily described. Define the complex number $\eta_{n,s}$ by $\eta_{n,s} = 0$ for $n \notin \mathcal{Z}_s(T;N)$, and when $n \in \mathcal{Z}_s(T;N)$ by means of the equation

$$\Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha \Big| = \eta_{n,s} \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha.$$

Plainly, one has $|\eta_{n,s}| = 1$ whenever $\eta_{n,s} \ne 0$. From (3.1) and (3.2) we now find that

$$ZT \le \sum_{n \le N} \eta_{n,s} \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha = \int_{\mathfrak{m}} f(\alpha)^s K_s(-\alpha;T)\, d\alpha, \tag{3.3}$$

where

$$K_s(\alpha;T) = \sum_{n \le N} \eta_{n,s} e(\alpha n).$$

Our object now is to estimate the integral on the right hand side of (3.3).

Before advancing further, we introduce some notation. When r is a natural number, and λ and μ are real numbers, we say that the triple $(\lambda, \mu; r)$ is *tailored for sums of s cubes* when one has the estimate

$$\int_0^1 |f(\alpha)^r K_s(\alpha;T)|^2\, d\alpha \ll P^\lambda Z + P^\mu Z^2. \tag{3.4}$$

LEMMA 3.1. *The triples* $(1, \varepsilon; 1)$, $(\frac{4}{3} + \varepsilon, 0; 1)$ *and* $(2, \frac{11}{6} + \varepsilon; 2)$ *are all tailored for sums of s cubes.*

Proof. The assertion of the lemma concerning the first triple follows from the argument of the proof of Lemma 6.1 of Wooley [15], on noting the ε-free exponent of U in the final display of that proof. The assertion concerning the second triple is immediate from the estimate (9.10) of Wooley [15], and the third is a consequence of Lemma 10.3 of [15], on noting that the hypothesis $R(\frac{11}{6})$ follows from the methods of Hooley [10] (see also the estimate for J_4 presented prior to equation (21) of Wooley [16]).

The estimation of Z is now readily completed. Let $s \geq 5$, suppose that $(\lambda, \mu; r)$ is a triple tailored for sums of s cubes, and assume that $t = 2s - 2r$ satisfies the lower bound $t \geq 4$. On applying Schwarz's inequality to (3.3), and then appealing to (3.4) and Lemma 2.3, we find that

$$\begin{aligned} ZT &\leq \int_{\mathfrak{m}} |f(\alpha)^s K_s(-\alpha; T)| \, d\alpha \\ &\leq \Big(\int_0^1 |f(\alpha)^r K_s(\alpha; T)|^2 \, d\alpha \Big)^{1/2} \Big(\int_{\mathfrak{m}} |f(\alpha)|^t \, d\alpha \Big)^{1/2} \\ &\ll (P^{\lambda} Z + P^{\mu} Z^2)^{1/2} (P^{\frac{3}{4}t-1} L^{\varepsilon - \tau})^{1/2}. \end{aligned}$$

Here τ is the function of t introduced in Lemma 2.3. Consequently, whenever

$$T > P^{\frac{1}{2}\mu + \frac{3}{8}t - \frac{1}{2}} L^{\varepsilon - \frac{1}{2}\tau}, \tag{3.5}$$

and P is sufficiently large, it follows that

$$Z \ll P^{\lambda + \frac{3}{4}t - 1} L^{\varepsilon - \tau} T^{-2}. \tag{3.6}$$

Let $\mathcal{Z}_s^*(T; N) = \mathcal{Z}_s(T; N) \backslash \mathcal{Z}_s(2T; N)$. Then, by (3.1) and (3.6), it follows that for each real number $h \geq 2$, one has

$$\sum_{n \in \mathcal{Z}_s^*(T;N)} \Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n) \, d\alpha \Big|^h \ll P^{\lambda + \frac{3}{4}t - 1} L^{\varepsilon - \tau} T^{h-2}. \tag{3.7}$$

An alternative estimate is available through Bessel's inequality. This asserts that

$$\sum_{n \leq N} \Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n) \, d\alpha \Big|^2 \leq \int_{\mathfrak{m}} |f(\alpha)|^{2s} \, d\alpha.$$

We may estimate the right hand side of this inequality by means of Lemma 2.3, and in this way it follows that for $s \geq 5$, one has

$$\sum_{n \in \mathcal{Z}_s^*(T;N)} \Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n) \, d\alpha \Big|^h \ll P^{\frac{3}{2}s - 1} L^{\frac{1}{2}s - 5 + \varepsilon} T^{h-2}. \tag{3.8}$$

We now apply the observation leading to (3.7) so as to estimate moments more general than those covered by Theorem 1.4. In order to describe our conclusions, it is useful to introduce some additional notation. When s is a natural number with $s \geq 5$ and h is a real number with $h \geq 2$, define

$$M_{s,h}(N) = \sum_{n \leq N} \Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n) \, d\alpha \Big|^h.$$

THEOREM 3.2. *When $h \geq 2$, one has*

$$M_{5,h}(N) \ll N^{\frac{5}{6}h + \frac{1}{2}} L^{\varepsilon - \frac{3}{2}h + \frac{1}{2}} + N^{\frac{8}{9}h + \frac{2}{9} + \varepsilon} + N^{\frac{11}{12}h} L^{\varepsilon - \frac{3}{4}h}, \tag{3.9}$$

$$M_{6,h}(N) \ll N^{\frac{13}{12}h + \frac{1}{2}} L^{\varepsilon - \frac{5}{4}h + \frac{1}{2}} + N^{\frac{41}{36}h + \frac{2}{9} + \varepsilon} + N^{\frac{7}{6}h} L^{\varepsilon - \frac{3}{2}h}, \tag{3.10}$$

$$M_{7,h}(N) \ll N^{\frac{4}{3}h + \frac{1}{2}} L^{\varepsilon - h + \frac{1}{2}} + N^{\frac{25}{18}h + \frac{2}{9} + \varepsilon} + N^{\frac{17}{12}h} L^{\varepsilon - \frac{9}{4}h + 2}. \tag{3.11}$$

Proof. We observe first that when $h = 2$, the estimates (3.9), (3.10) and (3.11) follow directly from Lemma 2.3 via Bessel's inequality. Henceforth, therefore, we are at liberty to assume that h exceeds 2. We initiate proceedings with (3.9), and thus take $s = 5$ in the preparatory analysis. By Lemma 2.3, one has

$$\int_{\mathfrak{m}} f(\alpha)^5 e(-\alpha n)\, d\alpha \ll P^{\frac{11}{4}} L^{\varepsilon - \frac{3}{4}}. \tag{3.12}$$

