THE CUBIC CASE OF THE MAIN CONJECTURE IN VINOGRADOV'S MEAN VALUE THEOREM

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Abstract. We apply a variant of the multigrade efficient congruencing method to estimate Vinogradov's integral of degree 3 for moments of order $2s$, establishing strongly diagonal behaviour for $1 \leqslant s \leqslant 6$. Consequently, the main conjecture is now known to hold for the first time in a case of degree exceeding 2.

1. Introduction

When k and s are natural numbers, and X is a large real number, denote by $J_{s,k}(X)$ the number of integral solutions of the system

$$x_1^j + \ldots + x_s^j = y_1^j + \ldots + y_s^j \quad (1 \leqslant j \leqslant k), \tag{1.1}$$

with $1 \leqslant x_i, y_i \leqslant X$ $(1 \leqslant i \leqslant s)$. The *main conjecture* in Vinogradov's mean value theorem asserts that for each $\varepsilon > 0$, one has

$$J_{s,k}(X) \ll X^{\varepsilon}(X^s + X^{2s-\frac{1}{2}k(k+1)}), \tag{1.2}$$

an estimate that, but for the presence of the factor X^{ε}, would be best possible (see [5, equation (7.4)]). Despite eighty years of intense investigation, such an estimate has been established only in two cases, namely the (trivial) linear case with $k = 1$, and the quadratic case with $k = 2$ in which the elementary theory of quadratic forms can be brought to bear. Our goal in this paper is the first proof of the main conjecture (1.2) in a case with $k > 2$.

Theorem 1.1. *For each $\varepsilon > 0$, one has $J_{s,3}(X) \ll X^{\varepsilon}(X^s + X^{2s-6})$.*

The estimate for $J_{s,3}(X)$ recorded in this theorem, which establishes the main conjecture in Vinogradov's mean value theorem in the cubic case $k = 3$, goes substantially beyond the estimates available hitherto. By means of Newton's formulae concerning the roots of polynomials, it is apparent that $J_{s,3}(X) = s!X^s + O(X^{s-1})$ for $1 \leqslant s \leqslant 3$, since the solutions of (1.1) are then simply the diagonal ones with $\{x_1, \ldots, x_s\} = \{y_1, \ldots, y_s\}$. Moreover, from [6, Theorem 1.5] one has

$$J_{4,3}(X) = 4!X^4 + O(X^{10/3}(\log 2X)^{35}).$$

These estimates confirm (1.2) for $1 \leqslant s \leqslant 4$ in a particularly strong form when $k = 3$, though in the latter range the estimate (1.2) has been known since at

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least the time of Hua [3]. Meanwhile, it follows from [3, Theorem 7] that when $s \geqslant 8$, then one has

$$J_{s,3}(X) \ll X^{2s-6+\varepsilon}, \tag{1.3}$$

a conclusion very recently improved in [10, Corollary 1.2] to the extent that (1.3) is now known to hold for $s \geqslant 7$. The situations with $s = 5$ and 6 have, however, thus far defied resolution.

Our strategy for proving Theorem 1.1 is based on the multigrade efficient congruencing method introduced in our recent work [10], and further developed in [11]. Indeed, the second of these papers shows that, when k is sufficiently large, one has the bound $J_{s,k}(X) \ll X^{s+\varepsilon}$ for $1 \leqslant s \leqslant \frac{1}{2}k(k+1) - \frac{1}{3}k + o(k)$, narrowly missing a proof of the main conjecture (1.2) throughout the critical interval $1 \leqslant s \leqslant \frac{1}{2}k(k+1)$. A careful inspection of the methods underlying the proof of this result shows, however, that these methods can be adapted to the case $k = 3$, and would narrowly miss a proof of the estimate

$$J_{6,3}(X) \ll X^{6+\varepsilon}. \tag{1.4}$$

Suitable application of Hölder's inequality in fact leads from such an estimate to the proof of the main conjecture in full for $k = 3$. In this paper, we are able to devise some modifications to the basic method that circumvent these implicit difficulties, leading to a proof of the estimate (1.4), and hence the proof of Theorem 1.1. We consequently economise in our exposition by reference to [11] in several places, though we aim to be transparent where confusion might otherwise occur.

Our account of the proof of Theorem 1.1 is split up into digestible stages spanning §§2–7. Aficionados of recent developments concerning Vinogradov's mean value theorem will recognise the basic structural features of this plan of attack, although novel elements must be incorporated as we proceed. We finish in §8 by noting a couple of applications of our new estimate. Further applications are available associated with the related exponential sums

$$\sum_{1 \leqslant x \leqslant X} e(\alpha x^3 + \beta x) \quad \text{and} \quad \sum_{1 \leqslant x \leqslant X} e(\alpha x^3 + \beta x^2),$$

where, as usual, we write $e(z)$ for $e^{2\pi iz}$. However, these applications require somewhat elaborate arguments that preclude their inclusion in this paper, and so we defer accounts of such developments to forthcoming papers [12, 13] elsewhere. The proof of the cubic case of the main conjecture seems worthy in its own right as the highlight of this memoir.

Finally, we note that a modification of the argument that we engineer here to establish Theorem 1.1 can in fact be adapted so as to establish a new bound for $J_{s,k}(X)$ when $k > 3$. We take this opportunity to announce this new result.

Theorem 1.2. *Suppose that $k \geqslant 3$ and $s \geqslant k(k-1)$. Then for each $\varepsilon > 0$, one has $J_{s,k}(X) \ll X^{2s-\frac{1}{2}k(k+1)+\varepsilon}$.*

This estimate improves on [10, Corollary 1.2], where we show that the estimate presented in Theorem 1.2 holds for $s \geqslant k^2 - k + 1$. Details of the proof of this new estimate will appear in a forthcoming paper.

2. The basic infrastructure

We prepare for the proof of Theorem 1.1 by introducing the notation and apparatus required in the iterative method that we ultimately engineer. This is based on our recent work [11], though we deviate somewhat in order to circumvent a number of technical difficulties. We abbreviate $J_{s,3}(X)$ to $J_s(X)$, and also $J_{6,3}(X)$ to $J(X)$, without further comment, and we define $\lambda \in \mathbb{R}$ by means of the relation

$$\lambda = \limsup_{X\to\infty} \frac{\log J(X)}{\log X}.$$

It follows that for each $\varepsilon > 0$, and any $X \in \mathbb{R}$ sufficiently large in terms of ε, one has $J(X) \ll X^{\lambda+\varepsilon}$.

Next we recall some standard notational conventions. The letter ε denotes a sufficiently small positive number. Our basic parameter is X, a large real number depending at most on ε, unless otherwise indicated. Whenever ε appears in a statement, we assert that the statement holds for each $\varepsilon > 0$. As usual, we write $\lfloor \psi \rfloor$ to denote the largest integer no larger than ψ, and $\lceil \psi \rceil$ to denote the least integer no smaller than ψ. We make sweeping use of vector notation. Thus, with t implied from the ambient environment, we write $\mathbf{z} \equiv \mathbf{w} \pmod p$ to denote that $z_i \equiv w_i \pmod p$ $(1 \leqslant i \leqslant t)$, or $\mathbf{z} \equiv \xi \pmod p$ to denote that $z_i \equiv \xi \pmod p$ $(1 \leqslant i \leqslant t)$. Finally, we employ the convention that whenever $G : [0,1)^3 \to \mathbb{C}$ is integrable, then

$$\oint G(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} = \int_{[0,1)^3} G(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}.$$

Thus, on writing

$$f(\boldsymbol{\alpha}; X) = \sum_{1\leqslant x\leqslant X} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3), \tag{2.1}$$

it follows from orthogonality that

$$J_s(X) = \oint |f(\boldsymbol{\alpha}; X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha}. \tag{2.2}$$

We next introduce the parameters appearing in our iterative method. We consider a positive number Δ with $12\Delta < 1$ to be chosen in due course. Put

$$\mathfrak{a} = \tfrac{2}{3}(7 + 2\Delta) \quad \text{and} \quad \mathfrak{b} = \tfrac{8}{3}(1 + \Delta), \tag{2.3}$$

and then define

$$\theta_+ = \tfrac{1}{2}(\mathfrak{a} + \sqrt{\mathfrak{a}^2 - 4\mathfrak{b}}) \quad \text{and} \quad \theta_- = \tfrac{1}{2}(\mathfrak{a} - \sqrt{\mathfrak{a}^2 - 4\mathfrak{b}}). \tag{2.4}$$

Notice here that

$$\theta_\pm = \tfrac{1}{3}\left(7 + 2\Delta \pm \sqrt{25 + 4\Delta + 4\Delta^2}\right),$$

so that our choice of Δ ensures that

$$\theta_+ > 4 + \tfrac{2}{3}\Delta \quad \text{and} \quad \theta_- < \tfrac{2}{3} + \tfrac{2}{3}\Delta < 1. \tag{2.5}$$

Our goal is to establish that $\lambda \leqslant 6+\Delta$. Since we are at liberty to take Δ to be an arbitrarily small positive number, it then follows that one has

$$J_6(X) \ll X^{6+\varepsilon}. \tag{2.6}$$

By applying Hölder's inequality to the right hand side of (2.2), we deduce from this estimate that whenever $1 \leqslant t \leqslant 6$, one has

$$J_t(X) \leqslant \Big(\oint |f(\boldsymbol{\alpha}; X)|^{12} \, \mathrm{d}\boldsymbol{\alpha} \Big)^{t/6} \ll X^{t+\varepsilon}.$$

Moreover, by applying the trivial estimate $|f(\boldsymbol{\alpha}; X)| \leqslant P$ in combination with (2.2) and (2.6), we find that when $t > 6$, one has

$$J_t(X) \leqslant X^{2t-12} \oint |f(\boldsymbol{\alpha}; X)|^{12} \, \mathrm{d}\boldsymbol{\alpha} \ll X^{2t-6+\varepsilon}.$$

Thus the main conjecture in the cubic case of Vinogradov's mean value theorem does indeed follow from (2.6).

