ON A CERTAIN NONARY CUBIC FORM AND RELATED EQUATIONS

R. C. Vaughan[1,2] and T. D. Wooley[1,3]

Dedicated to Professor Wolgang Schmidt on the occasion of his 60th birthday

1. Introduction

In analytic number theory there are many situations in which, for certain exponents k_j satisfying $1 \le k_1 < k_2 < \cdots < k_t$, one requires good estimates for the number, $S_s(P;\mathbf{k})$, of solutions to the system of equations

$$\sum_{i=1}^{s} \left(x_i^{k_j} - y_i^{k_j} \right) = 0 \quad (j = 1, \dots, t), \tag{1.1}$$

with $x_i, y_i \in [1, P] \cap \mathbb{Z}$. Thus, in the case $k_j = j$, estimates for $S_s(P;\mathbf{k})$, usually referred to as "Vinogradov's Mean Value Theorem", are central to the establishment of rather general estimates for exponential sums. These in turn lead to a number of theorems which, in the current state of knowledge, provide the best results available to us. For example, the best zero free region for the Riemann zeta function is obtained in this way, as is the best upper bound, when the exponent k is large, for the smallest number $\widetilde{G}(k)$ of variables for which the asymptotic formula holds in Waring's problem.

Besides their rôle in analytic number theory, estimates for the number of solutions of such systems as (1.1) provide useful insights into the distribution of rational points on certain algebraic varieties. For while the Hardy-Littlewood method will establish an asymptotic formula for the number of rational points, up to a given large height, lying on a variety satisfying suitable conditions (see Schmidt [29], and also Birch [3] for weaker results), such conditions are usually somewhat restrictive. In particular, the number of variables in the defining equations must be sufficiently large in terms of their degrees, and also sufficiently large in terms of the dimension of the singular locus. Thus in the present state of knowledge, the Hardy-Littlewood method fails, by a considerable margin, to resolve the conjectures of Manin et al.

[1] Research supported in part by The Isaac Newton Institute for Mathematical Sciences
[2] Supported by a SERC Senior Fellowship
[3] Supported by NSF grant DMS-9303505, an Alfred P. Sloan Research Fellowship and a Fellowship from the David and Lucile Packard Foundation.

concerning the distribution of rational points on algebraic varieties, and in particular Fano varieties (see [1], [12], [27] and also [26, Chapter X, Conjecture 4.3]). Moreover, as will be apparent from a brief perusal of the latter references, the classes of varieties which supply evidence for such conjectures are mostly of a special type. In the light of such observations, we wish to promote the systems (1.1) as generating interesting varieties with which to test ideas concerning the distribution of rational points. While standard conjectures on exponential sums lead, via the circle method, to an asymptotic formula for the number of integer solutions, when $s > k_1 + \cdots + k_t$, of the system (1.1) inside a large box (which can be proved unconditionally when s is sufficiently large compared to k_t), the situation in the contrary case is far less clear. One strongly suspects that the diagonal solutions, in which the x_i are a permutation of the y_j, should provide the majority of the solutions. We are therefore led to consider the quasi-projective variety defined by the system of equations (1.1) under the condition that $x_i = y_j$ for no i and j. In general such varieties have interesting and non-trivial behaviour in the context of the above-mentioned conjectures, and yet their inherent symmetry enhances our prospects of providing a useful analysis of the distribution of rational points. In particular, these varieties remain at least partially accessible to examination through the Hardy-Littlewood method (as will become clear in the Appendix).

The majority of this paper will be devoted to an analysis of one of the simplest non-trivial systems of the form (1.1), namely the case with $s = 3$, $t = 2$, $k_1 = 1$ and $k_2 = 3$. Thus we shall be concerned with estimates for $S_3(P; 1, 3)$, the number of solutions of the system of equations

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 &= y_1^3 + y_2^3 + y_3^3, \\ x_1 + x_2 + x_3 &= y_1 + y_2 + y_3, \end{aligned} \tag{1.2}$$

with

$$1 \le x_i, y_i \le P \quad (1 \le i \le 3). \tag{1.3}$$

Hua [24, Lemma 5.2] has shown that

$$S_3(P; 1, 3) \ll P^3 (\log P)^9, \tag{1.4}$$

and Heath-Brown [17] makes fundamental use of a slightly weaker estimate in important work leading to improvements in classical results of Weyl (see [32, Lemma 2.4]) and Hua (see [32, Lemma 2.5]). In particular, by obtaining new estimates for the exponential sum $\sum_{1 \le x \le P} e(\alpha x^k)$, Heath-Brown establishes the upper bound $\tilde{G}(k) \le \frac{7}{8} 2^k + 1$ for $k \ge 6$. More recently Boklan [4], in work showing inter alia that $\tilde{G}(k) \le \frac{7}{8} 2^k$ when $k \ge 6$, has required an inequality sharper than (1.4).

We shall say that a solution $\mathbf{x}, \mathbf{y}$ to the system (1.2) is *trivial* when (y_1, y_2, y_3) is a permutation of (x_1, x_2, x_3). It follows that the number, $T(P)$, of trivial solutions of (1.2) with (1.3) satisfies

$$T(P) = 6P^3 + O(P^2). \tag{1.5}$$

Our interest, therefore, lies in bounding the number, $U(P)$, of non-trivial solutions of (1.2) with (1.3). In §§2 and 3 we provide a comparatively easy argument which shows that $U(P)$ is substantially smaller than $T(P)$, an immediate consequence of which is the following theorem.

Theorem 1.1. $S_3(P;1,3) = 6P^3 + O\left(P^{7/3}(\log 2P)^{11}\right)$.

Having established an asymptotic formula for $S_3(P;1,3)$, we are naturally led to speculate concerning the nature of an asymptotic formula (should one exist) for the number of non-trivial solutions, $U(P)$. In line with our general philosophy on the nature of the Hardy-Littlewood method, we conjecture that the number of non-trivial solutions of a system of equations inside a large box should be given by the major arc contribution from the circle method. Precisely what we mean by "non-trivial" and "major arc contribution" is made clear in the Appendix, where we show that our conjecture implies that for large P,

$$U(P) \asymp P^2(\log P)^5. \tag{1.6}$$

In §§4, 5 and 6 we develop a treatment which establishes an upper bound of the same order of magnitude as that predicted in (1.6). Moreover, in §7 we are able to establish a lower bound for $U(P)$, also of the same order of magnitude as that predicted in (1.6). These two estimates we summarise in the following theorem.

Theorem 1.2. *The number $U(P)$ of non-trivial solutions of (1.2) with (1.3) satisfies*

$$P^2(\log P)^5 \ll U(P) \ll P^2(\log P)^5. \tag{1.7}$$

It seems likely that the methods we develop to establish (1.7) could be refined so as to obtain an asymptotic formula for $U(P)$. However, such an undertaking is likely to be one of great complexity, and must be deferred to another occasion in the interest of containing our circumlocutions. The determination of an asymptotic formula of the form (1.6) would, nonetheless, be of considerable importance in ascertaining the extent to which our conjectures concerning the behaviour of the circle method may be true. In this context we note that it transpires that the non-trivial solutions of (1.2) are related to the solutions of the nonary cubic form

$$\det\left(a_{ij}\right)_{1\le i,j\le 3} = 0, \tag{1.8}$$

with the entries constrained to lie in peculiar hyperbolical regions. Moreover Katznelson [25] has developed a method which establishes an asymptotic formula for the number of solutions of (1.8) with the variables constrained to lie in a homothetically expanding convex set. As it stands, unfortunately, Katznelson's treatment is insufficient to obtain (1.7), but there may well be improvements which might lead to an explicit asymptotic formula.

We are able to apply the ideas of §§2 and 3 to certain systems of equations more general than (1.2), thereby showing that the number of non-trivial solutions is of smaller order of magnitude than the number of trivial solutions. By developing a method which better exploits the intermediate equations, we are able to draw stronger conclusions. Thus, in §8, we consider the system of the form (1.1) with $\mathbf{k} = \mathbf{a}(s)$, where $\mathbf{a}(s)$ denotes the $(s-1)$-tuple $(k_1, \dots, k_{s-1})$ with $k_j = j$ $(1 \le j \le s-2)$, and with $k_{s-1} = s$. We are able to establish the asymptotic formula contained in the following theorem.

Theorem 1.3. *Let $T_s(P)$ denote the number of solutions of (1.1) counted by $S_s(P; \mathbf{a}(s))$ in which the x_i are a permutation of the y_j, so that, in particular, $T_s(P) = s!P^s + O_s(P^{s-1})$. Then for each $s \ge 3$,*

$$S_s(P; \mathbf{a}(s)) - T_s(P) \ll_{\varepsilon,s} P^{s-1+1/s+\varepsilon}, \tag{1.9}$$

and

$$S_s(P; \mathbf{a}(s)) - T_s(P) \ll_{\varepsilon,s} P^{(s+3)/2+\varepsilon}. \tag{1.10}$$

We remark that the estimate (1.10) is superior to (1.9) when $s \ge 5$. For comparison, Hua [24, Lemma 5.2] had previously obtained the upper bound

$$S_s(P; \mathbf{a}(s)) \ll_s P^s (\log P)^{s(2^{s-1}-1)}.$$

We remark that with some additional effort, the factors of P^ε occurring in the estimates (1.9) and (1.10) could be replaced by a power of $\log P$.

A second class of systems of the type (1.1) in which our methods apply is a special case of Vinogradov's Mean Value Theorem. Consider the system of the form (1.1) with $\mathbf{k} = \mathbf{b}(s)$, where $\mathbf{b}(s)$ denotes the $(s-1)$-tuple $(1, 2, \dots, s-1)$. In §9 we establish the following asymptotic formula.

Theorem 1.4. *Let $T_s(P)$ denote the number of solutions of (1.1) counted by $S_s(P; \mathbf{b}(s))$ in which the x_i are a permutation of the y_j, so that, in particular, $T_s(P) = s!P^s + O_s(P^{s-1})$. Then for each $s \ge 4$,*

$$S_s(P; \mathbf{b}(s)) - T_s(P) \ll_{\varepsilon,s} P^{(s+4)/2+\varepsilon}.$$

For comparison, Hua [24, Lemma 5.4] had previously obtained the upper bound

$$S_s(P; \mathbf{b}(s)) \ll_s P^s (\log P)^{2^{s-1}-1}.$$

In addition, Rogovskaya [28], improving on work of Bykovskii, has shown that

$$S_3(P; 1, 2) = \frac{18}{\pi^2} P^3 \log P + O(P^3).$$

Once again, the P^ε occurring in the statement of Theorem 1.4 may be replaced by a power of $\log P$ with little difficulty. In §10 we obtain sharper conclusions in the case $s = 4$.

Theorem 1.5. *Let $T_4(P)$ denote the number of solutions of (1.1) counted by $S_4(P;1,2,3)$ in which the x_i are a permutation of the y_j. Then*

$$P^2 \log P \ll S_4(P;1,2,3) - T_4(P) \ll P^{10/3}(\log 2P)^{35}.$$

There are other situations in which one might expect that the number of non-trivial solutions should be relatively small, and possibly zero. Perhaps the simplest example to which our methods are not obviously applicable is the system of the form (1.1) with $s = 3$, $t = 2$, $k_1 = 1$ and $k_2 = 4$. In an interesting development, so as to tackle this and associated situations, Greaves [15] has developed a quite different method, related to the sieving methods of Greaves [14] and Hooley [18, 19, 20, 21]. For each $h \geq 4$, Greaves obtains the asymptotic formula

$$S_3(P;1,h) = 6P^3 + O_{\varepsilon,h}\left(P^{17/6+\varepsilon}\right).$$

It is possible that in the systems of the type (1.1) with $\mathbf{k} = \mathbf{a}(s)$ or $\mathbf{b}(s)$, there are many values of s for which there are no non-diagonal solutions, so that the x_i are simply a permutation of the y_j in any such solution of (1.1). Indeed, it is possible that such is the case for all large s, but this is not at all clear. It is plain that each non-diagonal solution $(\mathbf{x},\mathbf{y}) = (\mathbf{u},\mathbf{v})$ generates $V(P;\mathbf{u},\mathbf{v})$ non-diagonal solutions in a box of the form (1.3), where $V(P;\mathbf{u},\mathbf{v}) \gg_{\mathbf{u},\mathbf{v}} P$, simply by considering the points $(\lambda\mathbf{u},\lambda\mathbf{v})$ $(\lambda \in \mathbb{N})$. Thus the current state of knowledge concerning the problem of Prouhet and Tarry (see Theorem 411 and the note on page 339 of [16]) suffices to show that the number of non-diagonal solutions counted by $S_s(P;\mathbf{b}(s))$ is $\gg_s P$ when $3 \leq s \leq 10$. In §11 we make some further comments concerning lower bounds for the number of non-diagonal solutions of several other systems of the type (1.1).

The authors thank the many people with whom they have had stimulating discussions concerning rational points on algebraic varieties, and in particular Professors E. Bombieri, Z. Rudnick and P. Sarnak. The authors also thank Professor Kent Boklan, whose work initially drew us into the topics described above, and Professor J. Thunder, who brought to our attention the work of Katznelson.

Throughout, $\ll$ and $\gg$ denote Vinogradov's well-known notation, and we write $f \asymp g$ to denote that $f \ll g$ and $f \gg g$. For the sake of conciseness, we make frequent use of vector notation. Thus, for example, we abbreviate $(c_1, \ldots, c_t)$ to $\mathbf{c}$. We write $d(\cdot)$ for the divisor function, $\phi(\cdot)$ for the Euler totient function, and $(a_1, \ldots, a_s)$ for the greatest common divisor of $a_1, \ldots, a_s$. Finally, we write $e(u)$ for $e^{2\pi i u}$, and $e_q(u)$ for $e(u/q)$.

2. The nonary cubic form

As our first step in the analysis of the non-trivial solutions of the system (1.2), we make a transformation which generates the "nonary cubic form" of the title. In addition to merely defining this transformation, in this section we shall also determine precisely the region in which the transformed variables lie.

Let us first observe that if in (1.2) we have $x_i = y_j$ for some i and j with $1 \le i, j \le 3$, then it follows that (x_1, x_2, x_3) is a permution of (y_1, y_2, y_3). Thus, in the analysis of the non-trivial solutions of the system (1.2), we may suppose that $x_i = y_j$ for no i and j. Next, on factorising the polynomial $x_1^3 + x_2^3 + x_3^3 - (x_1 + x_2 + x_3)^3$, we deduce that the system (1.2) is equivalent to the pair of equations

$$(x_2 + x_3)(x_3 + x_1)(x_1 + x_2) = (y_2 + y_3)(y_3 + y_1)(y_1 + y_2),$$
$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3.$$

Write $s(\mathbf{z}) = z_1 + z_2 + z_3$, and make the substitutions $X_i = s(\mathbf{x}) - x_i$ and $Y_i = s(\mathbf{y}) - y_i$ $(i = 1, 2, 3)$. Then we find that the solutions of (1.2) subject to (1.3) are in one-to-one correspondence with the solutions of the system

$$\begin{aligned} X_1X_2X_3 &= Y_1Y_2Y_3, \\ X_1 + X_2 + X_3 &= Y_1 + Y_2 + Y_3 \equiv 0 \pmod 2 \end{aligned} \tag{2.1}$$

subject to

$$0 < s(\boldsymbol{\alpha}) - 2\alpha_i \le 2P \quad (i = 1, 2, 3; \alpha = X, Y). \tag{2.2}$$

Moreover the condition that a solution $\mathbf{x}, \mathbf{y}$ be non-trivial corresponds to the condition that $X_i = Y_j$ for no i and j with $1 \le i, j \le 3$. For if $X_i = Y_j$ for some i and j, then it follows that (X_1, X_2, X_3) is a permutation of (Y_1, Y_2, Y_3), and hence that $x_i = y_j$ for some i and j.

The multiplicative structure of the cubic equation in (2.1) leads us to a very natural reduction, namely the removal of common factors between the variables. Let us therefore write

$$\left.\begin{array}{lll} d_1 = (Y_1, X_1), & d_2 = (Y_1/d_1, X_2), & d_3 = Y_1/(d_1d_2), \\ e_1 = (Y_2, X_1/d_1), & e_2 = (Y_2/e_1, X_2/d_2), & e_3 = Y_2/(e_1e_2), \\ f_1 = (Y_3, X_1/(d_1e_1)), & f_2 = (Y_3/f_1, X_2/(d_2e_2)), & f_3 = Y_3/(f_1f_2). \end{array}\right\} \tag{2.3}$$

Then the first equation of (2.1) becomes

$$\frac{X_1X_2}{d_1e_1f_1d_2e_2f_2}X_3 = d_3e_3f_3.$$

But we have

$$\left(\frac{X_1X_2}{d_1e_1f_1d_2e_2f_2}, d_3e_3f_3\right) = 1,$$

and thus we deduce that

$$X_1 = d_1e_1f_1, \quad X_2 = d_2e_2f_2, \quad X_3 = d_3e_3f_3. \tag{2.4}$$

Furthermore, from (2.3),

$$Y_1 = d_1d_2d_3, \quad Y_2 = e_1e_2e_3, \quad Y_3 = f_1f_2f_3. \tag{2.5}$$

On substitution, the linear equation in (2.1) reduces to the nonary cubic form

$$d_1e_1f_1 + d_2e_2f_2 + d_3e_3f_3 = d_1d_2d_3 + e_1e_2e_3 + f_1f_2f_3 \equiv 0 \pmod 2, \tag{2.6}$$

and the conditions (2.2) lead to the inequalities

$$0 < d_1e_1f_1 + d_2e_2f_2 + d_3e_3f_3 - 2d_ie_if_i \le 2P \quad (i = 1, 2, 3), \tag{2.7}$$

and

$$0 < d_1d_2d_3 + e_1e_2e_3 + f_1f_2f_3 - 2\alpha_1\alpha_2\alpha_3 \le 2P \quad (\alpha = d, e, f), \tag{2.8}$$

on the positive variables d_i, e_i and f_i $(i = 1, 2, 3)$. Moreover the condition that a solution be non-trivial is equivalent to the condition that

$$d_ie_if_i \ne \alpha_1\alpha_2\alpha_3 \quad (i = 1, 2, 3; \alpha = d, e, f), \tag{2.9}$$

and the definitions (2.3) imply that

$$(d_2d_3, e_1f_1) = (d_3, e_2f_2) = (e_2e_3, f_1) = (e_3, f_2) = 1. \tag{2.10}$$

It is a simple matter to check that the solutions of (2.6) satisfying the conditions (2.7)-(2.10) are in one-to-one correpsondence with the non-trivial solutions of (1.2) subject to (1.3). As we shall explain in the next section, this observation alone is sufficient to obtain a non-trivial estimate for $U(P)$.

As our concluding remark in this section, we note that the equation in (2.6) may be written in the form

$$\det \begin{pmatrix} d_1 & f_2 & e_3 \\ f_3 & e_1 & d_2 \\ e_2 & d_3 & f_1 \end{pmatrix} = 0,$$

justifying our comment in the introduction concerning the equation (1.8).