Next, we make use of the tailored triple $(2, \frac{11}{6} + \varepsilon; 2)$ provided by Lemma 3.1, and apply (3.7) with $t = 6$. By Lemma 2.3, one has $\tau = \frac{3}{2}$ in this case, and so on dividing the range of summation into dyadic intervals, when N is sufficiently large, it follows from (3.5) and (3.7) that

$$\sum_{\substack{n \in \mathcal{Z}_5^*(T;N) \\ P^{8/3+\varepsilon} \le T \le P^{11/4} L^{\varepsilon - 3/4}}} \Big| \int_{\mathfrak{m}} f(\alpha)^5 e(-\alpha n)\, d\alpha \Big|^h \ll P^{\frac{11}{2}} L^{\varepsilon - \frac{3}{2}} (P^{\frac{11}{4}} L^{\varepsilon - \frac{3}{4}})^{h-2}. \tag{3.13}$$

When $P^{5/2+\varepsilon} \le T \le P^{8/3+2\varepsilon}$, we use the tailored triple $(1, \varepsilon; 1)$ and take $t = 8$ with $\tau = 3$ in Lemma 2.3. Then, again by (3.5) and (3.7), we have

$$\sum_{\substack{n \in \mathcal{Z}_5^*(T;N) \\ P^{5/2+\varepsilon} \le T \le P^{8/3+\varepsilon}}} \Big| \int_{\mathfrak{m}} f(\alpha)^5 e(-\alpha n)\, d\alpha \Big|^h \ll P^6 L^{\varepsilon - 3} (P^{\frac{8}{3}+\varepsilon})^{h-2}. \tag{3.14}$$

When $P^{5/2} L^{\varepsilon - 3/2} \le T < P^{5/2+\varepsilon}$, we proceed similarly, but now with the tailored triple $(\frac{4}{3} + \varepsilon, 0; 1)$, and with $t = 8$ and $\tau = 3$, in (3.7). This time we obtain

$$\sum_{\substack{n \in \mathcal{Z}_5^*(T;N) \\ P^{5/2} L^{\varepsilon - 3/2} \le T \le P^{5/2+\varepsilon}}} \Big| \int_{\mathfrak{m}} f(\alpha)^5 e(-\alpha n)\, d\alpha \Big|^h \ll P^{\frac{19}{3}+\varepsilon} (P^{\frac{5}{2}+\varepsilon})^{h-2}. \tag{3.15}$$

Finally, when $0 \le T \le P^{5/2} L^{\varepsilon - 3/2}$, we again divide the range of summation into dyadic intervals and apply (3.8) to confirm the bound

$$\sum_{\substack{n \in \mathcal{Z}_5^*(T;N) \\ 0 \le T \le P^{5/2} L^{\varepsilon - 3/2}}} \Big| \int_{\mathfrak{m}} f(\alpha)^5 e(-\alpha n)\, d\alpha \Big|^h \ll P^{\frac{13}{2}} L^{\varepsilon - \frac{5}{2}} (P^{\frac{5}{2}} L^{\varepsilon - \frac{3}{2}})^{h-2}. \tag{3.16}$$

In order to deduce (3.9), it suffices to recall (3.12) and sum (3.13), (3.14), (3.15) and (3.16).

Our treatment of the estimate (3.10) follows the previous argument closely. On this occasion, Lemma 2.3 yields

$$\int_{\mathfrak{m}} f(\alpha)^6 e(-\alpha n)\, d\alpha \ll P^{\frac{7}{2}} L^{\varepsilon - \frac{3}{2}}.$$

Hence, applying (3.5) and (3.7) with $t = 8$ as in the argument leading to (3.13), we find that

$$\sum_{\substack{n \in \mathcal{Z}_6^*(T;N) \\ P^{41/12+\varepsilon} \le T \le P^{7/2} L^{\varepsilon - 3/2}}} \Big| \int_{\mathfrak{m}} f(\alpha)^6 e(-\alpha n)\, d\alpha \Big|^h \ll P^7 L^{\varepsilon - 3} (P^{\frac{7}{2}} L^{\varepsilon - \frac{3}{2}})^{h-2}.$$

Further, on taking $t = 10$, the argument used to derive (3.14) provides the bound

$$\sum_{\substack{n\in\mathcal{Z}_6^*(T;N)\\ P^{13/4+\varepsilon}\le T\le P^{41/12+\varepsilon}}} \Big| \int_{\mathfrak{m}} f(\alpha)^6 e(-\alpha n)\, d\alpha\Big|^h \ll P^{\frac{15}{2}}(\log P)^{\varepsilon-\frac{5}{2}}(P^{\frac{41}{12}+\varepsilon})^{h-2}.$$

Next, we find as in (3.15) that

$$\sum_{\substack{n\in\mathcal{Z}_6^*(T;N)\\ P^{13/4}L^{\varepsilon-5/4}\le T\le P^{13/4+\varepsilon}}} \Big| \int_{\mathfrak{m}} f(\alpha)^6 e(-\alpha n)\, d\alpha\Big|^h \ll P^{\frac{47}{6}+\varepsilon}(P^{\frac{13}{4}+\varepsilon})^{h-2}.$$

Finally, just as in the derivation of (3.16), we find that (3.8) yields the estimate

$$\sum_{\substack{n\in\mathcal{Z}_6^*(T;N)\\ 0\le T\le P^{13/4}L^{\varepsilon-5/4}}} \Big| \int_{\mathfrak{m}} f(\alpha)^6 e(-\alpha n)\, d\alpha\Big|^h \ll P^8(\log P)^{\varepsilon-2}(P^{\frac{13}{4}}L^{\varepsilon-\frac{5}{4}})^{h-2}.$$

The previous four bounds may now be summed to confirm (3.10).

When it comes to establishing (3.11), the argument is almost identical. Lemma 2.3 shows that

$$\int_{\mathfrak{m}} f(\alpha)^7 e(-\alpha n)\, d\alpha \ll P^{\frac{17}{4}}L^{\varepsilon-\frac{9}{4}}.$$

Consequently, on taking $t = 10$, the argument employed to confirm (3.13) now implies that

$$\sum_{\substack{n\in\mathcal{Z}_7^*(T;N)\\ P^{25/6+\varepsilon}\le T\le P^{17/4}L^{\varepsilon-9/4}}} \Big| \int_{\mathfrak{m}} f(\alpha)^7 e(-\alpha n)\, d\alpha\Big|^h \ll P^{\frac{17}{2}}L^{\varepsilon-\frac{5}{2}}(P^{\frac{17}{4}}L^{\varepsilon-\frac{9}{4}})^{h-2}.$$