Let R be a natural number sufficiently large in terms of Δ. Specifically, we choose R as follows. Since $\theta_+ > 4$, we may put $\nu = \theta_+ - 4 > 0$. Then we have

$$4^n = \theta_+^n (1 - \nu/\theta_+)^n \leqslant \theta_+^n e^{-\nu n/\theta_+}.$$

Consequently, if we take $R = \lceil W\theta_+/\nu \rceil$, with W a large enough integer, then we ensure that

$$4^R \leqslant e^{-W} \theta_+^R < \frac{\theta_+^{R+1} - \theta_-^{R+1}}{\theta_+ - \theta_-} - \tfrac{1}{2}\theta_+\theta_- \left(\frac{\theta_+^R - \theta_-^R}{\theta_+ - \theta_-} \right). \tag{2.7}$$

The significance of this condition will become apparent in due course (see the discussion surrounding (6.1) below). Having fixed R satisfying this condition, we take N to be a natural number sufficiently large in terms of R, and put

$$B = 3^N N, \quad \theta = (200N^2)^{-3RN}, \quad \delta = (10N)^{-12RN}\theta. \tag{2.8}$$

In view of the definition of λ, there exists a sequence of natural numbers $(X_l)_{l=1}^{\infty}$, tending to infinity with l, and with the property that $J(X_l) > X_l^{\lambda-\delta}$ $(l \in \mathbb{N})$. Also, provided that X_l is sufficiently large, one has the corresponding upper bound $J(Y) < Y^{\lambda+\delta}$ for $Y \geqslant X_l^{1/2}$. We consider a fixed element $X = X_l$ of the sequence $(X_l)_{l=1}^{\infty}$, which we may assume to be sufficiently large in terms of N. We put $M = X^{\theta}$, and note from (2.8) that $X^{\delta} < M^{1/N}$. Throughout, implicit constants may depend on N and ε, but not on any other variable.

We next introduce the cast of exponential sums and mean values appearing in our arguments. Let p be a prime number with $M < p \leqslant 2M$ to be fixed in due course. When c and ξ are non-negative integers, and $\boldsymbol{\alpha} \in [0,1)^3$, we define

$$\mathfrak{f}_c(\boldsymbol{\alpha}; \xi) = \sum_{\substack{1 \leqslant x \leqslant X \\ x \equiv \xi \ (\mathrm{mod}\ p^c)}} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3). \tag{2.9}$$

When $m \in \{1, 2\}$, denote by $\Xi_c^m(\xi)$ the set of integral m-tuples $(\xi_1, \dots, \xi_m)$, with $1 \leqslant \boldsymbol{\xi} \leqslant p^{c+1}$ and $\boldsymbol{\xi} \equiv \xi \ (\mathrm{mod}\ p^c)$, and in the case $m = 2$ satisfying the

property that $\xi_1 \not\equiv \xi_2 \pmod{p^{c+1}}$. We then put

$$\mathfrak{F}_c^m(\boldsymbol{\alpha};\xi) = \sum_{\boldsymbol{\xi}\in \Xi_c^m(\xi)} \prod_{i=1}^m \mathfrak{f}_{c+1}(\boldsymbol{\alpha};\xi_i).$$

Next, when a and b are positive integers, we define

$$I_{a,b}^m(X) = \max_{1\leqslant \xi\leqslant p^a} \max_{\substack{1\leqslant \eta\leqslant p^b\\ \eta\not\equiv \xi \pmod p}} \oint |\mathfrak{F}_a^m(\boldsymbol{\alpha};\xi)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{12-2m}| \,\mathrm{d}\boldsymbol{\alpha},$$

$$K_{a,b}^m(X) = \max_{1\leqslant \xi\leqslant p^a} \max_{\substack{1\leqslant \eta\leqslant p^b\\ \eta\not\equiv \xi \pmod p}} \oint |\mathfrak{F}_a^m(\boldsymbol{\alpha};\xi)^2 \mathfrak{F}_b^2(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_b(\boldsymbol{\alpha};\eta)^{8-2m}| \,\mathrm{d}\boldsymbol{\alpha}.$$

The implicit dependence on p in the above notation will be rendered irrelevant in §4, since we fix the choice of this prime following Lemma 4.2.

We next align the definition of $K_{a,b}^m(X)$ when $a = 0$ with the conditioning idea. When ξ is an integer and $\boldsymbol{\zeta}$ is a tuple of integers, we denote by $\Xi^m(\boldsymbol{\zeta})$ the set of m-tuples $(\xi_1,\dots,\xi_m) \in \Xi_0^m(0)$ such that $\xi_i \not\equiv \zeta_j \pmod p$ for all i and j. Recalling (2.9), we put

$$\mathfrak{F}^m(\boldsymbol{\alpha};\boldsymbol{\zeta}) = \sum_{\boldsymbol{\xi}\in \Xi^m(\boldsymbol{\zeta})} \prod_{i=1}^m \mathfrak{f}_1(\boldsymbol{\alpha};\xi_i),$$

and then define

$$K_{0,c}^m(X) = \max_{1\leqslant \eta\leqslant p^c} \oint |\mathfrak{F}^m(\boldsymbol{\alpha};\eta)^2 \mathfrak{F}_c^2(\boldsymbol{\alpha};\eta)^2 \mathfrak{f}_c(\boldsymbol{\alpha};\eta)^{8-2m}| \,\mathrm{d}\boldsymbol{\alpha}.$$

As in our earlier work, we make use of an operator that indicates the size of a mean value in relation to its anticipated magnitude. In the present circumstances, we adopt the convention that

$$[\![J(X)]\!] = J(X)/X^{6+\Delta}, \tag{2.10}$$

$$[\![I_{a,b}^m(X)]\!] = \frac{I_{a,b}^m(X)}{(X/M^a)^{m+\Delta}(X/M^b)^{6-m}}, \tag{2.11}$$

$$[\![K_{a,b}^m(X)]\!] = \frac{K_{a,b}^m(X)}{(X/M^a)^{m+\Delta}(X/M^b)^{6-m}}. \tag{2.12}$$

Using this notation, our earlier bounds for $J(X)$ may be written in the form

$$[\![J(X)]\!] > X^{\Lambda-\delta} \quad \text{and} \quad [\![J(Y)]\!] < Y^{\Lambda+\delta} \quad (Y \geqslant X^{1/2}), \tag{2.13}$$

where Λ is defined by $\Lambda = \lambda - (6+\Delta)$.

Finally, we recall a simple estimate associated with the system (1.1).

Lemma 2.1. *Suppose that c and d are non-negative integers with $c \leqslant \theta^{-1}$ and $d \leqslant \theta^{-1}$. Then whenever $u, v \in \mathbb{N}$ satisfy $u + v = 6$, and $\xi, \zeta \in \mathbb{Z}$, one has*

$$\oint |\mathfrak{f}_c(\boldsymbol{\alpha};\xi)^{2u} \mathfrak{f}_d(\boldsymbol{\alpha};\zeta)^{2v}| \,\mathrm{d}\boldsymbol{\alpha} \ll (J(X/M^c))^{u/6}(J(X/M^d))^{v/6}.$$

Proof. This is immediate from [2, Corollary 2.2]. □

3. Auxiliary systems of congruences

We must modify slightly our previous work concerning auxiliary congruences so as to accommodate behaviour that deviates slightly from the diagonal. When a and b are integers with $1 \leqslant a < b$, we denote by $\mathcal{B}^n_{a,b}(\mathbf{m};\xi,\eta)$ the set of solutions of the system of congruences

$$\sum_{i=1}^{n}(z_i-\eta)^j \equiv m_j \ (\mathrm{mod}\ p^{jb}) \quad (1 \leqslant j \leqslant 3), \tag{3.1}$$

with $1 \leqslant \mathbf{z} \leqslant p^{3b}$ and $\mathbf{z} \equiv \xi \ (\mathrm{mod}\ p^{a+1})$ for some $\boldsymbol{\xi} \in \Xi^n_a(\xi)$. We define an equivalence relation $\mathcal{R}(\lambda)$ on integral n-tuples by declaring $\mathbf{x}$ and $\mathbf{y}$ to be $\mathcal{R}(\lambda)$-equivalent when $\mathbf{x} \equiv \mathbf{y} \ (\mathrm{mod}\ p^{\lambda})$. We then write $\mathcal{C}^{n,h}_{a,b}(\mathbf{m};\xi,\eta)$ for the set of $\mathcal{R}(hb)$-equivalence classes of $\mathcal{B}^n_{a,b}(\mathbf{m};\xi,\eta)$, and define $B^{n,h}_{a,b}(p)$ by putting

$$B^{n,h}_{a,b}(p) = \max_{1\leqslant \xi\leqslant p^a} \max_{\substack{1\leqslant \eta\leqslant p^b\\ \eta\not\equiv\xi \ (\mathrm{mod}\ p)}} \max_{1\leqslant \mathbf{m}\leqslant p^{3b}} \mathrm{card}(\mathcal{C}^{n,h}_{a,b}(\mathbf{m};\xi,\eta)). \tag{3.2}$$