3. The simplest upper bound

In this section we prove Theorem 1.1 by establishing an upper bound for $U(P)$. Consider first the number of non-trivial solutions, $U_1(P)$, of the system (2.1) satisfying (2.2), subject to the additional conditions

$$Y_1 = \max_{i=1,2,3} \{X_i, Y_i\}, \quad (Y_1, X_1) = \max_{j=1,2,3} (Y_1, X_j), \tag{3.1}$$

and

$$\left(\frac{Y_1}{(X_1, Y_1)}, X_2 \right) \ge \left(\frac{Y_1}{(X_1, Y_1)}, X_3 \right). \tag{3.2}$$

By relabelling variables, we find that

$$U(P) \ll U_1(P). \tag{3.3}$$

Further, by following through the reduction (2.3), it follows that

$$U_1(P) \le U_2(P), \tag{3.4}$$

where $U_2(P)$ denotes the number of solutions of (2.6) with the parity condition discarded, and with the variables satisfying the inequalities

$$1 \le d_i e_i f_i \le 2P \quad (i = 1, 2, 3) \quad \text{and} \quad 1 \le \alpha_1\alpha_2\alpha_3 \le 2P \quad (\alpha = d, e, f). \tag{3.5}$$

Moreover the conditions (3.1) and (3.2), together with (2.9), impose the additional restrictions

$$d_1 d_2 d_3 > d_i e_i f_i \quad (i = 1, 2, 3), \tag{3.6}$$

and

$$d_1 \ge d_2 \ge d_3. \tag{3.7}$$

We take P to be a real number with $P \ge 1$, and write $Q = 2P$. On multiplying the equation in (2.6) through by d_3 and rearranging terms, we obtain

$$(d_1d_3 - e_2f_2)(d_2d_3 - e_1f_1) = (e_3d_3 - f_1f_2)(f_3d_3 - e_1e_2). \tag{3.8}$$

In view of (3.6), given d_1,d_2, d_3, e_1, e_2, f_1 and f_2 in any solution $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ counted by $U_2(P)$, the number of choices for e_3 and f_3 is at most

$$d\left((d_1d_3 - e_2f_2)(d_2d_3 - e_1f_1)\right) \le d(d_1d_3 - e_2f_2) \cdot d(d_2d_3 - e_1f_1).$$

We now note that since $d_1d_2d_3 \le Q$, it follows from (3.7) that

$$d_3 \le Q^{1/3}. \tag{3.9}$$

Therefore, from (3.3) and (3.4) we have

$$U(P) \ll \sum_{d_1e_1f_1 \le Q} \sum_{d_2e_2f_2 \le Q} \sum_{d_3 \le Q^{1/3}} d(d_1d_3 - e_2f_2) \cdot d(d_2d_3 - e_1f_1),$$

where the summation is restricted to those values of d_3 satisfying (3.6). Then by Cauchy's inequality, and symmetry,

$$\begin{aligned} U(P) &\ll \sum_{d_1e_1f_1 \le Q} \sum_{d_2e_2f_2 \le Q} \sum_{\substack{d_3 \le Q^{1/3} \\ d_1d_3 > e_2f_2}} d(d_1d_3 - e_2f_2)^2 \\ &\le \sum_{e_1f_1 \le Q} \sum_{d_2e_2f_2 \le Q} \sum_{e_2f_2 < m \le Q^{4/3}/(e_1f_1)} d(m)d(m - e_2f_2)^2. \end{aligned}$$

Thus, by Hölder's inequality, symmetry, and standard estimates for divisor sums,

$$\begin{aligned} U(P) &\ll \sum_{e_1f_1 \le Q} \sum_{d_2e_2f_2 \le Q} \sum_{m \le Q^{4/3}/(e_1f_1)} d(m)^3 \\ &\ll \sum_{e_1f_1 \le Q} \sum_{n \le Q} d_3(n) \frac{Q^{4/3}}{e_1f_1} (\log Q)^7 \\ &\ll Q^{7/3} (\log Q)^9 \sum_{e_1f_1 \le Q} (e_1f_1)^{-1}. \end{aligned}$$

Thus $U(P) \ll P^{7/3}(\log 2P)^{11}$, and Theorem 1.1 follows immediately.

4. A refined upper bound: preliminaries

The object of the next three sections will be to refine the argument of §§2 and 3 so as to provide the upper bound contained in Theorem 1.2. We first note that it follows from those sections that $U(P) \ll U_3(2P)$, where $U_3(Q)$ denotes the number of solutions of (3.8) subject to (3.5), (3.6), (3.7), (3.9), (2.10), and the additional conditions

$$2d_i e_i f_i < d_1 e_1 f_1 + d_2 e_2 f_2 + d_3 e_3 f_3, \tag{4.1}$$

and

$$d_1 d_2 d_3 < \min\{e_1 e_2 e_3 + f_1 f_2 f_3,\ Q\}. \tag{4.2}$$

Note that the last two conditions follow from (2.7) and (2.8). For each solution **d**, **e**, **f** counted by $U_3(Q)$, write

$$u = d_1 d_3 - e_2 f_2, \quad v = d_2 d_3 - e_1 f_1, \quad w = e_3 d_3 - f_1 f_2, \quad x = f_3 d_3 - e_1 e_2. \tag{4.3}$$

Then the equation (3.8) becomes

$$uv = wx, \tag{4.4}$$

and (2.10) may be replaced by the condition

$$(d_3, e_1 e_2 f_1 f_2) = (v, e_1 f_1) = (w, f_1 f_2) = (e_2, f_1) = 1. \tag{4.5}$$

Moreover the first of these coprimality conditions implies that

$$(uv, d_3) = 1. \tag{4.6}$$

We now use the idea of removing common factors between the variables in equation (4.4). First observe that (3.6) implies that u and v are both positive. Let

$$r = (u, w), \quad y = u/r, \quad z = |w|/r, \quad \text{and} \quad z^* = w/r. \tag{4.7}$$

On substituting into (4.4), we deduce that $yv = z|x|$. But $(y, z) = 1$, and so $z|v$. We therefore write

$$s = v/z, \quad \text{and} \quad s^* = v/z^*, \tag{4.8}$$

in view of which $|x| = ys$ and $x = ys^*$. By (4.7) and (4.8), since u and v are both positive, the variables y, r, s and z are also positive. Moreover from (4.3), the latter variables satisfy the equations

$$yr + e_2 f_2 = d_1 d_3, \quad zs + e_1 f_1 = d_2 d_3, \quad z^* r + f_1 f_2 = e_3 d_3, \quad ys^* + e_1 e_2 = f_3 d_3. \tag{4.9}$$

On substituting from (4.9), we deduce from (4.5) and (4.6) that

$$(d_3, e_1 e_2 f_1 f_2 yrzs) = (zs, e_1 f_1) = (zr, f_1 f_2) = (e_2, f_1) = 1. \tag{4.10}$$

Furthermore, the inequality (4.2) implies that

$$(yr+e_2f_2)(zs+e_1f_1)<f_1f_2(ys^*+e_1e_2)+e_1e_2(z^*r+f_1f_2),$$

and consequently

$$yrzs+(re_1-s^*f_2)(yf_1-z^*e_2)<e_1e_2f_1f_2. \tag{4.11}$$

Finally, we observe that the condition (3.6) together with (4.9) leads to the inequality

$$(yr+e_2f_2)(zs+e_1f_1)>(z^*r+f_1f_2)(ys^*+e_1e_2).$$

Thus,

$$yre_1f_1+zse_2f_2>z^*re_1e_2+ys^*f_1f_2,$$

and we deduce that

$$(re_1-s^*f_2)(yf_1-z^*e_2)>0. \tag{4.12}$$

It is clear from the last line of the preceding paragraph that the solutions counted by $U_3(Q)$ may be classified according to the signs of s^* and z^*. Thus we deduce that

$$U(P)\ll U^+(2P)+U^-(2P), \tag{4.13}$$

where $U^+(Q)$ denotes the number of positive solutions $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$, y, r, z, s of (4.9) satisfying (3.5), (3.6), (3.7), (3.9), (4.1), (4.2), (4.10), (4.11), (4.12), and s^* and z^* *positive*, and $U^-(Q)$ denotes the corresponding number of solutions with s^* and z^* *negative*.

The argument we use to bound $U^\pm(Q)$ depends for its success on suitable estimates for certain divisor sums of special type, these arising through the equations (4.9). We record here a number of estimates for such sums.

Lemma 4.1. *Suppose that $X\ge 1$, $Y\ge 1$, $Z\ge 1$, $a\in\mathbb{Z}$, $b\in\mathbb{Z}$ and $d\in\mathbb{N}$. Let $S(d)$ denote the number of ordered pairs of natural numbers x and y with $xy\le Z$, $d\le x\le X$, $d\le y\le Y$, $x\equiv a \pmod d$ and $y\equiv b \pmod d$. Then*

$$S(d)\ll d^{-2}\min\{Z,XY\}\log\left(\frac{2\min\{Zd^{-1},X\}\min\{Zd^{-1},Y\}}{\min\{Z,XY\}}\right). \tag{4.14}$$

Proof. We may plainly suppose that $0\le a,b<d$. Then on writing $u=(x-a)/d$ and $v=(y-b)/d$, we have

$$1\le u\le Xd^{-1},\quad 1\le v\le Yd^{-1}\quad\text{and}\quad uv\le Zd^{-2}. \tag{4.15}$$

A comparison of the definition of $S(d)$ with (4.15) reveals that (4.14) holds for arbitrary d so long as it holds for $d=1$, which we henceforth assume. Also, since when $Z\ge XY$ and $d=1$ the estimate (4.14) is trivial, we may suppose that

$Z < XY$. Moreover, by dividing into cases according to whether $x \le Z/Y$ or $x > Z/Y$, in this latter case we have

$$S(1) = \sum_{1 \le x \le Z/Y} \sum_{1 \le y \le Y} 1 + \sum_{Z/Y < x \le \min\{Z,X\}} \sum_{1 \le y \le Z/x} 1 \ll Z + Z \log\left(\frac{2\min\{Z,X\}}{\max\{Z/Y,1\}}\right),$$

whence the lemma follows once again.

Lemma 4.2. *Suppose that $X > 0$, $Y > 0$, $Z \ge 1$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ and $d \in \mathbb{N}$. Let $T(d)$ denote the number of ordered pairs of natural numbers x and y with $xy \le Z$, $x > X$, $y > Y$, $\min\{x,y\} \ge d$, $x \equiv a \pmod d$ and $y \equiv b \pmod d$. Then*

$$T(d) \ll Zd^{-2} \log\left(\frac{2\max\{Z,XY\}}{XY}\right). \tag{4.16}$$

Proof. Again we may suppose that $0 \le a, b < d$. On writing $u = (x-a)/d$ and $v = (y-b)/d$, we have

$$uv \le Zd^{-2}, \quad (1+u)d > X, \quad (1+v)d > Y, \quad u \ge 1 \quad \text{and} \quad v \ge 1. \tag{4.17}$$

Thus

$$u > X/(2d) \quad \text{and} \quad v > Y/(2d), \tag{4.18}$$

and a comparison of the definition of $T(d)$ with (4.17) and (4.18) reveals that (4.16) holds for d so long as it holds for $d = 1$, which we henceforth assume. Also, since when $Z \le XY$ the number $T(1)$ is zero, we may suppose that $XY < Z$. Then in order to be counted by $T(1)$, the pair (x,y) must satisfy $X < x < Z/Y$ and $Y < y \le Z/x$. Hence

$$T(1) \le \sum_{X < x < Z/Y} Z/x \ll Z \log \frac{2Z}{XY},$$

and the proof of the lemma is complete.

Lemma 4.3. *When n is a natural number, let $U(n,X)$ denote the number of positive integral solutions of the equation $xy + zt = n$ with $xy \le X$. Then*

$$U(n,X) \ll (Xn)^{1/2} + (\log 2X)(\log 2n)\min\{n,X\} \sum_{u|n} u^{-1}.$$

Proof. By symmetry we may suppose that $x \le \sqrt{X}$ and $z \le \sqrt{n}$. Thus, on taking $u = (x,z)$, we have

$$U(n,X) \ll \sum_{u|n} \sum_{v \le \sqrt{X}/u} \sum_{\substack{w \le \sqrt{n}/u \\ (v,w)=1}} M(n;u,v,w), \tag{4.19}$$

where $M(n;u,v,w)$ denotes the number of integral solutions of the equation

$$vy + wt = n/u \tag{4.20}$$

with

$$1 \le y \le X/(uv) \quad \text{and} \quad 1 \le t \le n/uw. \tag{4.21}$$

If a solution of (4.20) exists, say (y_0, t_0), then any other solution satisfies $v(y-y_0) = w(t_0 - t)$. Since $(v,w) = 1$, the solutions are therefore given by $y = y_0 + \lambda w$ and $t = t_0 - \lambda v$, for some integer λ. In view of (4.21), λ must lie in an interval of length at most $(uvw)^{-1}\min\{n, X\}$, and consequently

$$M(n;u,v,w) \ll 1 + (uvw)^{-1}\min\{n, X\}. \tag{4.22}$$

The lemma now follows immediately from (4.19) and (4.22).

Lemma 4.4. *Let*

$$V = \sum_{\substack{x,y,z,t\\ xy+zt=n}} \log\left(\frac{2n}{xy}\right).$$

Then

$$V \ll n(\log 2n)^2 \sum_{u|n} u^{-1}.$$

Proof. By using Riemann-Stieltjes integration, and integrating by parts, we obtain, in the notation of Lemma 4.3,

$$V = \int_1^{2n} U(n,X)X^{-1}dX.$$

Hence, by that lemma,

$$\begin{aligned} V &\ll n + \sum_{u|n} u^{-1} \int_1^{2n} (\log 2X)(\log 2n)\min\{n,X\}X^{-1}dX \\ &\ll n(\log 2n)^2 \sum_{u|n} u^{-1}, \end{aligned}$$

which completes the proof.

Before we proceed to estimate $U^{\pm}(Q)$, we shall record a useful inequality which is satisfied by solutions counted by $U^{\pm}(Q)$. By (3.5) and (3.6) we have

$$e_1e_2f_1f_2 = \frac{(d_1e_1f_1)(d_2e_2f_2)d_3}{d_1d_2d_3} < Qd_3. \tag{4.23}$$

5. A refined upper bound: the negative case

We now estimate the number of solutions counted by $U^-(Q)$, so that both s^* and z^* are negative. In this case the inequality (4.11) becomes

$$yrzs + (re_1 + sf_2)(yf_1 + ze_2) < e_1e_2f_1f_2, \tag{5.1}$$

and consequently

$$yr < e_2f_2, \quad zs < e_1f_1, \quad ys < e_1e_2, \quad zr < f_1f_2. \tag{5.2}$$

Also, in view of (4.9),

$$e_2f_2 < d_1d_3 < 2e_2f_2 \quad \text{and} \quad e_1f_1 < d_2d_3 < 2e_1f_1. \tag{5.3}$$

The proof now divides into a number of subcases.

(A)(i) $y > d_3$ and $s > d_3$, or (ii) $r > d_3$ and $z > d_3$. Since the analysis in each case is similar, we shall restrict attention to the solutions of type (i). From (4.9) and (4.10), we have

$$y \equiv -r^{-1}e_2f_2 \pmod{d_3} \quad \text{and} \quad s \equiv -z^{-1}e_1f_1 \pmod{d_3}.$$

Further, on combining (5.2) with our initial hypothesis,

$$d_3 < y < e_2f_2/r, \quad d_3 < s < e_1f_1/z \quad \text{and} \quad sy < e_1e_2.$$

Then given d_3, r, z, e_1, e_2, f_1, f_2, we may apply Lemma 4.1, making use of the last inequality in (5.2), to show that the number of possible choices for s and y is

$$\ll d_3^{-2}e_1e_2 \log\left(\frac{2f_1f_2}{rz}\right). \tag{5.4}$$

On recalling (4.9) and (4.23) we deduce from (5.4) that the number of solutions in this case, $U_A^-(Q)$, satisfies

$$U_A^-(Q) \ll \sum_{f_1 \le Q}\sum_{f_2 \le Q}\sum_{\substack{d_3,e_3,r,z\\ d_3e_3+rz=f_1f_2}} \log\left(\frac{2f_1f_2}{rz}\right)\sum_{e_1e_2 < \frac{Qd_3}{f_1f_2}} e_1e_2d_3^{-2}. \tag{5.5}$$

The last sum in (5.5) is

$$\ll \sum_{e_1 < \frac{Qd_3}{f_1f_2}} \frac{Q^2}{e_1(f_1f_2)^2} \ll (f_1f_2)^{-2}Q^2 \log\left(\frac{Qd_3}{f_1f_2}\right).$$

Therefore, on applying Lemma 4.4, and noting (3.9), we deduce that

$$U_A^-(Q) \ll Q^2(\log Q)^3 \sum_{f_1\le Q}\sum_{f_2\le Q}(f_1f_2)^{-1}\sum_{u|f_1f_2} u^{-1}. \tag{5.6}$$

But

$$\begin{aligned}\sum_{f_1\le Q}\sum_{f_2\le Q}(f_1f_2)^{-1}\sum_{u|f_1f_2} u^{-1} &\ll \sum_{u\le Q^2}\sum_{t|u}\sum_{g_1\le Q/t}\sum_{g_2\le Qt/u}\left(g_1g_2u^2\right)^{-1}\\ &\ll (\log Q)^2\sum_{u\le Q^2} d(u)u^{-2},\end{aligned}$$

and so by (5.6), we have $U_A^-(Q)\ll Q^2(\log Q)^5$.

(B)(i) $r>d_3$ and $s>d_3$, or (ii) $y>d_3$ and $z>d_3$. The analysis in each case is again similar, and so we restrict attention to the solutions of type (i). By (5.1) and our initial hypothesis,

$$d_3<r<\frac{e_2f_1f_2}{yf_1+ze_2}\quad\text{and}\quad d_3<s<\frac{e_1e_2f_1}{yf_1+ze_2}.$$

Further, from (4.9) and (4.10) we have

$$r\equiv -y^{-1}e_2f_2 \pmod{d_3}\quad\text{and}\quad s\equiv -z^{-1}e_1f_1 \pmod{d_3}.$$

Then given d_3, e_1, e_2, f_1, f_2, y, z, the number of possible choices for r and s is

$$\ll \frac{e_1f_2e_2^2f_1^2}{d_3^2(yf_1+ze_2)^2}. \tag{5.7}$$

Next we note that by (4.9) we have

$$f_1(yr+e_2f_2)+e_2(zr-f_1f_2)=d_3(d_1f_1-e_2e_3).$$

Thus by (4.10), $d_3|yf_1+ze_2$, and moreover on applying (3.5) we obtain $(yf_1+ze_2)/d_3\le Q$. Therefore, on recalling (4.23) we deduce from (5.7) and (4.9) that the number of solutions in this case, $U_B^-(Q)$, satisfies

$$\begin{aligned}U_B^-(Q) &\ll \sum_{d_3\le Q}\sum_{m\le Q}\sum_{\substack{y,z,f_1,e_2\\ yf_1+ze_2=d_3m}}\sum_{e_1f_2<Qd_3(e_2f_1)^{-1}}\frac{e_1f_2e_2^2f_1^2}{d_3^4m^2}\\ &\ll Q^2\log Q\sum_{d_3\le Q}\sum_{m\le Q}(d_3m)^{-2}\sum_{\substack{y,z,f_1,e_2\\ yf_1+ze_2=d_3m}} 1.\end{aligned}$$

By Lemma 4.4, therefore, we may conclude that

$$U_B^-(Q)\ll Q^2(\log Q)^3\sum_{d_3\le Q}\sum_{m\le Q}(d_3m)^{-1}\sum_{u|d_3m}u^{-1},$$

whence, as in case (A), we find that $U_B^-(Q)\ll Q^2(\log Q)^5$.

