RATIONAL POINTS ON COMPLETE INTERSECTIONS OF HIGHER DEGREE, AND MEAN VALUES OF WEYL SUMS

PER SALBERGER AND TREVOR D. WOOLEY*

Abstract. We establish upper bounds for the number of rational points of bounded height on complete intersections. When the degree of the intersection is sufficiently large in terms of its dimension, and the contribution arising from appropriate linear spaces is removed, these bounds are smaller than those arising from the expectation of "square-root cancellation". In particular, there is a paucity of non-diagonal solutions to the equation $x_1^d + \cdots + x_s^d = x_{s+1}^d + \cdots + x_{2s}^d$ provided that $d \geqslant (2s)^{4s}$. There are consequences for the approximate distribution function of Weyl sums of higher degree, and also for quasi-diagonal behaviour in mean-values of smooth Weyl sums.

1. Introduction

Upper bounds for even moments of exponential sums lie at the heart of modern applications of the Hardy-Littlewood (circle) method to diagonal diophantine problems. By orthogonality, such moments count the number of integral solutions of associated symmetric diagonal equations with the variables constrained to lie in suitable boxes, and here one may hope to apply methods from arithmetic geometry to good effect. Hitherto, the latter techniques have been successfully applied to only the lowest moments, with the sixth moment of the classical Weyl sum of higher degree tackled within the past five years by Browning and Heath-Brown [4] (see [20] for the sharpest available conclusions). Our purpose in this paper is first to establish rather general estimates for the number of rational points of bounded height on complete intersections. We then apply these estimates to demonstrate that diagonal solutions dominate the asymptotic formula in arbitrarily high even moments of the classical Weyl sum over d^{th} powers, provided that one takes the degree d of the underlying exponential sum sufficiently large in terms of the order of the moment. The fine control provided by the associated asymptotic formulae is then exploited in order to establish quasi-diagonal behaviour in even moments of smooth Weyl sums, and also in the analysis of the distribution function of the underlying exponential sum.

2000 *Mathematics Subject Classification.* 11D72, 11G35, 14G05, 11E76, 11L15, 11P55.

Key words and phrases. Rational points, paucity problems, exponential sums.

*The second author was supported in part in the initial phases of this work by an NSF grant, and subsequently by a Royal Society Wolfson Research Merit Award. Research on this project was conducted at the Mathematical Sciences Research Institute at Berkeley, and at the Hausdorff Research Institute for Mathematics at Bonn.

In order to describe our main conclusions concerning rational points on hypersurfaces, we require some notation. When $V \subset \mathbb{P}^n$ is a quasi-projective variety defined over $\mathbb{Q}$, we denote the number of rational points of height at most B on V by $N(V;B)$. In this context, given a rational point $\mathbf{x} \in \mathbb{P}^n$, we define its height $H(\mathbf{x})$ by taking a primitive integral $(n+1)$-tuple $(x_0, x_1, \dots, x_n)$ representing $\mathbf{x}$, and then we put $H(\mathbf{x}) = \max\{|x_0|, \dots, |x_n|\}$. Our conclusions involve a certain exponent $\kappa = \kappa(d,k,m)$ which we define, for positive integers d, k and m with $k < m$, by

$$\kappa(d,k,m) = \sum_{r=k+1}^{m} (r+1)/\sqrt[r]{d}.$$

Following some preliminary geometric discussion in section 2, we establish in section 3 our most general conclusions concerning the number of rational points on hypersurfaces of higher degree. It may be helpful to recall at this point that homogeneous polynomials are referred to as *forms*, and that the singular locus of a hypersurface defined by a form $F(\mathbf{x})$ is the variety defined by the simultaneous vanishing of its partial derivatives. By convention, a non-singular projective hypersurface has singular locus of dimension -1.

Theorem 1.1. *Let n and d be integers with $n \geqslant 3$ and $d \geqslant 3$. Let $F(\mathbf{x}) \in \mathbb{Q}[x_1, \dots, x_n]$ be a form of degree d that defines a projective hypersurface $X \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ having a singular locus of dimension $\mathfrak{s}$. Write $k = [(n+\mathfrak{s}-1)/2]$, and let $U \subset X$ be the complement of all the projective k-planes in $\mathbb{P}^{n-1}_{\mathbb{Q}}$ contained in X. Then for each positive number ε, one has*

$$N(U;B) \ll_{d,n,\varepsilon} B^{k-1+2/\sqrt{3}+\kappa(d,k,n-2)+\varepsilon}.$$

This conclusion is a special case of Theorem 3.5 below, where we establish an analogous estimate for complete intersections. Notice that when $k \geqslant 1$ and $d \geqslant (2k+2)^{4k+2}$, then in the situation described in the statement of the above theorem, one has

$$\kappa(d,k,n-2) \leqslant \kappa(d,k,2k+1) < (k+1)\frac{2k+2}{(2k+2)^2} = \frac{1}{2}.$$

We therefore obtain the following simple corollary to Theorem 1.1.

Corollary 1.2. *Under the hypotheses of Theorem 1.1, the estimate*

$$N(U;B) \ll_d B^{\frac{1}{2}(n+\mathfrak{s}+1)-\frac{1}{3}}$$

holds provided that $d \geqslant (n+\mathfrak{s}+1)^{2n+2\mathfrak{s}}$.

If one interprets the quantity $N(U;B)$ using the language of exponential sums via the circle method, then provided that the degree is large enough in terms of the dimension, Corollary 1.2 yields an estimate consistent with the expectation of square-root cancellation, modified to reflect the dimension of the singular locus. In the case of a diagonal form, we may be both more explicit, and slightly more precise.

Theorem 1.3. *Suppose that d and s are natural numbers with $d \geqslant 2s-1 \geqslant 3$. Let $X \subset \mathbb{P}_{\mathbb{Q}}^{2s-1}$ be the non-singular hypersurface defined by the equation $a_1x_1^d + \cdots + a_{2s}x_{2s}^d = 0$, wherein the coefficients $a_1, \ldots, a_{2s}$ are fixed non-zero rational numbers. Define W to be the union of the closed subsets defined by the systems of binary equations*

$$a_{j_i}x_{j_i}^d + a_{k_i}x_{k_i}^d = 0 \quad (1 \leqslant i \leqslant s),$$

for partitions $\{1, \ldots, n\} = \{j_1, k_1\} \cup \cdots \cup \{j_s, k_s\}$ into s pairs $\{j, k\}$. In addition, let U be the open subset given by the complement of W within X. Then for each positive number ε, one has

$$N(U; B) \ll_{d,\varepsilon} B^{s-2+2/\sqrt{3}+\kappa(d,s-1,2s-2)+\varepsilon}.$$

If, in addition, one has $d \geqslant (2s-1)^2$, then

$$N(U; B) \ll_{d,\varepsilon} B^{s-1+\kappa(d,s-1,2s-2)+\varepsilon}.$$

In particular, the estimate $N(U; B) \ll_d B^{s-1/2}$ holds whenever $d \geqslant (2s)^{4s}$.

The special case of Theorem 1.3 equivalent to that in which $a_i = (-1)^i$ $(1 \leqslant i \leqslant 2s)$ yields an important corollary. When B is a positive number, let $M_{d,s}(B)$ denote the number of $(2s)$-tuples $(x_1, \ldots, x_{2s})$ of positive integers not exceeding B for which

$$x_1^d + \cdots + x_s^d = x_{s+1}^d + \cdots + x_{2s}^d. \tag{1.1}$$

In addition, write $T_s(B)$ for the number of $(2s)$-tuples $(x_1, \ldots, x_{2s})$ of positive integers not exceeding B for which the s-tuple $(x_1, \ldots, x_s)$ is a permutation of $(x_{s+1}, \ldots, x_{2s})$. Thus, in particular, one has $T_s(B) \sim s!B^s$.

Corollary 1.4. *Suppose that d and s are natural numbers with $d \geqslant 2s-1 \geqslant 3$. Then*

$$M_{d,s}(B) - T_s(B) \ll_{d,\varepsilon} B^{s-2+2/\sqrt{3}+\kappa(d,s-1,2s-2)+\varepsilon}.$$

If, in addition, one has $d \geqslant (2s-1)^2$, then one has

$$M_{d,s}(B) - T_s(B) \ll_{d,\varepsilon} B^{s-1+\kappa(d,s-1,2s-2)+\varepsilon}.$$

In particular, whenever $d \geqslant (2s)^{4s}$, one has $M_{d,s}(B) - T_s(B) \ll_d B^{s-1/2}$, and hence $M_{d,s}(B) \sim s!B^s$.

This corollary shows that there is a paucity of non-diagonal solutions to the diophantine equation (1.1) whenever $d \geqslant (2s)^{4s}$, extending previous such conclusions restricted to the case $s = 3$ due to Browning and Heath-Brown [4] and the first author [20]. In particular, the diagonal solutions dominate the solutions of (1.1) for $s \leqslant \left(\frac{1}{4} + o(1)\right) \log d / \log\log d$. The generalised ABC-conjecture[1] (see the epilogue of Schmidt [25] and Brownawell and Masser [3]) would yield such a conclusion for $s \ll d^{1/3}$ or thereabouts. We refer the reader to the appendix of [35] for a discussion of this issue, and its application to quasi-diagonal behaviour.

[1]named the "alphabet conjecture" by Pomerance

We next introduce the exponential sum

$$f_d(\alpha;B)=\sum_{1\leqslant x\leqslant B} e(\alpha x^d),$$

where, as usual, we write $e(z)$ for $e^{2\pi iz}$. On considering the underlying diophantine equation, it follows in particular from Corollary 1.4 that whenever s is a natural number and $d\geqslant (2s)^{4s}$, then for large B one has

$$\int_0^1 |f_d(\alpha;B)|^{2s}\,d\alpha = s!B^s+O_d(B^{s-1/2}). \tag{1.2}$$

By employing the work of Vaughan and the second author [32], conclusions of this type may be wrought to gain insight concerning the distribution function of the normalised Weyl sum

$$g_d(\alpha;B)=B^{-1/2}|f_d(\alpha;B)|$$

for $\alpha\in[0,1)$. Let $\chi_{\mathcal{A}}(x)$ denote the indicator function of the set $\mathcal{A}$, and let

$$\phi_d(\Lambda;B)=\int_0^1 \chi_{[0,\Lambda]}(g_d(\alpha;B))\,d\alpha.$$

In section 6, we obtain the following unconditional version of Theorem 1 of [32].

Theorem 1.5. *Suppose that d is a large natural number, and $\Lambda>0$. Then one has*

$$\begin{aligned}\phi_d(\Lambda;B)=\,&1-e^{-\Lambda^2}+O_d\left((\Lambda+\Lambda^{-1})B^{-1/2}\right)\\ &+O\left(\Lambda^{1/2}e^{-\Lambda^2/2}\left(\frac{\log\log d}{\log d}\right)^{1/4}+(\Lambda+\Lambda^{-1})\left(\frac{\log\log d}{\log d}\right)^{1/2}\right).\end{aligned}$$

The control on the distribution function of $g_d(\alpha)$ gained via Theorem 1.5 permits one to interpolate between the even moments of $f_d(\alpha;B)$. Thus, again in section 6, we derive the following consequence of Theorem 2 of [32].

Corollary 1.6. *Suppose that d is a large natural number and $0<s\leqslant \frac{1}{2}\log d/\log\log d$. Then one has*

$$\int_0^1 |f_d(\alpha;B)|^s\,d\alpha=\Gamma(s/2+1)(1+O(\omega_{s,d}))B^{s/2}+O_d(B^{(s-1)/2}), \tag{1.3}$$

where $\omega_{s,d}=1/\log\log d$. Moreover, when

$$0<s\leqslant \frac{1}{2}\left(\frac{\log d}{\log\log d}\right)-\left(\frac{\log d}{\log\log d}\right)^{3/4},$$

the asymptotic relation (1.3) *holds with*

$$\omega_{s,d}=(s+1)^{1/2}2^{s/2}\left(\frac{\log\log d}{\log d}\right)^{1/4}.$$

The second author has conjectured that whenever $d \geqslant 2$, one has

$$\lim_{B\to\infty} B^{-1/2} \int_0^1 |f_d(\alpha;B)|\, d\alpha = \sqrt{\pi}/2.$$

Unpublished computations of Doug Covert in the cases $d = 2$ and 3 provide evidence in favour of this conjecture. Immediate consequences of Corollary 1.6 are the relations

$$\lim_{d\to\infty} \limsup_{B\to\infty} B^{-1/2} \int_0^1 |f_d(\alpha;B)|\, d\alpha = \sqrt{\pi}/2$$

and

$$\lim_{d\to\infty} \liminf_{B\to\infty} B^{-1/2} \int_0^1 |f_d(\alpha;B)|\, d\alpha = \sqrt{\pi}/2.$$

We offer some further remarks concerning approximate distribution functions in section 6 below.