Next setting $t = 12$, the argument leading to (3.14) now gives

$$\sum_{\substack{n\in\mathcal{Z}_7^*(T;N)\\ P^{4+\varepsilon}\le T\le P^{25/6+\varepsilon}}} \Big| \int_{\mathfrak{m}} f(\alpha)^7 e(-\alpha n)\, d\alpha\Big|^h \ll P^9 L^{\varepsilon-2}(P^{\frac{25}{6}+\varepsilon})^{h-2}.$$

Meanwhile, as in (3.15) we deduce that

$$\sum_{\substack{n\in\mathcal{Z}_7^*(T;N)\\ P^4L^{\varepsilon-1}\le T\le P^{4+\varepsilon}}} \Big| \int_{\mathfrak{m}} f(\alpha)^7 e(-\alpha n)\, d\alpha\Big|^h \ll P^{\frac{28}{3}+\varepsilon}(P^{4+\varepsilon})^{h-2}.$$

Finally, the bound (3.8) on this occasion yields the inequality

$$\sum_{\substack{n\in\mathcal{Z}_7^*(T;N)\\ 0\le T\le P^4L^{\varepsilon-1}}} \Big| \int_{\mathfrak{m}} f(\alpha)^7 e(-\alpha n)\, d\alpha\Big|^h \ll P^{\frac{19}{2}}L^{\varepsilon-\frac{3}{2}}(P^4L^{\varepsilon-1})^{h-2}.$$

The estimate (3.11) follows at once on combining these four bounds. This completes the proof of Theorem 3.2.

The estimates in Theorem 3.2 are readily transformed into bounds for moments of the error terms (1.8), that appear in Theorems 1.1 and 1.3. To see this, we employ (1.7) and Lemma 2.1 to infer that whenever $n \le N$, one has

$$E_s(n) = \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha + O(N^{\frac{s}{3}-\frac{13}{12}+\varepsilon}).$$

For a fixed real number h with $h \ge 2$, and any complex numbers z and w, the inequality $|z+w|^h \ll |z|^h + |w|^h$ holds, whence

$$|E_s(n)|^h \ll \Big| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n)\, d\alpha \Big|^h + N^{\frac{1}{3}sh-\frac{13}{12}h+\varepsilon}.$$

Summing over n reveals that

$$\sum_{n\le N} |E_s(n)|^h \ll M_{s,h}(N) + N^{1+\frac{1}{3}sh-\frac{13}{12}h+\varepsilon}.$$

On substituting the upper bounds provided by Theorem 3.2 into the latter estimate, we may conclude as follows.

THEOREM 3.3. *Let h be a real number with $h \ge 2$. Then*

$$\begin{aligned}
\sum_{n\le N} |E_5(n)|^h &\ll N^{\frac{7}{12}h+1+\varepsilon} + N^{\frac{5}{6}h+\frac{1}{2}} L^{\varepsilon-\frac{3}{2}h+\frac{1}{2}} + N^{\frac{8}{9}h+\frac{2}{9}+\varepsilon} + N^{\frac{11}{12}h} L^{\varepsilon-\frac{3}{4}h},\\
\sum_{n\le N} |E_6(n)|^h &\ll N^{\frac{11}{12}h+1+\varepsilon} + N^{\frac{13}{12}h+\frac{1}{2}} L^{\varepsilon-\frac{5}{4}h+\frac{1}{2}} + N^{\frac{41}{36}h+\frac{2}{9}+\varepsilon} + N^{\frac{7}{6}h} L^{\varepsilon-\frac{3}{2}h},\\
\sum_{n\le N} |E_7(n)|^h &\ll N^{\frac{5}{4}h+1+\varepsilon} + N^{\frac{4}{3}h+\frac{1}{2}} L^{\varepsilon-h+\frac{1}{2}} + N^{\frac{25}{18}h+\frac{2}{9}+\varepsilon} + N^{\frac{17}{12}h} L^{\varepsilon-\frac{9}{4}h+2}.
\end{aligned}$$

The conclusion of Theorem 1.1 is now immediate from (1.8) and the first estimate of Theorem 3.3. The bounds claimed in Theorem 1.3, meanwhile, follow from (1.8) and the second and third estimates of Theorem 3.3.

4. The third moment of $R_5(n)$.

It remains to establish Theorem 1.2. In this and the next section, we evaluate asymptotically the third moment of $R_5(n)$ by a method akin to our recent work on pairs of diagonal cubic forms [5]. In this context, we write $\mathcal{N}(P)$ for the number of solutions of the diophantine system

$$x_1^3 + x_2^3 - x_3^3 - x_4^3 = y_1^3 + y_2^3 - y_3^3 - y_4^3 = z_1^3 + z_2^3 + z_3^3 - z_4^3 - z_5^3 - z_6^3,$$

subject to $1 \le x_i, y_i \le P$ $(1 \le i \le 4)$ and $1 \le z_j \le P$ $(1 \le j \le 6)$. The following observation will be relevant.

LEMMA 4.1. *One has $\mathcal{N}(P) \ll P^{8+\varepsilon}$.*

Proof. By orthogonality, it is apparent that

$$\mathcal{N}(P) = \int_0^1 \int_0^1 |f(\alpha)^4 f(\beta)^4 f(\alpha+\beta)^6|\, d\alpha\, d\beta.$$

From the inequality $|f(\alpha)f(\beta)|^4 \le |f(\alpha)|^2|f(\beta)|^6 + |f(\alpha)|^6|f(\beta)|^2$, we conclude that

$$\mathcal{N}(P) \le 2\int_0^1\int_0^1 |f(\alpha)^2 f(\beta)^6 f(\alpha+\beta)^6|\,d\alpha\,d\beta.$$

But it follows from Theorem 2.2 of Brüdern and Wooley [3] that the integral on the right hand side here is $O(P^{8+\varepsilon})$, and the conclusion of the lemma follows at once.

We now study the sum $\sum_{n\le N} R_5(n)^3$ through its interpretation as the number of solutions to the system (1.5). Let

$$J(\gamma) = \sum_{n\le N} e(\gamma n). \tag{4.1}$$

Then, by orthogonality,

$$\sum_{n\le N} R_5(n)^3 = \int_0^1\int_0^1\int_0^1 f(\alpha)^5 f(\beta)^5 f(\gamma-\alpha-\beta)^5 J(-\gamma)\,d\alpha\,d\beta\,d\gamma. \tag{4.2}$$

We apply the Hardy-Littlewood method to evaluate this integral. Let $\mathfrak{N}$ denote the union of the boxes

$$\mathfrak{N}(q,a,b) = \big\{(\alpha,\beta)\in[0,1]^2\,:\,|q\alpha-a|\le P^{-9/4},\ |q\beta-b|\le P^{-9/4}\big\},$$

subject to $(q,a,b)=1$ and $0\le a,b\le q\le P^{3/4}$. Let $\mathfrak{n}=[0,1]^2\backslash\mathfrak{N}$. We proceed by estimating the contribution to the integral on the right hand side of (4.2) arising from the minor arcs $\mathfrak{n}$.

