

ARTIN'S CONJECTURE AND SYSTEMS OF DIAGONAL EQUATIONS

TREVOR D. WOOLEY

ABSTRACT. We show that Artin's conjecture concerning p -adic solubility of Diophantine equations fails for infinitely many systems of r homogeneous diagonal equations whenever $r \geq 2$.

1. INTRODUCTION

A modern formulation of Artin's Conjecture (AC) asserts that in the p -adic field \mathbb{Q}_p , any collection of forms $F_1, \dots, F_r \in \mathbb{Q}_p[x_1, \dots, x_s]$ of respective degrees d_1, \dots, d_r must possess a common non-trivial zero provided only that $s > d_1^2 + \dots + d_r^2$. This is in fact a variant of a special case of the conjecture made by E. Artin (see [2, Preface page x] and [16] for more on the subtext to this statement). The history of this conjecture is remarkably rich. For the purpose at hand, it suffices to note that AC is known to hold whenever p is sufficiently large in terms of the degrees of the forms at hand (see [3, 4]), yet fails spectacularly for each prime p (see [1, 5, 18, 19]). The conjecture is known to hold, however, for systems consisting of a single quadratic or cubic form, a pair of quadratic forms, and also in the case of a single diagonal form of any degree (see [8, 9, 10, 17]). The latter conclusion has prompted speculation that AC might be salvaged for systems of diagonal forms. Workers intimately familiar with the argument underlying [18] have long been aware that AC cannot hold for certain systems of diagonal forms in which sufficiently many distinct degrees of suitable size occur, although this observation does not seem to be particularly well-documented in the literature. However, for larger degrees such counter-examples to AC necessarily contain very many forms. Our goal in this note is to show that counter-examples to AC exist in abundance for systems of two or more diagonal forms of differing degrees.

Theorem 1.1. *Let p be a prime number. Then whenever $r \geq 2$, there are infinitely many r -tuples (d_1, \dots, d_r) having the property that Artin's Conjecture fails over \mathbb{Q}_p for a system of diagonal forms of respective degrees d_1, \dots, d_r .*

There has been much work in the recent literature concerning systems of diagonal equations and their relation to AC (see [6, 7, 11, 12, 13, 14, 15]). In particular, it is known that AC holds for pairs of diagonal forms of equal degree k , except possibly when k takes the form $p^\tau(p-1)$ or $3 \cdot 2^\tau$ ($p=2$), and that AC holds also for pairs of diagonal forms of distinct odd degrees.

2010 *Mathematics Subject Classification.* 11D88, 11D72, 11E76.

Key words and phrases. Local solubility, diagonal equations, Artin's Conjecture.

While our theorem rules out the validity of AC in general for systems of two or more diagonal forms, it remains conceivable that AC holds for systems of odd degree.

In order to establish our theorem, we adapt the strategy of Lewis and Montgomery [18] to obtain a strengthened p -adic interpolation lemma. Let p be a prime number and h a natural number, and write $\phi(p^h) = p^{h-1}(p-1)$. When $N \geq 1$, define $S_{h,m}(\mathbf{x})$ by putting

$$S_{h,m}(x_1, \dots, x_N) = \sum_{i=1}^N x_i^{\phi(p^h)m}.$$

Then by enhancing the argument of [18, Lemma 2], we are able to establish the following conclusion.

Theorem 1.2. *Let p be a prime number and $h \in \mathbb{N}$. When $p = 2$, suppose in addition that $h \geq 3$. Let M be a positive integer, and consider a set \mathcal{M} of r integers in the interval $[M, 2M)$. Write $W = [\phi(p^h)/(h+1)]$. Finally, suppose that there is a non-zero solution $\mathbf{x} \in \mathbb{Q}_p^{Ws}$ to the system of equations*

$$\sum_{l=0}^{W-1} p^{l(h+1)M} S_{h,m}(x_{1l}, \dots, x_{sl}) = 0 \quad (m \in \mathcal{M}). \quad (1.1)$$

Then one has $s \geq p^{rh}$.