The final consequence of Corollary 1.4 which we discuss here concerns quasi-diagonal behaviour. We require some notation in order to describe our conclusions. When P and R are positive numbers, we denote by $\mathcal{A}(P,R)$ the set of R-smooth numbers of size at most P, that is

$$\mathcal{A}(P,R) = \{n \in [1,P]\cap\mathbb{Z} \,:\, p|n \text{ and } p \text{ prime} \Rightarrow p \leqslant R\}.$$

Let d be a fixed positive integer exceeding 2, and define the smooth Weyl sum $h(\alpha) = h_d(\alpha;P,R)$ by

$$h_d(\alpha;P,R) = \sum_{x\in\mathcal{A}(P,R)} e(\alpha x^d).$$

Further, when s is a positive real number, define the mean value $U_s(P,R)$ by

$$U_s(P,R) = \int_0^1 |h_d(\alpha;P,R)|^s\, d\alpha.$$

We shall say that an exponent $\mu_s = \mu_{s,d}$ is *permissible* whenever the exponent has the property that for each $\varepsilon > 0$, there is a positive number $\eta = \eta(\varepsilon,s,d)$ such that whenever $R \leqslant P^\eta$, one has $U_s(P,R) \ll_{\varepsilon,s,d} P^{\mu_{s,d}+\varepsilon}$. It is not difficult to show that permissible exponents $\mu_{s,d}$ exist with $\mu_{s,d} \leqslant s$ for each positive number s, and moreover that whenever $\mu_{s,d}$ is permissible, one necessarily has $\mu_{s,d} \geqslant \max\{s/2, s-d\}$.

Now consider a sequence $(\mathcal{S}_d)_{d=1}^\infty$ of finite sets of positive real numbers. We say that the mean value $U_s(P,R)$ exhibits *quasi-diagonal behaviour* for this sequence when there exists a set of real numbers $\{\delta_{s,d} \,:\, d\in\mathbb{N},\, s\in\mathcal{S}_d\}$ with the property that

$$\lim_{d\to\infty} \max_{s\in\mathcal{S}_d} \delta_{s,d} = 0,$$

and for each $d\in\mathbb{N}$ and $s\in\mathcal{S}_d$, the exponent $\mu_{s,d} = s/2 + \delta_{s,d}$ is permissible. The concept of quasi-diagonal behaviour was introduced first in the context of Vinogradov's mean value theorem by the second author [35], building on earlier work of Arkhipov and Karatsuba [1] and Tyrina [27]. So far as smooth Weyl sums are concerned, work of Vaughan [28] establishes that one may bound $\delta_{s,d}$ roughly in the shape

$$\delta_{s,d} \ll (d/s^2)^{-(\log(d/s^2))/(\log 16)+O(1)},$$

thereby establishing quasi-diagonal behaviour for $s = o(d^{1/2})$. In section 7 we establish estimates for permissible exponents conditional on paucity estimates of the type provided by Corollary 1.4.

Theorem 1.7. *Let d and u be large natural numbers with $2u \leqslant d$, and suppose that for some positive number λ, the relation*

$$M_{d,w}(B) - T_w(B) \ll_{d,w} B^{w-\lambda}$$

holds for every integral exponent w with $1 \leqslant w \leqslant 2u$. Then the exponent $\mu_{s,d} = s/2 + \delta_{s,d}$ is permissible for $4u < s \leqslant \min\{\sqrt{du}, \sqrt{d\lambda}\}$, where

$$\delta_{s,d} = \sqrt{d/s^2}\exp\left(u - \frac{4du^2}{es^2}\right).$$

It follows from Theorem 1.7 that whenever one is able to take u to be a function of d increasing to $+\infty$, then the mean value $U_s(P,R)$ exhibits quasi-diagonal behaviour for the sequence $(\mathcal{S}_d)_{d=2}^\infty$, with $\mathcal{S}_d$ equal to the interval $(0, \min\{\sqrt{du}, \sqrt{d\lambda}\}]$. By making use of Corollary 1.4, we obtain the following unconditional conclusion.

Corollary 1.8. *For each large integer d, the exponent $\mu_{s,d} = s/2 + \delta_{s,d}$ is permissible for $0 < s \leqslant \sqrt{d/2}$, where*

$$\delta_{s,d} = \begin{cases} 0, & \text{for } 0 < s \leqslant \frac{1}{2}\left(\frac{\log d}{\log\log d}\right), \\ \exp\left(-\frac{d(\log d)^2}{44s^2(\log\log d)^2}\right), & \text{for } \frac{1}{2}\left(\frac{\log d}{\log\log d}\right) < s \leqslant \sqrt{d/2}. \end{cases}$$

We note that the constants appearing in the above theorem and its corollary, even in the arguments of the exponential functions, can certainly be improved with greater effort. However, the qualitative features of these estimates would seem to be the best that are accessible to our methods. Hitherto, quasi-diagonal behaviour for the mean-value $U_s(P,R)$ had been established only for the sequence $(\mathcal{S}_d)_{d=2}^\infty$, with $\mathcal{S}_d = (0, 4e^{-1}d^{1/2}]$ $(d \geqslant 2)$. Indeed, Theorem 1.3 of [36] establishes a conclusion similar to that of Corollary 1.8 in which the range of s is restricted to $4 \leqslant s \leqslant 4e^{-1}d^{1/2}$, and the permissible value of $\delta_{s,d}$ is given by

$$\delta_{s,d} = \frac{8d^{1/2}}{es}\exp\left(-\frac{16d}{e^2s^2}\right).$$

Thus, provided that one may take λ to be sufficiently large, the conclusion of Theorem 1.7 extends the domain in which quasi-diagonal behaviour is known to hold. Unfortunately, the conclusion of Corollary 1.4 shows only that $\lambda = 1/2$ is permissible, and the methods of this paper would not yield a permissible value exceeding 1. Thus the unconditional corollary to the theorem provides permissible exponents $\mu_{s,d}$ substantially closer to $s/2$ than previously available, but does not extend the domain of quasi-diagonal behaviour. This is a topic to which we intend to return on a future occasion.

As experts will instantly recognise, the new permissible exponents supplied by Corollary 1.8 permit sharper conclusions than available hitherto concerning the number of integers represented as the sum of s dth powers. Let $\mathcal{N}_{s,d}(X)$ denote the number of integers n with $1 \leqslant n \leqslant X$ that are represented as the sum of s dth powers of positive integers.

Corollary 1.9. *Suppose that d is a large natural number and*

$$1 \leqslant s \leqslant \left\lceil \frac{1}{4}\left(\frac{\log d}{\log\log d}\right)\right\rceil .$$

Then one has

$$\mathcal{N}_{s,d}(X) \sim (s!)^{-1}\frac{\Gamma(1+1/d)^s}{\Gamma(1+s/d)}X^{s/d}.$$

For comparison, the corollary to the main theorem of Browning and Heath-Brown [4] establishes such a conclusion in the special case $s = 3$, and earlier results in the case $s = 2$ may be found in work of Hooley (see Theorem 2 of [16], and also [17]).

Corollary 1.10. *Suppose that d is a large natural number and $1 \leqslant s \leqslant \sqrt{d/12}$. Then one has $\mathcal{N}_{s,d}(X) \gg_d X^{s/d-\nu_{s,d}}$, where*

$$\nu_{s,d} = \frac{1}{d}\exp\left(-\frac{d(\log d)^2}{176 s^2(\log\log d)^2}\right).$$

Earlier work of the second author (see Theorem 1.3 of [36]) would yield an analogous conclusion with $\nu_{s,d}$ replaced by the exponent

$$4(d^{1/2}es)^{-1}\exp(-4d/(es)^2).$$

This in turn improved on the aforementioned work of Vaughan [28]. Work predating the advent of Vaughan's new iterative method, meanwhile, was cruder. See, in particular, Chapter 6 of [29] for the output of Davenport's methods.

It follows from Corollary 1.4 that almost all solutions of the equation (1.1) are diagonal whenever $d \geqslant (2s)^{4s}$. In particular, when B is large enough in terms of d and s, aside from a set of integers numbering at most $O(B^{s-1/2})$, the integers n with $1 \leqslant n \leqslant B^d$ represented in the shape $x_1^d + x_2^d + \cdots + x_s^d$, are represented essentially uniquely (which is to say, uniquely up to permutations of variables). It therefore follows that in these circumstances one has

$$\mathcal{N}_{s,d}(X) = \sum_{\substack{1\leqslant x_1<x_2<\cdots<x_s\leqslant X^{1/d}\\ x_1^d+x_2^d+\cdots+x_s^d\leqslant X}} 1 + o(X^{s/d}).$$

The asymptotic formula in Corollary 1.9 therefore follows by approximating the above sum via an integral, and making use of familiar Beta-function formulae. The conclusion of Corollary 1.10, meanwhile, follows from a familiar application of Cauchy's inequality, making use of Corollary 1.8. It is conjectured that when $d > 2$, one has $\mathcal{N}_{s,d}(X) \gg_d X^{s/d}$ for $1 \leqslant s \leqslant d$.

Throughout, we reserve the letters ε and η to denote sufficiently small positive numbers, and we use P to denote a large positive number sufficiently large in terms of the ambient parameters (usually ε, η, d and s). The implicit constants in Vinogradov's well-known notation $\ll$ and $\gg$ will depend at most on d, s, ε and η, unless listed explicitly as a subscript to the notation. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Note that the "value" of ε may consequently change from statement to statement. Finally, when β is a real number, we write $[\beta]$ for the largest integer not exceeding β, and $\lceil \beta \rceil$ for the least integer no smaller than β.

The authors are grateful to the referee for useful comments.

2. Geometric preliminaries

In deriving our estimates for the number of rational points of bounded height, it is critical that good control be exercised over the potential existence of subvarieties of small degree. We establish in this section the geometric tools required in our subsequent deliberations. In this context, we note that it may be useful for the reader to refer to Lemma 5.1 below for a characterisation of the singular locus of a complete intersection, as well as for a means of calculating some of the relevant data associated with this intersection.

Theorem 2.1. *Let n be a positive even integer, and write $s = n/2$. Suppose that $F(x_1, \dots, x_n)$ is a form of degree $d \geqslant 3$ over an algebraically closed field of characteristic 0 which defines a non-singular projective hypersurface $X \subset \mathbb{P}^{n-1}$. Then the following hold.*
(a) There exists a number N depending only on d and n with the property that the number of projective $(s-1)$-planes on $X \subset \mathbb{P}^{n-1}$ is at most N.
(b) Suppose that $d \geqslant n-1$ and $F(x_1, \dots, x_n) = a_1 x_1^d + \cdots + a_n x_n^d$, where $a_1 \dots a_n \neq 0$. Then for any projective $(s-1)$-plane Π on $X \subset \mathbb{P}^{n-1}$, a partition of $\{1, 2, \dots, n\}$ into s pairs $\{j, k\}$ exists with the property that the corresponding s binary forms $a_j x_j^d + a_k x_k^d$ vanish on Π.