(C)(i) $y > d_3$ and $\max\{z, s\} \le d_3$, or (ii) $r > d_3$ and $\max\{s, z\} \le d_3$. In the second case we can proceed in the same manner as in the first case, but with r and y interchanged. Thus we shall restrict attention to the solutions of type (i). On recalling equations (3.9), (4.23), (5.3), (3.7), (4.10), (5.2), (4.9), and the initial hypothesis, we find that the number of solutions in this case, $U_C^-(Q)$, satisfies

$$U_C^-(Q) \ll \sum_{d_3 \le Q^{1/3}} \sum_{\substack{e_1e_2f_1f_2<Qd_3 \\ 2e_1f_1>d_3^2 \\ (e_1e_2f_1f_2,d_3)=1}} \sum_{r<e_2f_2} \sum_{\substack{d_3<y<e_2f_2/r \\ d_3|ry+e_2f_2}} \sum_{\substack{z\le d_3 \\ d_3|zr-f_1f_2}} \sum_{\substack{s\le d_3 \\ d_3|sy-e_1e_2}} 1.$$

On observing that the innermost triple sum here is $\ll e_2f_2(d_3r)^{-1}$, we find that

$$\begin{aligned} U_C^-(Q) &\ll \log Q \sum_{d_3\le Q^{1/3}} d_3^{-1} \sum_{\substack{e_1e_2f_1f_2<Qd_3 \\ 2e_1f_1>d_3^2}} e_2f_2 \\ &\ll Q^2(\log Q)^2 \sum_{d_3\le Q^{1/3}} d_3 \sum_{\frac{1}{2}d_3^2<e_1f_1<Qd_3} (e_1f_1)^{-2} \\ &\ll Q^2(\log Q)^3 \sum_{d_3\le Q^{1/3}} d_3^{-1}. \end{aligned}$$

Thus $U_C^-(Q) \ll Q^2(\log Q)^4$.

(D)(i) $z > d_3$ and $\max\{y, r\} \le d_3$, or (ii) $s > d_3$ and $\max\{r, y\} \le d_3$. In the first case we may argue in the same way as in (C)(i) after interchanging z and y, and r and s, and reversing the rôles of e_1f_1 and e_2f_2. In the second case we may proceed likewise on interchanging z and s. Thus we deduce that the number of solutions in this case, $U_D^-(Q)$, is $\ll Q^2(\log Q)^4$.

(E) $\max\{r, s, y, z\} \le d_3$. By (3.9), (4.23), (4.9), (4.10) and our initial hypothesis, the number of solutions in this case, $U_E^-(Q)$, satisfies

$$\begin{aligned} U_E^-(Q) &\ll \sum_{d_3\le Q^{1/3}} \sum_{\substack{e_1e_2f_1f_2<Qd_3 \\ (e_1e_2f_1f_2,d_3)=1}} \sum_{r\le d_3} \sum_{\substack{y\le d_3 \\ d_3|yr+e_2f_2}} \sum_{\substack{s\le d_3 \\ d_3|sy-e_1e_2}} \sum_{\substack{z\le d_3 \\ d_3|zs+e_1f_1}} 1 \\ &\ll Q(\log Q)^3 \sum_{d_3\le Q^{1/3}} d_3^2, \end{aligned}$$

and thus $U_E^-(Q) \ll Q^2(\log Q)^3$.

We now observe that every solution counted by $U^-(Q)$ may be classified as one of the types (A)-(E), and thus we may summarise the deliberations of this section in the following lemma.

Lemma 5.1. *The number of solutions in the negative case satisfies*

$$U^-(Q) \ll Q^2(\log Q)^5.$$

6. A refined upper bound: the positive case

In this section we estimate the number of solutions counted by $U^+(Q)$. Again the proof divides into a series of subcases. We observe, for future reference, that in this case

$$s = s^* \quad \text{and} \quad z = z^*, \tag{6.1}$$

so that by (4.11),

$$yrzs + yre_1f_1 + zse_2f_2 < ysf_1f_2 + e_1e_2zr + e_1e_2f_1f_2. \tag{6.2}$$

We first explore the situation in which either two at least of r, s, y and z exceed d_3, or two at least of e_1, e_2, f_1 and f_2 exceed d_3.

(A)(i) $sf_2 > re_1$ *and either (a)* $\min\{s,z\} > d_3$ *or (b)* $\min\{e_2, f_2\} > d_3$, *or (ii)* $re_1 > sf_2$ *and either (a)* $\min\{r,y\} > d_3$ *or (b)* $\min\{e_1, f_1\} > d_3$.

In the second case we can proceed in the same manner as in the first case, following a suitable interchange of variables. Furthermore, the case (i)(b) may be disposed of in the same manner as (i)(a), again following a suitable interchange of variables. Thus we restrict attention to the solutions of type (i)(a). By our initial hypothesis, (4.12) and (6.1), we have $ze_2 > yf_1$, and hence

$$s > re_1/f_2 \quad \text{and} \quad z > yf_1/e_2. \tag{6.3}$$

Further, from (4.9) and (4.10) we have

$$s \equiv -y^{-1}e_1e_2 \pmod{d_3} \quad \text{and} \quad z \equiv -r^{-1}f_1f_2 \pmod{d_3}. \tag{6.4}$$

Moreover, (4.9) yields $zs(yr + e_2f_2) = (d_2d_3 - e_1f_1)d_1d_3$, and hence by using (3.5) we obtain

$$zs < Qd_3(yr + e_2f_2)^{-1}. \tag{6.5}$$

Then given d_3, e_1, e_2, f_1, f_2, r, y, we may apply Lemma 4.2 with (6.3), (6.4), (6.5) and the initial hypothesis, to show that the number of possible choices for s and z is

$$\ll \frac{Q}{d_3(yr + e_2f_2)} \log\left(\frac{2Qd_3e_2f_2}{rye_1f_1(yr + e_2f_2)}\right). \tag{6.6}$$

Next we note that by (6.3) and (6.5),

$$Qd_3e_2f_2 > e_2f_2zs(yr + e_2f_2) > yre_1f_1(yr + e_2f_2), \tag{6.7}$$

and by (3.5) and (4.9),

$$e_1f_1 < Qd_3(d_1d_3)^{-1} = Qd_3(yr + e_2f_2)^{-1}. \tag{6.8}$$

Thus, on recalling (6.6), we deduce that given d_3, e_2, f_2, y and r, the number of possible choices, S_1, for e_1, f_1, s and z satisfies

$$S_1 \ll \sum_{e_1,f_1} \frac{Q}{d_3(yr + e_2f_2)} \left(\log\left(\frac{\mathcal{M}^*Qd_3}{e_1f_1(yr + e_2f_2)}\right) + \log\left(\frac{2e_2f_2}{yr\mathcal{M}^*}\right)\right),$$

where, by (6.7) and (6.8), the summation is over e_1 and f_1 satisfying

$$e_1 f_1 < \mathcal{M}^* Q d_3 (yr + e_2 f_2)^{-1},$$

and $\mathcal{M}^*$ is defined by

$$\mathcal{M}^* = \min\left\{1, e_2 f_2 (yr)^{-1}\right\}. \tag{6.9}$$

Consequently, on using (3.5) we obtain

$$\begin{aligned} S_1 &\ll \sum_{e_1 \le Q} \frac{Q^2 \mathcal{M}^*}{e_1 (yr + e_2 f_2)^2} \log\left(\frac{2e_2 f_2}{\min\{yr, e_2 f_2\}}\right) \\ &\ll \frac{Q^2 \log Q}{(yr + e_2 f_2)^2} \log\left(\frac{2(yr + e_2 f_2)}{yr}\right). \end{aligned} \tag{6.10}$$

Now, on substituting from (4.9) into (6.10), we find that the number of solutions in this case, $U_A^+(Q)$, satisfies

$$U_A^+(Q) \ll Q^2 \log Q \sum_{d_1 \le Q} \sum_{d_3 \le Q} (d_1 d_3)^{-2} \sum_{\substack{y, r, e_2, f_2 \\ yr + e_2 f_2 = d_1 d_3}} \log\left(\frac{2d_1 d_3}{yr}\right).$$

By Lemma 4.4, therefore,

$$U_A^+(Q) \ll Q^2 (\log Q)^3 \sum_{d_1 \le Q} \sum_{d_3 \le Q} (d_1 d_3)^{-1} \sum_{u | d_1 d_3} u^{-1},$$

and thus $U_A^+(Q) \ll Q^2 (\log Q)^5$.

(B)(i) $re_1 > sf_2$ *and either (a)* $\min\{s, z\} > d_3$ *or (b)* $\min\{e_2, f_2\} > d_3$,
or (ii) $sf_2 > re_1$ *and either (a)* $\min\{r, y\} > d_3$ *or (b)* $\min\{e_1, f_1\} > d_3$.
Once again, by a suitable interchange of variables, we may restrict attention to the solutions of type (i)(a). By our initial hypothesis, (4.12) and (6.1), we have $ze_2 < yf_1$, and hence

$$s < re_1/f_2 \quad \text{and} \quad z < yf_1/e_2. \tag{6.11}$$

Moreover, as in (A), we have (6.4) and (6.5). Then on recalling our intial hypothesis and (6.11), we may apply Lemma 4.1 to deduce that given d_3, e_1, e_2, f_1, f_2, r, y, the number of possible choices, S_1, for s and z satisfies

$$S_1 \ll \frac{Q}{d_3 (yr + e_2 f_2)} \log\left(\frac{2(yr + e_2 f_2) rye_1 f_1}{Qd_3 e_2 f_2}\right), \tag{6.12}$$

when $Qd_3 e_2 f_2 < rye_1 f_1 (yr + e_2 f_2)$, and otherwise satisfies

$$S_1 \ll rye_1 f_1 (d_3^2 e_2 f_2)^{-1}. \tag{6.13}$$

We now sum S_1 over the possible choices of e_1 and f_1, noting that the inequality (6.8) remains valid in this case. On recalling the notation defined in (6.9), we find from (6.13) and (3.5) that the contribution, S_2, of those terms with

$$e_1 f_1 \le Q d_3 (yr + e_2 f_2)^{-1} \mathcal{M}^*$$

satisfies

$$\begin{aligned} S_2 &\ll \sum_{e_1 \le Q} \left(\frac{Q d_3 \mathcal{M}^*}{e_1 (yr + e_2 f_2)} \right)^2 \frac{r y e_1}{d_3^2 e_2 f_2} \\ &\ll Q^2 \log Q \frac{\mathcal{M}^{*2}}{(yr + e_2 f_2)^2} \frac{ry}{e_2 f_2}. \end{aligned}$$

Thus

$$S_2 \ll \frac{Q^2 \log Q}{(yr + e_2 f_2)^2}. \tag{6.14}$$

Meanwhile, when $e_2 f_2 < yr$, there may be an additional contribution, S_3, from those terms with

$$\frac{Q d_3 e_2 f_2}{yr(yr + e_2 f_2)} < e_1 f_1 \le \frac{Q d_3}{yr + e_2 f_2}.$$

In this circumstance, by (6.12) and (3.5) we have

$$\begin{aligned} S_3 &\ll \sum_{e_1 \le Q} \frac{Q^2}{(yr + e_2 f_2)^2 e_1} \log \left(\frac{2yr}{e_2 f_2} \right) \\ &\ll \frac{Q^2 \log Q}{(yr + e_2 f_2)^2} \log \left(\frac{2yr}{e_2 f_2} \right). \end{aligned} \tag{6.15}$$

Collecting together (6.15), (6.14) and (4.9), we conclude that the total number of solutions in this case, $U_B^+(Q)$, satisfies

$$U_B^+(Q) \ll Q^2 \log Q \sum_{d_1 \le Q} \sum_{d_3 \le Q} (d_1 d_3)^{-2} \sum_{\substack{y,r,e_2,f_2 \\ yr + e_2 f_2 = d_1 d_3}} \log \left(\frac{2 d_1 d_3}{e_2 f_2} \right),$$

whence, using the same argument as in case (A), $U_B^+(Q) \ll Q^2 (\log Q)^5$.

(C)(i) $s f_2 < r e_1$ *and either (a)* $\min\{s, y\} > d_3$ *or (b)* $\min\{f_1, f_2\} > d_3$, *or (ii)* $r e_1 < s f_2$ *and either (a)* $\min\{r, z\} > d_3$ *or (b)* $\min\{e_1, e_2\} > d_3$.

Again, by a suitable interchange of variables, we may restrict attention to the solutions of type (i)(a). By (4.9) and (3.5), we have $y r e_1 f_1 < (d_1 e_1 f_1) d_3 \le Q d_3$, which together with the initial hypothesis gives

$$s < r e_1 / f_2 \quad \text{and} \quad y < Q d_3 (r e_1 f_1)^{-1}. \tag{6.16}$$

Further, from (4.9) and (4.10) we have

$$s \equiv -z^{-1}e_1f_1 \pmod{d_3} \quad \text{and} \quad y \equiv -r^{-1}e_2f_2 \pmod{d_3}. \tag{6.17}$$

Moreover, (4.9) and (6.1) yield $sy(zr + f_1f_2) < e_3d_3(f_3d_3 - e_1e_2)$, and hence by using (3.5) we obtain

$$sy < Qd_3/(zr + f_1f_2). \tag{6.18}$$

Then on recalling our initial hypothesis, we may apply Lemma 4.1 with (6.16), (6.17) and (6.18) to deduce that given d_3, e_1, e_2, f_1, f_2, r, z, the number, S_1, of permissible s and y satisfies

$$S_1 \ll Qd_3^{-1}(zr + f_1f_2)^{-1}\log\left(\frac{2(zr + f_1f_2)}{f_1f_2}\right). \tag{6.19}$$

In view of (3.5) the permissible e_1 and e_2 satisfy $e_1e_2 < Q/e_3$, and so by (3.5), (4.9) and (6.19), the total number of solutions in this case, $U_C^+(Q)$, satisfies

$$U_C^+(Q) \ll Q^2 \log Q \sum_{d_3 \le Q} \sum_{e_3 \le Q} (d_3e_3)^{-2} \sum_{\substack{z,r,f_1,f_2 \\ zr+f_1f_2=d_3e_3}} \log\left(\frac{2d_3e_3}{f_1f_2}\right).$$

Therefore, by applying the same argument as in case (A), we obtain $U_C^+(Q) \ll Q^2(\log Q)^5$.

(D)(i) $sf_2 > re_1$ *and either (a)* $\min\{s, y\} > d_3$ *or (b)* $\min\{f_1, f_2\} > d_3$*,*
or (ii) $re_1 > sf_2$ *and either (a)* $\min\{r, z\} > d_3$ *or (b)* $\min\{e_1, e_2\} > d_3$*.*
Once again we may restrict attention to the solutions of type (i)(a). By our initial hypothesis, (4.12) and (6.1), we have $yf_1 < ze_2$. Moreover by (4.9) and (3.5), $sze_2f_2 < (d_2e_2f_2)d_3 \le Qd_3$. Thus

$$y < ze_2/f_1 \quad \text{and} \quad s < Qd_3(ze_2f_2)^{-1}. \tag{6.20}$$

Further, following the same argument as in (C) we find that (6.17) and (6.18) remain valid. Then on recalling our initial hypothesis, we may apply Lemma 4.1 with (6.20) to deduce that given d_3, e_1, e_2, f_1, f_2, r, z, the number, S_1, of permissible s and y satisfies (6.19). Thus we may argue as in (C) to deduce that the number of solutions in this case, $U_D^+(Q)$, satisfies $U_D^+(Q) \ll Q^2(\log Q)^5$.

To summarise the progress so far, we have treated the contribution from all solutions in which two at least of r, s, y, z exceed d_3, or two at least of e_1, e_2, f_1, f_2 exceed d_3, except those solutions with

$$\min\{r, s\} > d_3, \quad \min\{y, z\} > d_3, \quad \min\{e_1, f_2\} > d_3 \quad \text{or} \quad \min\{e_2, f_2\} > d_3. \tag{6.21}$$

We now attend to those solutions for which (6.21) holds, dividing into two cases.

(E)(i) $\min\{r,s,e_1,f_2\} > d_3$ *and* $\max\{y,z,e_2,f_1\} \le d_3$,
or (ii) $\min\{y,z,e_2,f_1\} > d_3$ *and* $\max\{r,s,e_1,f_2\} \le d_3$.
Once again we may restrict attention to solutions of type (i), without loss of generality. We may also assume that $sf_2 < re_1$, since if $re_1 < sf_2$, then we may apply a kindred argument, but with the rôles of r, e_1 and s, f_2 reversed, and likewise y, f_1 and z, e_2. Consider then those solutions of type (i) with

$$sf_2 < re_1. \tag{6.22}$$

By (4.9) and (3.5), $yrsz < d_1d_2d_3^2 \le Qd_3$, so that on recalling (4.23) we obtain

$$s < Qd_3(yrz)^{-1} \quad \text{and} \quad f_2 < Qd_3(e_1e_2f_1)^{-1}. \tag{6.23}$$

Further, from (4.9) and (4.10) we have

$$s \equiv -z^{-1}e_1f_1 \pmod{d_3} \quad \text{and} \quad f_2 \equiv -e_2^{-1}yr \pmod{d_3}. \tag{6.24}$$

We note also that by (4.9) and (6.1) we have $r^2e_1^2yze_2f_1 < (d_1e_1f_1)(e_1e_2e_3)d_3^2$, and hence

$$re_1 < \frac{Q^2d_3^2}{yrze_1e_2f_1}. \tag{6.25}$$

Then on recalling our initial hypothesis, we may apply Lemma 4.1 with (6.22), (6.23) and (6.24), noting (6.25), to deduce that given d_3, e_1, e_2, f_1, r, y, z, the number, S_1, of permissible s and f_2 satisifes

$$S_1 \ll re_1d_3^{-2}\log\left(\frac{2Q^2d_3^2}{r^2e_1^2yze_2f_1}\right). \tag{6.26}$$

Next we observe that by (4.9) and (3.5), $re_1 < Qd_3(yf_1)^{-1}$. Moreover, by (6.22), (4.12) and (6.1), we have $yf_1 - ze_2 > 0$. Then in view of (6.26) and (3.5), given d_3, e_2, f_1, y, z, the number, S_2, of permissible r, s, e_1, f_2 satisfies

$$\begin{aligned} S_2 &\ll \sum_{re_1<Qd_3/(yf_1)} \frac{re_1}{d_3^2}\left(\log\left(\frac{2Q^2d_3^2}{r^2e_1^2y^2f_1^2}\right) + \log\left(\frac{yf_1}{ze_2}\right)\right) \\ &\ll \sum_{e_1\le Q} \frac{Q^2}{y^2f_1^2e_1}\left(1+\log\left(\frac{yf_1}{ze_2}\right)\right) \\ &\ll \frac{Q^2\log Q}{(y_1f_1)^2}\log\left(\frac{2yf_1}{ze_2}\right). \end{aligned} \tag{6.27}$$

Next we note that by (4.9) and (6.1) we have

$$f_1(yr+e_2f_2) - e_2(zr+f_1f_2) = d_3(d_1f_1 - e_2e_3).$$

Thus by (4.10),

$$d_3 | yf_1 - ze_2. \tag{6.28}$$

Therefore, by (4.9), (6.27) and (3.5), we deduce that the number of solutions in this case, $U_E^+(Q)$, satisfies

$$U_E^+(Q) \ll Q^2 \log Q \sum_{y \le Q} \sum_{f_1 \le Q} (yf_1)^{-2} \sum_{\substack{a,d_3,z,e_2 \\ ze_2+ad_3=yf_1}} \log\left(\frac{2yf_1}{ze_2}\right),$$

and by applying Lemma 4.4 once again, we find that $U_E^+(Q) \ll Q^2(\log Q)^5$.

(F)(i) $\min\{r,s,e_2,f_1\} > d_3$ *and* $\max\{y,z,e_1,f_2\} \le d_3$,
or (ii) $\min\{y,z,e_1,f_2\} > d_3$ *and* $\max\{r,s,e_2,f_1\} \le d_3$.