Note that the number of variables in the system (1.1) is Ws . Also, when $h \geq 5$, the integer W occurring in the statement of Theorem 1.2 satisfies

$$W \geq (p^{h-1} - h - 1)/(h + 1) \geq p^{h-2}/(h + 1).$$

The validity of AC for the system (1.1) would imply that a non-zero solution $\mathbf{x} \in \mathbb{Q}_p^{Ws}$ exists whenever

$$Ws \geq 4M^3 p^{2h} > \sum_{m \in \mathcal{M}} (p^{h-1}(p-1)m)^2.$$

However, the conclusion of Theorem 1.2 implies that no solution exists when $s < p^{rh}$. Consequently, whenever

$$p^{rh} > \frac{4M^3 p^{2h}}{p^{h-2}/(h+1)} = 4(h+1)M^3 p^{h+2},$$

one finds that AC fails for the system (1.1), and such is the case whenever $r > 1$ and h is sufficiently large in terms of M . The conclusion of Theorem 1.1 follows at once.

Throughout this note, we write $[\theta]$ for the greatest integer not exceeding θ .

2. THE PROOF OF THEOREM 1.2.

We begin by describing a variant of a result on p -adic interpolation provided by Lewis and Montgomery (see [18, Lemma 1]). The latter authors established the case $h = 1$ of the following conclusion. Here, as usual, when $\alpha = p^k a/b \in \mathbb{Q}$, with $(a, p) = (b, p) = 1$ and $k \in \mathbb{Z}$, we write $\text{ord } \alpha$ for k .

Lemma 2.1. *Let a be an integer and let n_1, \dots, n_K be distinct integers with $n_k \equiv a \pmod{p^h}$ ($1 \leq k \leq K$), for some $h \in \mathbb{N}$. Let $f \in \mathbb{Z}[z]$, and suppose that*

$$f(n_k) \equiv 0 \pmod{p^M} \quad (1 \leq k \leq K).$$

Then one has

$$\text{ord } f(a) \geq \min\{Kh, (K-1)h - L + M\},$$

where

$$L = \max_{1 \leq k \leq K} \left\{ \text{ord} \left(\prod_{\substack{j=1 \\ j \neq k}}^K (n_j - n_k) \right) \right\}.$$

Proof. One may follow the argument of the proof of [18, Lemma 1] without serious modification to accommodate arbitrary values of h in place of the special case $h = 1$. \square

We apply this conclusion to deduce a generalisation of [18, Lemma 2]. The argument of our proof is very similar to that of the latter, but differs in enough detail that a complete account seems warranted.

Lemma 2.2. *Suppose that p is a prime number and $h \in \mathbb{N}$. When $p = 2$, suppose in addition that $h \geq 3$. Let M be a positive integer, and let \mathcal{M} be a set of K integers in the interval $[M, 2M)$. Finally, suppose that there are N integers x_1, \dots, x_N , not all divisible by p , such that*

$$S_{h,m}(x_1, \dots, x_N) \equiv 0 \pmod{p^{(h+1)M}} \quad (m \in \mathcal{M}). \quad (2.1)$$

Then one has $N \geq p^{Kh}$.

Proof. Write $q = \phi(p^h)$ and suppose that $\mathcal{M} = \{m_1, \dots, m_K\}$. We begin by considering the situation in which p is an odd prime. Since x_n plays no role in the congruences (2.1) when $p|x_n$, we may suppose without loss that $(x_n, p) = 1$ for $1 \leq n \leq N$. Let g be a primitive root modulo p^2 , whence g is primitive for all powers of p . For each index n , there exists an integer a_n with $0 \leq a_n < \phi(p^{(h+1)M})$ having the property that $x_n \equiv g^{a_n} \pmod{p^{(h+1)M}}$. Put $f(z) = \sum_{n=1}^N z^{a_n}$, and note that $f(1) = N$. By the primitivity of g , one sees that for $1 \leq k \leq K$ the integers $n_k = g^{qm_k}$ are distinct and satisfy the congruence $n_k \equiv 1 \pmod{p^h}$. Furthermore, by hypothesis, one finds that $f(g^{qm}) \equiv 0 \pmod{p^{(h+1)M}}$ for $m \in \mathcal{M}$. We therefore deduce from Lemma 2.1 that

$$\text{ord } N = \text{ord } f(1) \geq \min\{Kh, (K-1)h - L + (h+1)M\}, \quad (2.2)$$

where

$$L = \max_{m \in \mathcal{M}} \text{ord} \left(\prod_{\substack{r \in \mathcal{M} \\ r \neq m}} (g^{qr} - g^{qm}) \right).$$