Proof. The conclusion of part (a) follows directly from the appendix by Starr to the paper [6]. Turning to part (b) of the theorem, we begin by observing that in view of the substitution $y_i = a_i^{1/d} x_i$ $(1 \leqslant i \leqslant n)$, there is no loss in supposing that $a_i = 1$ for $1 \leqslant i \leqslant n$. Next, let $F(X)$ denote the Fano scheme of lines on X, and likewise let $F(\Pi)$ denote the Fano scheme of lines on Π. Then $F(\Pi)$ is a closed subscheme of $F(X)$ of dimension $n-4$. But all irreducible components of $F(X)$ are of dimension $n-4$ (see the discussion on page 54 of [8]), and thus we see that $F(\Pi)$ must be an irreducible component of $F(X)$. Then on making use of the explicit description of all the components of $F(X)$ given in the reference just cited, it is apparent that there is a partition of $\{1, 2, \dots, n\}$ into s pairs $\{j, k\}$ such that the corresponding s binary forms $x_j^d + x_k^d$ vanish on Π. This completes the proof of the theorem. $\square$

We shall also need the following consequence of the Lefschetz theorem (see Theorem 5.2.6 of [9], and also section 3.1 of [19]). We refer the reader to page 19 of [11] for an introduction to some of the basic properties of singular homology and cohomology of which we will make use.

Lemma 2.2. *Let $i : X \to \mathbb{P}^{n-1}_{\mathbb{C}}$ be the inclusion of a non-singular complete intersection of dimension m and degree d. Let $\mathrm{cl}_{\mathbb{P}^{n-1}}(X) \in H_{2m}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$ be the homology class of $X \subset \mathbb{P}^{n-1}_{\mathbb{C}}$, and let r be an integer in the interval $(m/2, m]$. Then any element in the image of $i_* : H_{2r}(X, \mathbb{Z}) \to H_{2r}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$ may be written as a cap-product $\omega \cap \mathrm{cl}_{\mathbb{P}^{n-1}}(X)$, for some $\omega \in H^{2(m-r)}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$. In particular, any such element is divisible by d in $H_{2r}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$.*

Proof. It follows from Poincaré duality that the cap product $\cap$ with $\mathrm{cl}_X(X) \in H_{2m}(X, \mathbb{Z})$ gives an isomorphism from $H^{2(m-r)}(X, \mathbb{Z})$ to $H_{2r}(X, \mathbb{Z})$, and from Lefschetz's theorem that the functorial map

$$i^* : H^{2(m-r)}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z}) \to H^{2(m-r)}(X, \mathbb{Z})$$

is an isomorphism. We may therefore represent any element in $H_{2r}(X, \mathbb{Z})$ as $i^*\omega \cap \mathrm{cl}_X(X)$ for some $\omega \in H^{2(m-r)}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$. The first assertion of the lemma now follows from the projection formula, since $i_*(i^*\omega \cap \mathrm{cl}_X(X)) = \omega \cap i_*\mathrm{cl}_X(X) = \omega \cap \mathrm{cl}_{\mathbb{P}^{n-1}}(X)$. The final assertion of the lemma follows on observing that $\mathrm{cl}_{\mathbb{P}^{n-1}}(X) = d\,\mathrm{cl}_{\mathbb{P}^{n-1}}(H)$ for any m-plane $H \subset \mathbb{P}^{n-1}_{\mathbb{C}}$. □

We are now equipped to say something concerning the degrees of subvarieties of complete intersections, provided in the first instance that the intersection is non-singular.

Theorem 2.3. *Let $X \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ be a non-singular complete intersection of dimension m. Then the degree of any closed equi-dimensional subscheme Z of X of dimension $r > m/2$ is divisible by the degree of X.*

Proof. Let $\mathrm{cl}_X(z)$ denote the homology class in $H_{2r}(X, \mathbb{Z})$ of the effective r-cycle z defined by Z, and let $\mathrm{cl}_{\mathbb{P}^{n-1}}(z)$ denote the corresponding homology class in $H_{2r}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$. In addition, let $\Pi \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ be an $(n-1-r)$-plane and write $\mathrm{cl}_{\mathbb{P}^{n-1}}(\Pi)$ for its homology class in $H_{2(n-1-r)}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$. By Poincaré duality, there is a unique element τ in $H^{2r}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$ with the property that $\mathrm{cl}_{\mathbb{P}^{n-1}}(\Pi) = \tau \cap \mathrm{cl}_{\mathbb{P}^{n-1}}(\mathbb{P}^{n-1}_{\mathbb{C}})$. The degree of Z may now be defined topologically as $\tau \cap \mathrm{cl}_{\mathbb{P}^{n-1}}(z) \in H_0(X, \mathbb{Z}) = \mathbb{Z}$. But $\mathrm{cl}_{\mathbb{P}^{n-1}}(z) = i_*(\mathrm{cl}_X(z))$, and hence from Lemma 2.2 we see that $\mathrm{cl}_{\mathbb{P}^{n-1}}(z)$ is divisible by $\deg X$ in $H_{2r}(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathbb{Z})$. We therefore deduce that $\tau \cap \mathrm{cl}_{\mathbb{P}^{n-1}}(z) \in H_0(X, \mathbb{Z})$ is also divisible by $\deg X$, and hence conclude that the degree of Z is divisible by the degree of X. This completes the proof of the theorem. □

We finish this section by extending the previous conclusion to singular complete intersections.

Theorem 2.4. *Let $X \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ be a complete intersection of dimension $m \geqslant 2$ having a singular locus of dimension $\mathfrak{s}$. Then the degree of any closed equi-dimensional subscheme Z of X of dimension $r > (m + \mathfrak{s} + 1)/2$ is divisible by the degree of X.*

Proof. We establish the conclusion of the theorem by induction on $\mathfrak{s}$. When $\mathfrak{s} = -1$ the intersection X is non-singular, and so the desired conclusion follows from the previous theorem. Suppose then that $\mathfrak{s} \geqslant 0$, and that the conclusion of the theorem has been established for intersections having a singular locus of dimension smaller than $\mathfrak{s}$. Since $(m+\mathfrak{s}+1)/2 < r \leqslant m$, one has $\mathfrak{s} \leqslant m-2$, and so it follows from the arguments applied in the proof of Proposition 5.4(b) of [22] that X is necessarily integral. In addition, since $\mathfrak{s} \geqslant 0$, we may suppose that $r \geqslant 2$. We may therefore apply the theorem of Bertini (see Theorem 17.16 of [13]) to any integral component Z_i of Z. In this way, we find that there is a hyperplane $H \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ with the property that the singular locus of $X \cap H$ is of dimension at most $\mathfrak{s}-1$, and such that $Z_i \cap H$ is integral. But then, by applying the inductive hypothesis to $X \cap H \subset \mathbb{P}^{n-1}_{\mathbb{C}}$, we see that $\deg(X \cap H)$ divides $\deg(Z_i \cap H)$, whence $\deg X$ divides $\deg Z_i$. This completes the proof of the theorem. □

3. Rational points of bounded height

We now exploit the geometrical preparation of the previous section within the framework of Heath-Brown's determinant method, and thereby obtain the principal conclusions of this paper concerning estimates for the number of rational points of bounded height. In this section, the word *subvariety* will always mean *integral closed subscheme*. Thus, a subvariety of $\mathbb{P}^{n-1}_{\mathbb{Q}}$ will always be geometrically reduced, but not necessarily geometrically irreducible. The following result is essentially due to Broberg, and is established by means of the aforementioned method of Heath-Brown [15].

Theorem 3.1. *Let $Z \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ be a subvariety of dimension r and degree d. Then there exists a set of $O_{d,n,\varepsilon}(B^{(r+1)/\sqrt[r]{d}+\varepsilon})$ hypersurfaces over $\mathbb{Q}$, having degree $O_{d,n,\varepsilon}(1)$ and not containing Z, such that any rational point of height at most B on Z lies on one of these hypersurfaces.*

Proof. It follows from Lemmata 1.3 and 1.4 of [21] that Z is defined by forms of degree $O_{d,n}(1)$. The desired conclusion is therefore a consequence of Theorem 1 of [2]. □

We next provide a uniform bound for the number of rational points of bounded height on subvarieties.

Theorem 3.2. *Let $Z \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ be a subvariety of dimension r and degree d. Then*
(a) $N(Z;B) \ll_{d,n} B^{r+1}$,
(b) $N(Z;B) \ll_{d,n,\varepsilon} B^{r+\varepsilon}$, when $d=2$ or $d \geqslant 4$,
(c) $N(Z;B) \ll_{d,n,\varepsilon} B^{r-1+2/\sqrt{3}+\varepsilon}$, when $d=3$.

Proof. We apply induction on r. The case $r=0$ is trivial. Suppose then that $\dim Z = r > 0$, and that the estimate (a) holds for all varieties of dimension smaller than r. If Z is not geometrically integral, then the rational points on Z lie on $O_{d,n}(1)$ subvarieties of dimension smaller than r (see the proof of Theorem 2.1 of [20] for the necessary ideas). By applying the estimate (a) to varieties of dimension smaller than r, therefore, we find that $N(Z;B) \ll_{d,n} B^r$ when Z is not geometrically integral. When Z is geometrically integral, on the other hand, the estimate (a) may be found in Theorem 1 of [5]. By induction, therefore, the estimate (a) holds for every dimension r. The upper bounds (b) and (c), meanwhile, follow from Theorem 2 of [15] when $d = 2$, from Theorem 0.7 of [23] when $d = 3, 4, 5$, and from Corollary 2 of [7] when $d \geqslant 6$. This completes the proof of the theorem. $\square$

We remark that the paper [23] is not yet in print. In the absence of the estimates stemming from that paper, one could instead employ the weaker bound

$$N(Z;B) \ll_{d,n,\varepsilon} B^{r-3/4+5/(3\sqrt{3})+\varepsilon}$$

that is established in Corollary 2 of [7] when $d \geqslant 2$. The substitution of the latter bound in place of that of parts (b) and (c) of Theorem 3.2 would lead to conclusions analogous to those contained in this paper, save that the fraction $1/3$ would be replaced by $1/4$ in Corollary 1.2, with similar adjustments elsewhere.

We now show how to cover rational points of bounded height with subvarieties of codimension 1.

Lemma 3.3. *Let n be a positive integer and $X \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ be an m-dimensional complete intersection of degree d having a singular locus of dimension $\mathfrak{s}$. Let $Z \subset X$ be a closed equi-dimensional subscheme of dimension $r > (m+\mathfrak{s}+1)/2$ and degree e. Then*

(a) one has $d|e$, and

(b) there exists a set of $O_{e,n,\varepsilon}(B^{(r+1)/\sqrt[r]{d}+\varepsilon})$ subvarieties of dimension $r-1$ and degree $O_{e,n,\varepsilon}(1)$ with the property that any rational point on Z of height at most B lies on one of these subschemes.

Proof. The assertion (a) follows from Theorem 2.4, since both the dimension and the degree of Z remain unchanged under base field extensions. Turning our attention next to part (b), we begin by noting that there is no loss of generality in supposing Z to be integral, since there are at most e irreducible components of Z. Next we apply Theorem 3.1 in combination with the conclusion of part (a) of the present lemma. Thus we find that any rational point on Z of height at most B lies on at least one of a certain set of $O_{e,n,\varepsilon}(B^{(r+1)/\sqrt[r]{d}+\varepsilon})$ hypersurfaces W_i over $\mathbb{Q}$ of degree $O_{e,n,\varepsilon}(1)$ not containing Z. It therefore suffices to prove that the rational points of height at most B on $Z \cap W_i$ lie on $O_{e,n,\varepsilon}(1)$ subvarieties Y_{ij} of dimension $r-1$ and degree $O_{e,n,\varepsilon}(1)$. In order to establish this, let Y_{ij} be the irreducible components of $Z \cap W_i$ with their reduced scheme structures. These components are all of dimension $r-1$, and it follows from the theorem of Bézout (see Example 8.4.6 of [10]) that $\sum_j \deg Y_{ij} \leqslant (\deg Z)(\deg W_i)$. Consequently, there are $O_{e,n,\varepsilon}(1)$ components Y_{ij} of $Z \cap W_i$, and each of these components has degree $O_{e,n,\varepsilon}(1)$. This completes the proof of the lemma. $\square$

By repeated application of Lemma 3.3, we are able to cover rational points of bounded height with subschemes of more general codimension.