When $a = 0$ we modify these definitions, so that $\mathcal{B}^n_{0,b}(\mathbf{m};\xi,\eta)$ denotes the set of solutions of the system of congruences (3.1) with $1 \leqslant \mathbf{z} \leqslant p^{3b}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \ (\mathrm{mod}\ p)$ for some $\boldsymbol{\xi} \in \Xi^n_0(\xi)$, and for which in addition $\mathbf{z} \not\equiv \eta \ (\mathrm{mod}\ p)$. As in the situation in which one has $a \geqslant 1$, we write $\mathcal{C}^{n,h}_{0,b}(\mathbf{m};\xi,\eta)$ for the set of $\mathcal{R}(hb)$-equivalence classes of $\mathcal{B}^n_{0,b}(\mathbf{m};\xi,\eta)$, but we define $B^{n,h}_{0,b}(p)$ by putting

$$B^{n,h}_{0,b}(p) = \max_{1\leqslant \eta\leqslant p^b} \max_{1\leqslant \mathbf{m}\leqslant p^{3b}} \mathrm{card}(\mathcal{C}^{n,h}_{0,b}(\mathbf{m};0,\eta)). \tag{3.3}$$

We recall a version of Hensel's lemma made available in [8].

Lemma 3.1. *Let $f_1,\dots,f_d$ be polynomials in $\mathbb{Z}[x_1,\dots,x_d]$ with respective degrees $k_1,\dots,k_d$, and write*

$$J(\mathbf{f};\mathbf{x}) = \det\left(\frac{\partial f_j}{\partial x_i}(\mathbf{x})\right)_{1\leqslant i,j\leqslant d}.$$

When ϖ is a prime number, and l is a natural number, let $\mathcal{N}(\mathbf{f};\varpi^l)$ denote the number of solutions of the simultaneous congruences

$$f_j(x_1,\dots,x_d) \equiv 0 \ (mod\ \varpi^l) \quad (1 \leqslant j \leqslant d),$$

with $1 \leqslant x_i \leqslant \varpi^l$ $(1 \leqslant i \leqslant d)$ and $(J(\mathbf{f};\mathbf{x}),\varpi) = 1$. Then $\mathcal{N}(\mathbf{f};\varpi^l) \leqslant k_1\dots k_d$.

Proof. This is [8, Theorem 1]. □

We now present the key result on congruences utilised in this paper.

Lemma 3.2. *Suppose that a and b are integers with $0 \leqslant a < b$, and that h is a natural number with $2b-a \leqslant h \leqslant 2b-a+\Delta(b-a)$. Then one has*

$$B^{1,3}_{a,b}(p) \leqslant 6 \quad \textit{and} \quad B^{2,h/b}_{a,b}(p) \leqslant 6p^{h-2b+a}.$$

Proof. The estimate $B_{a,b}^{1,3}(p) \leqslant 6$ is immediate from the case $h = 3b$, $k = 3$ of [11, Lemma 3.1]. We therefore focus on establishing the second estimate asserted in the statement of the lemma. We begin by considering the situation with $a \geqslant 1$, the remaining cases with $a = 0$ being easily accommodated within our argument for the former case. Consider fixed natural numbers a, b and h with $1 \leqslant a \leqslant b$ and

$$2b - a \leqslant h \leqslant 2b - a + \Delta(b - a),$$

and fixed integers ξ and η with $1 \leqslant \xi \leqslant p^a$, $1 \leqslant \eta \leqslant p^b$ and $\eta \not\equiv \xi \pmod p$. Write $\omega = h - (2b - a)$, so that $0 \leqslant \omega \leqslant \Delta(b - a)$. We denote by $\mathcal{D}_1(\mathbf{n})$ the set of $\mathcal{R}(h)$-equivalence classes of solutions of the system of congruences

$$(z_1 - \eta)^j + (z_2 - \eta)^j \equiv n_j \pmod{p^{2b+\omega}} \quad (j = 2, 3), \tag{3.4}$$

with $1 \leqslant \mathbf{z} \leqslant p^{3b}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \pmod{p^{a+1}}$ for some $\boldsymbol{\xi} \in \Xi_a^2(\xi)$. Fix an integral triple $\mathbf{m}$. To any solution $\mathbf{z}$ of (3.4) there corresponds a unique pair $\mathbf{n} = (n_2, n_3)$ with $1 \leqslant \mathbf{n} \leqslant p^{2b+\omega}$ for which (3.4) holds and

$$n_j \equiv m_j \pmod{p^{\sigma(j)}} \quad (j = 2, 3),$$

where $\sigma(j) = \min\{jb, 2b + \omega\}$. We therefore infer that

$$\mathcal{C}_{a,b}^{2,h/b}(\mathbf{m}; \xi, \eta) \subseteq \bigcup_{\substack{1 \leqslant n_2 \leqslant p^{2b+\omega} \\ n_2 \equiv m_2 \pmod{p^{2b}}}} \bigcup_{\substack{1 \leqslant n_3 \leqslant p^{2b+\omega} \\ n_3 \equiv m_3 \pmod{p^{2b+\omega}}}} \mathcal{D}_1(\mathbf{n}).$$

The number of pairs $\mathbf{n}$ in the union is equal to p^ω. Consequently, one has

$$\operatorname{card}(\mathcal{C}_{a,b}^{2,h/b}(\mathbf{m}; \xi, \eta)) \leqslant p^\omega \max_{1 \leqslant \mathbf{n} \leqslant p^{2b+\omega}} \operatorname{card}(\mathcal{D}_1(\mathbf{n})). \tag{3.5}$$

Observe that for any solution $\mathbf{z}'$ of (3.4) there is an $\mathcal{R}(h)$-equivalent solution $\mathbf{z}$ satisfying $1 \leqslant \mathbf{z} \leqslant p^{2b+\omega}$. We next rewrite each variable z_i in the shape $z_i = p^a y_i + \xi$. One finds from the hypothesis $\mathbf{z} \equiv \boldsymbol{\xi} \pmod{p^{a+1}}$ for some $\boldsymbol{\xi} \in \Xi_a^2(\xi)$ that $y_1 \not\equiv y_2 \pmod p$. Write $\zeta = \xi - \eta$, note that $p \nmid \zeta$, and write the multiplicative inverse of ζ modulo $p^{2b+\omega}$ as ζ^{-1}. Then we deduce from (3.4) that $\operatorname{card}(\mathcal{D}_1(\mathbf{n}))$ is bounded above by the number of $\mathcal{R}(h - a)$-equivalence classes of solutions of the system of congruences

$$(p^a y_1 \zeta^{-1} + 1)^j + (p^a y_2 \zeta^{-1} + 1)^j \equiv n_j (\zeta^{-1})^j \pmod{p^{2b+\omega}} \quad (j = 2, 3), \tag{3.6}$$

with $1 \leqslant \mathbf{y} \leqslant p^{h-a}$. Recall that $h = 2b - a + \omega$, and let $\mathbf{y} = \mathbf{w}$ be any solution of the system (3.6), if any one such exists. Then we find that all other solutions $\mathbf{y}$ satisfy the system

$$\sum_{i=1}^{2} \big((p^a y_i \zeta^{-1} + 1)^j - (p^a w_i \zeta^{-1} + 1)^j \big) \equiv 0 \pmod{p^{2b+\omega}} \quad (j = 2, 3). \tag{3.7}$$

When $1 \leqslant j \leqslant 3$, write

$$s_j(\mathbf{y}, \mathbf{w}) = y_1^j + y_2^j - w_1^j - w_2^j.$$

Then by applying the Binomial theorem, it follows that the system (3.7) is equivalent to the new system

$$\left.\begin{aligned} 2(\zeta^{-1}p^a)s_1(\mathbf{y},\mathbf{w})+(\zeta^{-1}p^a)^2s_2(\mathbf{y},\mathbf{w}) &\equiv 0 \ (\mathrm{mod}\ p^{2b+\omega}) \\ 3(\zeta^{-1}p^a)s_1(\mathbf{y},\mathbf{w})+3(\zeta^{-1}p^a)^2s_2(\mathbf{y},\mathbf{w}) + (\zeta^{-1}p^a)^3s_3(\mathbf{y},\mathbf{w}) &\equiv 0 \ (\mathrm{mod}\ p^{2b+\omega}) \end{aligned}\right\}.$$

By employing the quadratic congruence to eliminate the linear term in the cubic congruence here, one finds that this system is in turn equivalent to

$$\left.\begin{aligned} s_1(\mathbf{y},\mathbf{w}) + (2\zeta)^{-1}p^a s_2(\mathbf{y},\mathbf{w}) &\equiv 0 \ (\mathrm{mod}\ p^h) \\ s_2(\mathbf{y},\mathbf{w}) + 2(3\zeta)^{-1}p^a s_3(\mathbf{y},\mathbf{w}) &\equiv 0 \ (\mathrm{mod}\ p^{h-a}) \end{aligned}\right\}.$$