Once again we may restrict attention to solutions of type (i), and start by considering those such solutions with $sf_2 < re_1$. Thus, on recalling (4.12) and (6.1), we have also $ze_2 < yf_1$. Hence, on substitution into (6.2),

$$zsyr + zse_2f_2 + yre_1f_1 < 2yre_1f_1 + e_1e_2f_1f_2,$$

and so $sz < e_1f_1$. Moreover, (4.9), (6.1) and (3.5) yield $(sy + e_1e_2)f_1f_2 = (f_1f_2f_3)d_3 \le Qd_3$. Thus, in view of our initial hypothesis, we have

$$d_3 < s < \min\left\{e_1f_1z^{-1}, Qd_3(yf_1f_2)^{-1}\right\}. \tag{6.29}$$

Also, (4.9) and (3.5) yield $(yr + e_2f_2)e_1f_1 = (d_1e_1f_1)d_3 \le Qd_3$, and so on recalling our initial hypothesis we obtain

$$d_3 < r < Qd_3(e_1f_1y)^{-1}. \tag{6.30}$$

Further, from (4.9) and (4.10) we have

$$s \equiv -z^{-1}e_1f_1 \pmod{d_3} \quad \text{and} \quad r \equiv -y^{-1}e_2f_2 \pmod{d_3}. \tag{6.31}$$

Hence, on collecting together (6.29), (6.30) and (6.31), we deduce that given d_3, e_1, e_2, f_1, f_2, y, z, the number, S_1, of permissible choices for r and s with $r \le f_1f_2z^{-1}$ satisfies

$$S_1 \ll d_3^{-2} \min\left\{e_1f_1^2f_2z^{-2}, Q^2d_3^2(e_1f_1^2f_2y^2)^{-1}\right\}. \tag{6.32}$$

Similarly, the corresponding number, S_2, of permissible choices with $r > f_1f_2z^{-1}$, which in view of (6.30) occur only when

$$e_1f_2 < Qd_3z(f_1^2y)^{-1}, \tag{6.33}$$

satisfies

$$S_2 \ll Q(d_3zy)^{-1}. \tag{6.34}$$

We now estimate the number, S_3, of permissible choices for s, r, e_1, f_2 for given d_3, e_2, f_1, y, z. By summing (6.34) over e_1 and f_2 satisfying (6.33), we find that the contribution to S_3 due to S_2 is

$$\ll Q^2 \log Q(yf_1)^{-2}.$$

Similarly from (6.32), the contribution to S_3 due to S_1 when (6.33) is satisfied is, by (3.5),

$$\begin{aligned}&\ll \sum_{e_1\le Q} \frac{f_1^2e_1}{z^2d_3^2}\left(\frac{Qd_3z}{f_1^2ye_1}\right)^2\\ &\ll Q^2\log Q(yf_1)^{-2}.\end{aligned}$$

In view of (4.23), it remains only to estimate the contribution to S_3 due to S_1 when

$$Qd_3z(f_1^2y)^{-1}\le e_1f_2<Qd_3(f_1e_2)^{-1}.$$

Since the latter contribution is

$$\begin{aligned}&\ll \sum_{e_1\le Q}\frac{Q^2}{e_1f_1^2y^2}\left(\log\left(\frac{Qd_3}{f_1e_1e_2}\right)-\log\left(\frac{Qd_3z}{e_1f_1^2y}\right)+1\right)\\ &\ll Q^2\log Q(f_1y)^{-2}\log\left(\frac{2yf_1}{ze_2}\right),\end{aligned}\tag{6.35}$$

we conclude that S_3 is bounded by (6.35). We now observe that (6.28) remains valid, and hence by (3.5), (4.9) and the initial hypothesis, the number of solutions of this type, $U_{F_1}^+(Q)$, satisfies

$$U_{F_1}^+(Q)\ll Q^2\log Q\sum_{f_1\le Q}\sum_{y\le Q}(yf_1)^{-2}\sum_{\substack{z,e_2,a,d_3\\ ze_2+ad_3=yf_1}}\log\left(\frac{2yf_1}{ze_2}\right).$$

An application of Lemma 4.4 once again leads to the conclusion that $U_{F_1}^+(Q)\ll Q^2(\log Q)^5$.

It remains to consider the solutions of type (i) for which $re_1<sf_2$. But for such solutions we may deduce, from (4.12) and (6.1), that $yf_1<ze_2$. Thus, on substitution into (6.2),

$$yrzs+yre_1f_1+zse_2f_2<2sze_2f_2+e_1e_2f_1f_2,$$

and so $yr<e_2f_2$. We may now follow an argument similar to that above, after an interchange of variables which swaps r and s. Thus the number of solutions of this type is also $\ll Q^2(\log Q)^5$, and the total number of solutions of type (F), $U_F^+(Q)$, satisfies $U_F^+(Q)\ll Q^2(\log Q)^5$.

In order to complete the analysis in the positive case, we have merely to dispose of those solutions in which at most one of r, s, y and z exceed d_3, or likewise for e_1, e_2, f_1 and f_2.

(G)(i) $\max\{s,z\} \le d_3$ *or (ii)* $\max\{e_1,f_1\} \le d_3$. By a suitable interchange of variables we may restrict attention to the solutions of type (i). We note that by (4.9), (4.10) and (6.1) we have

$$s \equiv -y^{-1}e_1e_2 \pmod{d_3} \quad \text{and} \quad z \equiv -r^{-1}f_1f_2 \pmod{d_3},$$

and hence, in view of our initial hypothesis, s and z are determined uniquely. Thus by (4.9), (3.5) and (3.7), the total number of solutions in this case, $U_G^+(Q)$, satisfies

$$\begin{aligned} U_G^+(Q) &\ll \sum_{d_3 \le Q} \sum_{d_1 \le Qd_3^{-2}} \sum_{\substack{e_1,f_1 \\ e_1f_1 < Qd_1^{-1}}} \sum_{\substack{y,r,e_2,f_2 \\ yr+e_2f_2=d_1d_3}} 1 \\ &\ll Q\log Q \sum_{d_3 \le Q} \sum_{d_1 \le Qd_3^{-2}} d_1^{-1} \sum_{\substack{y,r,e_2,f_2 \\ yr+e_2f_2=d_1d_3}} 1. \end{aligned}$$

By Lemma 4.4, therefore,

$$U_G^+(Q) \ll Q(\log Q)^3 \sum_{d_3 \le Q} d_3 \sum_{d_1 \le Qd_3^{-2}} \sum_{u \mid d_1d_3} u^{-1}. \tag{6.36}$$

But

$$\begin{aligned} \sum_{d_3 \le Q} d_3 \sum_{d_1 \le Qd_3^{-2}} \sum_{u \mid d_1d_3} u^{-1} &= \sum_{u \le Q} \sum_{t \mid u} \sum_{g_3 \le Qt^{-1}} \sum_{g_1 \le Q(g_3^2tu)^{-1}} g_3tu^{-1} \\ &\le \sum_{u \le Q} \sum_{t \mid u} \sum_{g_3 \le Qt^{-1}} Q(g_3u^2)^{-1} \\ &\ll Q\log Q \sum_{u \le Q} d(u)u^{-2}. \end{aligned}$$

Then we may conclude from (6.36) that $U_G^+(Q) \ll Q^2(\log Q)^4$.

If at most one of r, s, y, z exceeds d_3, and $\max\{s,z\} > d_3$, then necessarily $\max\{r,y\} \le d_3$. Thus to complete our treatment of the positive case we have only to consider the solutions of the latter type, together with those for which $\max\{e_2,f_2\} \le d_3$.

(H)(i) $\max\{r,y\} \le d_3$ *or (ii)* $\max\{e_2,f_2\} \le d_3$. By a suitable interchange of variables we may restrict attention to the solutions of type (i). By imitating the argument at the start of case (G), we deduce that the total number of solutions in this case, $U_H^+(Q)$, satisfies

$$\begin{aligned} U_H^+(Q) &\ll \sum_{d_3 \le Q^{1/3}} \sum_{d_2 \le (Q/d_3)^{1/2}} \sum_{\substack{e_2,f_2 \\ e_2f_2 < Qd_2^{-1}}} \sum_{\substack{z,s,e_1,f_1 \\ zs+e_1f_1=d_2d_3}} 1 \\ &\ll Q(\log Q)^3 \sum_{d_3 \le Q^{1/3}} d_3 \sum_{d_2 \le (Q/d_3)^{1/2}} \sum_{u \mid d_2d_3} u^{-1}, \end{aligned} \tag{6.37}$$

again by employing Lemma 4.4. But

$$\sum_{d_3\le Q^{1/3}} d_3 \sum_{d_2\le (Q/d_3)^{1/2}} \sum_{u|d_2d_3} u^{-1} = \sum_{u\le Q}\sum_{t|u}\sum_{g_3\le Q^{1/3}t^{-1}}\sum_{g_2\le (Qtg_3^{-1})^{1/2}u^{-1}} g_3tu^{-1}$$
$$\le \sum_{u\le Q}\sum_{t|u}\sum_{g_3\le Q^{1/3}t^{-1}} (Qt^3g_3)^{1/2}u^{-2}$$
$$\ll Q\sum_{u\le Q} d(u)u^{-2}.$$

Then we may conclude from (6.37) that $U_H^+(Q)\ll Q^2(\log Q)^3$.

We may now summarise the conclusions of this section in the following lemma.

Lemma 6.1. *The number of solutions in the positive case satisfies*

$$U^+(Q)\ll Q^2(\log Q)^5.$$

Plainly, Lemmata 5.1 and 6.1, together with (4.13), yield the upper bound of Theorem 1.2.

7. A lower bound

In this section we derive a lower bound for $U(P)$ by establishing a lower bound for the number of non-trivial solutions of (2.1) subject to (2.2). We begin by making a series of changes of variable which convert the original problem into one which is easier to handle. We satisfy the condition (2.2) by restricting the variables to satisfy the inequalities

$$\tfrac45 P<\alpha_i\le \tfrac65 P\quad (i=1,2,3;\alpha=X,Y).$$

We may also conveniently satisfy the parity condition in (2.1) by further restricting our variables to be even. Thus we put $X_i=2x_i$ and $Y_i=2y_i$ $(1\le i\le 3)$, and estimate from below the number of solutions of the system

$$\begin{aligned} x_1x_2x_3&=y_1y_2y_3\\ x_1+x_2+x_3&=y_1+y_2+y_3 \end{aligned}\tag{7.1}$$

with $x_i=y_j$ for no i and j, and satisfying

$$\tfrac25 P<\alpha_i\le \tfrac35 P\quad (i=1,2,3;\alpha=x,y).\tag{7.2}$$

We now follow the analysis of §2, but applied to our new variables. In this way we find that the non-trivial solutions of the system (7.1) with (7.2) are in one-to-one correspondence with the solutions of the nonary cubic equation

$$d_1e_1f_1+d_2e_2f_2+d_3e_3f_3=d_1d_2d_3+e_1e_2e_3+f_1f_2f_3,\tag{7.3}$$

subject to (2.9), (2.10) and

$$\begin{aligned} \tfrac{2}{5}P < d_ie_if_i \le \tfrac{3}{5}P \quad (i = 1, 2, 3), \\ \tfrac{2}{5}P < \alpha_1\alpha_2\alpha_3 \le \tfrac{3}{5}P \quad (\alpha = d, e, f). \end{aligned} \tag{7.4}$$

We now proceed as at the start of §5, our aim at this stage being to remove any explicit reference to d_1, d_2, e_3 and f_3. We rewrite (7.3) in the form (3.8) with (4.3), and then make the substitutions (4.7) and (4.8). Since we are interested only in a lower bound for $U(P)$, we may restrict attention to the situation in which y, r, z^* and s^* are all positive. Then (4.9) reduces to

$$ry + e_2f_2 = d_1d_3, \quad sz + e_1f_1 = d_2d_3, \quad rz + f_1f_2 = d_3e_3, \quad sy + e_1e_2 = d_3f_3, \tag{7.5}$$

and the conditions (2.10), and those implicit in the substitutions (4.7) and (4.8), become

$$(d_3, e_1e_2f_1f_2) = (e_2, f_1) = (sz, e_1f_1) = (rz, f_1f_2) = (y, z) = 1. \tag{7.6}$$

On substituting from (7.5), we find that (7.4) becomes the condition that each of the expressions

$$\begin{aligned} &(rz + f_1f_2)(sy + e_1e_2), \quad e_1f_1(ry + e_2f_2), \quad e_1e_2(rz + f_1f_2), \\ &(ry + e_2f_2)(sz + e_1f_1), \quad f_1f_2(sy + e_1e_2), \quad e_2f_2(sz + e_1f_1) \end{aligned} \tag{7.7}$$

lies in the interval $(\frac{2}{5}Q, \frac{3}{5}Q]$, where for the sake of brevity here, and in what follows, we write

$$Q = Pd_3. \tag{7.8}$$

Also, in view of (7.5) and the positivity of y, r, z and s, the condition (2.9) may be replaced by the five inequalities

$$\begin{aligned} &(sz + e_1f_1)f_2 \ne e_1(rz + f_1f_2), \quad (sz + e_1f_1)e_2 \ne f_1(sy + e_1e_2), \\ &(ry + e_2f_2)f_1 \ne e_2(rz + f_1f_2), \quad (ry + e_2f_2)e_1 \ne f_2(sy + e_1e_2), \end{aligned}$$

and

$$(rz + f_1f_2)(sy + e_1e_2) \ne (ry + e_2f_2)(sz + e_1f_1).$$

Fortunately, the latter inequalities reduce to

$$sf_2 \ne re_1 \quad \text{and} \quad ze_2 \ne yf_1. \tag{7.9}$$

Thus, in order to establish a lower bound for $U(P)$, it suffices to bound from below the number of d_3, e_1, e_2, f_1, f_2, y, r, z, s with

$$d_3 | (ry + e_2f_2, sz + e_1f_1, rz + f_1f_2, sy + e_1e_2), \tag{7.10}$$

for which (7.6) and (7.9) hold, and such that each of the expressions in (7.7) lies in the interval $(\frac{2}{5}Q, \frac{3}{5}Q]$.

It is the last condition above which is most awkward to handle. Our strategy is to choose e_1, e_2, f_1, f_2 so that $e_1e_2f_1f_2$ is close to $\frac{1}{2}Q$, and then to choose y, r, z, s so that the other terms arising in (7.7) are somewhat smaller. To achieve this aim we proceed as follows. Let δ be a sufficiently small positive constant, and suppose that P is sufficiently large in terms of δ. Then we suppose that

$$d_3 \le P^\delta, \quad P^{1/4} < e_1, e_2, f_1 \le P^{2/7}, \tag{7.11}$$

$$\left(\tfrac{1}{2} - \delta\right) Q < e_1e_2f_1f_2 \le \left(\tfrac{1}{2} + \delta\right) Q, \tag{7.12}$$

$$\left(\frac{\delta Q}{e_1f_1}\right)^{\frac{1}{2}-\delta} < y \le \left(\frac{\delta Q}{e_1f_1}\right)^{\frac{1}{2}+\delta}, \tag{7.13}$$

$$\frac{\delta Q}{2ye_1f_1} < r \le \frac{\delta Q}{ye_1f_1}, \quad \frac{\delta Q}{20re_1e_2} < z \le \frac{\delta Q}{10re_1e_2}, \quad 0 < s \le \frac{\delta Q}{100ze_2f_2}, \tag{7.14}$$

$$yr \equiv -e_2f_2 \pmod{d_3}, \quad zr \equiv -f_1f_2 \pmod{d_3}, \quad sz \equiv -e_1f_1 \pmod{d_3}, \tag{7.15}$$

and

$$(d_3, e_1e_2f_1f_2) = (e_2, f_1) = (y, d_3) = (r, f_1f_2) = (z, e_1f_1f_2y) = (s, e_1f_1) = 1. \tag{7.16}$$

Lemma 7.1. *Let $N_0(P)$ denote the number of d_3, e_1, e_2, f_1, f_2, y, r, z, and s satisfying (7.11)-(7.16). Then $U(P) \gg N_0(P)$.*

Proof. First note that the coprimality conditions (7.16) immediately imply those of (7.6). Next we observe that (7.15) implies that

$$yrzs \equiv e_1e_2f_1f_2 \equiv -e_1e_2rz \pmod{d_3}. \tag{7.17}$$

Then by (7.16) we have $(d_3, yrzs) = (d_3, e_1e_2f_1f_2) = 1$, and hence $ys \equiv -e_1e_2 \pmod{d_3}$. Consequently (7.15) and (7.16) together imply (7.10). We now consider the expressions (7.7). By (7.12) and (7.14),

$$ry \le \frac{2\delta}{1-2\delta} \cdot e_2f_2, \quad rz \le \frac{2\delta}{1-2\delta} \cdot \frac{f_1f_2}{10}, \quad sz \le \frac{2\delta}{1-2\delta} \cdot \frac{e_1f_1}{100}. \tag{7.18}$$

Further, by (7.14) we have $y \le \delta Q(e_1f_1r)^{-1}$, so that by incorporating (7.12) we obtain

$$sy \le \frac{\delta Q}{100ze_2f_2} \cdot \frac{\delta Q}{e_1f_1r} < \frac{2\delta}{1-2\delta} \cdot \frac{e_1e_2}{5}. \tag{7.19}$$

On applying (7.18) and (7.19), we find that when δ is sufficiently small, each of the expressions in (7.7) lies in the interval $(\frac{2}{5}Q, \frac{3}{5}Q]$. Further, again by (7.14),

$$sf_2 \le \frac{\delta Q}{100ze_2} < \tfrac{1}{5}re_1 \quad \text{and} \quad ze_2 \le \frac{\delta Q}{10re_1} < \tfrac{1}{5}yf_1,$$

so that (7.9) is satisfied. This completes the proof of the lemma.

In order to exploit this lemma, we must provide estimates for certain counting functions.