Next we observe that since $q = \phi(p^h)$, one has $\text{ord}(g^{qs} - 1) = h + \text{ord } s$ for every natural number s , whence $\text{ord}(g^{qr} - g^{qm}) = h + \text{ord}(r - m)$. But when

$m \in \mathcal{M}$, just as in the argument of the proof of [18, Lemma 2], a consideration of binomial coefficients reveals that

$$\text{ord} \left(\prod_{\substack{r \in \mathcal{M} \\ r \neq m}} (r - m) \right) \leq \text{ord} ((m - M)!(2M - m - 1)!) \leq \text{ord} ((M - 1)!).$$

Hence we obtain

$$L \leq (K - 1)h + \sum_{j=1}^{\infty} \left[\frac{M - 1}{p^j} \right] \leq (K - 1)h + \frac{M - 1}{p - 1}.$$

One therefore has

$$(K - 1)h - L + (h + 1)M \geq (h + 1 - 1/(p - 1))M \geq hM \geq hK.$$

Thus we conclude from (2.2) that $\text{ord } N \geq hK$, so that $p^{hK} | N$.

When $p = 2$, we modify the above argument by using the fact that the reduced residues are generated by -1 and 5 modulo any power of 2 . Thus, for each index n , there exists an integer a_n with $0 \leq a_n < \frac{1}{2}\phi(2^{(h+1)M})$ having the property that $x_n^2 \equiv 5^{a_n} \pmod{2^{(h+1)M}}$. Put $f(z) = \sum_{n=1}^N z^{a_n}$ and note once more that $f(1) = N$. For $1 \leq k \leq K$, the integers $n_k = 5^{m_k q/2}$ are distinct and satisfy the congruence $n_k \equiv 1 \pmod{2^h}$. Further, by hypothesis, for $1 \leq k \leq K$ one has

$$f(5^{m_k q/2}) = \sum_{n=1}^N x_n^{q m_k} \equiv 0 \pmod{2^{(h+1)M}}.$$

It therefore follows that $f(n_k) \equiv 0 \pmod{2^{(h+1)M}}$ for $1 \leq k \leq K$. As before, we find from Lemma 2.1 that the lower bound (2.2) holds, though in the present circumstances in which $p = 2$, one has $L \leq (K - 1)h + M - 1$. Consequently, we deduce on this occasion that since $h \geq 3$, one has

$$(K - 1)h - L + (h + 1)M \geq hM \geq hK.$$

Thus we conclude that $2^{hK} | N$, thereby completing the proof of the lemma. \square

We are now equipped to prove Theorem 1.2. Suppose if possible that $s < p^{rh}$ and that the equations (1.1) have a non-zero simultaneous solution $\mathbf{x} \in \mathbb{Q}_p^{W_s}$. By homogeneity, we may suppose that $\mathbf{x} \in \mathbb{Z}_p^{W_s}$, and further that $p \nmid x_{ju}$ for some indices j and u with $1 \leq j \leq s$ and $0 \leq u < W$. For some index l with $0 \leq l < W$, one has $p | x_{iu}$ for $u < l$ and $1 \leq i \leq s$, and further $p \nmid x_{jl}$ for some index j with $1 \leq j \leq s$. For each $m \in \mathcal{M}$, we thus have

$$S_{h,m}(x_{1u}, \dots, x_{su}) \equiv 0 \pmod{p^{\phi(p^h)M}} \quad (u < l).$$

On the other hand, since

$$\phi(p^h) - (h + 1)l \geq \phi(p^h) - (h + 1)(W - 1) \geq h + 1,$$

we find from (1.1) that there is a solution $\mathbf{x}_l \in \mathbb{Z}_p^s$ of the simultaneous congruences