Denote by $\mathcal{D}_2(\mathbf{u})$ the set of $\mathcal{R}(h-a)$-equivalence classes of solutions of the system of congruences

$$\left.\begin{aligned} y_1 + y_2 + (2\zeta)^{-1}p^a(y_1^2+y_2^2) &\equiv u_2 \ (\mathrm{mod}\ p^{h-a}) \\ y_1^2 + y_2^2 + 2(3\zeta)^{-1}p^a(y_1^3+y_2^3) &\equiv u_3 \ (\mathrm{mod}\ p^{h-a}) \end{aligned}\right\},$$

with $1 \leqslant y_1, y_2 \leqslant p^{h-a}$ satisfying $y_1 \not\equiv y_2 \ (\mathrm{mod}\ p)$. Then we have shown thus far that

$$\mathrm{card}(\mathcal{D}_1(\mathbf{n})) \leqslant \max_{1\leqslant \mathbf{u}\leqslant p^{h-a}} \mathrm{card}(\mathcal{D}_2(\mathbf{u})). \tag{3.8}$$

Next define the determinant

$$J(\mathbf{y}) = \det \begin{pmatrix} 1 + 2(2\zeta)^{-1}p^a y_1 & 1 + 2(2\zeta)^{-1}p^a y_2 \\ 2y_1 + 6(3\zeta)^{-1}p^a y_1^2 & 2y_2 + 6(3\zeta)^{-1}p^a y_2^2 \end{pmatrix}.$$

One has

$$J(\mathbf{y}) \equiv 2(y_2 - y_1) \not\equiv 0 \ (\mathrm{mod}\ p),$$

and hence we deduce from Lemma 3.1 that $\mathrm{card}(\mathcal{D}_2(\mathbf{u})) \leqslant 6$. In combination with (3.5) and (3.8), this estimate delivers the bound

$$\mathrm{card}(\mathcal{C}_{a,b}^{2,h/b}(\mathbf{m};\xi,\eta)) \leqslant 6p^{\omega}.$$

We thus conclude from (3.2) that $B_{a,b}^{n,h/b}(p) \leqslant 6p^{h-2b+a}$, and this completes the proof of the lemma when $a \geqslant 1$.

The proof presented above requires little modification to handle the situation in which $a = 0$. In this case, we denote by $\mathcal{D}_1(\mathbf{n};\eta)$ the set of solutions of the system of congruences (3.4) with $1 \leqslant \mathbf{z} \leqslant p^{3b}$ and $\mathbf{z} \equiv \boldsymbol{\xi} \ (\mathrm{mod}\ p)$ for some $\boldsymbol{\xi} \in \Xi_0^2(0)$, and for which in addition $z_i \not\equiv \eta \ (\mathrm{mod}\ p)$ for $i = 1, 2$. Then as in the opening paragraph of our proof, it follows from (3.4) that

$$\mathrm{card}(\mathcal{C}_{0,b}^{2,h/b}(\mathbf{m};0,\eta)) \leqslant p^{\omega} \max_{1\leqslant \mathbf{n}\leqslant p^{2b+\omega}} \mathrm{card}(\mathcal{D}_1(\mathbf{n};\eta)). \tag{3.9}$$

But $\mathrm{card}(\mathcal{D}_1(\mathbf{n};\eta)) = \mathrm{card}(\mathcal{D}_1(\mathbf{n};0))$, and $\mathrm{card}(\mathcal{D}_1(\mathbf{n};0))$ counts the solutions of the system of congruences

$$\left.\begin{aligned} y_1^3 + y_2^3 &\equiv n_3 \ (\mathrm{mod}\ p^{2b+\omega}) \\ y_1^2 + y_2^2 &\equiv n_2 \ (\mathrm{mod}\ p^{2b+\omega}) \end{aligned}\right\},$$

with $1 \leqslant \mathbf{y} \leqslant p^{2b+\omega}$ satisfying $y_1 \not\equiv y_2 \pmod p$ and $p \nmid y_i$ $(i = 1, 2)$. Write

$$J(\mathbf{y}) = \det \begin{pmatrix} 3y_1^2 & 3y_2^2 \\ 2y_1 & 2y_2 \end{pmatrix}.$$

Then since $p > 3$, we have

$$J(\mathbf{y}) = 6y_1y_2(y_1 - y_2) \not\equiv 0 \pmod p.$$

We therefore conclude from Lemma 3.1 that $\mathrm{card}(\mathcal{D}_1(\mathbf{n}; 0)) \leqslant 6$. In view of (3.3), the conclusion of the lemma therefore follows from (3.9) when $a = 0$. □

4. The conditioning and pre-congruencing processes

We recall a consequence of a lemma from [11] which permits the mean value $I^2_{a,b}(X)$ to be bounded in terms of $K^2_{c,d}(X)$, for suitable parameters c and d.

Lemma 4.1. *Let a and b be integers with $1 \leqslant a < b$, and let H be any integer with $H \geqslant 15$. Suppose that $b + H \leqslant (2\theta)^{-1}$. Then there exists an integer h with $0 \leqslant h < H$ having the property that*

$$I^2_{a,b}(X) \ll (M^h)^{8/3} K^2_{a,b+h}(X) + M^{-H}(X/M^b)^4(X/M^a)^{\lambda-4}.$$

Proof. This is simply a special case of [11, Lemma 4.2]. □

Next we recall a lemma from [11] which initiates the iterative process.

Lemma 4.2. *There exists a prime number p, with $M < p \leqslant 2M$, and an integer h with $0 \leqslant h \leqslant 4B$, for which one has*

$$J(X) \ll M^{8B+8h/3} K^2_{0,B+h}(X).$$

Proof. Again, this is simply a special case of [11, Lemma 5.1]. □

We now fix the prime number p, once and for all, in accordance with the conclusion of Lemma 4.2.

5. Efficient congruencing and the multigrade combination

We adapt the treatment of [11, §6] to the present cubic situation.

Lemma 5.1. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant \theta^{-1}$, and suppose further that $b \geqslant (1 + \frac{2}{3}\Delta)a$. Then one has*

$$K^1_{a,b}(X) \ll M^{3b-a}(I^2_{b,3b}(X))^{1/4}(J(X/M^b))^{3/4}. \tag{5.1}$$

Moreover, whenever b' is an integer with

$$2b - a \leqslant b' \leqslant 2b - a + \Delta(b - a),$$

one has

$$K^2_{a,b}(X) \ll M^{b'-2b+a}(M^{b'-a})^{4/3}(I^2_{b,b'}(X))^{1/3}(K^1_{a,b}(X))^{2/3}. \tag{5.2}$$

Proof. The estimate (5.1) is the special case $s = 4$, $m = 0$ of [11, Lemma 6.1] corresponding to exponent $k = 3$, in which one takes $b' = 3b$. We focus, therefore, on the proof of the estimate (5.2). Even in this situation, however, the argument of the proof of [11, Lemma 6.1] applies without serious modification. Applying the latter with $s = 4$ and $m = 1$, we find that the final conclusion must be modified only to reflect the fact that, in view of Lemma 3.2, one has in present circumstances the bound

$$\mathrm{card}(\mathcal{C}_{a,b}^{2,b'/b}(\mathbf{m};\xi,\eta)) \leqslant 6p^{b'-2b+a},$$

whereas in the discussion following [11, equation (6.5)] one had the sharper bound $\mathrm{card}(\mathcal{C}_{a,b}^{2,b'/b}(\mathbf{m};\xi,\eta)) \leqslant 6$, owing to the stronger constraint on b' therein. On accounting for the presence of the additional factor $p^{b'-2b+a}$ in the analogue of the discussion leading from [11, equation (6.6)] to the conclusion of the proof of [11, Lemma 6.1], the upper bound (5.2) follows at once. This completes the proof of the lemma. □

We note that when a and b are sufficiently large in terms of Δ, then the hypothesis $b \geqslant (1 + \frac{2}{3}\Delta)a$ in the statement of Lemma 5.1 ensures that

$$\begin{aligned} 2b - a + \Delta(b-a) = (2+\Delta)b - (1+\Delta)a &\geqslant \left(2 + \Delta - \frac{1+\Delta}{1+\frac{2}{3}\Delta}\right) b \\ &= \left(1 + \Delta - \frac{\frac{1}{3}\Delta}{1+\frac{2}{3}\Delta}\right) b \geqslant \lceil (1+\tfrac{2}{3}\Delta)b \rceil. \end{aligned}$$

We are therefore at liberty to apply Lemma 5.1 with a choice for b' satisfying the condition $b' \geqslant (1 + \frac{2}{3}\Delta)b$, thereby preparing appropriately for subsequent applications of Lemma 5.1.

We next combine the estimates supplied by Lemma 5.1 so as to bound $K_{a,b}^2(X)$ in terms of the mean values $I_{b,k_m b}^2(X)$ $(m = 0, 1)$, in which $k_0 = 3$ and

$$2 - a/b \leqslant k_1 \leqslant 2 - a/b + \Delta(1 - a/b).$$

Lemma 5.2. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant \theta^{-1}$, and suppose further that $b \geqslant (1 + \frac{2}{3}\Delta)a$. Then whenever d is an integer with $0 \leqslant d \leqslant \Delta(b-a)$, one has*

$$[[K_{a,b}^2(X)]] \ll \big((X/M^b)^{\Lambda+\delta}\big)^{1/2} [[I_{b,3b}^2(X)]]^{1/6} [[I_{b,b'}^2(X)]]^{1/3},$$

where $b' = 2b - a + d$.