Lemma 4.2. *The estimate*

$$\iint_{\mathfrak{n}} |f(\alpha)f(\beta)f(\gamma-\alpha-\beta)|^5\,d\alpha\,d\beta \ll P^{35/4+\varepsilon}$$

holds uniformly for real numbers γ.

Proof. We dissect the set $\mathfrak{n}$ into the subsets

$$\begin{aligned}\mathfrak{k}_1 &= \{(\alpha,\beta)\in\mathfrak{n}:\ \alpha\in\mathfrak{m}\},\\ \mathfrak{k}_2 &= \{(\alpha,\beta)\in\mathfrak{n}:\ \beta\in\mathfrak{m}\},\\ \mathfrak{K} &= \{(\alpha,\beta)\in\mathfrak{n}:\ (\alpha,\beta)\in\mathfrak{M}\times\mathfrak{M}\},\end{aligned} \tag{4.3}$$

and treat the contribution arising from each one separately. By Lemma 2.2 and Schwarz's inequality, one finds that

$$\iint_{\mathfrak{k}_1} |f(\alpha)f(\beta)f(\gamma-\alpha-\beta)|^5\,d\alpha\,d\beta \ll P^{3/4+\varepsilon}(I_1I_2)^{1/2},$$

where

$$I_1 = \int_0^1\int_0^1 |f(\alpha)^4 f(\beta)^4 f(\gamma-\alpha-\beta)^6|\,d\alpha\,d\beta,$$

and

$$I_2 = \int_0^1 \int_0^1 |f(\alpha)^4 f(\gamma-\alpha-\beta)^4 f(\beta)^6|\, d\alpha\, d\beta.$$

On considering the underlying diophantine systems, it follows from orthogonality that $I_1 \le \mathcal{N}(P)$ and $I_2 \le \mathcal{N}(P)$. From Lemma 4.1 we may therefore conclude that the estimate

$$\iint_{\mathfrak{k}_1} |f(\alpha)f(\beta)f(\gamma-\alpha-\beta)|^5\, d\alpha\, d\beta \ll P^{35/4+\varepsilon}$$

indeed holds uniformly in γ. By symmetry in α and β, this bound is also valid when $\mathfrak{k}_2$ is substituted for $\mathfrak{k}_1$.

The treament of the set $\mathfrak{K}$ depends on the bound

$$\sup_{(\alpha,\beta)\in\mathfrak{K}} |f(\alpha)f(\beta)| \ll P^{7/4+\varepsilon}, \tag{4.4}$$

which we now derive. For $\alpha \in \mathfrak{M}$ and $\beta \in \mathfrak{M}$, it follows from (2.1) that there exist $a, b \in \mathbb{Z}$ and $q_1, q_2 \in \mathbb{N}$ with $\max\{q_1, q_2\} \le P^{3/4}$, $(a, q_1) = (b, q_2) = 1$,

$$|q_1\alpha - a| \le P^{-9/4} \quad \text{and} \quad |q_2\beta - b| \le P^{-9/4}.$$

By (2.6), the hypothesis that $|f(\alpha)f(\beta)| \ge P^{7/4+\varepsilon}$ implies that the upper bound

$$q_1q_2(1 + P^3|\alpha - a/q_1|)(1 + P^3|\beta - b/q_2|) \le P^{3/4}$$

holds for all large values of P. It follows that

$$q_1q_2 \le P^{3/4}, \quad |q_1q_2\alpha - aq_2| \le P^{-9/4} \quad \text{and} \quad |q_1q_2\beta - bq_1| \le P^{-9/4}.$$

We are therefore forced to conclude that $(\alpha, \beta) \in \mathfrak{N}$, which in view of (4.3) is not the case whenever $(\alpha, \beta) \in \mathfrak{K}$. This establishes (4.4).

Making use of (4.4) and the trivial estimate $|f(\gamma - \alpha - \beta)| = O(P)$, followed by two applications of (2.7), we find that

$$\begin{aligned}\iint_{\mathfrak{K}} |f(\alpha)f(\beta)f(\gamma-\alpha-\beta)|^5\, d\alpha\, d\beta &\ll P^{27/4+\varepsilon} \int_{\mathfrak{M}}\int_{\mathfrak{M}} |f(\alpha)f(\beta)|^4\, d\alpha\, d\beta, \\ &\ll P^{35/4+\varepsilon}.\end{aligned}$$

This completes the proof of Lemma 4.2.

We can now dispense of the minor arcs within the integral (4.2). From the definition (4.1) of $J(\gamma)$, we have $J(\gamma) \ll \min\{N, \|\gamma\|^{-1}\}$, and hence

$$\int_0^1 |J(\gamma)|\, d\gamma \ll \log N.$$

This simple bound may be coupled with the estimate provided by Lemma 4.2 to conclude from (4.2) that

$$\begin{aligned}\sum_{n\le N} R_5(n)^3 &= \int_0^1 J(-\gamma) \iint_{\mathfrak{N}} f(\alpha)^5 f(\beta)^5 f(\gamma-\alpha-\beta)^5\, d\alpha\, d\beta\, d\gamma + O(P^{35/4+\varepsilon}) \\ &= \iint_{\mathfrak{N}} f(\alpha)^5 f(\beta)^5 F(\alpha+\beta)\, d\alpha\, d\beta + O(P^{35/4+\varepsilon}),\end{aligned} \tag{4.5}$$

where

$$F(\nu) = \int_0^1 J(-\gamma)f(\gamma - \nu)^5 \, d\gamma = \sum_{n \le N} R_5(n)e(-\nu n). \tag{4.6}$$

5. A twofold major arc analysis.

Further progress on the last integral in (4.5) now depends on approximations for the product of generating functions $f(\alpha)f(\beta)$. When $(\alpha, \beta) \in \mathfrak{N}$, there is a uniquely defined integral triple (a, b, q) with $0 \le a, \, b \le q \le P^{3/4}$, $(a, b, q) = 1$,

$$|q\alpha - a| \le P^{-9/4}, \quad \text{and} \quad |q\beta - b| \le P^{-9/4}.$$