Lemma 3.4. *Let n be a positive integer and $X \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ be an m-dimensional complete intersection of degree d having a singular locus of dimension $\mathfrak{s}$. Let k be an integer with $\frac{1}{2}(m+\mathfrak{s}) \leqslant k < m$. Then there is a set of $O_{d,n,\varepsilon}(B^{\kappa(d,k,m)+\varepsilon})$ subvarieties of dimension k, with degree $O_{d,n,\varepsilon}(1)$, such that any point on X of height at most B lies on one of these subvarieties.*

Proof. Let $S(X;B)$ denote the set of rational points on X of height at most B. We show by induction that when $1 \leqslant l \leqslant m-k$, then any point in $S(X;B)$ lies on one at least of a set of $O_{d,n,\varepsilon}(B^{\kappa(d,m-l,m)+l\varepsilon})$ subvarieties of dimension $m-l$, with degree $O_{d,n,\varepsilon}(1)$. When $l=1$, this assertion follows from an application of Lemma 3.3(b) with $Z=X$. Suppose then that this assertion has been established for $1 \leqslant l < m-k$. We apply Lemma 3.3 to all of the $O_{d,n,\varepsilon}(B^{\kappa(d,m-l,m)+l\varepsilon})$ subvarieties of dimension $m-l$, with degree $O_{d,n,\varepsilon}(1)$, supplied by the inductive hypothesis. Notice here that $m-l>k$, whence $m-l>\frac{1}{2}(m+\mathfrak{s}+1)$. We deduce that any point in $S(X;B)$ lies on one at least of a set of

$$O_{d,n,\varepsilon}(B^{\kappa(d,m-l,m)+(m-l+1)/\sqrt[m-l]{d}+(l+1)\varepsilon}) = O_{d,n,\varepsilon}(B^{\kappa(d,m-l-1,m)+(l+1)\varepsilon})$$

subvarieties of dimension $m-l-1$, with degree $O_{d,n,\varepsilon}(1)$. Thus the inductive hypothesis holds with l replaced by $l+1$, and by induction the above hypothesis holds with $l=m-k$. Consequently, any point in $S(X;B)$ lies on one at least of a set of $O_{d,n,\varepsilon}(B^{\kappa(d,k,m)+(m-k)\varepsilon})$ subvarieties of dimension k, with degree $O_{d,n,\varepsilon}(1)$. This completes the proof of the lemma. □

Our extensive preparations now complete, we establish the most general estimate for the number of rational points of bounded height of this paper.

Theorem 3.5. *Let n be a positive integer and $X \subset \mathbb{P}^{n-1}_{\mathbb{Q}}$ be an m-dimensional complete intersection of degree d having a singular locus of dimension $\mathfrak{s}$. Let $k=[(m+\mathfrak{s}+1)/2]$, and let $U \subset X$ be the complement of all the projective k-planes in $\mathbb{P}^{n-1}_{\mathbb{Q}}$ contained in X. Then for each positive number ε, one has*

$$N(U;B) \ll_{d,n,\varepsilon} B^{k-1+2/\sqrt{3}+\kappa(d,k,m)+\varepsilon}.$$

Proof. According to Lemma 3.4, the rational points of X of height at most B lie on one at least of a set of $O_{d,n,\varepsilon}(B^{\kappa(d,k,m)+\varepsilon})$ subvarieties of dimension k, with degree $O_{d,n,\varepsilon}(1)$. Any one of these rational points lying on such a subvariety of degree 1 is not counted by $N(U;B)$, since the subvariety in question is a projective k-plane. Meanwhile, the number of rational points of height at most B lying on any one of the remaining subvarieties is $O_{d,n,\varepsilon}(B^{k-1+2/\sqrt{3}+\varepsilon})$, as a consequence of Theorem 3.2. The upper bound for $N(U;B)$ claimed in the theorem now follows on multiplying this estimate by our earlier bound on the number of subvarieties of dimension k. □

The conclusion of Theorem 1.1 is just the special case of Theorem 3.5 in which $m=n-2$. We remark also that as a consequence of Theorem 2.1(a), one finds that U is non-empty for any non-singular projective hypersurface $X \subset \mathbb{P}^{n-1}$ of positive even dimension and degree at least three. The first conclusion of Theorem 1.3 follows from Theorem 2.1(b) in combination with Theorem 1.1, whilst the second follows in like manner on making use of the refinement made available in Theorem 4.4 below. Amongst the first four results presented in the introduction, it now remains only to establish Corollary 1.4. Let $M^*_{d,s}(B)$ denote the number of $(2s)$-tuples $(x_1,\dots,x_{2s})$ of positive integers not exceeding B, subject to the coprimality condition $(x_1,\dots,x_{2s})=1$, and satisfying the equation (1.1). Also, let $T^*_s(B)$ denote the number of $(2s)$-tuples $(x_1,\dots x_{2s})$ of positive integers not exceeding B for which $(x_1,\dots x_s)$ is a permutation of $(x_{s+1},\dots,x_{2s})$ and $(x_1,\dots,x_{2s})=1$. Then one has the transparent relation

$$M_{d,s}(B)-T_s(B)=\sum_{1\leqslant f\leqslant B}(M^*_{d,s}(B/f)-T^*_s(B/f)).$$

We now apply Theorem 1.3 to estimate $M^*_{d,s}(B/f)-T^*_s(B/f)$. Put $a_i=1$ for $(1\leqslant i\leqslant s)$, and $a_i=-1$ for $s+1\leqslant i\leqslant 2s$. Then in the statement of Theorem 1.3, we find that $N(U;B)$ counts primitive solutions of the equation (1.1) with $|x_i|\leqslant B$ $(1\leqslant i\leqslant 2s)$, in which $(x_1,\dots,x_s)$ is not a permutation of $(x_{s+1},\dots,x_{2s})$. Consequently, one has

$$M^*_{d,s}(B/f)-T^*_s(B/f)\leqslant N(U;B/f)\ll_{d,\varepsilon}(B/f)^{s-2+2/\sqrt{3}+\kappa(d,s-1,2s-2)+\varepsilon},$$

with the sharper estimate

$$M^*_{d,s}(B/f)-T^*_s(B/f)\leqslant N(U;B/f)\ll_{d,\varepsilon}(B/f)^{s-1+\kappa(d,s-1,2s-2)+\varepsilon}$$

available when $d\geqslant (2s-1)^2$. The conclusion of Corollary 1.4 therefore follows at once when $s\geqslant 2$, as we may suppose to be the case.

4. Subvarieties of degree at most three on Fermat hypersurfaces

Our objective in this section is a refinement of Theorem 3.5, in which the exponent $k-1+2/\sqrt{3}+\kappa(d,k,m)+\varepsilon$ is replaced by $k+\kappa(d,k,m)+\varepsilon$, valid whenever the complete intersection in question has a diagonal equation of suitably high degree as one of its defining equations. This we achieve by considering more carefully the subvarieties of X that might potentially exist with degree 3. By means of an application of Green's theorem (see [12]), we are able to demonstrate that such subvarieties are forced to have structure that severely restricts their contribution within the associated counting function $N(U;B)$. The details are not entirely pedestrian. We begin by considering curves on diagonal hypersurfaces.

Lemma 4.1. *Suppose that n and d are integers with $n\geqslant 3$ and $d\geqslant (n-1)^2$. Let K be an algebraically closed field of characteristic 0, and let $(a_1,\dots,a_n)\in (K^\times)^n$. In addition, let $X\subset\mathbb{P}^{n-1}$ be the hypersurface given by the equation $a_1x_1^d+\cdots+a_nx_n^d=0$, and let $\mathcal{C}$ be a closed integral curve on X of degree* *at most* 3*. Then, for some proper non-empty subset I of $\{1,\ldots,n\}$, the form $\sum_{i\in I} a_i x_i^d$ vanishes on $\mathcal{C}$.*

Proof. By an appropriate coordinate change, there is no loss of generality in supposing that $a_1 = \cdots = a_n = 1$. Moreover, by the Lefschetz principle, we may suppose without loss that $K = \mathbb{C}$. Let $\widetilde{\mathcal{C}}$ be the normalisation of $\mathcal{C}$. Then $\widetilde{\mathcal{C}}$ is of genus 0 or 1. We therefore obtain the conclusion of the lemma by applying Green's theorem (see [12] and page 205 of [18]) to the universal covering space of $\widetilde{\mathcal{C}}$ when the genus of $\widetilde{\mathcal{C}}$ is 1, and to the complement of a point when $\widetilde{\mathcal{C}} = \mathbb{P}^1$. □

We next extend our conclusions on curves to analogous results for higher dimensional varieties.

Lemma 4.2. *Suppose that n and d are integers with $n \geqslant 3$ and $d \geqslant (n-1)^2$. Let K be an algebraically closed field of characteristic 0, and let $(a_1,\ldots,a_n) \in (K^\times)^n$. In addition, let $X \subset \mathbb{P}^{n-1}$ be the hypersurface given by the equation $a_1x_1^d + \cdots + a_nx_n^d = 0$, and let Y be a closed integral subvariety on X of positive dimension and of degree at most 3. Then, for some proper non-empty subset I of $\{1,\ldots,n\}$, the form $\sum_{i\in I} a_i x_i^d$ vanishes on Y.*

Proof. We use induction with respect to the dimension of Y, applying the previous lemma in order to establish the base of the induction wherein the dimension of Y is 1. Suppose then that the dimension of Y exceeds 1, and assume that the conclusion of the lemma holds for subvarieties of degree at most 3 of lower dimension than that of Y. Since Y is integral, by Bertini's theorem there exists a non-empty open subset $\mathcal{O}$ of the dual projective space $(\mathbb{P}^{n-1})^\vee$ such that, for each hyperplane $H \subset \mathbb{P}^{n-1}$ represented by a point in $\mathcal{O}$, the intersection $Y \cap H$ is integral. Consequently, by the inductive hypothesis for the intersection corresponding to each such hyperplane H, there exists a non-empty proper subset $I(H)$ of $\{1,\ldots,n\}$ such that $\sum_{i\in I(H)} a_i x_i^d$ vanishes on $Y \cap H$. Let $V \subseteq Y$ be the union of all the sections $Y \cap H$, with hyperplanes H parameterised by points in $\mathcal{O}$. Also, when $H \in \mathcal{O}$, let Y_H denote the closed subset of Y defined by the equation $\sum_{i\in I(H)} a_i x_i^d = 0$. Then

$$V \subseteq \bigcup_{H\in\mathcal{O}} Y_H \subseteq Y,$$

and V is dense in Y. Therefore, since Y is irreducible and there are only finitely many subsets $I(H)$ and associated closed subsets Y_H of Y, we conclude that $Y = Y_H$ for some hyperplane $H \in \mathcal{O}$. But then one concludes that $\sum_{i\in I(H)} a_i x_i^d$ vanishes on Y, thereby confirming the inductive hypothesis for Y. This completes the proof of the lemma. □

By repeated application of the previous lemma, we obtain a detailed structure theorem for subvarieties of degree at most 3 contained in diagonal hypersurfaces.

Theorem 4.3. *Suppose that n and d are integers with $n \geqslant 3$ and $d \geqslant (n-1)^2$. Let K be an algebraically closed field of characteristic 0, and let $(a_1,\ldots,a_n) \in$*

$(K^\times)^n$. In addition, let $X \subset \mathbb{P}^{n-1}$ be the hypersurface given by the equation $a_1x_1^d + \cdots + a_nx_n^d = 0$, and let Y be a closed integral subvariety on X of positive dimension and of degree at most 3. Then there exists a partition $I_1 \cup \cdots \cup I_l$ of $\{1, \ldots, n\}$ with the following properties.
(a) The form $\sum_{i \in I_j} a_i x_i^d$ vanishes on Y for $1 \leqslant j \leqslant l$.
(b) Let $\Lambda_j \subset \mathbb{P}^{n-1}$ be the linear subspace defined by the vanishing of all the variables x_k for $k \in \{1, \ldots, n\} \setminus I_j$. Then for $1 \leqslant j \leqslant l$, the intersection $Y \cap \Lambda_j$ is either empty or a point.