Proof. By substituting the estimate for $K_{a,b}^1(X)$ provided by equation (5.1) of Lemma 5.1 into (5.2), we find that

$$K_{a,b}^2(X) \ll M^d \big((M^{b'-a})^4 I_{b,b'}^2(X)\big)^{1/3} \big((M^{3b-a})^4 I_{b,3b}^2(X)\big)^{1/6} \big(J(X/M^b)\big)^{1/2}.$$

On recalling (2.10) to (2.12), therefore, we deduce that

$$[[K_{a,b}^2(X)]] \ll M^{\Omega} [[I_{b,3b}^2(X)]]^{1/6} [[I_{b,b'}^2(X)]]^{1/3} \big((X/M^b)^{\Lambda+\delta}\big)^{1/2},$$

where

$$\Omega = d + \Delta(a-b) \leqslant \Delta(b-a) + \Delta(a-b) = 0.$$

Since $\Omega \leqslant 0$, the conclusion of the lemma is now immediate. $\square$

We next study a multistep multigrade combination stemming from Lemma 5.2. We begin by introducing some additional notation. We recall that R is a positive integer sufficiently large in terms of Δ. We consider R-tuples of integers $(m_1, \dots, m_R) \in \{0,1\}^R$, to each of which we associate an R-tuple of integers $\mathbf{h} = (h_1(\mathbf{m}), \dots, h_R(\mathbf{m})) \in [0,\infty)^R$. The integral tuples $\mathbf{h}(\mathbf{m})$ will be fixed as the iteration proceeds, with $h_n(\mathbf{m})$ depending at most on the first n coordinates of $(m_1, \dots, m_R)$. We may abuse notation in some circumstances by writing $h_n(\mathbf{m}, m_n)$ or $h_n(m_1, \dots, m_{n-1}, m_n)$ in place of $h_n(m_1, \dots, m_R)$, reflecting the latter implicit dependence. We suppose that a positive integer b has already been fixed. We then define the sequences $(a_n) = (a_n(\mathbf{m};\mathbf{h}))$ and $(b_n) = (b_n(\mathbf{m};\mathbf{h}))$ by putting

$$a_0 = \lfloor b/(1+\tfrac{2}{3}\Delta)\rfloor \quad \text{and} \quad b_0 = b, \tag{5.3}$$

and then applying the iterative relations, for $1 \leqslant n \leqslant R$, given by

$$a_n = b_{n-1} \tag{5.4}$$

and

$$b_n = \begin{cases} 3b_{n-1} + h_n(\mathbf{m}), & \text{when } m_n = 0, \\ 2b_{n-1} - a_{n-1} + \lfloor \Delta(b_{n-1} - a_{n-1})\rfloor + h_n(\mathbf{m}), & \text{when } m_n = 1. \end{cases} \tag{5.5}$$

Next, we define the quantity $\Theta_n(\mathbf{m};\mathbf{h})$ for $0 \leqslant n \leqslant R$ by writing

$$\Theta_n(\mathbf{m};\mathbf{h}) = (X/M^b)^{-\Lambda-\delta}[[K^2_{a_n,b_n}(X)]] + M^{-12\cdot 3^R b}. \tag{5.6}$$

Finally, we put

$$\phi_0 = 1/6 \quad \text{and} \quad \phi_1 = 1/3.$$

Lemma 5.3. *Suppose that a and b are integers with $0 < a < b \leqslant (16\cdot 3^{2R}R\theta)^{-1}$, and suppose further that $a \leqslant b/(1+\frac{2}{3}\Delta)$. Then there exists a choice for $\mathbf{h}(\mathbf{m}) \in \{0,1\}^R$, satisfying the condition that $0 \leqslant h_n(\mathbf{m}) \leqslant 15\cdot 3^R b$ $(1 \leqslant n \leqslant R)$, and for which one has*

$$(X/M^b)^{-\Lambda-\delta}[[K^2_{a,b}(X)]] \ll \prod_{\mathbf{m}\in\{0,1\}^R} \Theta_R(\mathbf{m};\mathbf{h})^{\phi_{m_1}\cdots\phi_{m_R}}.$$

Proof. A comparison of Lemma 5.2 above with [11, Lemma 7.2] reveals that the argument of the proof of [11, Lemma 7.3] applies in the present situation, mutatis mutandis, to establish the conclusion of the lemma. We note here that our Lemma 4.1 above serves as a substitute for [11, Lemma 4.2] for this purpose. $\square$

6. The latent monograde process

We next convert the block estimate encoded in Lemma 5.3 into a single monograde estimate that can be incorporated into our iterative method. We begin by recalling an elementary lemma from our previous work [10].

Lemma 6.1. *Suppose that $z_0, \dots, z_l \in \mathbb{C}$, and that β_i and γ_i are positive real numbers for $0 \leqslant i \leqslant l$. Put $\Omega = \beta_0\gamma_0 + \dots + \beta_l\gamma_l$. Then one has*

$$|z_0^{\beta_0} \dots z_l^{\beta_l}| \leqslant \sum_{i=0}^{l} |z_i|^{\Omega/\gamma_i}.$$

Proof. This is [10, Lemma 8.1]. □

Before proceeding further, we introduce some additional notation. Define the positive number s_0 by means of the relation

$$s_0^R = \frac{\theta_+^{R+1} - \theta_-^{R+1}}{\theta_+ - \theta_-} - \frac{\theta_+\theta_-}{2(1 + \frac{2}{3}\Delta)} \left(\frac{\theta_+^R - \theta_-^R}{\theta_+ - \theta_-} \right), \tag{6.1}$$

in which $\theta_\pm$ are defined as in (2.4). We recall that, in view of (2.7), one has $s_0 > 4$. Next we make use of a new pair of sequences $(\tilde{a}_n) = (\tilde{a}_n(\mathbf{m}))$ and $(\tilde{b}_n) = (\tilde{b}_n(\mathbf{m}))$ defined by means of the relations

$$\tilde{a}_0 = 1/(1 + \tfrac{2}{3}\Delta) \quad \text{and} \quad \tilde{b}_0 = 1, \tag{6.2}$$

and then, when $1 \leqslant n \leqslant R$, by

$$\tilde{a}_n = \tilde{b}_{n-1} \tag{6.3}$$

and

$$\tilde{b}_n = \begin{cases} 3\tilde{b}_{n-1}, & \text{when } m_n = 0, \\ 2\tilde{b}_{n-1} - \tilde{a}_{n-1} + \Delta(\tilde{b}_{n-1} - \tilde{a}_{n-1}), & \text{when } m_n = 1. \end{cases} \tag{6.4}$$

We then define

$$k_{\mathbf{m}} = \tilde{b}_R(\mathbf{m}) \quad \text{and} \quad \rho_{\mathbf{m}} = \tilde{b}_R(\mathbf{m})(4/s_0)^R \quad \text{for} \quad \mathbf{m} \in \{0,1\}^R. \tag{6.5}$$

Lemma 6.2. *Suppose that $\Lambda \geqslant 0$, let a and b be integers with*

$$0 \leqslant a < b \leqslant (20 \cdot 3^{2R} R\theta)^{-1},$$

and suppose further that $a \leqslant b/(1 + \frac{2}{3}\Delta)$. Suppose in addition that there are real numbers ψ, c and γ, with

$$0 \leqslant c \leqslant (2\delta)^{-1}\theta, \quad \gamma \geqslant -4b \quad \text{and} \quad \psi \geqslant 0,$$

such that

$$X^\Lambda M^{\Lambda\psi} \ll X^{c\delta} M^{-\gamma} [[K_{a,b}^2(X)]]. \tag{6.6}$$

Then, for some $\mathbf{m} \in \{0,1\}^R$, there is a real number h with $0 \leqslant h \leqslant 16 \cdot 3^{2R} b$, and positive integers a' and b' with $a' \leqslant b'/(1 + \frac{2}{3}\Delta)$, such that

$$X^\Lambda M^{\Lambda\psi'} \ll X^{c'\delta} M^{-\gamma'} [[K_{a',b'}^2(X)]], \tag{6.7}$$

where ψ', c', γ' and b' are real numbers satisfying the conditions

$$\psi' = \rho_{\mathbf{m}}(\psi + \tfrac{1}{2}b), \quad c' = \rho_{\mathbf{m}}(c+1), \quad \gamma' = \rho_{\mathbf{m}}\gamma, \quad b' = k_{\mathbf{m}} b + h.$$

Moreover, the real number $k_{\mathbf{m}}$ satisfies $(1 + \frac{2}{3}\Delta)^R \leqslant k_{\mathbf{m}} \leqslant 3^R$.