Lemma 7.2. *Suppose that d_3, e_1, e_2, f_1, f_2 and y satisfy (7.11), (7.12), (7.13) and*

$$(d_3, e_1e_2f_1f_2) = (e_2, f_1) = (y, d_3) = 1.$$

Then the number, $N_1(P)$, of possible choices for r, z and s subject to (7.14), (7.15) and (7.16) satisfies

$$N_1(P) \gg \frac{\phi(e_1f_1)\phi(e_1f_1f_2y)\phi(f_1f_2)}{d_3e_1^3e_2f_1^4f_2^3y^2}P^2.$$

Proof. Suppose that we are given d_3, e_1, e_2, f_1, f_2, y, r and z satisfying the conditions of (7.11)-(7.16) independent of s. Write

$$R = \frac{\delta Q}{100ze_2f_2}, \tag{7.23}$$

and denote by N_2 the number of integers s with

$$0 < s \le R, \quad (s, e_1f_1) = 1 \quad \text{and} \quad sz \equiv -e_1f_1 \pmod{d_3}.$$

Then we have

$$N_2 = \sum_{m|e_1f_1} \mu(m) \sum_{\substack{1 \le t \le R/m \\ tmz \equiv -e_1f_1 \pmod{d_3}}} 1. \tag{7.24}$$

For the values of m occurring in the first summation, we have $m|e_1f_1$, so that in view of (7.16), the congruence condition in the final summation can be replaced by $t \equiv -(mz)^{-1}e_1f_1 \pmod{d_3}$. On substitution into (7.24) we deduce that

$$N_2 = \sum_{m|e_1f_1} \mu(m)\left(\frac{R}{md_3} + O(1)\right) = \frac{R\phi(e_1f_1)}{d_3e_1f_1} + O_\varepsilon\left((e_1f_1)^\varepsilon\right). \tag{7.25}$$

Moreover, by (7.23) and (7.12)-(7.14),

$$R \ge \frac{re_1}{10f_2} > \frac{\delta Qe_1e_2}{20(e_1e_2f_1f_2)y} \ge \frac{\delta^{\frac{1}{2}-\delta}}{20(\frac{1}{2}+\delta)}e_1e_2(e_1f_1/Q)^{\frac{1}{2}+\delta}.$$

Therefore, by (7.8) and (7.11),

$$R > \frac{\delta^{\frac{1}{2}-\delta}}{20(\frac{1}{2}+\delta)}\left(P^{1/4}\right)^{3+2\delta}\left(P^{1+\delta}\right)^{-\frac{1}{2}-\delta} > P^{2\delta}.$$

Consequently the error term in (7.25) is of smaller order than the main term, and so by (7.8),

$$N_2 \gg \frac{P\phi(e_1f_1)}{ze_1f_1e_2f_2}. \tag{7.26}$$

Suppose now that we are given d_3, e_1, e_2, f_1, f_2, y and r satisfying the conditions of (7.11)-(7.16) independent of s and z. Write

$$X = \frac{\delta Q}{10re_1e_2}, \tag{7.27}$$

and denote by N_3 the number of permissible choices for s and z. Thus by (7.26) and (7.14)-(7.16), we have

$$N_3 \gg \frac{P\phi(e_1f_1)}{e_1e_2f_1f_2}S_1, \tag{7.28}$$

where $S_1 = \sum_z z^{-1}$, and the summation is over those values of z satisfying

$$\tfrac{1}{2}X < z \le X, \quad (z, e_1f_1f_2y) = 1 \quad \text{and} \quad zr \equiv -f_1f_2 \pmod{d_3}.$$

But we have

$$S_1 = \sum_{m|e_1f_1f_2y} \frac{\mu(m)}{m} \sum_{\substack{\frac{X}{2m}<u\le\frac{X}{m} \\ umr\equiv -f_1f_2 \pmod{d_3}}} u^{-1}. \tag{7.29}$$

For the values of m occurring in the first summation, we have $m|e_1f_1f_2y$, so that in view of (7.16), the congruence condition in the final summation can be replaced by $u \equiv -(mr)^{-1}f_1f_2 \pmod{d_3}$. On substitution into (7.29) we deduce that

$$\begin{aligned} S_1 &= \sum_{m|e_1f_1f_2y} \frac{\mu(m)}{m}\left(d_3^{-1}\log 2 + O(mX^{-1})\right) \\ &= \frac{\phi(e_1f_1f_2y)}{d_3e_1f_1f_2y}\log 2 + O\left((e_1f_1f_2y)^{\varepsilon}X^{-1}\right). \end{aligned} \tag{7.30}$$

Moreover, by (7.27), (7.13), (7.14), together with (7.8) and (7.11),

$$X \ge \frac{yf_1}{10e_2} > \tfrac{1}{10}\left(\delta Q/e_1\right)^{\frac12-\delta} e_2^{-1}f_1^{\frac12+\delta} > \tfrac{1}{10}\left(\delta P^{5/7}\right)^{\frac12-\delta}P^{-2/7}\left(P^{1/4}\right)^{\frac12+\delta}.$$

Therefore $X > P^{2\delta}$, and so the error term in (7.30) is of smaller order than the main term. Thus, by (7.28) and (7.30),

$$N_3 \gg \frac{P\phi(e_1f_1)\phi(e_1f_1f_2y)}{d_3e_2(e_1f_1f_2)^2y}. \tag{7.31}$$

Next we suppose that we are given d_3, e_1, e_2, f_1, f_2 and y satisfying the conditions of (7.11)-(7.16) independent of s, z and r. Write

$$Y = \frac{\delta Q}{ye_1f_1},$$

and denote by N_4 the number of integers r with $\frac12 Y < r \le Y$, $(r, f_1f_2) = 1$ and $yr \equiv -e_2f_2 \pmod{d_3}$. Then by following the argument applied in the treatment of N_2 above, we obtain

$$N_4 \gg \frac{P\phi(f_1f_2)}{e_1f_1^2f_2y}. \tag{7.32}$$

On collecting together (7.31) and (7.32), the lemma follows immediately.

We shall find it useful to record here an estimate for certain sums involving the Euler function ϕ.

Lemma 7.3. *Suppose that $1 \le A \le B$, $q \in \mathbb{N}$ and $\alpha \ge 0$. Define the sum T_α by*

$$T_\alpha = \sum_{\substack{A < x \le B \\ (x,q)=1}} \frac{\phi(x)^\alpha}{x^{\alpha+1}}.$$

Further, take $g_\alpha(m)$ to be the multiplicative function of m which, when p is prime, is defined by

$$g_\alpha(p^t) = \begin{cases} (1-1/p)^\alpha - 1, & \text{when } t = 1, \\ 0, & \text{when } t > 1. \end{cases}$$

Then

$$T_\alpha = \frac{\phi(q)}{q} \log(B/A) \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{g_\alpha(m)}{m} + O_{\alpha,\varepsilon}\left((qB)^\varepsilon A^{-1}\right), \tag{7.33}$$

and there is a positive number γ_α, independent of q, such that

$$\sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{g_\alpha(m)}{m} \ge \gamma_\alpha.$$

Proof. We begin by observing that

$$0 < -g_\alpha(p) < \min\{1, \alpha p^{-1}\}. \tag{7.34}$$

Thus $g_\alpha(m) \ll_{\alpha,\varepsilon} m^{\varepsilon-1}$, and so the infinite series in (7.33) converges absolutely. Moreover, since

$$\left(\frac{\phi(x)}{x}\right)^\alpha = \sum_{m|x} g_\alpha(m),$$

we have

$$T_\alpha = \sum_{\substack{m \le B \\ (m,q)=1}} \frac{g_\alpha(m)}{m} \sum_{\substack{A/m < y \le B/m \\ (y,q)=1}} y^{-1}. \tag{7.35}$$

But an elementary argument shows that

$$\sum_{\substack{X < y \le Y \\ (y,q)=1}} y^{-1} = \frac{\phi(q)}{q} \log(Y/X) + O\left(q^\varepsilon X^{-1}\right). \tag{7.36}$$

The conclusion (7.33) therefore follows easily by substituting (7.36) into (7.35).

For the final assertion of the lemma we note that by the absolute convergence of the infinite series, together with the multiplicative property of g_α, we have

$$\sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{g_\alpha(m)}{m} = \prod_{p \nmid q} \left(1 + \frac{g_\alpha(p)}{p}\right) \ge \prod_{p} \left(1 + \frac{g_\alpha(p)}{p}\right), \tag{7.37}$$

the last inequality following from (7.34). Moreover, from the standard theory of such infinite products, together with (7.34) once again, the last product in (7.37) is positive (and evidently independent of q). This completes the proof of the lemma.

We now estimate $N_0(P)$ by summing $N_1(P)$ over the permissible choices of d_3, e_1, e_2, f_1, f_2 and y. During this process we shall frequently make use of the elementary inequality

$$\phi(nm) \geq \phi(n)\phi(m).$$

We start by considering the sum

$$S_2 = \sum_{\substack{Z^- < y \leq Z^+ \\ (y,d_3)=1}} \frac{\phi(y)}{y^2},$$

in which we have written

$$Z^{\pm} = \left(\frac{\delta Q}{e_1 f_1}\right)^{\frac{1}{2}\pm\delta}. \tag{7.38}$$

In view of (7.8) and (7.11) we have $Q/(e_1f_1) \geq P^{3/7}$, and hence by Lemma 7.3 together with (7.13) and (7.38),

$$\begin{aligned} S_2 &= \frac{\phi(d_3)}{d_3} \log(Z^+/Z^-) \sum_{\substack{m=1 \\ (m,d_3)=1}}^{\infty} \frac{g_1(m)}{m} + O_\varepsilon\left((d_3Z^+)^\varepsilon (Z^-)^{-1}\right) \\ &\gg \frac{\phi(d_3)}{d_3} \log P. \end{aligned}$$

Thus, by Lemma 7.2, for each d_3, e_1, e_2, f_1 and f_2 satisfying (7.11), (7.12) and $(d_3, e_1e_2f_1f_2) = (e_2, f_1) = 1$, the number, N_5, of permissible choices for y, r, z and s subject to (7.13)-(7.16) satisfies

$$N_5 \gg \frac{\phi(d_3)\phi(e_1)^2\phi(f_1)^3\phi(f_2)^2}{d_3^2 e_1^3 e_2 f_1^4 f_2^3} P^2 \log P. \tag{7.39}$$

Next we consider the sum

$$S_3 = \sum_{\substack{W^- < f_2 \leq W^+ \\ (f_2,d_3)=1}} \frac{\phi(f_2)^2}{f_2^3},$$

in which we have written

$$W^{\pm} = \left(\tfrac{1}{2} \pm \delta\right) Q(e_1e_2f_1)^{-1}. \tag{7.40}$$

In view of (7.11) we have $e_1e_2f_1 \le P^{6/7}$, and hence by Lemma 7.3 together with (7.8) and (7.40),

$$S_3 = \frac{\phi(d_3)}{d_3}\log\left(\frac{\frac{1}{2}+\delta}{\frac{1}{2}-\delta}\right)\sum_{\substack{m=1\\(m,d_3)=1}}^{\infty}\frac{g_2(m)}{m} + O_\varepsilon\left(P^\varepsilon e_1e_2f_1Q^{-1}\right) \gg \frac{\phi(d_3)}{d_3}.$$

Thus, by (7.39), for each d_3, e_1, e_2 and f_1 satisfying (7.11) and $(d_3, e_1e_2f_1) = (e_2, f_1) = 1$, the number, N_6, of permissible choices for y, r, z, s and f_2 subject to (7.12)-(7.16) satisfies

$$N_6 \gg \frac{\phi(d_3)^2\phi(e_1)^2\phi(f_1)^3}{d_3^3e_1^3e_2f_1^4}P^2\log P. \tag{7.41}$$

Following a similar argument using Lemma 7.3, we next show that the sum

$$S_4 = \sum_{\substack{P^{1/4}<f_1\le P^{2/7}\\(f_1,d_3e_2)=1}}\frac{\phi(f_1)^3}{f_1^4}$$

satisfies

$$S_4 \gg \frac{\phi(d_3e_2)}{d_3e_2}\log P.$$

Thus, by (7.41), for each d_3, e_1 and e_2 satisfying (7.11) and $(d_3, e_1e_2) = 1$, the number N_7, of permissible choices for y, r, z, s, f_1 and f_2 subject to (7.11)-(7.16) satisfies

$$N_7 \gg \frac{\phi(d_3)^3\phi(e_1)^2\phi(e_2)}{d_3^4e_1^3e_2^2}P^2(\log P)^2. \tag{7.42}$$

Similarly, the sums

$$S_5^{(i)} = \sum_{\substack{P^{1/4}<e_i\le P^{2/7}\\(d_3,e_i)=1}}\frac{\phi(e_i)^{3-i}}{e_i^{4-i}} \quad (i=1,2)$$

satisfy

$$S_5^{(i)} \gg \frac{\phi(d_3)}{d_3}\log P.$$

Therefore, by (7.42) and Lemma 7.1, we have

$$U(P) \gg \sum_{P^{\delta/2}<d_3\le P^\delta}\frac{\phi(d_3)^5}{d_3^6}P^2(\log P)^4,$$

so that by Lemma 7.3 once again, we have $U(P) \gg P^2(\log P)^5$. This completes the proof of Theorem 1.2.

8. The general case

In this section we prove Theorem 1.3, our first task being the proof of the estimate (1.9). Let $U_k(P)$ denote the number of solutions of the simultaneous diophantine equations

$$\sum_{i=1}^{k} X_i^j = \sum_{i=1}^{k} Y_i^j \quad (j = 1, 2, \ldots, k-2 \text{ and } k), \tag{8.1}$$

with

$$1 \le X_i, Y_i \le P \quad (1 \le i \le k), \tag{8.2}$$

and satisfying the additional condition that $(X_1, \ldots, X_k)$ is not a permutation of $(Y_1, \ldots, Y_k)$. Then on recalling Theorem 1.2, in order to establish estimate (1.9) of Theorem 1.3 it suffices to prove that

$$U_k(P) \ll_{\varepsilon,k} P^{k-1+1/k+\varepsilon} \tag{8.3}$$

whenever $k > 3$.

We start by observing that if, for some i and j with $1 \le i, j \le k$, we have $X_i = Y_j$, then by elementary properties of symmetric polynomials, it follows that $(X_1, \ldots, X_k)$ is a permutation of $(Y_1, \ldots, Y_k)$. We henceforth assume the contrary, that $X_i = Y_j$ for no i and j.

We next attend to the problem of constructing an identity analogous to that obtained at the start of §2. Let us write, for convenience, $l(\mathbf{Z}) = Z_1 + \cdots + Z_k$. Then as in the proof of [24, Lemma 5.2] (see, in particular, equation (4)), for each solution $\mathbf{X}$, $\mathbf{Y}$ of the system (8.1) we have

$$\prod_{i=1}^{k} (l(\mathbf{X}) - X_i) = \prod_{i=1}^{k} (l(\mathbf{Y}) - Y_i).$$

Make the substitution

$$x_i = l(\mathbf{X}) - X_i \quad \text{and} \quad y_i = l(\mathbf{Y}) - Y_i \quad (1 \le i \le k). \tag{8.4}$$

Then we find that the solutions of (8.1) subject to (8.2) are in one-to-one correspondence with a subset of the set of solutions of the system

$$\prod_{i=1}^{k} x_i = \prod_{i=1}^{k} y_i, \tag{8.5}$$

$$\sum_{i=1}^{k} (l(\mathbf{x}) - (k-1)x_i)^j = \sum_{i=1}^{k} (l(\mathbf{y}) - (k-1)y_i)^j \quad (1 \le j \le k-2), \tag{8.6}$$

subject to

$$0 < \alpha_i \le kP \quad (1 \le i \le k; \alpha = x, y). \tag{8.7}$$

Furthermore, on noting that $l(\mathbf{x}) = l(\mathbf{y})$ and expanding the equations (8.6), we discover that they are equivalent to the system

$$\sum_{i=1}^{k} x_i^j = \sum_{i=1}^{k} y_i^j \quad (1 \le j \le k-2). \tag{8.8}$$

Now observe that by solving the equations implicit in (8.4), if for some i and j we have $x_i = y_j$, then we also have $X_i = Y_j$. In consequence we have

$$U_k(P) \ll V_k(P), \tag{8.9}$$

where $V_k(P)$ denotes the number of solutions of the system (8.5) and (8.8), subject to (8.7), and

$$x_i \ne y_j \quad (1 \le i, j \le k). \tag{8.10}$$

At this stage in the proceedings we establish that the solutions of the system (8.5) and (8.8) satisfy an identity, this facilitating our later analysis.

Lemma 8.1. *Suppose that the integers* $\mathbf{x}, \mathbf{y}$ *satisfy the equations (8.5) and (8.8). Then*

$$x_1^{-1} \prod_{j=1}^{k-1} (y_j - x_1) = y_k^{-1} \prod_{i=2}^{k} (x_i - y_k). \tag{8.11}$$

Proof. We require some notation. When i and s are positive integers, we define the polynomial $\phi_{i,s}(\mathbf{u}) \in \mathbb{Z}[u_1, \ldots, u_s]$ by

$$\phi_{i,s}(\mathbf{u}) = u_1^i + \cdots + u_s^i. \tag{8.12}$$

Also, we define the polynomial $f(x, \mathbf{z}) \in \mathbb{Z}[x, \mathbf{z}]$ by

$$f(x, \mathbf{z}) = x^{-1} \left(\prod_{j=1}^{k-1} (z_j - x) - (z_1 \ldots z_{k-1}) \right). \tag{8.13}$$

By using elementary properties of symmetric polynomials, we can find a polynomial $\Psi(\boldsymbol{\xi}) \in \mathbb{Z}[\xi_1, \ldots, \xi_{k-2}]$ such that, for some non-zero integer A, the polynomial

$$A v_1 \ldots v_{k-2} + \Psi(\phi_{1,k-2}(\mathbf{v}), \ldots, \phi_{k-2,k-2}(\mathbf{v}))$$

is identically zero. Then by (8.12) and (8.13), for each i with $1 \le i \le k-1$, we have

$$A f(z_i, \mathbf{z}) - \Psi(\phi_{1,k-1}(\mathbf{z}) - z_i, \ldots, \phi_{k-2,k-1}(\mathbf{z}) - z_i^{k-2}) = 0,$$

and hence the polynomial

$$A f(x, \mathbf{z}) - \Psi(\phi_{1,k-1}(\mathbf{z}) - x, \ldots, \phi_{k-2,k-1}(\mathbf{z}) - x^{k-2}) \tag{8.14}$$

is divisible by $(x - z_1)\dots(x - z_{k-1})$. But in view of (8.13), and an argument based on homogeneity, the polynomial (8.14) has degree $k - 2$. Consequently, the latter polynomial must be zero, since it is divisible by a polynomial of degree $k-1$, which leads to a contradiction.

For the sake of convenience, write $\tilde{\mathbf{x}}$ for $(x_2, \dots, x_k)$, and $\tilde{\mathbf{y}}$ for $(y_1, \dots, y_{k-1})$. Then we observe that the equations (8.8) are equivalent to

$$\phi_{j,k-1}(\tilde{\mathbf{x}}) - y_k^j = \phi_{j,k-1}(\tilde{\mathbf{y}}) - x_1^j \quad (1 \le j \le k-2). \tag{8.15}$$

Thus, since the polynomial (8.14) is zero, we deduce from (8.15) that

$$\begin{aligned} Af(x_1, \tilde{\mathbf{y}}) &= \Psi(\phi_{1,k-1}(\tilde{\mathbf{y}}) - x_1, \dots, \phi_{k-2,k-1}(\tilde{\mathbf{y}}) - x_1^{k-2}) \\ &= \Psi(\phi_{1,k-1}(\tilde{\mathbf{x}}) - y_k, \dots, \phi_{k-2,k-1}(\tilde{\mathbf{x}}) - y_k^{k-2}) \\ &= Af(y_k, \tilde{\mathbf{x}}). \end{aligned}$$

In view of (8.13), this completes the proof of the lemma.