Proof. We consider the set of partitions $I_1 \cup \cdots \cup I_l$ of $\{1, \ldots, n\}$ satisfying (a). Since the trivial partition $I_1 = \{1, \ldots, n\}$ satisfies (a), this set of partitions is non-empty. We now introduce a partial ordering between such partitions as follows. If $I_1 \cup \cdots \cup I_l = \{1, \ldots, n\}$ and $J_1 \cup \cdots \cup J_k = \{1, \ldots, n\}$ are two such partitions, then we say that $\{J_1, \ldots, J_k\}$ is a refinement of $\{I_1, \ldots, I_l\}$ when, for $1 \leqslant i \leqslant k$, the set J_i is a subset of one of the sets $I_1, \ldots, I_l$. Next, let $\{I_1, \ldots, I_l\}$ be a partition of $\{1, \ldots, n\}$ which satisfies the condition (a), and of which no refinement satisfies (a). Such a partition exists, since the partition into n subsets, each containing just one element, does not satisfy (a). We now seek to prove that $Y \cap \Lambda_j$ is a point for any index $j \in \{1, \ldots, l\}$ for which $Y \cap \Lambda_j$ is non-empty.

In order to achieve the latter goal, we begin by observing that, on considering a suitable permutation of $(1, \ldots, n)$, there is no loss of generality in supposing that $I_j = \{1, \ldots, m\}$ for some positive integer m with $m \leqslant n$. Let U be the open subset of Y such that $(x_1, \ldots, x_m) \neq (0, \ldots, 0)$. Then U contains $Y \cap \Lambda_j$. Let $\pi : U \to \mathbb{P}^{m-1}$ be the morphism which sends $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_m)$. Then the closure Y' of $\pi(U)$ is an irreducible closed subvariety which lies on the hypersurface $X' \subset \mathbb{P}^{m-1}$ defined by the vanishing of the form $a_1x_1^d + \cdots + a_mx_m^d$. If $\dim Y' \geqslant 1$, then $m \geqslant 3$ and $\deg Y' \leqslant \deg Y \leqslant 3$. For such a subset I_j, we may therefore apply the previous lemma, with X' in place of X and Y' in place of Y, in order to conclude that there is a proper non-empty subset I of I_j with the property that the form $\sum_{i \in I} a_i x_i^d$ vanishes on Y', and hence also on Y. The condition (a), therefore, will still hold if we replace the partition $\{I_1, \ldots, I_l\}$ by the finer partition wherein I_j is decomposed into I and $I_j \setminus I$. This conclusion contradicts our assumption that no such refinement is possible, and thus Y' must be a point. We therefore find that $Y \cap \Lambda_j$ is a point for any index $j \in \{1, \ldots, l\}$ having the property that $Y \cap \Lambda_j$ is non-empty, and this completes the proof of the theorem. $\square$

We remark that, if we order the sets in the partition $I_1 \cup \cdots \cup I_l$ in such a way that $Y \cap \Lambda_j$ is a point y_j for $1 \leqslant j \leqslant k$, and empty for $k+1 \leqslant j \leqslant l$, then the subvariety Y will be contained in the projective linear $(k-1)$-subspace Π spanned by $y_1, \ldots, y_k$. Since there are $O_n(1)$ possible partitions of $\{1, \ldots, n\}$ of the above type, there can be at most $O_n(1)$ such projective $(k-1)$-subspaces. Also, since $\mathrm{card}(I_j) \geqslant 2$ for $1 \leqslant j \leqslant k$, we find that $2k \leqslant n$ and that $\dim Y < \dim \Pi \leqslant \frac{1}{2} \dim X$. Note also that, in view of the conclusions (a) and (b) of Theorem 4.3, one has $\Pi \subset X$.

Finally, we deduce the anticipated analogue of Theorem 3.5.

Theorem 4.4. *Suppose that n and D are integers with $n \geqslant 3$ and $D \geqslant (n-1)^2$. Let $(a_1, \ldots, a_n) \in (\mathbb{Q}^\times)^n$, and let $X \subset \mathbb{P}_{\mathbb{Q}}^{n-1}$ be an m-dimensional complete intersection of degree d in which one of the defining equations is $a_1 x_1^D + \cdots + a_n x_n^D = 0$. Suppose that X has a singular locus of dimension $\mathfrak{s}$. Let $k = [(m + \mathfrak{s} + 1)/2]$, and let $U \subset X$ be the complement of all the projective k-planes in $\mathbb{P}_{\mathbb{Q}}^{n-1}$ contained in X. Then for each positive number ε, one has*

$$N(U; B) \ll_{d,n,\varepsilon} B^{k+\kappa(d,k,m)+\varepsilon} + B^{n/2-1+\kappa(d,k,m)+\varepsilon}.$$

Proof. As in the proof of Theorem 3.5, we begin by applying Lemma 3.4 to show that the rational points of X of height at most B lie on one at least of a set of $O_{d,n,\varepsilon}(B^{\kappa(d,k,m)+\varepsilon})$ subvarieties of dimension k, with degree $O_{d,n,\varepsilon}(1)$. The rational points lying on such subvarieties of degree 1 are again not counted by $N(U; B)$, since these subvarieties are projective k-planes. Meanwhile, by Theorem 3.2, the number of rational points of height at most B lying on any one of the remaining subvarieties is $O_{d,n,\varepsilon}(B^{k+\varepsilon})$, except possibly when the subvariety has degree 3. We claim that such subvarieties together contribute at most $O(B^{n/2-1+\kappa(d,k,m)+\varepsilon})$ rational points to the number counted by $N(U; B)$. Granted this claim, the upper bound for $N(U; B)$ claimed in the theorem now follows on multiplying the previous estimate by our earlier bound on the number of subvarieties of dimension k.

Let us return to the claim concerning subvarieties of degree 3. By considering such a subvariety Y as a subvariety of the hypersurface defined by the equation $a_1 x_1^D + \cdots + a_n x_n^D = 0$, we find from Theorem 4.3 and the ensuing discussion that the cubic subvariety Y has dimension at most $n/2 - 2$. We therefore deduce from Theorem 3.2(a) that the number of rational points of height at most B lying on Y is $O(B^{n/2-1})$. Multiplying this bound by our earlier bound on the number of subvarieties of dimension k, we deduce that the contribution arising from such cubic subvarieties to $N(U; B)$ is at most $O(B^{n/2-1+\kappa(d,k,m)+\varepsilon})$, and this confirms our earlier claim. $\square$

We remark that the estimate obtained in the final paragraph of the proof of Theorem 4.4 is crude. While it suffices for our purposes within this paper, substantial refinement is plainly possible.

5. Systems of diagonal equations

The conclusion of Theorem 3.5 provides a means of establishing the paucity of non-diagonal solutions in systems of diagonal diophantine equations of large degree. Although somewhat technical in nature, these conclusions are of some interest in their own right, and moreover have potential applications within the theory of exponential sums over smooth numbers (see [33], [37]). We therefore permit ourselves some space to discuss the most immediate applications of Theorem 3.5 in this arena.

We begin by providing a simple criterion for determining the degree of a complete intersection.

Lemma 5.1. *Let $X \subset \mathbb{P}_K^{n-1}$ be the closed subscheme defined by the forms $F_i(x_1, \dots, x_n)$ $(1 \leqslant i \leqslant t)$ with coefficients in the field K. Let $\mathcal{S} \subset X$ be the closed subset defined by the vanishing of all the $(t \times t)$-determinants*

$$\det \left(\frac{\partial F_i}{\partial x_{j_l}} \right)_{1 \leqslant i,l \leqslant t},$$

in which $1 \leqslant j_1 < j_2 < \dots < j_t \leqslant n$. Then

$$\dim X \leqslant \max\{n - t - 1, \dim \mathcal{S}\}.$$

In particular, if $\dim \mathcal{S} \leqslant n - t - 1$, then X is a complete intersection with singular locus $\mathcal{S}$, and $\deg X = (\deg F_1)(\deg F_2) \dots (\deg F_t)$.

Proof. Let $x \in X \setminus \mathcal{S}$ and $A = \mathcal{O}_{X,x}$ be the local ring at x, and let $\mathfrak{m} = \mathfrak{m}_{X,x}$ be its maximal ideal. Then by Chapter 1 of [14], one has

$$\text{Krull } \dim A \leqslant \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = n - t - 1.$$

Hence $\dim(X \setminus \mathcal{S}) \leqslant n - t - 1$, and the desired conclusions follow. □

Next we introduce an analogue of the counting function $M_{d,s}(B)$. Consider natural numbers $d_1, \dots, d_t$ with

$$1 \leqslant d_1 < d_2 < \dots < d_t. \tag{5.1}$$

When B is a positive number, let $M_{\mathbf{d},s}(B)$ denote the number of $(2s)$-tuples $(x_1, \dots, x_{2s})$ of positive integers not exceeding B for which

$$\sum_{i=1}^{s} x_i^{d_j} = \sum_{i=s+1}^{2s} x_i^{d_j} \quad (1 \leqslant j \leqslant t). \tag{5.2}$$

Theorem 5.2. *Let s and t be natural numbers with $s \geqslant t + 1 \geqslant 3$, and let $d_1, \dots, d_t$ be natural numbers satisfying the condition (5.1) with $d_t \geqslant 2s - 1$. Then one has*

$$M_{\mathbf{d},s}(B) - T_s(B) \ll_{\mathbf{d},\varepsilon} B^{s-2+2/\sqrt{3}+\kappa(\widetilde{d},s-1,2s-t-1)+\varepsilon},$$

where $\widetilde{d} = d_1 d_2 \dots d_t$. If, in addition, one has $d_t \geqslant (2s - 1)^2$, then

$$M_{\mathbf{d},s}(B) - T_s(B) \ll_{\mathbf{d},\varepsilon} B^{s-1+\kappa(\widetilde{d},s-1,2s-t-1)+\varepsilon}.$$

In particular, whenever $\widetilde{d} \geqslant (2s-t)^{4s-2t}$, one has $M_{\mathbf{d},s}(B) = s!B^s + O(B^{s-1/2})$, and so there is a paucity of non-diagonal solutions to the diophantine system (5.2).

Proof. By restricting attention to the equation in (5.2) of degree d_t alone, one finds from Theorem 2.1(b) that the only $(s-1)$-planes on the complete intersection defined by (5.2) are those corresponding to the diagonal solutions counted by $T_s(B)$. Let $X \subset \mathbb{P}_{\mathbb{Q}}^{2s-1}$ denote the closed subscheme defined by the forms defining the system (5.2). Consider the closed subset $\mathcal{S} \subset X$ defined by the vanishing of all the $(t \times t)$-determinants $\det(x_{j_l}^{d_i - 1})_{1 \leqslant i,l \leqslant t}$, in which $1 \leqslant j_1 < j_2 < \dots < j_t \leqslant 2s$. The elements $\mathbf{x}$ of $\mathcal{S}$ may be classified according to the dimension of the linear space spanned by the column vectors $(x_j^{d_i - 1})_{1 \leqslant i \leqslant t}$ for $1 \leqslant j \leqslant 2s$. This space must have affine dimension at most $t - 1$ for every element $\mathbf{x}$ of $\mathcal{S}$, for if the dimension were larger, then one could find a non-vanishing $(t\times t)$-determinant $\det(x_{j_l}^{d_i-1})_{1\leqslant i,l\leqslant t}$, contradicting the definition of $\mathcal{S}$.

Let m be an integer with $1\leqslant m\leqslant t-1$, consider indices j_l $(1\leqslant l\leqslant m)$ with

$$1\leqslant j_1<j_2<\cdots<j_m\leqslant 2s, \tag{5.3}$$

and suppose that the column vectors $(x_{j_l}^{d_i-1})_{1\leqslant i\leqslant t}$ are linearly independent for $1\leqslant l\leqslant m$. Write $\mathcal{T}_m(\mathbf{j})$ for the set of points $\mathbf{x}\in\mathcal{S}$ satisfying the property that for $1\leqslant j\leqslant 2s$, all of the column vectors $(x_j^{d_i-1})_{1\leqslant i\leqslant t}$ belong to the linear space spanned by the above vectors[2], and let $\mathcal{S}_m$ denote the union of the sets $\mathcal{T}_m(\mathbf{j})$ over all choices of $\mathbf{j}$ satisfying (5.3). Then it is apparent that $\mathcal{S}$ is the union of $\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_{t-1}$. Moreover, for $1\leqslant m\leqslant t-1$, the set $\mathcal{T}_m(\mathbf{j})$ is determined by the non-vanishing of at least one $(m\times m)$-determinant involving the variables $x_{j_1},\dots,x_{j_m}$, together with the vanishing of all $((m+1)\times(m+1))$-determinants obtained by adjoining another variable x_j with $j\notin\{j_1,\dots,j_m\}$. Here, the determinants in question are of submatrices of the matrix

$$(x_j^{d_i-1})_{\substack{1\leqslant i\leqslant t\\ 1\leqslant j\leqslant 2s}}.$$

It follows that each x_j with $j\notin\{j_1,\dots,j_m\}$ satisfies a non-trivial polynomial equation determined by $x_{j_1},\dots,x_{j_m}$, and hence that $\mathcal{S}_m$ has affine dimension at most m. In this way, we may conclude that the projective dimension of $\mathcal{S}$ is at most $t-2$.