Proof. We deduce from the postulated bound (6.6) and Lemma 5.3 that there exists a choice of the tuple $\mathbf{h} = \mathbf{h}(\mathbf{m})$, with $0 \leqslant h_n(\mathbf{m}) \leqslant 15 \cdot 3^R b$ $(1 \leqslant n \leqslant R)$, such that

$$X^{\Lambda} M^{\Lambda\psi} \ll X^{(c+1)\delta} M^{-\gamma} (X/M^b)^{\Lambda} \prod_{\mathbf{m} \in \{0,1\}^R} \Theta_R(\mathbf{m}; \mathbf{h})^{\phi_{m_1} \cdots \phi_{m_R}}.$$

Consequently, one has

$$\prod_{\mathbf{m} \in \{0,1\}^R} \Theta_R(\mathbf{m}; \mathbf{h})^{\phi_{m_1} \cdots \phi_{m_R}} \gg X^{-(c+1)\delta} M^{\Lambda(\psi+b)+\gamma}.$$

Note that $\phi_0 + \phi_1 = \frac{1}{2}$, so that

$$\sum_{\mathbf{m} \in \{0,1\}^R} \phi_{m_1} \dots \phi_{m_R} = \big(\tfrac{1}{2}\big)^R \leqslant \tfrac{1}{2}.$$

Then we deduce from the definition (5.6) of $\Theta_n(\mathbf{m}; \mathbf{h})$ that

$$\prod_{\mathbf{m} \in \{0,1\}^R} \Big(X^{-\Lambda} [\![K^2_{a_R, b_R}(X)]\!] + M^{-12 \cdot 3^R b} \Big)^{\phi_{m_1} \cdots \phi_{m_R}} \gg X^{-(c+1)\delta} M^{\Lambda(\psi+\frac{1}{2}b)+\gamma}. \tag{6.8}$$

In preparation for our application of Lemma 6.1, we examine the exponents $\phi_{m_1} \dots \phi_{m_R}$. Put

$$\beta^{(n)}_{\mathbf{m}} = \phi_{m_1} \dots \phi_{m_n} \quad \text{and} \quad \gamma^{(n)}_{\mathbf{m}} = \tilde{b}_n(\mathbf{m}) \quad (\mathbf{m} \in \{0,1\}^n).$$

In addition, we define

$$B_n = \sum_{\mathbf{m} \in \{0,1\}^n} \beta^{(n)}_{\mathbf{m}} \tilde{b}_n(\mathbf{m}) \quad \text{and} \quad A_n = \sum_{\mathbf{m} \in \{0,1\}^n} \beta^{(n)}_{\mathbf{m}} \tilde{a}_n(\mathbf{m}),$$

and then put $\Omega = B_R$. From the iterative formulae (6.2) to (6.4), we obtain

$$\begin{aligned} B_{n+1} = &\frac{1}{6} \sum_{\mathbf{m} \in \{0,1\}^n} 3\tilde{b}_n(\mathbf{m}) \phi_{m_1} \dots \phi_{m_n} \\ &+ \frac{1}{3} \sum_{\mathbf{m} \in \{0,1\}^n} (2\tilde{b}_n(\mathbf{m}) - \tilde{a}_n(\mathbf{m}) + \Delta(\tilde{b}_n(\mathbf{m}) - \tilde{a}_n(\mathbf{m})) \phi_{m_1} \dots \phi_{m_n}, \end{aligned}$$

so that

$$\begin{aligned} B_{n+1} &= \tfrac{1}{2} B_n + (\tfrac{2}{3} + \tfrac{1}{3}\Delta) B_n - (\tfrac{1}{3} + \tfrac{1}{3}\Delta) A_n \\ &= (\tfrac{7}{6} + \tfrac{1}{3}\Delta) B_n - (\tfrac{1}{3} + \tfrac{1}{3}\Delta) A_n. \end{aligned}$$

Similarly, one finds that

$$A_{n+1} = \frac{1}{2} \sum_{\mathbf{m} \in \{0,1\}^n} \tilde{b}_n(\mathbf{m}) \phi_{m_1} \dots \phi_{m_n} = \tfrac{1}{2} B_n.$$

Thus we conclude via (2.3) that

$$4^2 B_{n+2} = \mathfrak{a}(4B_{n+1}) - \mathfrak{b} B_n \quad (n \geqslant 1). \tag{6.9}$$

In addition, one has the initial data

$$4B_1 = 4\Big(\tfrac{1}{6}(3\tilde{b}_0) + \tfrac{1}{3}(2\tilde{b}_0 - \tilde{a}_0 + \Delta(\tilde{b}_0 - \tilde{a}_0))\Big) = \mathfrak{a} - \tfrac{1}{2}\mathfrak{b}/(1 + \tfrac{2}{3}\Delta), \tag{6.10}$$

$$4A_1 = 4(\tfrac{1}{2}\tilde{b}_0) = 2,$$

and hence

$$4^2B_2 = 4^2\big((\tfrac{7}{6} + \tfrac{1}{3}\Delta)B_1 - \tfrac{1}{3}(1 + \Delta)A_1\big) = \mathfrak{a}(\mathfrak{a} - \tfrac{1}{2}\mathfrak{b}/(1 + \tfrac{2}{3}\Delta)) - \mathfrak{b}. \tag{6.11}$$

The recurrence formula (6.9) has a solution of the shape

$$4^nB_n = \sigma_+\theta_+^n + \sigma_-\theta_-^n \quad (n \geqslant 1),$$

where, in view of (6.10) and (6.11), one has

$$\sigma_+\theta_+ + \sigma_-\theta_- = 4B_1 = \mathfrak{a} - \tfrac{1}{2}\mathfrak{b}/(1 + \tfrac{2}{3}\Delta)$$

and

$$\sigma_+\theta_+^2 + \sigma_-\theta_-^2 = 4^2B_2 = \mathfrak{a}(\mathfrak{a} - \tfrac{1}{2}\mathfrak{b}/(1 + \tfrac{2}{3}\Delta)) - \mathfrak{b}.$$

Since $\mathfrak{a} = \theta_+ + \theta_-$ and $\mathfrak{b} = \theta_+\theta_-$, we therefore deduce that

$$4^nB_n = \frac{\theta_+^{n+1} - \theta_-^{n+1}}{\theta_+ - \theta_-} - \frac{\theta_+\theta_-}{2(1 + \frac{2}{3}\Delta)}\left(\frac{\theta_+^n - \theta_-^n}{\theta_+ - \theta_-}\right).$$

In particular, on recalling (6.1), we find that $4^RB_R = s_0^R$, so that $B_R = (s_0/4)^R$. Also, therefore, it follows from (2.7) that $B_R > 1$.

Returning now to the application of Lemma 6.1, we note first that $\Omega = B_R$, and hence (6.8) yields the relation

$$\sum_{\mathbf{m}\in\{0,1\}^R}\Big(X^{-\Lambda}[[K^2_{a_R,b_R}(X)]] + M^{-12\cdot 3^Rb}\Big)^{B_R/\tilde{b}_R(\mathbf{m})} \gg X^{-(c+1)\delta}M^{\Lambda(\psi+\frac{1}{2}b)+\gamma}.$$

But in view of (6.5), one has $\tilde{b}_R(\mathbf{m})/B_R = \rho_{\mathbf{m}}$, and thus we find that for some tuple $\mathbf{m} \in \{0,1\}^R$, one has

$$X^{-\Lambda}[[K^2_{a_R,b_R}(X)]] + M^{-12\cdot 3^Rb} \gg X^{-\rho_{\mathbf{m}}(c+1)\delta}M^{\Lambda\rho_{\mathbf{m}}(\psi+\frac{1}{2}b)+\rho_{\mathbf{m}}\gamma},$$

whence

$$X^{-\Lambda}[[K^2_{a_R,b_R}(X)]] + M^{-12\cdot 3^Rb} \gg X^{-c'\delta}M^{\Lambda\psi'+\gamma'}. \tag{6.12}$$

We next remove the term $M^{-12\cdot 3^Rb}$ on the left hand side of (6.12). We observe that the relations (6.4) ensure that $\tilde{b}_R(\mathbf{m}) \leqslant 3^R$, and hence (2.7) and (6.5) together reveal that $\rho_{\mathbf{m}} \leqslant \tilde{b}_R(\mathbf{m}) \leqslant 3^R$. By hypothesis, we have $X^{c\delta} < M^{1/2}$, whence $X^{c'\delta} \ll M^{3^R}$. Thus we deduce from (2.8) that

$$X^{-c'\delta}M^{\Lambda\psi'+\gamma'} \geqslant M^{-3^R+\rho_{\mathbf{m}}\gamma} \geqslant M^{-3^R-4\cdot 3^Rb}.$$

Since

$$M^{-12\cdot 3^Rb} < M^{-3^R-8\cdot 3^Rb},$$

it follows from (6.12) that

$$X^{-\Lambda}[[K^2_{a_R,b_R}(X)]] \gg X^{-c'\delta}M^{\Lambda\psi'+\gamma'}. \tag{6.13}$$

Our final task consists of extracting appropriate constraints on the parameters a_R and b_R. Here, a comparison of (5.3) to (5.5) with (6.2) to (6.4) reveals

that we may follow the argument leading from [11, equation (8.16)] to the conclusion of the proof of [11, Lemma 8.2], but substituting $1+\frac{2}{3}\Delta$ in place of $\sqrt{k}$ throughout. The reader should experience little difficulty in adapting the argument given therein to show that

$$k_{\mathbf{m}}b \leqslant b_R \leqslant k_{\mathbf{m}}b + 16\cdot 3^{2R}b,$$

and further that

$$a_R = b_{R-1} < b_R/(1+\tfrac{2}{3}\Delta).$$

Moreover, one may also verify that $(1+\frac{2}{3}\Delta)^R \leqslant k_{\mathbf{m}} \leqslant 3^R$, just as in the conclusion of the proof of [11, Lemma 8.2]. The estimate (6.7), with all associated conditions, therefore follows from (6.13) on taking $a' = a_R$ and $b' = b_R$. This completes our account of the proof of the lemma. □

7. The iterative process

We begin with a crude estimate of use at the conclusion of our argument.