We write a_1/q_1 for the fraction a/q written in lowest terms, and likewise b_2/q_2 for b/q in lowest terms. By standard properties of Gauss sums, one may confirm that $q_1^{-1}S(q_1, a_1) = q^{-1}S(q, a)$, and similarly $q_2^{-1}S(q_2, b_2) = q^{-1}S(q, b)$. Define $f^*(\alpha, \beta)$ for $(\alpha, \beta) \in \mathfrak{N}(q, a, b) \subseteq \mathfrak{N}$ by taking

$$f^*(\alpha, \beta) = q^{-2}S(q, a)S(q, b)v(\alpha - a/q)v(\beta - b/q).$$

Utilising (2.3) and (2.4), we confirm the upper bound

$$f^*(\alpha, \beta) \ll \psi(q_1)\psi(q_2)P^2(1 + N|\alpha - a/q|)^{-1/3}(1 + N|\beta - b/q|)^{-1/3} \tag{5.1}$$

that we shall use later. By combining two applications of (2.2) together with (2.3) and (2.4), moreover, we see that

$$f(\alpha)f(\beta) = f^*(\alpha, \beta) + O(P^{11/8+\varepsilon}) \tag{5.2}$$

uniformly on $\mathfrak{N}$. Next we observe that as a consequence of (5.2) and the binomial expansion, we have

$$f(\alpha)^5 f(\beta)^5 - f^*(\alpha, \beta)^5 \ll P^{55/8+\varepsilon} + P^{11/8+\varepsilon}|f^*(\alpha, \beta)|^4.$$

We multiply this relation by $F(\alpha+\beta)$ and integrate over $\mathfrak{N}$. In view of (4.6), the estimate $F(\nu) \ll P^5$ holds uniformly in ν. Thus, since the measure of $\mathfrak{N}$ is $O(P^{9/4}N^{-2})$, we obtain the preliminary formula

$$\iint_{\mathfrak{N}} f(\alpha)^5 f(\beta)^5 F(\alpha + \beta) \, d\alpha \, d\beta = M + O(P^{65/8+\varepsilon} + P^{51/8+\varepsilon}K), \tag{5.3}$$

where

$$K = \iint_{\mathfrak{N}} |f^*(\alpha, \beta)|^4 \, d\alpha \, d\beta \quad \text{and} \quad M = \iint_{\mathfrak{N}} f^*(\alpha, \beta)^5 F(\alpha + \beta) \, d\alpha \, d\beta. \tag{5.4}$$

Lemma 5.1. *The integral K defined in* (5.4) *satisfies $K \ll P^{2+\varepsilon}$.*

Proof. On recalling (5.1), we find that

$$K \ll P^8 \sum_{q \le P^{3/4}} \Psi(q)\Upsilon(N),$$

where

$$\Psi(q)=\sum_{\substack{a=1\\(q,a,b)=1}}^{q}\sum_{b=1}^{q}\psi(q_1)^4\psi(q_2)^4$$

and

$$\Upsilon(N)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+N|\zeta|)^{-4/3}(1+N|\xi|)^{-4/3}\,d\zeta\,d\xi.$$

But $\Upsilon(N)\ll N^{-2}$, and thus

$$K\ll P^8N^{-2}\sum_{q\le P^{3/4}}\Psi(q). \tag{5.5}$$

Note that $q_1=q/(q,a)$ and $q_2=q/(q,b)$. It follows, in particular, that $\Psi(q)$ is multiplicative. In order to obtain an upper bound for $\Psi(q)$ when $q=p^l$ is a positive power of the prime p, we observe that the relation $(a,b,p^l)=1$ implies that at least one of a and b is coprime to p. By symmetry in a and b, this confirms the inequality

$$\Psi(p^l)\le 2\psi(p^l)^4\sum_{\substack{a=1\\(a,p)=1}}^{p^l}\sum_{b=1}^{p^l}\psi(p^l/(p^l,b))^4.$$

Moreover, from the definition of $\psi(q)$ one deduces that

$$\begin{aligned}\sum_{b=1}^{p^l}\psi(p^l/(b,p^l))^4 &\le \sum_{k=0}^{l}p^k\psi(p^k)^4\le\sum_{k=0}^{\infty}p^k\psi(p^k)^4\\ &= \frac{p}{p-1}(1+\psi(p)^4+\psi(p^2)^4).\end{aligned}$$

Consequently, one has the upper bound $\Psi(p^l)\le 24p^l\psi(p^l)^4$, and hence

$$\Psi(q)\ll q^{1+\varepsilon}\psi(q)^4. \tag{5.6}$$

The lemma follows by incorporating (5.6) and the second inequality of (2.7) within (5.5).

We briefly pause to collect the results obtained so far. On substituting (5.3) and the conclusion of Lemma 5.1 into (4.5), we obtain the relation

$$\sum_{n\le N}R_5(n)^3=M+O(P^{35/4+\varepsilon}), \tag{5.7}$$

and we progress to the evaluation of the integral M in (5.4). Here, we require a major arc approximation to $F(\nu)$ provided by the following lemma.

Lemma 5.2. *Let $F(\nu)$ be defined as in* (4.6)*. Suppose that $c\in\mathbb{Z}$, $q\in\mathbb{N}$ and $\eta\in\mathbb{R}$ satisfy $1\le q\le\frac{1}{6}P$ and $|\eta|\le\frac{1}{12}P^{-2}$. Then*

$$F(c/q+\eta)=q^{-5}S(q,-c)^5V(\eta)+O(P^{19/4+\varepsilon}),$$

where

$$V(\eta) = \int_{-\infty}^{\infty} \int_0^N e(-\vartheta\gamma) v(\gamma - \eta)^5 \, d\vartheta \, d\gamma. \tag{5.8}$$

Proof. It is convenient within this proof to write $\Gamma = \frac{1}{12}P^{-2}$. By (4.1), one has $J(\gamma) \ll \|\gamma\|^{-1} \ll P^2$ for $\|\gamma\| \geq \Gamma$, whence

$$\int_\Gamma^{1/2} |J(-\gamma) f(\nu - \gamma)^5| \, d\gamma \ll P^2 \int_0^1 |f(\nu - \gamma)|^5 \, d\gamma.$$

The integrand on the right hand side has period 1, so that an application of Hölder's inequality together with Hua's lemma (see [14], Lemma 2.5) yields

$$\int_0^1 |f(\nu - \gamma)|^5 \, d\gamma = \int_0^1 |f(\gamma)|^5 \, d\gamma \ll P^{11/4+\varepsilon}. \tag{5.9}$$

Substituting the last bound into the previous one, and this in turn into (4.6), we now infer the initial approximation

$$F(\nu) = \int_{-\Gamma}^{\Gamma} J(-\gamma) f(\gamma - \nu)^5 \, d\gamma + O(P^{19/4+\varepsilon}). \tag{5.10}$$