At this point we may conclude that the dimension of $\mathcal{S}$ is at most $2s-t-1$ provided only that $s\geqslant t$, and this is already ensured by the hypotheses of the theorem. Then from Lemma 5.1 we find that X is a complete intersection of degree $d_1d_2\dots d_t=\widetilde{d}$, with a singular locus of dimension at most $t-2$. Put

$$k=\left[\frac{(2s-t-1)+(t-2)+1}{2}\right]=s-1,$$

and let $U\subset X$ denote the complement of all the projective k-planes on $X\subset\mathbb{P}_{\mathbb{Q}}^{2s-1}$. Then Theorem 3.5 delivers the estimate

$$N(U;B)\ll_{\widetilde{d},s,\varepsilon} B^{k-1+2/\sqrt{3}+\kappa(\widetilde{d},k,2s-t-1)+\varepsilon}.$$

The first estimate of the theorem now follows on applying an argument paralleling that employed in the proof of Corollary 1.4 at the end of section 3. The second estimate of the theorem follows in like manner, though employing Theorem 4.4 in place of Theorem 3.5.

For the final conclusion of the theorem, we merely note that when $\widetilde{d}\geqslant(2s-t)^{4s-2t}$, then

$$\kappa(\widetilde{d},s-1,2s-t-1)\leqslant(s-t)\frac{2s-t}{(2s-t)^2}=\frac{\frac{1}{2}(2s-t)-\frac{1}{2}t}{2s-t}.$$

[2]Equivalently, $\mathcal{T}_m(\mathbf{j})$ is the set of points $\mathbf{x}\in\mathcal{S}$ satisfying the property that the column vectors $(x_{j_l}^{d_i-1})_{1\leqslant i\leqslant t}$, for $1\leqslant l\leqslant m$, form a basis for the column space of the vectors $(x_j^{d_i-1})_{1\leqslant i\leqslant t}$ $(1\leqslant j\leqslant 2s)$.

Then since by hypothesis $t \geqslant 2$, one finds that $\kappa(\widetilde{d}, s-1, 2s-t-1) < \frac{1}{2}$, and this completes the proof of Theorem 5.2. □

We note that Steinig [26] has shown that when a t-tuple $\mathbf{d}$ satisfies (5.1) and $s = t$, then all the solutions of (5.2) counted by $M_{\mathbf{d},s}(B)$ are diagonal, which is to say that $M_{\mathbf{d},s}(B) = T_s(B)$. Hence the condition $s \geqslant t+1$ in the statement of the theorem does not amount to a serious restriction. The paucity of non-diagonal solutions in systems of symmetric diagonal equations has been investigated by several authors, but essentially only in the case wherein $s = t+1$. In the latter situation the second author [34] has established a quasi-paucity estimate of the shape $M_{\mathbf{d},t+1}(B) \ll_{\mathbf{d},t,A} B^{t+1}(\log B)^A$ for a suitable $A = A(\mathbf{d}, t) > 0$. There are also strong estimates for the difference $M_{\mathbf{d},t+1}(B) - T_{t+1}(B)$ when $\mathbf{d}$ is equal to $(1, 2, \ldots, t)$ or $(1, 2, \ldots, t-1, t+1)$ (see [31]). A number of related results can also be found in [30]. The vast majority of the remaining literature concerns the situation with $t = 2$ and $s = 3$, and here one has $M_{\mathbf{d},3}(B) - T_3(B) = o(B^3)$ whenever $d_2 > d_1 \geqslant 1$ and $d_2 \geqslant 3$ (see work of the first author [22] for details and also a summary of earlier results).

In order to illustrate the kind of conclusions now made available through the medium of Theorem 5.2, we point out that straightforward calculations reveal that

$$M_{\mathbf{d},s}(B) = s!B^s + O(B^{s-1/3})$$

when:

(i) $\mathbf{d} = (1, 2, \ldots, t-1, d)$ and $s = t+1$, provided that $d \geqslant (et)^{t+6}$;

(ii) $\mathbf{d} = (d, d+1, \ldots, d+t-1)$ and $s = t+1$, provided that t is large and one has $d \geqslant (1 + o(1))(t+2)^2$;

(iii) $\mathbf{d} = (d, 2d, \ldots, td)$ and $s \leqslant 2t$, provided that $d \geqslant (5t)^5$;

(iv) any $\mathbf{d}$ satisfying (5.1) provided that $s < (\frac{1}{4} + o(1)) \log \widetilde{d} / \log\log \widetilde{d}$.

It is unfortunate that our methods shed no new light in the special case of Vinogradov's mean value theorem, wherein we have $\mathbf{d} = (1, 2, \ldots, t)$ and $s \geqslant t+1$.

We finish this section by remarking that analogous conclusions with coefficients different from ± 1 can be obtained with no greater effort, and indeed that systems of diagonal equations of the same degree can also be successfully investigated using the methods of this section.

6. Approximate distribution functions

Our conclusions concerning the approximate distribution function of the normalised Weyl sum $g_d(\alpha; B)$, and the allied consequences for the moments of $f_d(\alpha; B)$, are immediate from the main theorems of Vaughan and the second author [32]. We therefore provide only an abbreviated discussion of the proofs of Theorem 1.5 and Corollary 1.6.

The starting point for the application of Theorem 1 of [32] is the hypothesis that the formula (1.2) holds for $s = 0, 1, \ldots, n$. One is then able to infer that

$$\phi_d(\Lambda; B) = 1 - e^{-\Lambda^2} + O(\Lambda^{1/2} e^{-\Lambda^2/2} n^{-1/4} + (\Lambda + \Lambda^{-1}) n^{-1/2}) + O_n((\Lambda + \Lambda^{-1}) B^{-1/2}). \tag{6.1}$$

But Corollary 1.4 ensures that this hypothesis holds whenever $d \geqslant (2n)^{4n}$. When d is large, the latter conditions are satisfied on taking

$$n = \left\lceil \frac{1}{4} \left(\frac{\log d}{\log \log d} \right) \right\rceil . \tag{6.2}$$

In this way, therefore, the conclusion of Theorem 1.5 is obtained by substituting (6.2) into (6.1).

With the aforementioned hypothesis, it is a consequence of Theorem 2 of [32] that when $0 < s \leqslant 2n - n^{2/3}$, one has

$$\int_0^1 |g_d(\alpha; B)|^s \, d\alpha = \Gamma(s/2 + 1) \left(1 + O((s+1)^{1/2} 2^{s/2} n^{-1/4})\right) + O_n(B^{-1/2}),$$

and that when $0 \leqslant s \leqslant 2n$, then

$$\int_0^1 |g_d(\alpha; B)|^s \, d\alpha = \Gamma(s/2 + 1) \left(1 + O(1/\log(2n))\right) + O_n(B^{-1/2}).$$

Both conclusions of Corollary 1.6 therefore follow on noting that the choice for n given by (6.2) is permissible, and then recalling that

$$|f_d(\alpha; B)| = |g_d(\alpha; B)| B^{1/2}.$$

Theorem 1.5 shows that when d is large enough in terms of Λ, and B is sufficiently large in terms of d and Λ, then $\phi_d(\Lambda; B) \sim 1 - e^{-\Lambda^2}$. The underlying proof may be extended to exploit the structural features of solutions provided via Theorem 1.3, and this theme we explore more thoroughly within a forthcoming paper [24]. As an illustration of the kind of conclusion made available through this circle of ideas, consider non-zero integers a and b. When a/b is not the dth power of any rational number, the joint cumulative distribution function

$$\phi_d(\Lambda, \mathrm{M}; B) = \int_0^1 \chi_{[0,\Lambda]}(g_d(a\alpha; B)) \chi_{[0,\mathrm{M}]}(g_d(b\alpha; B)) \, d\alpha$$

may be shown to satisfy $\phi_d(\Lambda, \mathrm{M}; B) \sim (1 - e^{-\Lambda^2})(1 - e^{-\mathrm{M}^2})$. Here, the asymptotic relation is claimed to hold when d is large enough in terms of Λ and M, and B is sufficiently large in terms of d, Λ and M. A moment's reflection reveals that this cumulative distribution function is the same as that arising from two independent variables, as in

$$\int_0^1 \int_0^1 \chi_{[0,\Lambda]}(g_d(\alpha; B)) \chi_{[0,\mathrm{M}]}(g_d(\beta; B)) \, d\alpha \, d\beta,$$

and thus $g_d(a\alpha)$ and $g_d(b\alpha)$ are asymptotically uncorrelated for almost all α.

Full details of the arguments underlying the above claim in a rather wider setting will be found in our forthcoming work [24]. For now we content ourselves with a sketch of the proof. Following the discussion of §2 of [32], our starting point is the definite integral

$$\int_0^\infty \frac{\sin(\xi y)}{y}\,dy = \begin{cases} \pi/2, & \text{when } \xi > 0,\\ 0, & \text{when } \xi = 0,\\ -\pi/2, & \text{when } \xi < 0.\end{cases}$$

When x is a positive real number with $x \ne \Lambda$, one obtains

$$\chi_{[0,\Lambda]}(x) = \frac{2}{\pi}\int_0^\infty y^{-1}\sin(y\Lambda)\cos(yx)\,dy.$$

Formally speaking, therefore, one finds that

$$\begin{aligned}&\int_0^1 \chi_{[0,\Lambda]}(g(a\alpha))\chi_{[0,\mathrm{M}]}(g(b\alpha))\,d\alpha\\ &\quad = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(y\Lambda)\sin(z\mathrm{M})}{yz}\int_0^1 \cos(yg(a\alpha))\cos(zg(b\alpha))\,d\alpha\,dy\,dz.\end{aligned} \tag{6.3}$$

Here, and in what follows, we find it convenient to abbreviate $g_d(\alpha;B)$ to $g(\alpha)$.

We next observe that, on considering the underlying diophantine equation, it follows from Theorem 1.3 that whenever d is sufficiently large in terms of s and t, and B is large enough in terms of d, s and t, then

$$\int_0^1 |g(a\alpha)^{2s}g(b\alpha)^{2t}|\,d\alpha = s!t! + O_{d,s,t}(B^{-1/2}). \tag{6.4}$$

Then on applying the power series expansion for cosine together with (6.4), it follows from (6.3) and the formula

$$m! = \int_0^\infty v^m e^{-v}\,dv$$

that

$$\phi_d(\Lambda,\mathrm{M};B) \sim \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{\sin(y\Lambda)\sin(z\mathrm{M})}{yz} I(y)I(z)\,dy\,dz,$$

where we have written

$$I(r) = \int_0^\infty e^{-u}\cos(ru^{1/2})\,du.$$

Consequently, one finds that

$$\phi_d(\Lambda,\mathrm{M};B) \sim \int_0^\infty\int_0^\infty J(\Lambda,v)J(\mathrm{M},w)e^{-v-w}\,dv\,dw,$$

where

$$J(\mathrm{N},u) = \frac{2}{\pi}\int_0^\infty x^{-1}\sin(x\mathrm{N})\cos(xu^{1/2})\,dx.$$

We hence deduce that

$$\phi_d(\Lambda,\mathrm{M};B)\sim\int_0^{\Lambda^2}\int_0^{\mathrm{M}^2}e^{-v-w}\,dv\,dw=(1-e^{-\Lambda^2})(1-e^{-\mathrm{M}^2}).$$

The extended argument of §2 of [32] can be adapted so as to justify the conclusion of the previous paragraph with error terms. In our forthcoming work [24], more general consequences are established, with rigorous treatment of error terms, and an analysis is also given of the situation wherein a/b is a dth power of a rational number.