Lemma 7.1. *Suppose that a and b are integers with $0 \leqslant a < b \leqslant (2\theta)^{-1}$. Then provided that $\Lambda \geqslant 0$, one has*

$$[[K_{a,b}^2(X)]] \ll X^{\Lambda+\delta}.$$

Proof. On considering the underlying Diophantine equations, we deduce from Lemma 2.1 that

$$K_{a,b}^2(X) \ll (J(X/M^a))^{1/3}(J(X/M^b))^{2/3},$$

whence

$$\begin{aligned}[[K_{a,b}^2(X)]] &\ll \frac{X^\delta\left((X/M^a)^{1/3}(X/M^b)^{2/3}\right)^{6+\Delta+\Lambda}}{(X/M^a)^{2+\Delta}(X/M^b)^4}\\ &\ll X^{\Lambda+\delta}M^{\frac{2}{3}\Delta(a-b)} \ll X^{\Lambda+\delta}.\end{aligned}$$

This completes the proof of the lemma. □

We now come to the crescendo of our argument.

Theorem 7.2. *Suppose that Δ is a positive number with $\Delta < \frac{1}{12}$. Then for each $\varepsilon > 0$, one has $J(X) \ll X^{6+\Delta+\varepsilon}$.*

Proof. We prove that $\Lambda \leqslant 0$, for then the conclusion of the lemma follows at once from (2.13). Assume then that $\Lambda \geqslant 0$, for otherwise there is nothing to prove. We begin by noting that as a consequence of Lemma 4.2, one finds from (2.10) and (2.12) that there exists an integer h_{-1} with $0 \leqslant h_{-1} \leqslant 4B$ such that

$$[[J(X)]] \ll M^{4B-4h_{-1}/3}[[K_{0,B+h_{-1}}^2(X)]].$$

We therefore deduce from (2.13) that

$$X^\Lambda \ll X^\delta[[J(X)]] \ll X^\delta M^{4B-4h_{-1}/3}[[K_{0,B+h_{-1}}^2(X)]]. \tag{7.1}$$

Next we define sequences (κ_n), (h_n), (a_n), (b_n), (c_n), (ψ_n) and (γ_n), for $0 \leqslant n \leqslant N$, in such a way that

$$(1+\tfrac{2}{3}\Delta)^R \leqslant \kappa_{n-1} \leqslant 3^R, \quad 0 \leqslant h_{n-1} \leqslant 16 \cdot 3^{2R} b_{n-1} \quad (n \geqslant 1), \tag{7.2}$$

and

$$X^{\Lambda} M^{\Lambda \psi_n} \ll X^{c_n \delta} M^{-\gamma_n} [[K^2_{a_n,b_n}(X)]]. \tag{7.3}$$

We note here that the sequences (a_n) and (b_n) are not directly related to our earlier use of these letters. Given a fixed choice for the sequences (a_n), (κ_n) and (h_n), the remaining sequences are defined by means of the relations

$$b_{n+1} = \kappa_n b_n + h_n, \tag{7.4}$$

$$c_{n+1} = (4/s_0)^R \kappa_n (c_n + 1), \tag{7.5}$$

$$\psi_{n+1} = (4/s_0)^R \kappa_n (\psi_n + \tfrac{1}{2} b_n), \tag{7.6}$$

$$\gamma_{n+1} = (4/s_0)^R \kappa_n \gamma_n. \tag{7.7}$$

We put

$$\kappa_{-1} = 3^R, \quad b_{-1} = 1, \quad a_0 = 0, \quad b_0 = B + h_{-1}$$
$$\psi_0 = 0, \quad c_0 = 1, \quad \gamma_0 = \tfrac{4}{3} h_{-1} - 4B,$$

so that both (7.2) and (7.3) hold with $n = 0$ as a consequence of our initial choice of κ_{-1} and b_{-1}, together with (7.1). We prove by induction that for each non-negative integer n with $n < N$, the sequences $(a_m)_{m=0}^n$, $(\kappa_m)_{m=0}^n$ and $(h_m)_{m=-1}^n$ may be chosen in such a way that

$$1 \leqslant b_n \leqslant \big(20 \cdot 3^{2R} R\theta\big)^{-1}, \quad \psi_n \geqslant 0, \quad \gamma_n \geqslant -4b_n, \quad 0 \leqslant c_n \leqslant (2\delta)^{-1}\theta, \tag{7.8}$$

$$0 \leqslant a_n \leqslant b_n/(1+\tfrac{2}{3}\Delta), \tag{7.9}$$

and so that (7.2) and (7.3) both hold with n replaced by $n+1$.

Let $0 \leqslant n < N$, and suppose that (7.2) and (7.3) both hold for the index n. We have already shown such to be the case for $n = 0$. We observe first that from (7.2) and (7.4), we find that $b_n \leqslant 4(17 \cdot 3^{2R})^n B$, whence by invoking (2.8), we find that for $0 \leqslant n \leqslant N$, one has $b_n \leqslant (20 \cdot 3^{2R} R\theta)^{-1}$. It is apparent from (7.5) and (7.6) that c_n and ψ_n are non-negative for all n. Observe also that since $s_0 \geqslant 4$ and $\kappa_m \leqslant 3^R$, then by iterating (7.5) we obtain the bound

$$c_n \leqslant 3^{Rn} + 3^R \left(\frac{3^{Rn} - 1}{3^R - 1} \right) \leqslant 3^{Rn+1}, \tag{7.10}$$

and by reference to (2.8), we discern that $c_n \leqslant (2\delta)^{-1}\theta$ for $0 \leqslant n < N$.

In order to bound γ_n, we recall that $s_0 \geqslant 4$ and iterate the relation (7.7) to deduce that

$$\gamma_m = (4/s_0)^{Rm} \kappa_0 \dots \kappa_{m-1} \gamma_0 \geqslant -4(4/s_0)^{Rm} \kappa_0 \dots \kappa_{m-1} B. \tag{7.11}$$

In addition, we find from (7.4) that for $m \geqslant 0$ one has $b_{m+1} \geqslant \kappa_m b_m$, so that an inductive argument yields the lower bound

$$b_m \geqslant \kappa_0 \dots \kappa_{m-1} b_0 \geqslant \kappa_0 \dots \kappa_{m-1} B. \tag{7.12}$$

Hence we deduce from (7.11) that $\gamma_m \geqslant -4(4/s_0)^{Rm}b_m > -4b_m$. Assembling this conclusion together with those of the previous paragraph, we have shown that (7.8) holds for $0 \leqslant n \leqslant N$.

At this point in the argument, we may suppose that (7.3), (7.8) and (7.9) hold for the index n. An application of Lemma 6.2 therefore reveals that there exist numbers κ_n, h_n and a_n satisfying the constraints implied by (7.2) with n replaced by $n+1$, for which the upper bound (7.3) holds for some a_n with $0 \leqslant a_n \leqslant b_n/(1+\frac{2}{3}\Delta)$, also with n replaced by $n+1$. This completes the inductive step, so that in particular (7.3) holds for $0 \leqslant n \leqslant N$.

We now exploit the bound just established. Since we have the upper bound $b_N \leqslant 4(17 \cdot 3^{2R})^N \leqslant (2\theta)^{-1}$, it is a consequence of Lemma 7.1 that

$$[\![K^2_{a_N,b_N}(X)]\!] \ll X^{\Lambda+\delta}.$$

By combining this with (7.3) and (7.11), we obtain the bound

$$X^{\Lambda}M^{\Lambda\psi_N} \ll X^{\Lambda+(c_N+1)\delta}M^{4\kappa_0\dots\kappa_{N-1}B(4/s_0)^{RN}}. \tag{7.13}$$

Meanwhile, an application of (7.10) in combination with (2.8) shows that $X^{(c_N+1)\delta} < M$. We therefore deduce from (7.13) that

$$\Lambda\psi_N \leqslant 4(4/s_0)^{RN}\kappa_0\dots\kappa_{N-1}B + 1.$$

On recalling (2.5) and (6.1), we see that

$$s_0^R \leqslant \frac{\theta_+^{R+1}}{\theta_+ - \theta_-} < \frac{(4+\frac{2}{3}\Delta)\theta_+^R}{(4+\frac{2}{3}\Delta) - (\frac{2}{3}+\frac{2}{3}\Delta)} < \tfrac{4}{3}\theta_+^R.$$

Thus, since R is sufficiently large, one finds that $s_0 < 4+2\Delta$. Notice here that $\kappa_n \geqslant (1+\frac{2}{3}\Delta)^R$ and

$$4/s_0 \geqslant 4/(4+2\Delta) = 1/(1+\tfrac{1}{2}\Delta).$$

Hence we deduce that

$$4(4/s_0)^{RN}\kappa_0\dots\kappa_{N-1}B \geqslant 4\left(\frac{1+\frac{2}{3}\Delta}{1+\frac{1}{2}\Delta}\right)^{RN}B \geqslant 1,$$

so that

$$\Lambda\psi_N \leqslant 9(4/s_0)^{RN}\kappa_0\dots\kappa_{N-1}B. \tag{7.14}$$

A further application of the lower bound $b_n \geqslant \kappa_0\dots\kappa_{n-1}B$, available from (7.12), leads from (7.6) and the bound $s_0 \geqslant 4$ to the relation

$$\begin{aligned}\psi_{n+1} &= (4/s_0)^R(\kappa_n\psi_n + \tfrac{1}{2}\kappa_n b_n)\\ &\geqslant (4/s_0)^R\kappa_n\psi_n + \tfrac{1}{2}(4/s_0)^R\kappa_0\dots\kappa_n B\\ &\geqslant (4/s_0)^R\kappa_n\psi_n + \tfrac{1}{2}(4/s_0)^{R(n+1)}\kappa_0\dots\kappa_n B.\end{aligned}$$