Next write

$$I(\gamma) = \int_0^N e(\vartheta\gamma) \, d\vartheta, \tag{5.11}$$

and recall that $N = P^3$. Applying Euler's summation formula to (4.1), we find that $J(\gamma) = I(\gamma) + O(1 + N|\gamma|)$. For $|\gamma| \leq \Gamma$ this reduces to $J(\gamma) = I(\gamma) + O(P)$. Hence, by another application of (5.9), we may replace the term $J(-\gamma)$ occurring in (5.10) by $I(-\gamma)$ without affecting the error term. In the resulting formula, we take $\nu = c/q + \eta$. Then, for $|\gamma| \leq \Gamma$ and $|\eta| \leq \Gamma$, we may apply (2.2) to deduce that

$$f(\gamma - \nu) = q^{-1} S(q, -c) v(\gamma - \eta) + O(P^{1/2+\varepsilon}).$$

Use of the binomial expansion, coupled with the trivial bound $O(P)$ for the right hand side here suffices to confirm the approximate identity

$$f(\gamma - \nu)^5 = q^{-5} S(q, -c)^5 v(\gamma - \eta)^5 + O(P^{9/2+\varepsilon}).$$

We now multiply this by $I(-\gamma)$ and integrate over $[-\Gamma, \Gamma]$. Since the estimate $I(\gamma) \ll |\gamma|^{-1}$ is readily obtained by partial integration, we arrive at the relation

$$F(c/q + \eta) - q^{-5} S(q, -c)^5 \int_{-\Gamma}^{\Gamma} I(-\gamma) v(\gamma - \eta)^5 \, d\gamma \ll P^{19/4+\varepsilon}. \tag{5.12}$$

On noting that $I(\gamma) \ll P^2$ for $|\gamma| > \Gamma$, and recalling the definition (5.11), it follows from (5.8) that

$$\begin{aligned} V(\eta) - \int_{-\Gamma}^{\Gamma} I(-\gamma) v(\gamma - \eta)^5 \, d\gamma &\ll \int_{|\gamma| \geq \Gamma} |I(-\gamma) v(\gamma - \eta)^5| \, d\gamma \\ &\ll P^2 \int_{-\infty}^{\infty} |v(\gamma - \eta)|^5 \, d\gamma. \end{aligned}$$

On making use of (2.4), we therefore find that

$$V(\eta) - \int_{-\Gamma}^{\Gamma} I(-\gamma)v(\gamma-\eta)^5\,d\gamma \ll P^4,$$

and the conclusion of Lemma 5.2 follows at once from (5.12).

We return to the main theme and compute the integral M defined in (5.4) by replacing the expression $F(\alpha+\beta)$ occurring in the integrand with the approximation suggested by Lemma 5.2. When $(\alpha,\beta) \in \mathfrak{N}(q,a,b)$, let

$$F^*(\alpha,\beta) = q^{-5}S(q,-a-b)^5 V\Big(\alpha+\beta-\frac{a+b}{q}\Big).$$

Then by Lemma 5.2, one has

$$\sup_{(\alpha,\beta)\in\mathfrak{N}} |F(\alpha+\beta) - F^*(\alpha,\beta)| \ll P^{19/4+\varepsilon},$$

and hence a trivial estimate for $f^*(\alpha,\beta)$ leads from (5.4) to the formula

$$M = \iint_{\mathfrak{N}} f^*(\alpha,\beta)^5 F^*(\alpha,\beta)\,d\alpha\,d\beta + O(P^{27/4+\varepsilon}K).$$

On recalling Lemma 5.1, we find that the error term here is $O(P^{35/4+\varepsilon})$. On the other hand, using the definitions of f^* and F^*, we see that the integral on the right hand side of the last equation is equal to

$$\sum_{q\le P^{3/4}} T(q) \iint_{\mathfrak{B}(q^{-1}P^{-9/4})} v(\alpha)^5 v(\beta)^5 V(\alpha+\beta)\,d\alpha\,d\beta, \tag{5.13}$$

where $\mathfrak{B}(X) = \{(\alpha,\beta)\in\mathbb{R}^2 : |\alpha|\le X \text{ and } |\beta|\le X\}$, and

$$T(q) = \sum_{\substack{a=1 \\ (q,a,b)=1}}^{q} \sum_{b=1}^{q} q^{-15} S(q,a)^5 S(q,b)^5 S(q,-a-b)^5. \tag{5.14}$$

We compare the integral in (5.13) with the singular integral

$$\mathfrak{I}(P) = \iint_{\mathbb{R}^2} v(\alpha)^5 v(\beta)^5 V(\alpha+\beta)\,d\alpha\,d\beta. \tag{5.15}$$

In view of (2.4), this integral is absolutely convergent. Moreover, again by (2.4), for any $X \ge P^{-3}$ we have

$$\int_X^\infty |v(\alpha)|^5\,d\alpha \ll P^5 \int_X^\infty (1+N|\gamma|)^{-5/3}\,d\gamma \ll P^2(1+NX)^{-2/3} \ll X^{-2/3},$$

while from (5.8) we have the trivial estimate

$$V(\eta) \ll P^5 \int_{-\infty}^{\infty}\int_0^N (1+N|\gamma-\eta|)^{-5/3}\,d\theta\,d\gamma \ll P^2 N = P^5.$$

Combining these estimates with that implicit in (5.12), we infer that

$$\iint_{\mathbb{R}^2 \setminus \mathfrak{B}(X)} |v(\alpha)^5 v(\beta)^5 V(\alpha+\beta)|\, d\alpha\, d\beta \ll P^7 X^{-2/3}.$$

We therefore deduce from (5.13) that

$$M = \sum_{q \le P^{3/4}} T(q)(\mathfrak{J}(P) + O(q^{2/3} P^{17/2})) + O(P^{35/4+\varepsilon}). \tag{5.16}$$

The endgame in the evaluation of M begins with a crude bound for $T(q)$ that we derive from (5.14) and (2.3). With $q_1 = q/(q,a)$, $q_2 = q/(q,b)$ and $q_3 = q/(q,a+b)$, this yields

$$|T(q)| \le \sum_{\substack{a=1 \\ (q,a,b)=1}}^{q} \sum_{b=1}^{q} \psi(q_1)^5 \psi(q_2)^5 \psi(q_3)^5,$$

and much as in the proof of Lemma 5.1, it is readily seen that this upper bound for $|T(q)|$ is multiplicative in q. Now let $q = p^l$ be a positive power of the prime p. Then whenever $(p,a,b) = 1$, it follows that at least two of the numbers a, b and $a+b$ are coprime to p. Hence, in the current situation, one has $\psi(q_1)\psi(q_2)\psi(q_3) \le \psi(p^l)^2$. On recalling (2.3) and applying multiplicativity, it follows that $|T(q)| \le q^2 \psi(q)^{10} \ll q^{\varepsilon} \psi(q)^4$. Thus, we conclude that the series

$$\mathfrak{T} = \sum_{q=1}^{\infty} T(q)$$

converges absolutely, and moreover, from the definition of $\psi(q)$, one finds also that

$$\mathfrak{T} - \sum_{q \le P^{3/4}} T(q) \ll P^{-1/2} \sum_{q > P^{3/4}} q^{2/3+\varepsilon} \psi(q)^4 \ll P^{-1/2},$$

and also that

$$\sum_{q \le P^{3/4}} q^{2/3} |T(q)| \ll \sum_{q \le P^{3/4}} q^{2/3+\varepsilon} \psi(q)^4 \ll 1.$$