The multiplicative structure of the equation (8.5) may be exploited, as in the case $k = 3$, by the removal of common factors between the variables. When $1 \le i, j \le k$, we define the integer a_{ij} inductively by

$$a_{ij} = \left(\frac{x_i}{b_{ij}}, \frac{y_j}{c_{ij}} \right),$$

where

$$b_{ij} = \prod_{m=1}^{j-1} a_{im} \quad \text{and} \quad c_{ij} = \prod_{m=1}^{i-1} a_{mj},$$

and here we adopt the convention that the empty product is unity. A little consideration of the respective common factors reveals that

$$x_i = \prod_{m=1}^{k} a_{im} \quad \text{and} \quad y_j = \prod_{m=1}^{k} a_{mj} \quad (1 \le i, j \le k). \tag{8.16}$$

On substitution into (8.5), (8.7), (8.8) and (8.10), we deduce that

$$V_k(P) \ll W_k(P), \tag{8.17}$$

where $W_k(P)$ denotes the number of solutions of the equations

$$\sum_{i=1}^{k} a_{i*}^j = \sum_{i=1}^{k} a_{*i}^j \quad (1 \le j \le k-2), \tag{8.18}$$

subject to

$$0 < a_{i*}, a_{*j} \le kP \quad \text{and} \quad a_{i*} \ne a_{*j} \quad (1 \le i, j \le k). \tag{8.19}$$

Here, for the sake of transparency, we write

$$a_{i*} = \prod_{m=1}^{k} a_{im} \quad \text{and} \quad a_{*j} = \prod_{n=1}^{k} a_{nj}. \tag{8.20}$$

Furthermore, on substituting (8.16) and (8.20) into (8.11), we deduce from Lemma 8.1 that for each solution counted by $W_k(P)$, we have

$$\prod_{j=1}^{k-1} \left(\frac{a_{*j}}{a_{1j}} - \frac{a_{1*}}{a_{1j}} \right) = \prod_{i=2}^{k} \left(\frac{a_{i*}}{a_{ik}} - \frac{a_{*k}}{a_{ik}} \right). \tag{8.21}$$

Now observe, from inequality (8.19), that by a suitable rearrangement of variables we have

$$a_{11} \geq a_{12} \geq \cdots \geq a_{1k} \geq 1.$$

Thus

$$1 \leq a_{1k} \leq (kP)^{1/k}. \tag{8.22}$$

Consider any solution of (8.18) counted by $W_k(P)$. By (8.19), (8.20) and (8.22) there are at most $O_{k,\varepsilon}(P^{k-1+1/k+\varepsilon})$ possible choices for a_{ij} $(2 \leq i \leq k,\ 1 \leq j \leq k)$ and a_{1k}. Fix any such choice. Observe that a_{*j}/a_{1j} is independent of a_{1h} $(1 \leq h \leq k)$. Then for some non-zero integers $m_1, \ldots, m_{k-1}$, and an integer M, we have

$$\prod_{j=1}^{k-1} \left(m_j - \frac{a_{1*}}{a_{1j}} \right) = M.$$

Moreover by (8.19), M is non-zero. Then by using elementary estimates for the divisor function, there are $O_{\varepsilon,k}(P^{\varepsilon})$ possible choices for integers d_j $(1 \leq j \leq k-1)$ with

$$m_j - a_{1*}/a_{1j} = d_j.$$

But for each possible choice for $d_1, \ldots, d_{k-1}$, there are at most $O_{\varepsilon,k}(P^{\varepsilon})$ possible choices for a_{1j} $(1 \leq j \leq k-1)$, again by using elementary estimates for the divisor function. Thus

$$W_k(P) \ll_{\varepsilon,k} P^{k-1+1/k+\varepsilon},$$

and so (8.3) follows from (8.9) and (8.17). This completes the proof of estimate (1.9) of Theorem 1.3.

We now turn our attention to estimate (1.10). Initially, we argue as above, noting that it suffices to prove that when $k \geq 3$, we have

$$U_k(P) \ll_{\varepsilon,k} P^{(k+3)/2+\varepsilon}. \tag{8.23}$$

Observe that if the integers $\mathbf{x}$, $\mathbf{y}$ satisfy the equations (8.5) and (8.8), then by Lemma 8.1 and a rearrangement of variables, for any u with $1 \le u \le k$,

$$x_u^{-1} \prod_{j=1}^{k-1} (y_j - x_u) = y_k^{-1} \prod_{\substack{1 \le i \le k \\ i \ne u}} (x_i - y_k).$$

Consequently,

$$x_u^{-1} \prod_{j=1}^{k} (y_j - x_u) = -y_k^{-1} \prod_{i=1}^{k} (x_i - y_k),$$

and thus for each u and v with $1 \le u < v \le k$,

$$x_v \prod_{i=1}^{k} (y_i - x_u) = x_u \prod_{j=1}^{k} (y_j - x_v). \tag{8.24}$$

Consider first the solutions of (8.5) and (8.6) counted by $V_k(P)$ with $x_i = x_j$ for every i and j with $1 \le i < j \le k$. Fix any one such choice, say with $x_i = x$ $(1 \le i \le k)$, and consider the equation (8.5). By using standard estimates for the divisor function there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for the integers $y_1, \dots, y_k$. Consequently the total number of solutions of this type counted by $V_k(P)$ is $O_{\varepsilon,k}(P^{1+\varepsilon})$. We may therefore restrict attention to the situation with $x_1 \ne x_2$. But given x_1, x_2 and $y_1, \dots, y_k$ satisfying (8.7), we may combine standard estimates for the divisor function with (8.5) to deduce that there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for $x_3, \dots, x_k$. Thus, on substituting $u_0 = x_2$, $v_0 = x_1$,

$$u_i = y_i - x_1 \quad \text{and} \quad v_i = y_i - x_2 \quad (1 \le i \le k),$$

we deduce from (8.24) and (8.7) that

$$V_k(P) \ll_{\varepsilon,k} P^\varepsilon W_k(P) + P^{1+\varepsilon}, \tag{8.25}$$

where $W_k(P)$ denotes the number of solutions of the system

$$\prod_{i=0}^{k} u_i = \prod_{j=0}^{k} v_j, \tag{8.26}$$

with

$$v_0 + u_i = u_0 + v_i \quad (1 \le i \le k), \tag{8.27}$$

$$1 \le u_i, v_i \le kP \quad (0 \le i \le k), \tag{8.28}$$

and $u_0 \ne v_0$.

We now prosecute an argument parallel to that applied to reach (8.16) and (8.18). When $0 \le i, j \le k$, we define the integers α_{ij} inductively by

$$\alpha_{ij} = \left(\frac{u_i}{\beta_{ij}}, \frac{v_j}{\gamma_{ij}} \right),$$

where

$$\beta_{ij} = \prod_{m=0}^{j-1} \alpha_{im} \quad \text{and} \quad \gamma_{ij} = \prod_{n=0}^{i-1} \alpha_{nj},$$

and again we adopt the convention that the empty product is unity. Writing, for the sake of transparency,

$$\alpha_{i*} = \prod_{m=0}^{k} \alpha_{im} \quad \text{and} \quad \alpha_{*j} = \prod_{n=0}^{k} \alpha_{nj},$$

we find that $u_i = \alpha_{i*}$ and $v_j = \alpha_{*j}$ $(0 \le i, j \le k)$. Thus, on substitution into (8.26), (8.27) and (8.28), we deduce from (8.25) that

$$V_k(P) \ll_{\varepsilon,k} P^{\varepsilon} R_k(P) + P^{1+\varepsilon}, \tag{8.29}$$

where $R_k(P)$ denotes the number of solutions of the system

$$\alpha_{*0} + \alpha_{i*} = \alpha_{0*} + \alpha_{*i} \quad (1 \le i \le k), \tag{8.30}$$

with $\alpha_{0*} \ne \alpha_{*0}$, and

$$1 \le \alpha_{i*}, \alpha_{*i} \le kP \quad (0 \le i \le k). \tag{8.31}$$

Write

$$X = \prod_{1 \le i < j \le k} \alpha_{ij}, \quad Y = \prod_{1 \le j < i \le k} \alpha_{ij}, \quad \text{and} \quad Z = \prod_{i=1}^{k} \alpha_{ii}.$$

Then we have

$$\left(\alpha_{00}^{-1}\alpha_{*0}\alpha_{0*}\right)^2 XYZ = \alpha_{00}^{-1}\alpha_{*0}\alpha_{0*} \prod_{i=0}^{k}\prod_{j=0}^{k} \alpha_{ij}.$$

We therefore deduce from (8.31) that

$$\left(\alpha_{00}^{-1}\alpha_{*0}\alpha_{0*}\right)^2 XY \le (kP)^2 \prod_{i=0}^{k} \alpha_{i*} \le (kP)^{k+3},$$

and hence that in any solution counted by $R_k(P)$, one at least of the inequalities

$$\alpha_{00}^{-1}\alpha_{*0}\alpha_{0*} X \le (kP)^{(k+3)/2} \quad \text{and} \quad \alpha_{00}^{-1}\alpha_{*0}\alpha_{0*} Y \le (kP)^{(k+3)/2}$$

must hold. By symmetry, therefore, we have

$$R_k(P) \ll R_k^*(P), \tag{8.32}$$

where $R_k^*(P)$ denotes the number of solutions of the system (8.30) subject to (8.31), and the additional condition

$$\alpha_{*0} \prod_{0\le i<j\le k} \alpha_{ij} \le (kP)^{(k+3)/2}. \tag{8.33}$$

We claim that for a fixed choice of the variables α_{ij} with

$$0 \le i < j \le k \quad \text{or} \quad j = 0 \text{ and } 0 \le i \le k, \tag{8.34}$$

there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for the remaining variables satisfying (8.30) and (8.31). If such is the case, then by (8.32) and (8.33) we have

$$R_k(P) \ll_{\varepsilon,k} P^{(k+3)/2+\varepsilon},$$

and so estimate (1.10) of Theorem 1.3 follows from (8.9) and (8.29).

It remains to establish the latter claim, which we prove inductively as follows. For a fixed choice of the α_{ij} with i and j satisfying (8.34), we suppose at step t that there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for the variables α_{ij} with $i < t$ or $j < t$. This inductive hypothesis is plainly satisfied for $t = 1$, since α_{0*} and α_{*0} are fixed from the beginning. Suppose then that the hypothesis is satisfied for a $t \ge 1$, and consider one of the $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for the α_{ij} with $i < t$ or $j < t$. We consider the equation (8.30) with $i = t$ in the form

$$\alpha_{tt} \left(\prod_{\substack{0\le i\le k\\ i\ne t}} \alpha_{it} - \prod_{\substack{0\le j\le k\\ j\ne t}} \alpha_{tj} \right) = \alpha_{*0} - \alpha_{0*}. \tag{8.35}$$

Since $\alpha_{0*} \ne \alpha_{*0}$, a standard estimate for the divisor function shows that there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for α_{tt}. Fix any one such choice. On writing N_t for the fixed integer $(\alpha_{*0} - \alpha_{0*})/\alpha_{tt}$, we obtain from (8.35) the equation

$$\prod_{t<i\le k} \alpha_{it} = \left(\prod_{0\le i<t} \alpha_{it} \right)^{-1} \left(N_t + \prod_{\substack{0\le j\le k\\ j\ne t}} \alpha_{tj} \right). \tag{8.36}$$

Moreover, by hypothesis, the right hand side of the equation (8.36) is already fixed. Thus, by using standard estimates for the divisor function, there are at most $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for the variables α_{it} with $t < i \le k$. Consequently, there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for the α_{ij} with $i \le t$ or $j \le t$, and so the inductive hypothesis holds with $t + 1$ in place of t. This completes the proof of the claim, and hence also the proof of estimate (1.10) of Theorem 1.3.

9. A special case of Vinogradov's Mean Value Theorem

In this section, by an analysis somewhat similar to that applied in §8, we prove Theorem 1.4. Let $U_k(P)$ denote the number of solutions of the simultaneous diophantine equations

$$\sum_{i=1}^{k+1} X_i^j = \sum_{i=1}^{k+1} Y_i^j \quad (1 \le j \le k), \tag{9.1}$$

with

$$1 \le X_i, Y_i \le P \quad (1 \le i \le k+1), \tag{9.2}$$

and satisfying the additional condition that $(X_1, \dots, X_{k+1})$ is not a permutation of $(Y_1, \dots, Y_{k+1})$. Then Theorem 1.4 follows from the estimate

$$U_k(P) \ll_{\varepsilon,k} P^{(k+5)/2+\varepsilon} \quad (k \ge 3). \tag{9.3}$$

Moreover, if $\mathbf{X}$, $\mathbf{Y}$ is a solution of (9.1) satisfying the condition that for some i and j with $1 \le i, j \le k+1$, we have $X_i = Y_j$, then it follows that $(X_1, \dots, X_{k+1})$ is a permutation of $(Y_1, \dots, Y_{k+1})$. Thus it suffices to count only those solutions for which $X_i = Y_j$ for no i and j.

We now note that by rearranging variables we may suppose that

$$Y_{k+1} = \min_{i,j} \{X_i, Y_j\}.$$

Moreover an application of the binomial theorem reveals that the equations (9.1) are equivalent to

$$\sum_{i=1}^{k+1} (X_i - Y_{k+1})^j = \sum_{i=1}^{k+1} (Y_i - Y_{k+1})^j \quad (1 \le j \le k).$$

Thus, on observing that there are at most P choices for Y_{k+1}, we may substitute $x_i = X_i - Y_{k+1}$ and $z_i = Y_i - Y_{k+1}$, and conclude from (9.1) and (9.2) that

$$U_k(P) \ll_k PV_k(P), \tag{9.4}$$

where $V_k(P)$ denotes the number of solutions of the system

$$\sum_{i=1}^{k+1} x_i^j = \sum_{i=1}^{k} z_i^j \quad (1 \le j \le k), \tag{9.5}$$

with

$$1 \le x_i \le P \quad (1 \le i \le k+1) \quad \text{and} \quad 0 \le z_i \le P \quad (1 \le i \le k), \tag{9.6}$$

and $x_i = z_j$ for no i and j.

We now adopt the notation (8.12), and observe that by the properties of the elementary symmetric polynomials, there is a positive integer C and a polynomial $\Upsilon(\mathbf{w}) \in \mathbb{Z}[w_1, \dots, w_{k-1}]$ such that

$$C\phi_{k,k-1}(\mathbf{x}) = \Upsilon(\phi_{1,k-1}(\mathbf{x}), \dots, \phi_{k-1,k-1}(\mathbf{x})).$$

We therefore deduce from (8.12) that

$$\Upsilon\left(\phi_{1,k}(\mathbf{x}) - u, \dots, \phi_{k-1,k}(\mathbf{x}) - u^{k-1}\right) - C(\phi_{k,k}(\mathbf{x}) - u^k) \tag{9.7}$$

is a non-trivial polynomial divisible by $(u - x_1)\dots(u - x_k)$. Moreover, since the polynomial in (9.7) has degree at most k, it follows from (9.5) and (9.6) that for each solution counted by $V_k(P)$, we have

$$\prod_{i=1}^{k}(z_i - x_s) = \prod_{\substack{1\le j\le k+1\\ j\ne s}} x_j \quad (1 \le s \le k+1). \tag{9.8}$$

Furthermore, plainly, for each s and t with $1 \le s < t \le k+1$, we have

$$x_s \prod_{i=1}^{k}(z_i - x_s) = x_t \prod_{j=1}^{k}(z_j - x_t), \tag{9.9}$$

an equation strikingly similar to (8.24).

Consider first the solutions of (9.5) and (9.6) counted by $V_k(P)$ with $x_i = x_j$ for every i and j with $1 \le i < j \le k+1$. Fix any one such choice, say with $x_i = x$ $(1 \le i \le k+1)$, and consider the equation (9.8). By using standard estimates for the divisor function there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for integers $d_1, \dots, d_k$ satisfying $z_i = x + d_i$ $(1 \le i \le k)$. Consequently the total number of solutions of this type counted by $V_k(P)$ is $O_{\varepsilon,k}(P^{1+\varepsilon})$. We may therefore restrict attention to the situation with $x_1 \ne x_2$. But given x_1, x_2 and $z_1, \dots, z_k$ satisfying (9.6), we may combine standard estimates for the divisor function with (9.8) to deduce that there are $O_{\varepsilon,k}(P^\varepsilon)$ possible choices for $x_3, \dots, x_{k+1}$. Thus, on substituting $u_0 = x_1$, $v_0 = x_2$,

$$u_i = z_i - x_1 \quad \text{and} \quad v_i = z_i - x_2 \quad (1 \le i \le k),$$

we deduce from (9.9) and (9.6) that

$$V_k(P) \ll_{\varepsilon,k} P^\varepsilon W_k(P) + P^{1+\varepsilon}, \tag{9.10}$$

where $W_k(P)$ denotes the number of solutions of the system

$$\prod_{i=0}^{k} u_i = \prod_{j=0}^{k} v_j, \tag{9.11}$$

with

$$u_0 + u_i = v_0 + v_i \quad (1 \le i \le k), \tag{9.12}$$

$$1 \le u_0, v_0 \le P \quad \text{and} \quad 1 \le |u_i|, |v_i| \le P \quad (1 \le i \le k), \tag{9.13}$$

and $u_0 \neq v_0$.

We now proceed precisely as in the argument leading to (8.29)-(8.31). Employing the same notation, we deduce that

$$V_k(P) \ll_{\varepsilon,k} P^{\varepsilon} R_k(P) + P^{1+\varepsilon}, \tag{9.14}$$

where $R_k(P)$ denotes the number of solutions of the system

$$\alpha_{0*} + \lambda_i \alpha_{i*} = \alpha_{*0} + \mu_i \alpha_{*i} \quad (1 \le i \le k), \tag{9.15}$$

with

$$\alpha_{0*} \neq \alpha_{*0}, \quad 1 \le \alpha_{0*}, \alpha_{*0} \le P, \tag{9.16}$$

and

$$\lambda_i, \mu_i \in \{+1, -1\}, \quad 1 \le \alpha_{i*}, \alpha_{*i} \le P \quad (1 \le i \le k). \tag{9.17}$$

Moreover, as in the analysis leading to (8.32), we find that

$$R_k(P) \ll R_k^*(P), \tag{9.18}$$

where $R_k^*(P)$ denotes the number of solutions of the system (9.15) subject to (9.16), (9.17), and the additional condition

$$\alpha_{*0} \prod_{0 \le i < j \le k} \alpha_{ij} \le P^{(k+3)/2}. \tag{9.19}$$

It is now a simple matter to imitate the analysis concluding §8. We claim that for a fixed choice of the variables α_{ij} with

$$0 \le i < j \le k \quad \text{or} \quad j = 0 \text{ and } 0 \le i \le k,$$

there are $O_{\varepsilon,k}(P^{\varepsilon})$ possible choices for the remaining variables satisfying (9.15), (9.16) and (9.17). If such is the case, then by (9.18) and (9.19) we have

$$R_k(P) \ll_{\varepsilon,k} P^{(k+3)/2+\varepsilon},$$

and so Theorem 1.4 follows from (9.4) and (9.14). But the system (9.15) is identical to (8.30), save that the rôles of α_{0*} and α_{*0} have been interchanged, and there are 2^{2k} possible choices for the coefficients λ_i and μ_i. Such complications are easily accomodated in the argument concluding §8, the equation (8.35) simply being replaced by

$$\alpha_{tt}\left(\mu_t \prod_{\substack{0 \le i \le k \\ i \neq t}} \alpha_{it} - \lambda_t \prod_{\substack{0 \le j \le k \\ j \neq t}} \alpha_{tj}\right) = \alpha_{0*} - \alpha_{*0}.$$

Thus the above claim follows with little difficulty, and the proof of Theorem 1.4 is concluded.