An inductive argument therefore delivers the lower bound

$$\psi_N \geqslant \tfrac{1}{2}N(4/s_0)^{RN}\kappa_0\dots\kappa_{N-1}B.$$

Thus we deduce from (7.14) that

$$\Lambda \leqslant \frac{9(4/s_0)^{RN}\kappa_0 \dots \kappa_{N-1}B}{\frac{1}{2}N(4/s_0)^{RN}\kappa_0 \dots \kappa_{N-1}B} = \frac{18}{N}.$$

Since we are at liberty to take N as large as we please in terms of Δ, we are forced to conclude that $\Lambda \leqslant 0$. In view of our opening discussion, this completes the proof of the theorem. □

Corollary 7.3. *For each $\varepsilon > 0$, one has $J(X) \ll X^{6+\varepsilon}$.*

Proof. We apply Theorem 7.2 with $\Delta = \frac{1}{2}\varepsilon$. Then for each $\varepsilon' > 0$, one has

$$J(X) \ll X^{6+\frac{1}{2}\varepsilon+\varepsilon'},$$

and the desired conclusion follows by taking $\varepsilon' = \frac{1}{2}\varepsilon$. □

As we discussed following (2.6) above, the conclusion of Corollary 7.3 establishes the main conjecture in full for $J_{s,3}(X)$, and thus the proof of Theorem 1.1 is complete.

8. Applications

We take the opportunity to report on some immediate applications of Theorem 1.1, with brief notes on the necessary arguments. In all cases, the methods of proof are standard for those with a passing familiarity with the area, the hard work having been accomplished with the proof of Theorem 1.1.

We begin by discussing the anticipated asymptotic formula for $J_s(X)$. Define the singular series

$$\mathfrak{S}_s = \sum_{q=1}^{\infty} \sum_{\substack{a_1=1 \\ (q,a_1,a_2,a_3)=1}}^{q} \sum_{a_2=1}^{q} \sum_{a_3=1}^{q} \Big| q^{-1} \sum_{r=1}^{q} e((a_1 r + a_2 r^2 + a_3 r^3)/q) \Big|^{2s},$$

and the singular integral

$$\mathfrak{J}_s = \int_{\mathbb{R}^3} \Big| \int_0^1 e(\beta_1\gamma + \beta_2\gamma^2 + \beta_3\gamma^3)\, \mathrm{d}\gamma \Big|^{2s} \mathrm{d}\boldsymbol{\beta}.$$

Theorem 8.1. *When $s \geqslant 7$, one has $J_s(X) \sim \mathfrak{S}_s\mathfrak{J}_s X^{2s-6}$.*

Proof. On recalling (2.1), it follows from orthogonality that the bound presented in Theorem 1.1 delivers the estimate

$$\oint |f(\boldsymbol{\alpha}; X)|^{12}\, \mathrm{d}\boldsymbol{\alpha} \ll X^{6+\varepsilon},$$

we find that the argument of the proof of [9, Theorem 1.2] detailed in [9, §9] applies without modification to establish the claimed asymptotic formula. □

We note that the elementary lower bound $J_s(X) \gg X^{2s-6}$ (see [5, equation (7.4)]), suffices to confirm that $\mathfrak{S}_s > 0$ and $\mathfrak{J}_s > 0$, since one has also the estimates $\mathfrak{S}_s \ll 1$ and $\mathfrak{J}_s \ll 1$ for $s \geqslant 7$.

For comparison, the methods of [3, Chapter V] and [5, Chapter 7] would combine to yield a conclusion analogous to Theorem 8.1, but subject to the hypothesis $s \geqslant 9$. Our recent work [10, Corollary 1.2] would permit this condition to be sharpened slightly to $s \geqslant 8$. Meanwhile, one may conjecture that for $1 \leqslant s \leqslant 5$, one should have $J_s(X) \sim s!X^s$. Such is known for $1 \leqslant s \leqslant 4$ (see especially [6]), but remains unproven for $s = 5$. The remaining even moment would be expected to satisfy a different asymptotic formula. Here, the philosophy underlying [6, Appendix] would suggest that $J_6(X) \sim CX^6$, with $C = 6! + \mathfrak{S}_6\mathfrak{J}_6$, this corresponding to a sum of the anticipated major arc contribution together with the solutions on linear spaces accounted for by the expected minor arc contribution. This seems presently to be far beyond our reach. Perhaps it is worth emphasising in this context that one has

$$0 < \mathfrak{S}_6 \ll 1 \quad \text{and} \quad 0 < \mathfrak{J}_6 \ll 1.$$

The second of these estimates is plain from the standard theory. For the first, one should use the quasi-multiplicative property of

$$\sum_{r=1}^{q} e((a_1r + a_2r^2 + a_3r^3)/q)$$

in order to divide the problem into a consideration of the situation where q is a prime p, or a prime power p^h with $h \geqslant 2$. In the latter case, standard estimates (see the proof of [5, Theorem 7.1]) show that

$$\sum_{\substack{a_1=1\\(p^h,a_1,a_2,a_3)=1}}^{p^h}\sum_{a_2=1}^{p^h}\sum_{a_3=1}^{p^h}\Big|p^{-h}\sum_{r=1}^{p^h} e((a_1r + a_2r^2 + a_3r^3)/p^h)\Big|^{12} \ll p^{3h}(p^{-h/3})^{12} \ll p^{-h}.$$

Meanwhile, when $h = 1$, one finds from [7] that

$$p^{-1}\sum_{r=1}^{p} e((a_1r + a_2r^2 + a_3r^3)/p) \ll p^{-1/2}(p, a_1, a_2, a_3)^{1/2},$$

whence

$$\sum_{\substack{a_1=1\\(p,a_1,a_2,a_3)=1}}^{p}\sum_{a_2=1}^{p}\sum_{a_3=1}^{p}\Big|p^{-1}\sum_{r=1}^{p} e((a_1r + a_2r^2 + a_3r^3)/p)\Big|^{12} \ll p^{-3}.$$

Thus we deduce that for a suitable fixed $A > 0$ one has

$$\mathfrak{S}_6 \ll \prod_p (1 + Ap^{-2}) \ll 1.$$

Finally, we consider a diagonal Diophantine system consisting of a cubic, quadratic and linear equation. When s is a natural number, and a_{ij} are integers for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant s$, we write

$$\phi_i(\mathbf{x}) = \sum_{j=1}^{s} a_{ij}x_j^i \quad (1 \leqslant i \leqslant 3),$$

and we consider the Diophantine system

$$\phi_i(\mathbf{x}) = 0 \quad (1 \leqslant i \leqslant 3). \tag{8.1}$$

We write $N(B)$ for the number of integral solutions of the system (8.1) with $|\mathbf{x}| \leqslant B$. We next define the (formal) real and p-adic densities associated with the system (8.1), following Schmidt [4]. When $L > 0$, define

$$\lambda_L(\eta) = \begin{cases} L(1 - L|\eta|), & \text{when } |\eta| \leqslant L^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

We then put

$$\mu_L = \int_{|\boldsymbol{\xi}| \leqslant 1} \prod_{i=1}^{3} \lambda_L(\phi_i(\boldsymbol{\xi})) \, \mathrm{d}\boldsymbol{\xi}.$$

The limit $\sigma_\infty = \lim_{L \to \infty} \mu_L$, when it exists, is called the *real density*. Meanwhile, given a natural number q, we write

$$M(q) = \mathrm{card}\{\mathbf{x} \in (\mathbb{Z}/q\mathbb{Z})^s : \phi_i(\mathbf{x}) \equiv 0 \pmod{q} \ (1 \leqslant i \leqslant 3)\}.$$

For each prime number p, we then put

$$\sigma_p = \lim_{H \to \infty} p^{H(3-s)} M(p^H),$$

provided that this limit exists, and we refer to σ_p as the *p-adic density*.

Theorem 8.2. *Let s be a natural number with $s \geqslant 13$. Suppose that a_{ij} $(1 \leqslant i \leqslant 3, \, 1 \leqslant j \leqslant s)$ are non-zero integers. Suppose, in addition, that the system of equations (8.1) possess non-singular real and p-adic solutions for each prime number p. Then one has*

$$N(B) \sim \sigma_\infty \Big(\prod_p \sigma_p \Big) B^{s-6}.$$

In particular, the system (8.1) satisfies the Hasse principle.

The argument of the proof here is essentially standard, mirroring that of the proof of Theorem 8.1, and we therefore offer no details. Here, the work of [3, Chapter V] combines with the methods of [5, Chapter 7] to deliver such a conclusion for $s \geqslant 17$. Our present work, in which we require only $s \geqslant 13$, achieves the limit imposed by the convexity barrier in this problem (see [1]). The latter is a practical requirement in applications of the circle method for higher degree problems imposed by square-root cancellation considerations for exponential sums, and in this instance requires the number of variables s to exceed twice the sum of degrees in the problem.

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