Thus, on recalling (5.16), one obtains the upper bound

$$M - \mathfrak{T}\mathfrak{J}(P) \ll P^{-1/2} |\mathfrak{J}(P)| + P^{35/4+\varepsilon}.$$

Finally, obvious substitutions give

$$v(\alpha) = P \int_0^1 e(P^3 \alpha \gamma^3)\, d\gamma \quad \text{and} \quad I(\alpha) = P^3 \int_0^1 e(P^3 \alpha \gamma)\, d\gamma,$$

whence by (5.15), (5.8) and another obvious substitution, one arrives at the relation $\mathfrak{J}(P) = P^9 \mathfrak{J}(1)$. On substituting this formula and the last estimate into (5.7), we reach our final formula

$$\sum_{n \le N} R_5(n)^3 = \mathfrak{T}\mathfrak{J}(1) P^9 + O(P^{35/4+\varepsilon}). \tag{5.17}$$

This is the asymptotic formula for the cubic moment of $R_5(n)$ claimed in Theorem 1.2, though we must still show that the constant $C = \mathfrak{T}\mathfrak{J}(1)$ is positive. It would be possible to use standard methods for the analysis of the singular series and integrals in order to establish the desired conclusion. However, we prefer to give a different proof of this fact that also leads to the sign-sensitive moment estimate that is the second clause in Theorem 1.2.

6. The proof of Theorem 1.2.

Having completed most of the less conventional work already, Theorem 1.2 will now be deduced by an argument that is elementary save for a cubic singular series average. This is the subject of our next lemma.

Lemma 6.1. *There exists a positive number C_1 with the property that*

$$\sum_{n\le N}\mathfrak{S}_5(n)^3=C_1N+O(N^{8/9+\varepsilon}).$$

Proof. By employing (2.3) and (2.7), one may confirm the inequality

$$\sum_{q>P}q\psi(q)^5\ll P^{\varepsilon-1/3}, \tag{6.1}$$

and this estimate we apply twice in what follows. The difference between the truncated sum

$$\mathfrak{S}(n,P)=\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}q^{-5}S(q,a)^5e(-an/q) \tag{6.2}$$

and $\mathfrak{S}_5(n)$ is bounded in order by the sum in (6.1). On recalling (1.2), an application of (6.1) with P replaced by 1 confirms that $\mathfrak{S}_5(n)\ll 1$, and hence it follows that

$$\mathfrak{S}_5(n)^3=\mathfrak{S}(n,P)^3+O(P^{\varepsilon-1/3}). \tag{6.3}$$

Consequently, from (6.2) one obtains

$$\sum_{n\le N}\mathfrak{S}(n,P)^3=\sum_{\mathbf{a},\mathbf{q}}\Big(\frac{S(q_1,a_1)S(q_2,a_2)S(q_3,a_3)}{q_1q_2q_3}\Big)^5\sum_{n\le N}e\Big(-n\Big(\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{a_3}{q_3}\Big)\Big), \tag{6.4}$$

where, here and later, the summation over $\mathbf{a}$ and $\mathbf{q}$ is taken over a_j and q_j with $1\le a_j\le q_j\le P$ and $(a_j,q_j)=1$ $(1\le j\le 3)$.

Let $\mathcal{Q}$ denote the set of all tuples $(\mathbf{a},\mathbf{q})$ of natural numbers with $1\le a_j\le q_j$ and $(a_j,q_j)=1$ $(1\le j\le 3)$ for which

$$\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{a_3}{q_3}\in\mathbb{Z}, \tag{6.5}$$

and let $\mathcal{Q}(P)$ be the subset where additionally one has $q_j\le P$ for $1\le j\le 3$. The innermost sum over n in (6.4) is $N+O(1)$ when (6.5) holds, and in the complementary case is bounded in order of magnitude by

$$\Big\|\frac{a_1}{q_1}+\frac{a_2}{q_2}+\frac{a_3}{q_3}\Big\|^{-1}\le q_1q_2q_3,$$

as is readily confirmed by summing the geometric sum. By (2.3) and the above observations, we infer from (6.4) that

$$\begin{aligned}\sum_{n\le N}\mathfrak{S}(n,P)^3 &= \sum_{\mathcal{Q}(P)}\left(\frac{S(q_1,a_1)S(q_2,a_2)S(q_3,a_3)}{q_1q_2q_3}\right)^5(N+O(1))\\ &\quad +O\left(\sum_{\mathbf{a},\mathbf{q}}q_1q_2q_3\psi(q_1)^5\psi(q_2)^5\psi(q_3)^5\right).\end{aligned}\tag{6.6}$$

The rightmost term here factorizes, and thereby is bounded above via (2.3) and (2.7) by

$$\left(\sum_{q\le P}\sum_{\substack{a=1\\(a,q)=1}}^{q}q\psi(q)^5\right)^3\ll\left(P^{2/3+\varepsilon}\sum_{q\le P}q\psi(q)^4\right)^3\ll P^{2+3\varepsilon}.$$

An essentially identical argument shows, via (6.1) with P replaced by 1, that

$$\sum_{(\mathbf{a},\mathbf{q})\in\mathcal{Q}}\left|\frac{S(q_1,a_1)S(q_2,a_2)S(q_3,a_3)}{q_1q_2q_3}\right|^5\ll\left(\sum_{q\le P}q\psi(q)^5\right)^3\ll 1.$$

In this way, we see that the error term that arises from the term $O(1)$ in the leading sum on the right hand side of (6.6) amounts to $O(1)$ in total, and also the sum

$$C_1=\sum_{(\mathbf{a},\mathbf{q})\in\mathcal{Q}}\left(\frac{S(q_1,a_1)S(q_2,a_2)S(q_3,a_3)}{q_1q_2q_3}\right)^5\tag{6.7}$$

is recognised to be absolutely convergent. We compare C_1 with its finite analogue implicit in (6.7) in which the sum is taken over $\mathcal{Q}(P)$ only. These two sums differ by those terms in (6.7) where $\max\{q_1,q_2,q_3\}>P$. However, by (2.3) and (6.1), the contribution to (6.7) of those terms with $q_1>P$ is at most

$$\sum_{q_1>P}q_1\psi(q_1)^5\sum_{q_2\ge 1}q_2\psi(q_2)^5\sum_{q_3\ge 1}q_3\psi(q_3)^5\ll P^{\varepsilon-1/3}.$$

Exploiting the symmetry between q_1, q_2 and q_3, we may therefore deduce from (6.6) that

$$\sum_{n\le N}\mathfrak{S}(n,P)^3=C_1N+O(NP^{\varepsilon-1/3}).$$

On summing (6.3), we see that this final formula remains valid with $\mathfrak{S}(n,P)^3$ replaced by $\mathfrak{S}_5(n)^3$. This establishes the asymptotic formula presented in Lemma 6.1, though it remains to confirm that $C_1>0$. But by Theorem 4.6 of Vaughan [14], we have also $\mathfrak{S}_5(n)\gg 1$, whence

$$\sum_{n\le N}\mathfrak{S}_5(n)^3\gg N.$$

Such a lower bound is possible only if $C_1>0$, and thus the proof of the lemma is complete.