10. A sharper estimate

In this section we improve on the estimate of the previous section in the case $k = 3$, and indeed we are able to establish a lower bound. We follow an argument motivated by an identity to be found in [13]. Consider a solution $\mathbf{X}$, $\mathbf{Y}$ of (9.1) counted by $U_3(P)$. Let us write, for convenience, $l(\mathbf{Z}) = Z_1 + \cdots + Z_4$. Then an application of the binomial theorem reveals that when $k = 3$, the system (9.1) is equivalent to

$$\sum_{i=1}^{4} (4X_i - l(\mathbf{X}))^j = \sum_{i=1}^{4} (4Y_i - l(\mathbf{Y}))^j \quad (1 \le j \le 3). \tag{10.1}$$

We make the substitution

$$u_i = 4X_i - l(\mathbf{X}), \quad v_i = 4Y_i - l(\mathbf{Y}) \quad (1 \le i \le 4). \tag{10.2}$$

Since $0 < l(\mathbf{X}) = l(\mathbf{Y}) \le 4P$, on taking account of all the possible choices for $l(\mathbf{X})$, we are forced to conclude from (10.1) and (9.2) that

$$U_3(P) \ll PN_1(4P), \tag{10.3}$$

where $N_1(P)$ denotes the number of solutions of the system

$$\sum_{i=1}^{4} u_i^j = \sum_{i=1}^{4} v_i^j \quad (j = 2, 3), \tag{10.4}$$

$$u_1 + u_2 + u_3 + u_4 = 0 = v_1 + v_2 + v_3 + v_4, \tag{10.5}$$

subject to

$$0 \le |u_i|, |v_i| \le P \quad (1 \le i \le 4), \tag{10.6}$$

and with $u_i = v_j$ for no i and j with $1 \le i, j \le 4$. Moreover, to any solution $\mathbf{u}$, $\mathbf{v}$ counted by $N_1(P)$ there correspond at least P solutions $\mathbf{X}$, $\mathbf{Y}$ of the system (9.1) with $1 \le X_i, Y_i \le 3P$ and $X_i = Y_j$ for no i and j. For we may take any integer σ with $P < \sigma \le 2P$, and put $X_i = u_i + \sigma$, $Y_i = v_i + \sigma$ $(1 \le i \le 4)$, and once again apply the binomial theorem in (9.1). Thus we deduce that $U_3(3P) \gg PN_1(P)$. Then taking note of (10.3), we deduce that

$$PN_1(P/3) \ll U_3(P) \ll PN_1(4P). \tag{10.7}$$

We now substitute for u_4 and v_4 from (10.5) into (10.4), and write

$$x_i = u_1 + u_2 + u_3 - u_i, \quad y_i = v_1 + v_2 + v_3 - v_i \quad (i = 1, 2, 3). \tag{10.8}$$

Then a solution $\mathbf{u}$, $\mathbf{v}$ of the system (10.4) and (10.5) gives a solution $\mathbf{x}$, $\mathbf{y}$ of the system

$$x_1x_2x_3 = y_1y_2y_3, \tag{10.9}$$

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2. \tag{10.10}$$

Let us consider what happens to the diagonal solutions of the system (10.4) and (10.5) under this transformation. First we note that by an elementary argument, if for any i and j we have $u_i = v_j$ in (10.4) and (10.5), then necessarily

$$(u_1, u_2, u_3, u_4) \text{ is a permutation of } (v_1, v_2, v_3, v_4). \tag{10.11}$$

Moreover in view of (10.8) we have

$$2u_i = x_1 + x_2 + x_3 - 2x_i, \quad 2v_i = y_1 + y_2 + y_3 - 2y_i \quad (i = 1, 2, 3), \tag{10.12}$$

and

$$2u_4 = -(x_1 + x_2 + x_3), \quad \text{and} \quad 2v_4 = -(y_1 + y_2 + y_3). \tag{10.13}$$

When (10.11) holds with $u_4 = v_4$, by (10.8) we find that (x_1, x_2, x_3) is a permutation of (y_1, y_2, y_3). Otherwise for some i and j with $1 \le i, j \le 3$, we have $u_4 = v_j$ and $v_4 = u_i$. Then by (10.12) and (10.13),

$$-(x_1 + x_2 + x_3) = 2u_4 = 2v_j = y_1 + y_2 + y_3 - 2y_j,$$

and

$$-(y_1 + y_2 + y_3) = 2v_4 = 2u_i = x_1 + x_2 + x_3 - 2x_i,$$

and thus $x_i = y_j$. Then in either case we have $x_i = y_j$ for some i and j with $1 \le i, j \le 3$. But if $\mathbf{x}$, $\mathbf{y}$ is a solution of (10.9) and (10.10) in which the latter condition holds, then by a suitable rearrangement of variables we may suppose that $x_3 = y_3$, and it easily follows that

$$(x_1^2, x_2^2, x_3^2) \text{ is a permutation of } (y_1^2, y_2^2, y_3^2). \tag{10.14}$$

Now suppose that this last condition holds. If in fact (x_1, x_2, x_3) is a permutation of (y_1, y_2, y_3), then (10.11) follows from (10.12) and (10.13). Otherwise we have a situation typified by the case $(x_1, x_2, x_3) = (y_1, -y_2, -y_3)$. But then, by (10.12) and (10.13), we have

$$2u_4 = -(x_1 + x_2 + x_3) = y_2 + y_3 - y_1 = 2v_1.$$

Thus we deduce that the solutions of (10.4) and (10.5), in which $u_i = v_j$ for some i and j, are in one-to-one correspondence with the solutions of (10.9) and (10.10) in which $x_k^2 = y_l^2$ for some k and l.

Lemma 10.1. *We have*

$$S_4(P; 1, 2, 3) - 4!P^4 \ll P^{10/3}(\log 2P)^{35}.$$

Proof. In view of (10.7) it suffices to show that $N_1(P) \ll P^{7/3}(\log 2P)^{35}$. But by the above analysis we have $N_1(P) \ll N_2(P)$, where $N_2(P)$ denotes the number of solutions of the system (10.9) and (10.10) subject to $0 \le |x_i|, |y_i| \le 2P$ $(1 \le i \le 3)$, and with $x_i^2 = y_j^2$ for no i and j. Moreover, if any x_i vanishes, then by (10.9), so does some y_j, and then, in view of (10.10), we discover that the number of solutions of this type is $O(P^2 \log P)$. Thus

$$N_1(P) \ll N_3(P) + P^2 \log P, \tag{10.15}$$

where $N_3(P)$ denotes the number of solutions of (10.9) and (10.10) subject to

$$1 \le |x_i|, |y_i| \le 2P \quad (1 \le i \le 3), \tag{10.16}$$

and with $x_i^2 = y_j^2$ for no i and j.

At this stage we follow the same procedure that was adopted in §2 for the removal of common factors among the variables. Adopting the same notation, save with upper case X and Y replaced here by the corresponding lower case letters, and noting a possible ambiguity of sign in the current situation, we discover that

$$N_3(P) \ll N_4(P), \tag{10.17}$$

where $N_4(P)$ denotes the number of solutions of the equation

$$(d_1e_1f_1)^2 + (d_2e_2f_2)^2 + (d_3e_3f_3)^2 = (d_1d_2d_3)^2 + (e_1e_2e_3)^2 + (f_1f_2f_3)^2, \tag{10.18}$$

subject to (3.5) and (2.9). Moreover, as in the argument described in §3, we may once again assume, without any loss, that the inequalities (3.6) and (3.7) hold. Thus we may follow the analysis of §3, on this occasion replacing the identity (3.8) with a similar one obtained from (10.18), namely

$$(d_1^2d_3^2 - e_2^2f_2^2)(d_2^2d_3^2 - e_1^2f_1^2) = (e_3^2d_3^2 - f_1^2f_2^2)(f_3^2d_3^2 - e_1^2e_2^2). \tag{10.19}$$

Hence, on writing $Q = 2P$, we find that

$$N_4(P) \ll \sum_{d_1e_1f_1 \le Q} \sum_{d_2e_2f_2 \le Q} \sum_{d_3 \le Q^{1/3}} d(u)d(v)d(w)d(x),$$

where we have written

$$u = d_1d_3 - e_2f_2, \quad v = d_1d_3 + e_2f_2, \quad w = d_2d_3 - e_1f_1, \quad x = d_2d_3 + e_1f_1,$$

and the summation is restricted to those values of d_3 satisfying (3.6). An elementary argument, with only minor complications, now shows that

$$N_4(P) \ll Q^{7/3}(\log Q)^{35},$$

which, by (10.15) and (10.17), completes the proof of the lemma.

For the sake of simplicity, we establish a lower bound for $U_3(P)$ which is probably somewhat weaker than might be obtained through the use of the identity (10.19).

Lemma 10.2. *We have $U_3(P) \gg P^2 \log P$.*

Proof. Suppose that a, b, c are positive integers satisfying the equation

$$16c^2 = a^2 + 15b^2. \tag{10.20}$$

We put $x_1 = a$, $x_2 = 2b$, $x_3 = 7c$, $y_1 = 2a$, $y_2 = 7b$, $y_3 = c$, and write

$$u_i = x_1 + x_2 + x_3 - 2x_i, \quad v_i = y_1 + y_2 + y_3 - 2y_i \quad (i = 1, 2, 3),$$

and

$$u_4 = -(u_1 + u_2 + u_3), \quad v_4 = -(v_1 + v_2 + v_3).$$

Then with only a little effort we discover that the equations (10.4) and (10.5) are satisfied. Moreover, as in the analysis preceding Lemma 10.1, if $u_i = v_j$ for any i and j, then necessarily $x_i^2 = y_j^2$ for some i and j. If the latter does indeed occur, then we have a non-trivial equation relating two of the variables a, b and c, and moreover the equation (10.20) provides a further relation incorporating the third variable. Thus we deduce that

$$N_1(P) \gg N_5(P) + O(P), \tag{10.21}$$

where $N_5(P)$ denotes the number of solutions of the equation (10.20) with

$$1 \le a, b, c \le P/100. \tag{10.22}$$

An elementary argument shows that a parametric solution to (10.20) is given by

$$a = 4w(s^2 - 15t^2), \quad b = 8wst, \quad c = w(s^2 + 15t^2). \tag{10.23}$$

Furthermore, distinct triples (s, t, w) satisfying $(s, t) = 1$ determine distinct triples (a, b, c). Thus we deduce from (10.22) and (10.23) that $N_5(P) \gg N_6(P)$, where $N_6(P)$ denotes the number of triples (s, t, w) with

$$1 \le s, t \le \delta(P/w)^{1/2}, \quad (s, t) = 1 \quad \text{and} \quad 1 \le w \le P^{1/2},$$

in which δ is a sufficiently small, but fixed, positive number. On noting that when $X \gg P^\delta$ we have

$$\sum_{\substack{1 \le t \le X \\ (s,t)=1}} 1 = \sum_{m|s} \mu(m) \sum_{1 \le t \le X/m} 1 \gg X\phi(s)/s,$$

we deduce that

$$N_5(P) \gg P^{1/2} \sum_{1 \le w \le P^{1/2}} w^{-1/2} \sum_{1 \le s \le \delta(P/w)^{1/2}} \phi(s)/s. \tag{10.24}$$

A standard argument therefore shows that

$$N_5(P) \gg P \sum_{1 \le w \le P^{1/2}} w^{-1} \gg P \log P,$$

and thus the lemma follows immediately from (10.21) and (10.7).

The proof of Theorem 1.5 follows by combining Lemmata 10.1 and 10.2.

11. Further lower bounds

The problem of obtaining non-trivial lower bounds for the number of non-diagonal solutions of systems of the form (1.1) is probably of greatest interest in the case $s = t+1$. In a number of such cases in which the exponents are quite small one can show that there are many non-trivial solutions. In particular, Bremner [6, 7] has shown that there are infinitely many primitive solutions to the system (1.1) when $s = 3$, $t = 2$, and (k_1, k_2) is either $(1, 5)$ or $(2, 6)$. We refer the reader also to Gloden [13] and Choudhry [9] for a consideration of parametric solutions in a number of interesting cases. Note, however, that knowledge of parametric solutions to a system of equations is usually not, by itself, sufficient to obtain good lower bounds for the number of non-diagonal solutions inside a box. Thus, for example, the work of Bremner and Brudno [5] does not immediately imply the lower bound of Theorem 1.2. For now we content ourselves with brief proofs of two non-trivial lower bounds in Theorems 11.1 and 11.2 below.

Theorem 11.1. *Let $V_4(P)$ denote the number of non-diagonal solutions counted by $S_4(P; 1, 2, 4)$. Then $V_4(P) \gg P^2 \log P$.*

Proof. We consider the number, $V_4(P)$, of solutions of the system (1.1) with $s = 4$, $t = 3$ and $(k_1, k_2, k_3) = (1, 2, 4)$, and with the variables satisfying

$$1 \le x_i, y_i \le P \quad (1 \le i \le 4),$$

and $x_i = y_j$ for no i and j. Following an argument parallel to that of §8, we discover, by means of the substitution

$$x_i = X_1 + X_2 + X_3 + X_4 - 3X_i, \quad y_i = Y_1 + Y_2 + Y_3 + Y_4 - 3Y_i \quad (1 \le i \le 4),$$

that $V_4(P) \gg M_1(P)$, where $M_1(P)$ denotes the number of solutions of the system

$$X_1X_2X_3X_4 = Y_1Y_2Y_3Y_4, \tag{11.1}$$

$$\sum_{i=1}^{4} X_i^j = \sum_{i=1}^{4} Y_i^j \quad (j = 1, 2), \tag{11.2}$$

with $X_i = Y_j$ for no i and j, and subject to

$$0 < \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 3\alpha_i \le P \quad (1 \le i \le 4; \alpha = X, Y). \tag{11.3}$$

Let η be a sufficiently small, but fixed, positive number. Suppose that a, b, c and d are integers satisfying the equation

$$ab = cd \tag{11.4}$$

with

$$1 \le a, b, c, d \le \eta^2 P^{1/2}. \tag{11.5}$$

Suppose further that r and s are integers with

$$(\tfrac{1}{2}-\eta)P^{1/2} < r, s \le \tfrac{1}{2}P^{1/2}, \tag{11.6}$$

and satisfying the coprimality conditions

$$(s, bc) = (r, ad) = 1. \tag{11.7}$$

We consider the substitution

$$\begin{aligned} X_1 &= (r+a)(s+b), & X_2 &= (r-a)(s-b), \\ X_3 &= (r+d)(s-c), & X_4 &= (r-d)(s+c), \\ Y_1 &= (r+a)(s-b), & Y_2 &= (r-a)(s+b), \\ Y_3 &= (r-d)(s-c), & Y_4 &= (r+d)(s+c). \end{aligned} \tag{11.8}$$

Notice first that distinct 6-tuples (r, s, a, b, c, d) determine distinct 8-tuples $\mathbf{X}, \mathbf{Y}$. For if $(r_i, s_i, a_i, b_i, c_i, d_i)$ $(i = 1, 2)$ each determine the same values for $\mathbf{X}$ and $\mathbf{Y}$, then by elimination in (11.8), we obtain

$$\frac{s_1+b_1}{s_1-b_1} = \frac{X_1}{Y_1} = \frac{s_2+b_2}{s_2-b_2},$$

and hence $s_1b_2 = s_2b_1$. But $(s_i, b_i) = 1$ for $i = 1, 2$, and thus $s_1 = s_2$ and $b_1 = b_2$, and a similar argument applies for the remaining variables. A little calculation reveals that with the substitution (11.8), the equations (11.1) and (11.2) are satisfied, and further that the inequalities (11.3) hold. Thus we conclude that

$$M_1(P) \gg M_2(P) - M_3(P), \tag{11.9}$$

where $M_2(P)$ is the number of integers r, s, a, b, c, d satisfying (11.4)-(11.7), and $M_3(P)$ is the number of such integers for which $X_i = Y_j$ for some i and j.

Suppose first that r, s, a, b, c, d are integers counted by $M_3(P)$. If $X_i = Y_j$ for some i and j, then by (11.8) and (11.4) we necessarily have an equation of the type

$$(r+\alpha)(s+\beta) = (r+\gamma)(s+\delta), \tag{11.10}$$

in which $(\alpha^2, \beta^2, \gamma^2, \delta^2)$ is some permutation of (a^2, b^2, c^2, d^2), and $\alpha\beta = \gamma\delta$. The number of integers counted by $M_3(P)$ with $\alpha = \gamma$ and $\beta = \delta$ is $O(P^2)$. Meanwhile, if

$$\alpha \ne \gamma, \quad \text{or} \quad \beta \ne \delta, \tag{11.11}$$

then (11.10) provides a non-trivial equation between r and s. Thus, for any fixed choice of a, b, c and d satisfying the inequalities implied by (11.11), the number of possible choices for r and s is $O(P^{1/2})$. But the number of solutions of (11.4) with (11.5) is $O(P \log P)$, and hence the total number of solutions of this latter type is $O(P^{3/2} \log P)$. We therefore conclude that

$$M_3(P) \ll P^2. \tag{11.12}$$

Next suppose that r, s, a, b, c, d are integers counted by $M_2(P)$. In view of (11.4), we may write $u=(a,c)$, $v=a/u$, $w=c/u$. Then since $(v,w)=1$, we may define the integer x by $x=d/v$, and we have $a=uv$, $b=wx$, $c=uw$ and $d=vx$. Thus we deduce that $M_2(P)$ is the number of integers u, v, w, x, r, s satisfying

$$1\le uv,wx,uw,vx\le \eta^2P^{1/2},\quad (v,w)=1,$$

with r and s satisfying (11.6) and $(s,uwx)=(r,uvx)=1$. Following an argument similar to that applied in the proof of Lemma 7.2, we therefore deduce that

$$M_2(P)\gg P\sum_{u,v,w,x}\frac{\phi(uvx)\phi(uwx)}{u^2x^2vw},$$

in which the summation is over u, v, w and x satisfying

$$1\le u,x\le \eta^2P^{1/2},\quad 1\le v,w\le \eta^2P^{1/2}\left(\max\{u,x\}\right)^{-1},\quad (v,w)=1.$$

Thus a standard argument similar to that of the proof of Lemma 7.3 ultimately leads to the conclusion

$$M_2(P)\gg P^2\log P. \tag{11.13}$$

The theorem follows on combining (11.9), (11.12) and (11.13).

Theorem 11.2. *Let $V_3(P)$ denote the number of non-trivial solutions counted by $S_3(P;2,4)$. Then $V_3(P)\gg P^2\log P$.*

Proof. We exploit a remarkable identity to be found in [2], brought to our attention by G. Myerson (who notes that the identity in question is not the "remarkable identity" of the title of that paper). Suppose that a, b, c and d are positive integers satisfying $ad=bc$, and write

$$\begin{aligned} x_1&=a+b+c,\quad x_2=b+c+d,\quad x_3=|a-d|,\\ y_1&=c+d+a,\quad y_2=d+a+b,\quad y_3=|b-c|.\end{aligned} \tag{11.14}$$

Then, as may be verified on performing the somewhat tedious task of expansion, the x_i and y_i are positive integers satisfying the system

$$\sum_{i=1}^3(x_i^k-y_i^k)=0\quad (k=2,4). \tag{11.15}$$

Moreover, if in (11.15) we have $x_i^2=y_j^2$ for any i and j, then it follows that (x_1^2,x_2^2,x_3^2) is a permutation of (y_1^2,y_2^2,y_3^2), and hence by positivity, (x_1,x_2,x_3) is a permutation of (y_1,y_2,y_3). Consequently, in view of (11.14), as ordered pairs we have $(a,d)=(b,c)$ or $(a,d)=(c,b)$. Furthermore, distinct choices for a, b, c, d provide distinct choices for $\mathbf{x}$, $\mathbf{y}$. Then we find that $V_3(P)\gg W(P)+O(P^2)$, where $W(P)$ denotes the number of choices for a, b, c, d with $1\le a,b,c,d\le P/3$, and satisfying $ad=bc$. Thus $W(P)\gg P^2\log P$, and the theorem follows immediately.

Appendix. Quasi-Hardy-Littlewood systems and the circle method

We begin this section by discussing a conjecture which concerns the fundamental nature of the Hardy-Littlewood circle method. Since such a conjecture is likely to provide at most an approximation to the true state of affairs, as indeed will become apparent, we prefer to refer to this conjecture as a "model". In order to describe this model we shall require some notation.