7. Quasi-diagonal behaviour

With the proof of Corollary 1.4 in hand, we are equipped in this section to discuss mean value estimates for smooth Weyl sums. The sharpest conclusions are made available through the application of the methods underlying "breaking convexity" (see [36]), and consequently entail some notational overhead. Recall the discussion in the preamble to the statement of Theorem 1.7. When M is a positive number with $1\leqslant M\leqslant P$, define the exponential sum $F(\alpha)=F(\alpha;P,M,R)$ by

$$F(\alpha;P,M,R)=\sum_{M<u\leqslant MR}\ \sum_{\substack{1\leqslant y<x\leqslant P\\ x\equiv y\ (\mathrm{mod}\ u^d)}}e(\alpha u^{-d}(x^d-y^d)).$$

We write $H=PM^{-d}$ in order to assist with concision.

The starting point for our analysis is the following auxiliary mean value estimate.

Lemma 7.1. *Let d be a large positive integer, let ρ be a positive integer with $1\leqslant\rho\leqslant d$, and suppose that for some positive number λ, the relation*

$$M_{d,s}(B)-T_s(B)\ll_{d,s}B^{s-\lambda}$$

holds for every integral exponent s with $1\leqslant s\leqslant\rho$. Then for each integral exponent τ with $1\leqslant\tau\leqslant\rho/2$, one has

$$\int_0^1|F(\alpha;P,M,R)|^{2\tau}\,d\alpha\ll(MR)^{2\tau}((PH)^\tau+P^{2\tau-\lambda}).\tag{7.1}$$

Proof. We imitate the argument of the proof of Lemma 2.2 of [35] so far as is possible. We begin by observing that the mean value on the left hand side of (7.1) is bounded above by the number of solutions of the diophantine equation

$$\sum_{i=1}^{\tau}u_i^{-d}(x_i^d-y_i^d)=\sum_{i=\tau+1}^{2\tau}u_i^{-d}(x_i^d-y_i^d),$$

with $M<u_i\leqslant MR$, $1\leqslant y_i<x_i\leqslant P$ and $x_i\equiv y_i\ (\mathrm{mod}\ u_i^d)$ $(1\leqslant i\leqslant 2\tau)$. Given any such solution $\mathbf{x},\mathbf{y},\mathbf{u}$, for each index i there exists an integer h_i with

$1 \leqslant h_i < Pu_i^{-d} < H$ for which $x_i = y_i + h_i u_i^d$. On considering the underlying diophantine equations, therefore, we obtain the upper bound

$$\int_0^1 |F(\alpha; P, M, R)|^{2\tau}\, d\alpha \leqslant \int_0^1 \left| \sum_{M<u\leqslant MR} G_u(\alpha; P, H) \right|^{2\tau} d\alpha,$$

in which we have written

$$G_u(\alpha; P, H) = \sum_{1\leqslant y\leqslant P} \sum_{\substack{1\leqslant h\leqslant H\\ y+hu^d\leqslant P}} e\left(\alpha u^{-d}((y+hu^d)^d - y^d)\right).$$

A swift application of Hölder's inequality consequently yields the estimate

$$\int_0^1 |F(\alpha; P, M, R)|^{2\tau}\, d\alpha \ll (MR)^{2\tau-1} \sum_{M<u\leqslant MR} K_\tau(u), \tag{7.2}$$

where

$$K_\tau(u) = \int_0^1 |G_u(\alpha; P, H)|^{2\tau}\, d\alpha. \tag{7.3}$$

By orthogonality, the mean value on the right hand side of (7.3) counts the number of solutions of the diophantine equation

$$\sum_{i=1}^{\tau}((y_i + h_i u^d)^d - y_i^d) = \sum_{i=1}^{\tau}((z_i + g_i u^d)^d - z_i^d), \tag{7.4}$$

with $1 \leqslant y_i, z_i \leqslant P$ and $1 \leqslant g_i, h_i \leqslant H$ subject to $y_i + h_i u^d \leqslant P$ and $z_i + g_i u^d \leqslant P$ $(1 \leqslant i \leqslant \tau)$. Let $\mathcal{K}_1(u)$ denote the number of such solutions counted by $K_\tau(u)$ in which

$$(y_1 + h_1 u^d, \dots, y_\tau + h_\tau u^d, z_1, \dots, z_\tau) \tag{7.5}$$

is *not* a permutation of

$$(z_1 + g_1 u^d, \dots, z_\tau + g_\tau u^d, y_1, \dots, y_\tau), \tag{7.6}$$

and let $\mathcal{K}_2(u)$ denote the corresponding number of solutions in which the former is a permutation of the latter. Then one has

$$K_\tau(u) = \mathcal{K}_1(u) + \mathcal{K}_2(u). \tag{7.7}$$

Consider first any solution $\mathbf{y}, \mathbf{z}, \mathbf{g}, \mathbf{h}$ of (7.4) counted by $\mathcal{K}_1(u)$. Since

$$\max_{1\leqslant i\leqslant\tau} \{y_i, z_i, y_i + h_i u^d, z_i + g_i u^d\} \leqslant P,$$

one finds that $\mathcal{K}_1(u)$ is bounded above by $M_{d,2\tau}(P) - T_{2\tau}(P)$. On making use of the hypothesis of the statement of the lemma, therefore, we see that

$$\mathcal{K}_1(u) \ll P^{2\tau-\lambda}. \tag{7.8}$$

Next consider a solution $\mathbf{y}, \mathbf{z}, \mathbf{g}, \mathbf{h}$ of (7.4) counted by $\mathcal{K}_2(u)$. By relabelling indices if necessary, we may suppose that

$$y_\tau = \min_{1\leqslant i\leqslant\tau} y_i \quad \text{and} \quad z_\tau = \min_{1\leqslant i\leqslant\tau} z_i.$$

A comparison of (7.5) and (7.6) reveals that y_τ and z_τ must be equal. Let $\mathcal{K}_3(u)$ denote the number of such solutions of (7.4) counted by $\mathcal{K}_2(u)$ in which $g_\tau = h_\tau$, and let $\mathcal{K}_4(u)$ denote the corresponding number of solutions in which instead $g_\tau < h_\tau$. Then by symmetry, one has

$$\mathcal{K}_2(u) \ll \mathcal{K}_3(u) + \mathcal{K}_4(u). \tag{7.9}$$

For a solution $\mathbf{y}, \mathbf{z}, \mathbf{g}, \mathbf{h}$ of (7.4) counted by $\mathcal{K}_3(u)$, the final terms in the sums on the left and right hand sides of (7.4) cancel. On considering the underlying diophantine equation, one thus obtains the upper bound

$$\mathcal{K}_3(u) \leqslant PH \int_0^1 |G_u(\alpha; P, H)|^{2\tau-2}\, d\alpha. \tag{7.10}$$

Given such a solution counted by $\mathcal{K}_4(u)$, on the other hand, we may relabel $z_\tau + g_\tau u^d$ as y_τ, and in this way the equation (7.4) may be rewritten as

$$\sum_{i=1}^{\tau} ((y_i + h_i u^d)^d - y_i^d) = \sum_{i=1}^{\tau-1} ((z_i + g_i u^d)^d - z_i^d).$$

On accounting for the multiplicity with which the new variable y_τ is represented in the above shape, and considering the underlying diophantine equation, we arrive at the estimate

$$\mathcal{K}_4(u) \ll H \int_0^1 G_u(\alpha; P, H) |G_u(\alpha; P, H)|^{2\tau-2}\, d\alpha. \tag{7.11}$$

On collecting (7.9), (7.10) and (7.11), and then applying Hölder's inequality, we may now infer that

$$\begin{aligned}\mathcal{K}_2(u) &\ll PH \int_0^1 |G_u(\alpha)|^{2\tau-2}\, d\alpha + H \int_0^1 |G_u(\alpha)|^{2\tau-1}\, d\alpha \\ &\ll PH \left(\int_0^1 |G_u(\alpha)|^{2\tau}\, d\alpha \right)^{1-1/\tau} + H \left(\int_0^1 |G_u(\alpha)|^{2\tau}\, d\alpha \right)^{1-1/(2\tau)},\end{aligned}$$

where, for concision, we have abbreviated $G_u(\alpha; P, H)$ to $G_u(\alpha)$. Inserting the latter bound into (7.7) and recalling (7.3) and (7.8) leads us to the inequality

$$K_\tau(u) \ll P^{2\tau-\lambda} + PHK_\tau(u)^{1-1/\tau} + HK_\tau(u)^{1-1/(2\tau)},$$

whence we obtain

$$K_\tau(u) \ll P^{2\tau-\lambda} + (PH)^\tau + H^{2\tau}.$$

The conclusion of the lemma now follows on substituting this estimate into (7.2), and recalling that $H \leqslant P$. □

We note that the argument applied in the proof of Lemma 7.1 is susceptible to improvement so far as the exponent of MR is concerned. In the application to quasi-diagonal behaviour, however, such elaborations lead only to microscopic improvements in the ensuing mean value estimates.

By incorporating the estimate (7.1) into the efficient differencing process of §3 of [36], one is able to derive new permissible exponents. The next lemma provides a simplified conclusion useful in our later deliberations.

Lemma 7.2. *Under the hypotheses on d and ρ from the statement of Lemma 7.1, let σ and t be real numbers with $0 < t \leqslant 1$ and $\sigma + 2t > 2\rho$. Also, let τ be an integer with $1 \leqslant \tau \leqslant \rho/2$, and write $w = t/(2\tau)$ and $\nu = \sigma/(1-w)$. Finally, suppose that $\mu_{\sigma,d}$ and $\mu_{\nu,d}$ are permissible exponents, and put*

$$\theta = \frac{(1-w)\mu_\nu - \mu_\sigma}{\frac{1}{2}dt + (1-w)\mu_\nu - \mu_\sigma}.$$

Then provided that $0 \leqslant \theta \leqslant \lambda/(\tau d)$, the exponent $\mu_{\sigma+2t,d}$ is permissible, where

$$\mu_{\sigma+2t} = \mu_\sigma(1-\theta) + t + \sigma\theta.$$

Proof. Suppose that σ and ν satisfy the hypotheses of the statement of the lemma, and write $s = \sigma + 2t$. Take ϕ to be a real number with $0 \leqslant \phi \leqslant 1/d$ to be chosen later, and write

$$M = P^\phi, \quad H = PM^{-d} \quad \text{and} \quad Q = PM^{-1}. \tag{7.12}$$

We follow the argument of §4 of [36] surrounding equations (4.1), (4.2) and (4.3) of that paper, modifying the underlying application of Lemma 3.4 of [36] by replacing the second and fourth moments of equation (3.18) of [36] with the moment of order 2τ estimated in Lemma 7.1 above. In this way, the estimate for the quantity $\mathcal{U}_t$ occurring in Lemma 3.4 of [36] is replaced in the present context by

$$\mathcal{U}_t \ll D^{dt}\left((MR)^{2\tau}((PH)^\tau + P^{2\tau-\lambda})\right)^{t/(2\tau)} Q^{(1-w)\mu_\nu}.$$

Provided that

$$(PH)^\tau \geqslant P^{2\tau-\lambda}, \tag{7.13}$$

we find that our choice for ϕ is determined from the equation

$$(PM)^t Q^{\mu_\sigma} = (PH)^{t/2} M^t Q^{(1-w)\mu_\nu},$$

and then the definitions (7.12) imply that our choice for ϕ should be given by $\phi = \min\{\theta, 1/d\}$, where θ is defined as in the statement of the lemma. Provided that $0 \leqslant \theta \leqslant \lambda/(\tau d)$, this choice of ϕ ensures that the earlier condition (7.13) is met, and thus we may mimic the proof of Theorem 1.1 of [36] in order to conclude that the exponent

$$\mu^*_{\sigma+2t} = \mu_\sigma(1-\theta) + t + \sigma\theta$$

is permissible. The conclusion of the lemma follows immediately. □

We are now in a position to prove Theorem 1.7. We first recall some notation from [36]. For each real number s, we say that the exponent $\delta_s = \delta_{s,d}$ is an *associated exponent* if $\mu_{s,d} = s/2 + \delta_{s,d}$ is permissible. Suppose that u and λ satisfy the hypotheses of the statement of the theorem. Write $w = t/(2u)$, and suppose that δ_s and $\delta_{s/(1-w)}$ are associated exponents. In addition, write

$$\theta = \frac{(1-w)\delta_{s/(1-w)} - \delta_s}{\frac{1}{2}dt + (1-w)\delta_{s/(1-w)} - \delta_s}.$$