$$S_{h,m}(x_{1l}, \dots, x_{sl}) \equiv 0 \pmod{p^{(h+1)M}} \quad (m \in \mathcal{M})$$

in which $p \nmid x_{jl}$ for some index j with $1 \leq j \leq s$. In particular, there is a non-zero s -tuple $\mathbf{y} \in \mathbb{Z}^s$ with $\mathbf{y} \equiv \mathbf{x}_l \pmod{p^{(h+1)M}}$ such that

$$S_{h,m}(\mathbf{y}) \equiv 0 \pmod{p^{(h+1)M}} \quad (m \in \mathcal{M}).$$

Since $s < p^{rh}$, it follows from Lemma 2.2 that one must have $p|y_i$, and hence also $p|x_{il}$, for $1 \leq i \leq s$. This contradicts our earlier assumption, and so we reach a contradiction. We are therefore forced to conclude that $s \geq p^{rh}$, and this completes the proof of Theorem 1.2.

REFERENCES

- [1] G. I. Arkhipov and A. A. Karatsuba, *Local representation of zero by a form*, Izv. Akad. Nauk SSSR Ser. Mat. **45** (1981), 948–961, 1198.
- [2] E. Artin, *The collected papers of Emil Artin*, Addison-Wesley, Reading, MA, 1965.
- [3] J. Ax and S. Kochen, *Diophantine problems over local fields, I*, Amer. J. Math. **87** (1965), 605–630.
- [4] S. S. Brown, *Bounds on transfer principles for algebraically closed and complete discretely valued fields*, Mem. Amer. Math. Soc. **15** (1978), no. 204, iv+92 pp.
- [5] W. D. Brownawell, *On p -adic zeros of forms*, J. Number Theory **18** (1984), 342–349.
- [6] J. Brüdern and H. Godinho, *On Artin's conjecture, I: Systems of diagonal forms*, Bull. London Math. Soc. **31** (1999), 305–313.
- [7] J. Brüdern and H. Godinho, *On Artin's conjecture, II: Pairs of additive forms*, Proc. London Math. Soc. (3) **84**, 513–538.
- [8] H. Davenport and D. J. Lewis, *Homogeneous additive equations*, Proc. Roy. Soc. Ser. A **274** (1963), 443–460.
- [9] V. B. Dem'yanov, *Pairs of quadratic forms over a complete field with discrete norm with a finite field of residue classes*, Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 307–324.
- [10] H. Hasse, *Darstellbarkeit von Zahlen durch quadratische Formen in einem beliebigen algebraischen Zahlkörper*, J. Reine Angew. Math. **153** (1924), 113–130.
- [11] M. P. Knapp, *Systems of diagonal equations over p -adic fields*, J. London Math. Soc. (2) **63** (2001), 257–267.
- [12] M. P. Knapp, *Diagonal equations of different degrees over p -adic fields*, Acta Arith. **126** (2007), 139–154.
- [13] M. P. Knapp, *On systems of diagonal forms*, J. Austral. Math. Soc. **82** (2007), 221–236.
- [14] M. P. Knapp, *On systems of diagonal forms, II*, Math. Proc. Cambridge Philos. Soc. **147** (2009), 31–45.
- [15] M. P. Knapp, *Pairs of additive forms of odd degrees*, Michigan Math. J. **61** (2012), 493–505.
- [16] S. Lang, *On quasi algebraic closure*, Ann. of Math. (2) **55** (1952), 373–390.
- [17] D. J. Lewis, *Cubic homogeneous polynomials over p -adic number fields*, Ann. of Math. (2) **56** (1952), 473–478.
- [18] D. J. Lewis and H. L. Montgomery, *On zeros of p -adic forms*, Michigan Math. J. **30** (1983), 83–87.
- [19] G. Terjanian, *Un contre-exemple à une conjecture d'Artin*, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A612.

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, CLIFTON,
BRISTOL BS8 1TW, UNITED KINGDOM

E-mail address: matdw@bristol.ac.uk