Proof of Theorem 1.2. We apply Lemma 6.1 to show that $\mathfrak{T}\mathfrak{I}(1) > 0$. By (5.17), the latter suffices to establish the first assertion in Theorem 1.2. For notational simplicity, denote the main term in the putative asymptotic formula for $R_5(n)$ by

$$H(n) = \frac{\Gamma(4/3)^5}{\Gamma(5/3)}\mathfrak{S}_5(n)n^{2/3}.$$

By partial summation and Lemma 6.1, one has

$$\sum_{n\le N} H(n)^3 = CN^3 + O(N^{26/9+\varepsilon}), \tag{6.8}$$

where C is a *positive* real number. In what follows we use the shorthand $R = R_5(n)$ and $H = H(n)$. An upper bound is needed for the sum

$$\sum_{n\le N}(R^3 - H^3) = \sum_{n\le N}(R-H)(R^2 + RH + H^2).$$

Note that R and H are non-negative, and that $R^2 + RH + H^2 \le 2(R^2 + H^2)$. Then, since

$$|R^2 + H^2|^{3/2} \ll R^3 + H^3$$

uniformly in n, we deduce from Hölder's inequality that

$$\Big|\sum_{n\le N}(R^3 - H^3)\Big| \ll \Big(\sum_{n\le N}|R-H|^3\Big)^{1/3}\Big(\sum_{n\le N}(R^3 + H^3)\Big)^{2/3}.$$

We apply Theorem 1.1 to control the first factor on the right hand side, an upper bound for the second factor following from (6.8) and (5.17). Thus we obtain the upper bound

$$\sum_{n\le N}(R^3 - H^3) \ll N^3(\log N)^{\varepsilon-4/3}.$$

On comparing coefficients in (6.8) and (5.17), we therefore see that

$$(C - \mathfrak{T}\mathfrak{I}(1))N^3 + O(N^{35/12+\varepsilon}) \ll N^3(\log N)^{\varepsilon-4/3}.$$

We are therefore forced to conclude that $C = \mathfrak{T}\mathfrak{I}(1)$, and in view of (6.8), this constant is positive.

In order to establish the second assertion of Theorem 1.2, we sum the binomial expansion

$$(R-H)^3 = R^3 - H^3 - 3RH(R-H)$$

over n. With (6.8), (5.17) and the formula $C = \mathfrak{T}\mathfrak{I}(1)$ now at our disposal, the first two terms on the right hand side largely cancel, and it follows that

$$\sum_{n\le N}(R-H)^3 = 3\sum_{n\le N}RH(H-R) + O(N^{35/12+\varepsilon}). \tag{6.9}$$

Next we inject the trivial relation $R = (R-H) + H$ to obtain

$$\sum_{n\le N}RH|R-H| \le \sum_{n\le N}H|R-H|^2 + \sum_{n\le N}H^2|R-H|.$$

We have observed earlier that $\mathfrak{S}_5(n) \ll 1$, and so $H \ll n^{2/3}$ and

$$\sum_{n\le N} RH|R-H| \ll N^{2/3} \sum_{n\le N} |R-H|^2 + N^{4/3} \sum_{n\le N} |R-H|.$$

By (1.3), we have $\sum_{n\le N} |R-H|^2 \ll N^{13/6+\varepsilon}$. Yet another application of Cauchy's inequality for the second term on the right hand side here now suffices to confirm the bound

$$\sum_{n\le N} RH|R-H| \ll N^{35/12+\varepsilon},$$

and thus (6.9) implies the second assertion of Theorem 1.2.

Bibliography

[1] K. D. Boklan, A reduction technique in Waring's problem, I. Acta Arith. 65 (1993), 147–161.

[2] J. Brüdern, K. Kawada and T. D. Wooley, Additive representation in thin sequences, I: Waring's problem for cubes. Ann. Sci. École Norm. Sup. (4) 34 (2001), 471–501.

[3] J. Brüdern and T. D. Wooley, Asymptotic formulae for pairs of diagonal equations. Math. Proc. Cambridge Philos. Soc. 137 (2004), 227–235.

[4] J. Brüdern and T. D. Wooley, The Hasse principle for pairs of diagonal cubic forms. Ann. of Math. 166 (2007), 865–895.

[5] J. Brüdern and T. D. Wooley, Asymptotic formulae for pairs of diagonal cubic equations, to appear.

[6] R. Hall and G. Tenenbaum, Divisors. Cambridge University Press, Cambridge, 1988.

[7] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum': I. A new solution of Waring's Problem. Göttinger Nachrichten (1920), 33–54.

[8] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum': IV. The singular series in Waring's Problem and the value of the number $G(k)$. Math. Z. 12 (1922), 161–188.

[9] G. H. Hardy and J. E. Littlewood, Some problems of 'Partitio Numerorum' (VI): Further researches in Waring's Problem. Math. Z. 23 (1925), 1–37.

[10] C. Hooley, On the representations of a number as the sum of four cubes: I. Proc. London Math. Soc. (3) 36 (1978), 117–140.

[11] C. Hooley, On a new technique and its applications to the theory of numbers. Proc. London Math. Soc. (3) 38 (1979), 115–151.

[12] C. Hooley, On the numbers that are representable as the sum of two cubes. J. Reine Angew. Math. 314 (1980), 146–173.

[13] R. C. Vaughan, On Waring's problem for cubes. J. Reine Angew. Math. 365 (1986), 122–170.

[14] R. C. Vaughan, The Hardy-Littlewood method. Second edition. Cambridge University Press, Cambridge, 1997.

[15] T. D. Wooley, Slim exceptional sets for sums of cubes. Canad. J. Math. 54 (2002), 417–448.

[16] T. D. Wooley, Slim exceptional sets in Waring's problem: one square and five cubes. Quart. J. Math. 53 (2002), 111–118.

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