Then provided that $0 \leqslant \theta \leqslant \lambda/(du)$, the exponent δ'_{s+2t} is associated, where

$$\delta'_{s+2t} = \delta_s(1-\theta) + \tfrac{1}{2}s\theta.$$

By convexity, one may suppose that $(1-w)\delta_{s/(1-w)} \geqslant \delta_s$, and thus we have the upper bound

$$\delta'_{s+2t} \leqslant \delta_s + \frac{s}{dt}\left((1-w)\delta_{s/(1-w)} - \delta_s\right). \tag{7.14}$$

The relation (7.14) permits us to establish an iterative process by which new associated exponents may be established. Suppose that δ_s $(0 < s \leqslant d)$ are associated exponents. We define a new sequence of associated exponents $(\delta'_{s,t})$ in the following manner. We put $\delta'_{s,t} = 0$ for $0 < s \leqslant 4u$, this being justified by the upper bound $M_{d,l}(B) \ll B^l$ that follows from the first hypothesis of the statement of the theorem for integral values of l with $1 \leqslant l \leqslant 2u$. Next, when $s > 4u$, we define new associated exponents $\delta'_{s,t}$ in a two-step process for successively increasing values of s. First we define $\delta^*_{s+2t,t}$ by means of the relation

$$\delta^*_{s+2t,t} = \delta'_{s,t} + \frac{s}{dt}\left((1-w)\delta_{s/(1-w)} - \delta'_{s,t}\right). \tag{7.15}$$

Then we take $\delta'_{s+2t,t}$ to be a real number with $\delta^*_{s+2t,t} \leqslant \delta'_{s+2t,t} \leqslant \delta_{s+2t,t}$ chosen so that the quantity $(1-w)\delta'_{s/(1-w),t} - \delta'_{s,t}$ remains small. In this way we are able to ensure that the values of θ underlying our iterative argument satisfy the condition $0 \leqslant \theta \leqslant \lambda/(du)$. We note at this point that the latter holds provided that

$$0 \leqslant \frac{2}{dt}\left((1-w)\delta_{s/(1-w),t} - \delta'_{s,t}\right) \leqslant \lambda/(du). \tag{7.16}$$

We now find that the sequence $(\delta'_{s,t})$ consists of associated exponents. Optimising with respect to the parameter t subject to the condition $0 < t \leqslant 1$, we may repeat the whole process.

In the first instance we take $t = 1$, and claim that for each positive number γ with $1 \leqslant \gamma \leqslant u$, the exponents

$$\delta_s = \frac{u}{4}\left(\frac{s^2}{2du}\right)^{\gamma} \tag{7.17}$$

are associated provided only that

$$1 \leqslant s \leqslant \min\{\sqrt{d\lambda},\, \sqrt{du}\}. \tag{7.18}$$

Notice that the argument of sections 2, 3 and 4 of [36] permit one to show that whenever δ_s is an associated exponent, then so is δ'_{s+2}, where $\delta'_{s+2} = \delta_s(1-1/d) + s/(2d)$. It therefore follows by induction that $\delta_s = s^2/(8d)$ is an associated exponent for each positive number s, and this confirms (7.17) when $\gamma = 1$.

Suppose next that (7.17) holds for a fixed real number γ with $1 \leqslant \gamma \leqslant u$ whenever s is a real number satisfying (7.18). Take γ' to be a real number with $\gamma < \gamma' \leqslant \gamma + 1/d^5$. When d is sufficiently large and $1 \leqslant s \leqslant d$, the latter hypothesis ensures that

$$\left|\left(\frac{s^2}{2du}\right)^{\gamma'-\gamma} - 1\right| \leqslant \exp(\log d/d^5) - 1 < 1/d^4.$$

We claim that for each real number s satisfying (7.18), the exponent

$$\delta'_s = \frac{u}{4}\left(\frac{s^2}{2du}\right)^{\gamma'} \tag{7.19}$$

is associated. Such is true for $1 \leqslant s \leqslant 4u$, since we may suppose that the exponent $\delta_s = 0$ is associated for this range of s. Suppose then that the exponent δ'_s given by (7.19) has been shown to be associated for $1 \leqslant s \leqslant \sigma$. Let s_1 be any real number with $\sigma < s_1 \leqslant \sigma + 2$, and write $s = s_1 - 2$. We may suppose that the exponent given by (7.19) is associated, as is the exponent $\delta_{s/(1-w)}$ given by means of the formula (7.17).

Observe next that when $\gamma \leqslant u \leqslant d$ and u is large, one has

$$(1-w)^{1-2\gamma} = (1 - 1/(2u))^{1-2\gamma} < 1 + 2\gamma/u - 1/d^2.$$

We therefore deduce that

$$\begin{aligned} 0 \leqslant \frac{2}{dt}\left((1-w)\delta_{s/(1-w)} - \delta'_s\right) &= \frac{u}{2d}\left(\frac{s^2}{2du}\right)^{\gamma}\left((1-w)^{1-2\gamma} - \left(\frac{s^2}{2du}\right)^{\gamma'-\gamma}\right) \\ &\leqslant \frac{\gamma}{d}\left(\frac{s^2}{2du}\right)^{\gamma}. \end{aligned}$$

Under the constraints (7.18), the latter implies that

$$\frac{2}{dt}\left((1-w)\delta_{s/(1-w)} - \delta'_s\right) \leqslant \gamma 2^{-\gamma}\lambda/(du) \leqslant \lambda/(du),$$

and hence the condition (7.16) is satisfied. It follows that the formula (7.15) supplies an associated exponent δ^*_{s+2}. But

$$\begin{aligned} \delta^*_{s+2} &= \frac{u}{4}\left(\frac{s^2}{2du}\right)^{\gamma}\left(\left(\frac{s^2}{2du}\right)^{\gamma'-\gamma} + \frac{s}{d}\left((1-w)^{1-2\gamma} - \left(\frac{s^2}{2du}\right)^{\gamma'-\gamma}\right)\right) \\ &\leqslant \frac{u}{4}\left(\frac{s^2}{2du}\right)^{\gamma}\left(1 + \frac{s}{d}\left(\frac{2\gamma}{u}\right)\right). \end{aligned}$$

On noting that

$$\left(\frac{s+2}{s}\right)^{\gamma} \geqslant 1 + \frac{2\gamma}{s},$$

we deduce from (7.18) that

$$\begin{aligned} \frac{u}{4}\left(\frac{(s+2)^2}{2du}\right)^{\gamma'} &\geqslant \frac{u}{4}\left(1 + \frac{2\gamma}{s}\right)\left(\frac{s^2}{2du}\right)^{\gamma} \\ &\geqslant \frac{u}{4}\left(1 + \frac{2\gamma s}{du}\right)\left(\frac{s^2}{2du}\right)^{\gamma} \geqslant \delta^*_{s+2}. \end{aligned}$$

On recalling (7.19), we see that the associated exponent δ^*_{s+2} is bounded above by δ'_{s+2}, whence δ'_{s+2} is also an associated exponent. Then (7.19) provides associated exponents in the range $\sigma < s \leqslant \sigma + 2$, and so our earlier claim follows by induction.

In the next phase of our analysis, we claim that for each real number s satisfying (7.18), the exponent

$$\delta_{s,t}^{(r)}=\frac{r!}{4(u-1)!}\left(\frac{2u}{e}\right)^u\left(\frac{es^2}{4du^2}\right)^r \tag{7.20}$$

is associated for every natural number r with $r\geqslant u$. Notice that when $r=u$, the formula (7.20) yields the same exponent as the case $\gamma=u$ of (7.17). We may therefore proceed inductively, supposing that (7.20) holds for some fixed $r\geqslant u$, and for all real numbers s satisfying (7.18), and then seek to establish such with r replaced by $r+1$.

Take $t=u/(r+1)$, and note that the sequence $((r+1)\delta_{s,t}^{(r)})_{r\geqslant u}$ is decreasing for

$$r+1\leqslant 4du^2/(es^2), \tag{7.21}$$

as is easily verified with bare hands. It follows that when s satisfies the condition (7.18), and r satisfies (7.21), one has

$$\begin{aligned}\frac{2}{dt}(1-w)\delta_{s/(1-w),t}^{(r)}&\leqslant\frac{2}{du}\left(1-\frac{1}{2r+2}\right)^{1-2r}(r+1)\delta_{s,t}^{(r)}\\&\leqslant\frac{2e}{du}(u+1)\delta_{s,t}^{(u)}=\frac{e}{2d}(u+1)\left(\frac{s^2}{2du}\right)^u.\end{aligned}$$

But since $s^2<du$ and $s^2<d\lambda$, we find that when u is large one has

$$0\leqslant\frac{2}{dt}\left((1-w)\delta_{s/(1-w),t}^{(r)}-\delta_{s,t}^{(r)}\right)\leqslant(u+1)2^{-1-u}e\lambda/(du)<\lambda/(du).$$

Thus the condition (7.16) is met, and we may infer from (7.15) that the exponent $\delta_{s+2t,t}^*$ is associated, where

$$\delta_{s+2t,t}^*=\delta_{s,t}^{(r)}+\frac{s}{dt}(1-w)\delta_{s/(1-w),t}^{(r)}.$$

By iterating the use of this relation, and recalling that $t=u/(r+1)$, we find that the exponents $\delta'_{s,t}$ are associated, where

$$\delta'_{s,t}\leqslant\frac{r!}{4(u-1)!}\left(\frac{2u}{e}\right)^u\left(\frac{e}{4du^2}\right)^r\frac{1}{dt}\left(1-\frac{1}{2r+2}\right)^{1-2r}\sum_{1\leqslant l\leqslant[s/(2t)]}(s-2lt)^{2r+1}.$$

But $(1-1/(2r+2))^{1-2r}<e$ and

$$\sum_{1\leqslant l\leqslant[s/(2t)]}(s-2lt)^{2r+1}<\frac{s^{2r+2}}{2t(2r+2)}.$$

Hence we deduce that

$$\delta'_{s,t}<\frac{(r+1)!}{4(u-1)!}\left(\frac{2u}{e}\right)^u\left(\frac{es^2}{4du^2}\right)^{r+1},$$

and this establishes (7.20) with $r+1$ in place of r.

Finally, on taking

$$r=\left[\frac{4du^2}{es^2}\right]-1$$

in (7.20), and making use of the inequality $r! \leqslant r^{r+1/2}e^{-r}$, we find that the exponent δ_s is associated, where when u is large one has

$$\begin{aligned}\delta_s &< \frac{1}{4(u-1)!}\left(\frac{2u}{e}\right)^u (r+1)^{1/2}e^{-1-r}\\ &\leqslant \frac{e^{1/2}}{2}\left(\frac{u\sqrt{d}}{s(u-1)!}\right)\left(\frac{2u}{e}\right)^u \exp\left(-\frac{4du^2}{es^2}\right)\\ &< \sqrt{d/s^2}\exp\left(u-\frac{4du^2}{es^2}\right).\end{aligned}$$

This establishes the conclusion of Theorem 1.7.

Observe that Corollary 1.4 supplies the bounds for $M_{d,w}(B)-T_w(B)$ required by Theorem 1.7 unconditionally whenever d is large and

$$w \leqslant \left\lceil \frac{1}{4}\left(\frac{\log d}{\log\log d}\right)\right\rceil.$$

Consequently, the conclusion of Theorem 1.7 holds with

$$u = \frac{1}{2}\left\lceil \frac{1}{4}\left(\frac{\log d}{\log\log d}\right)\right\rceil,$$

and Corollary 1.8 follows at once.

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PS: Department of Mathematical Sciences, Chalmers University of Technology, S-412 96 Göteborg, Sweden
E-mail address: `salberg@chalmers.se`

TDW: School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, United Kingdom
E-mail address: `matdw@bristol.ac.uk`