Let $F_1, \dots, F_t \in \mathbb{Z}[x_1, \dots, x_s]$ be homogeneous polynomials with respective degrees $k_1 \dots, k_t$, satisfying $1 \le k_1 \le k_2 \le \dots \le k_t$, and let $\mathcal{B}$ be a convex region in $\mathbb{R}^s$ having finite positive s-volume. Our aim is to estimate, when P is large, the number, $N_{\mathcal{B}}(\mathbf{F}; P)$, of integral solutions of the system of equations

$$F_i(\mathbf{x}) = 0 \quad (1 \le i \le t), \tag{A.1}$$

with $\mathbf{x} \in P\mathcal{B}$. Let $\mathcal{V}$ denote the projective variety in $\mathbb{P}^{s-1}$ defined by the system of equations (A.1). We define a height function on the set of subvarieties of $\mathcal{V}$ as follows. First define the height of a polynomial, $G \in \mathbb{Z}[x_1, \dots, x_s]$, which we write $h(G)$, to be the maximum of the absolute values of the coefficients occurring in G. For each subvariety $\mathcal{W}$ of $\mathcal{V}$ there corresponds a collection $\mathcal{Z}(\mathcal{W})$ of finite sets of homogeneous polynomials in $\mathbb{Z}[x_1, \dots, x_s]$, which define $\mathcal{W}$ in the sense that if $\mathcal{A} \in \mathcal{Z}(\mathcal{W})$, then the point $\mathbf{y}$ lies on $\mathcal{W}$ if and only if $G(\mathbf{y}) = 0$ for each $G \in \mathcal{A}$. We define the *height* of $\mathcal{W}$, which we abbreviate to $H(\mathcal{W})$, by

$$H(\mathcal{W}) = \inf_{\mathcal{A} \in \mathcal{Z}(\mathcal{W})} \max_{G \in \mathcal{A}} h(G). \tag{A.2}$$

Further, we define the *integral degree* of $\mathcal{W}$, which we abbreviate to $\partial_{\mathbb{Z}}\mathcal{W}$, by

$$\partial_{\mathbb{Z}}\mathcal{W} = \inf_{\mathcal{A} \in \mathcal{Z}(\mathcal{W})} \sum_{G \in \mathcal{A}} \deg(G). \tag{A.3}$$

We note that it is by no means necessary that a set of defining equations for a subvariety $\mathcal{W}$ of $\mathcal{V}$ contains $F_1, \dots, F_t$. Moreover, since certain of the equations defining $\mathcal{V}$ may be redundant, it is not necessarily the case that $\partial_{\mathbb{Z}}\mathcal{V} = k_1 + \dots + k_t$, although there is plainly no loss of generality in assuming that the latter is the case.

We are now in a position to explain what we mean, in general, by the "trivial solutions" of a system of equations. When Q is a real number exceeding $H(\mathcal{V})$, and K is a natural number, we define $\mathcal{S}(\mathcal{V}; Q, K)$ to be the set of proper subvarieties, $\mathcal{W}$, of $\mathcal{V}$ with $H(\mathcal{W}) \le Q$ and $\partial_{\mathbb{Z}}\mathcal{W} \le K$. We then let $T_{\mathcal{B}}(\mathbf{F}; P; Q, K)$ denote the number of integral solutions of the system (A.1) with $\mathbf{x} \in P\mathcal{B}$, and satisfying the additional condition that for some $\mathcal{W} \in \mathcal{S}(\mathcal{V}; Q, K)$, and some $\mathcal{A} \in \mathcal{Z}(\mathcal{W})$, one has $G(\mathbf{x}) = 0$ for each $G \in \mathcal{A}$. Observe that when Q is not too large compared to $H(\mathcal{V})$, and K is not too large compared to $\partial_{\mathbb{Z}}\mathcal{V}$, then the points counted by $T_{\mathcal{B}}(\mathbf{F}; P; Q, K)$ lie on subvarieties of $\mathcal{V}$ on which we may expect to find an unusually high density of rational points, and such points we describe as "trivial".

In order to discuss the application of the Hardy-Littlewood method, we shall require some additional notation. When P is a large real number we define the exponential sum $\Phi(\boldsymbol{\alpha};\mathbf{F})$ by

$$\Phi(\boldsymbol{\alpha};\mathbf{F}) = \sum_{\mathbf{x}\in P\mathcal{B}} e\left(\Psi(\mathbf{x};\mathbf{F};\boldsymbol{\alpha})\right), \tag{A.4}$$

where

$$\Psi(\mathbf{x};\mathbf{F};\boldsymbol{\alpha}) = \sum_{i=1}^{t} \alpha_i F_i(\mathbf{x}). \tag{A.5}$$

Thus, by orthogonality one has

$$N_{\mathcal{B}}(\mathbf{F};P) = \int_{(0,1]^t} \Phi(\boldsymbol{\alpha};\mathbf{F})d\boldsymbol{\alpha}, \tag{A.6}$$

and the "Hardy-Littlewood method" simply comprises of an asymptotic analysis of the latter integral, based upon a suitable dissection of the range of integration. Our natural inclination is to define a dissection based on a cartesian product of Farey dissections, each one covering the entire interval. However, for the sake of simplicity of exposition, we proceed as follows. When $1 \le a_i \le q_i \le Q_i \le P^{k_i}$ $(1 \le i \le t)$, define the arc $\mathfrak{M}(\mathbf{q},\mathbf{a};\mathbf{Q})$ by

$$\mathfrak{M}(\mathbf{q},\mathbf{a};\mathbf{Q}) = \left\{\boldsymbol{\alpha} \in (0,1]^t \ : \ |q_i\alpha_i - a_i| \le Q_i P^{-k_i} \ (1 \le i \le t)\right\}. \tag{A.7}$$

We define the *major arcs of level* $\mathbf{Q}$, which we write as $\mathfrak{M}(\mathbf{Q})$, to be the union of the $\mathfrak{M}(\mathbf{q},\mathbf{a};\mathbf{Q})$ with $1 \le a_i \le q_i \le Q_i$ and $(a_i,q_i)=1$ $(1 \le i \le t)$. We then define the corresponding *minor arcs* $\mathfrak{m}(\mathbf{Q})$ to be the complement in $(0,1]^t$ of $\mathfrak{M}(\mathbf{Q})$. We note that when $Q_i \le \frac{1}{2}P^{k_i/2}$ $(1 \le i \le t)$, the arcs comprising $\mathfrak{M}(\mathbf{Q})$ are non-overlapping. Moreover, an easy application of Dirichlet's theorem on diophantine approximation shows that $\mathfrak{M}(\mathbf{Q}) = (0,1]^t$ whenever $Q_i \ge P^{k_i/2}$ $(1 \le i \le t)$.

When the Q_i are sufficiently small powers of P, and $\boldsymbol{\alpha} \in \mathfrak{M}(\mathbf{Q})$, it is usually possible, by using standard techniques from the theory of the circle method, to determine an asymptotic formula for $\Phi(\boldsymbol{\alpha};\mathbf{F})$. Motivated by the analysis of the major arcs in applications of the circle method, we make the following definitions. When q_i $(1 \le i \le t)$ are natural numbers, we define the *partial singular integral* $J(\mathbf{q})$ by

$$J(\mathbf{q}) = \int_{\mathcal{T}(\mathbf{q})} \int_{\mathcal{B}} e\left(\Psi(\boldsymbol{\gamma};\mathbf{F};\boldsymbol{\beta})\right) d\boldsymbol{\gamma} d\boldsymbol{\beta}, \tag{A.8}$$

where $\mathcal{T}(\mathbf{q})$ denotes the set of $\boldsymbol{\beta}$ satisfying the conditions $|\beta_i| \le q_i^{-1}Q_i$ $(1 \le i \le t)$. Next we define the *truncated product of local densities* as

$$\Delta(\mathbf{Q}) = \sum_{1\le q_1\le Q_1} \cdots \sum_{1\le q_t\le Q_t} J(\mathbf{q})v(\mathbf{q}), \tag{A.9}$$

where

$$v(\mathbf{q}) = q^{-s} \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} \cdots \sum_{\substack{a_t=1\\(a_t,q_t)=1}}^{q_t} \sum_{r_1=1}^{q} \cdots \sum_{r_s=1}^{q} e\left(\Psi\left(\mathbf{r};\mathbf{F};\frac{a_1}{q_1},\ldots,\frac{a_t}{q_t}\right)\right), \tag{A.10}$$

and $q = [q_1, \ldots, q_t]$ denotes the lowest common multiple of $q_1, \ldots, q_t$.

Now, at last, we arrive at the central object of our discussion. Let

$$\tilde{N}_{\mathcal{B}}(\mathbf{F}; P; Q, K) = N_{\mathcal{B}}(\mathbf{F}; P) - T_{\mathcal{B}}(\mathbf{F}; P; Q, K). \tag{A.11}$$

Thus $\tilde{N}$ counts the "non-trivial" solutions of the system (A.1). We shall say that the system of equations (A.1) is quasi-Hardy-Littlewood (and will refer to a variety, implicitly defined by a system of equations, as being QHL) for the domain $\mathcal{B}$, if, whenever P is sufficiently large, and δ is a sufficiently small positive number, we have

$$\tilde{N}_{\mathcal{B}}(\mathbf{F}; P; P^{\delta}, K) = (1 + o(1))\Delta(\mathbf{Q})P^{s-K}, \tag{A.12}$$

with $K = k_1 + \cdots + k_t$, and $Q_i = P^{k_i/2}$ $(1 \le i \le t)$. Here the implicit constant in the little o-notation is independent of P, but may depend on $\mathcal{B}$ and $\mathbf{F}$. (We remark that it is possible, in delicate circumstances, that Δ has to be modified by the insertion of weights corresponding to the degree of overlapping which may occur in the arcs with larger values of q_i). We now state our general conjecture.

QHL Model. *Suppose that $F_1, \ldots, F_t \in \mathbb{Z}[x_1, \ldots, x_s]$ are homogeneous polynomials with respective degrees $k_1, \ldots, k_t$, that $s > k_1 + \cdots + k_t$, and that the variety $\mathcal{V}$ defined by the system of equations (A.1) satisfies $\partial_{\mathbb{Z}}\mathcal{V} = k_1 + \cdots + k_t$. Then $\mathcal{V}$ is QHL for every convex domain.*

We note that there are certain instances in which the QHL Model is known to fail. For example, Cassels and Guy [8] have found a diagonal cubic form in four variables for which there are no rational solutions, despite the existence of solutions everywhere locally. Consequently one should modify the QHL model so as to exclude from consideration those systems in which there are known obstructions to the Hasse principle. It may be possible, for example, simply to strengthen the condition on s to $s > k_1 + \cdots + k_t + 1$ in order to achieve this end.

Let us spend a little time interpreting QHL varieties, and the QHL model, in the context of familiar ideas, and in particular, the work of Schmidt [29]. We first note that a standard analysis based on suitable convergence assumptions shows that $\Delta(\mathbf{Q})$ is asymptotically equal to $\mu_\infty(\mathcal{B})\mathfrak{S}$, where

$$\mu_\infty(\mathcal{B}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{\mathcal{B}} e\left(\Psi(\boldsymbol{\gamma}; \mathbf{F}; \boldsymbol{\beta})\right) d\boldsymbol{\gamma} d\boldsymbol{\beta}, \tag{A.13}$$

and

$$\mathfrak{S} = \prod_{p \text{ prime}} \mu_p(\mathbf{F}), \tag{A.14}$$

in which

$$\mu_p(\mathbf{F}) = \lim_{h\to\infty} p^{h(t-s)}\text{card}\left\{\mathbf{x} \in \left(\mathbb{Z}/p^h\mathbb{Z}\right)^s : \ F_i(\mathbf{x}) = 0 \ (1 \le i \le t)\right\}. \qquad (A.15)$$

The integral $\mu_\infty(\mathcal{B})$ may be shown to be the real density defined in [29, equation (3.7)], which is a normalised version of the familiar *singular integral* in the circle method. Further, the factors $\mu_p(\mathbf{F})$ are Schmidt's local factors (see [29, equation (3.1)]), the product of which is just the usual singular series from the circle method. Thus, under these convergence assumptions, our QHL varieties are, in the language of Schmidt, Hardy-Littlewood systems (see [29, equation (1.3)] and its preamble, and also [11] for some recent interesting developments). As is evident to experts in the subject, Schmidt's Hardy-Littlewood systems are those varieties for which the "major arc" contribution, arising from the application of the Hardy-Littlewood method, provides the main term in the asymptotic formula for the number of integral points inside a box $P\mathcal{B}$. For a generic variety in sufficiently many variables, the latter circumstance has long been conjectured to be the case by many experts, under the general philosophy that one expects the major and minor arcs in any Hardy-Littlewood dissection to correspond, roughly speaking, to the non-trivial and trivial solutions respectively. Thus our QHL model pursues this line of thought rather further than Schmidt's notion of a Hardy-Littlewood system. Indeed, there are QHL varieties which are not Hardy-Littlewood systems. As an example, consider the variety defined by $x_1x_2 = x_3x_4$. A simple analysis demonstrates that the number of integral points on this variety with $\mathbf{x} \in [0, P]^4$ is $CP^2 \log P$, for a certain positive constant C. Then with little effort, one deduces that this variety is QHL, but is not a Hardy-Littlewood system.

There are two fundamental differences between our model, and that of Schmidt. Firstly, whereas Schmidt takes an infinite product of local densities, ours is truncated, so that one can handle situations in which this product does not converge. Secondly, in our model we are interested only in "typical" points, and thus we are effectively investigating the quasi-projective variety obtained by excision of a certain number of subvarieties from that in which we are primarily interested. Thus systems of equations with many parametric solutions which fail to be Hardy-Littlewood systems may nonetheless be QHL.

We illustrate the preceding comments by considering $S_3(P; 1, 3)$, under the hypothesis that the system (1.2) is QHL. Write

$$f(\boldsymbol{\alpha}) = \sum_{1\le x\le P} e(\alpha_1 x^3 + \alpha_2 x), \qquad (A.16)$$

so that by orthogonality,

$$S_3(P; 1, 3) = \int_0^1 \int_0^1 |f(\boldsymbol{\alpha})|^6 d\boldsymbol{\alpha}. \qquad (A.17)$$

From (A.9),

$$\Delta(P^{3/2}, P^{1/2}) = \sum_{1\le q_1\le P^{3/2}} \sum_{1\le q_2\le P^{1/2}} J(\mathbf{q})v(\mathbf{q}), \qquad (A.18)$$

where

$$v(\mathbf{q}) = q^{-6} \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} \sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} \left| \sum_{r=1}^{q} e\left(\frac{a_1}{q_1} r^3 + \frac{a_2}{q_2} r \right) \right|^6, \tag{A.19}$$

$$J(\mathbf{q}) = \int_{|\beta_1| \le q_1^{-1} P^{3/2}} \int_{|\beta_2| \le q_2^{-1} P^{1/2}} |w(\boldsymbol{\beta})|^6 \, d\beta_2 d\beta_1, \tag{A.20}$$

and where we have written $q = [q_1, q_2]$, and

$$w(\boldsymbol{\beta}) = \int_0^1 e(\beta_1 \gamma^3 + \beta_2 \gamma) d\gamma. \tag{A.21}$$

We first consider $J(\mathbf{q})$.

Lemma A.1. *Let*

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |w(\boldsymbol{\beta})|^6 d\beta_2 d\beta_1.$$

Then $0 < J \ll 1$*, and moreover, when* q_1 *and* q_2 *satisfy* $1 \le q_1 \le P^{3/2}$ *and* $1 \le q_2 \le P^{1/2}$*, then*

$$J(\mathbf{q}) - J \ll q P^{-1/2}.$$

Proof. By [32, Theorem 7.3], we have

$$w(\boldsymbol{\beta}) \ll (1 + |\beta_1| + |\beta_2|)^{-1/3}. \tag{A.22}$$

Thus, on combining the latter estimate with van der Corput's estimates for exponential integrals (see, for example, [31, Lemma 4.2]), we obtain

$$w(\boldsymbol{\beta}) \ll \min\left\{ |\beta_2|^{-1}, |\beta_1|^{-1/3} \right\} \ll |\beta_1 \beta_2|^{-1/4}. \tag{A.23}$$

A straightforward analysis, using the estimates (A.22) and (A.23), establishes the convergence of the integral J, and the first assertion of the lemma follows immediately. Also, again using (A.22) and (A.23),

$$J(\mathbf{q}) - J \ll I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{-1}^{1} \int_{|\beta_2| > q_2^{-1} P^{1/2}} |\beta_2|^{-6} d\beta_2 d\beta_1, \quad I_2 = \int_{|\beta_1| > q_1^{-1} P^{3/2}} \int_{-1}^{1} |\beta_1|^{-2} d\beta_2 d\beta_1,$$

and

$$I_3 = \int_{|\beta_1| > q_1^{-1} P^{3/2}} \int_{|\beta_2| > q_2^{-1} P^{1/2}} |\beta_1 \beta_2|^{-3/2} d\beta_1 d\beta_2.$$

Thus,

$$J(\mathbf{q}) - J \ll q_1 P^{-3/2} + q_2^5 P^{-5/2} + (q_1 q_2)^{1/2} P^{-1},$$

and the lemma follows.

We investigate the sum (A.19) through estimates for $F(q)$, which we define by

$$F(q) = q^{-6} \sum_{a_1=1}^{q} \sum_{\substack{a_2=1 \\ (a_1,a_2,q)=1}}^{q} |S(q,a_1,a_2)|^6,$$

where

$$S(q,a_1,a_2) = \sum_{r=1}^{q} e_q(a_1 r^3 + a_2 r).$$

Lemma A.2. *Suppose that $p > 3$ is prime. Then $F(p) = 5(p-1)^2 p^{-3}$.*

Proof. We observe that

$$F(p) = p^{-6} \left(\sum_{a_1=1}^{p} \sum_{a_2=1}^{p} |S(p,a_1,a_2)|^6 - p^6 \right) = p^{-4} M(p) - 1, \tag{A.24}$$

where $M(p)$ denotes the number of solutions, over the finite field $\mathbb{F}_p$, of the system (1.2). When $p > 3$, an argument analogous to that applied at the start of §2 reveals that $M(p)$ is the number of solutions, over $\mathbb{F}_p$, of the simultaneous equations

$$X_1 X_2 X_3 = Y_1 Y_2 Y_3,$$
$$X_1 + X_2 + X_3 = Y_1 + Y_2 + Y_3.$$

Thus

$$\begin{aligned} M(p) &= p^{-2} \sum_{a=1}^{p} \sum_{b=1}^{p} \left| \sum_{x=1}^{p} \sum_{y=1}^{p} \sum_{z=1}^{p} e_p\left(axyz + b(x+y+z)\right) \right|^2 \\ &= \sum_{a=1}^{p} \sum_{b=1}^{p} \left| \sum_{x=1}^{p} \sum_{\substack{y=1 \\ p \mid axy+b}}^{p} e_p\left(b(x+y)\right) \right|^2 . \end{aligned} \tag{A.25}$$

The term with $a = b = p$ in (A.25) contributes p^4 to $M(p)$, those with $a \neq p$ and $b = p$ contribute $(p-1)(2p-1)^2$, and those with $a = p$ and $b \neq p$ have an empty innermost sum. Further, the contribution of the remaining terms in the innermost double sum is $M^*(p)$, where

$$\begin{aligned} M^*(p) &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left| \sum_{x=1}^{p-1} e_p\left(b(x - ba^{-1}x^{-1})\right) \right|^2 \\ &= \sum_{b=1}^{p-1} \sum_{r=1}^{p} \left| \sum_{x=1}^{p-1} e_p\left(b(x + rx^{-1})\right) \right|^2 - \sum_{b=1}^{p-1} \left| \sum_{x=1}^{p-1} e_p(bx) \right|^2 . \end{aligned}$$

Thus, by orthogonality we have $M^*(p) = (p-1)^2 p - (p-1)$, and hence $M(p) = p^4 + 5p(p-1)^2$. The lemma now follows from (A.24).

Lemma A.3. *Let p be prime and $t > 1$. Then*

$$F(p^t) \ll p^{-2t/3}.$$

Proof. Let $(p^t, a_1, a_2) = 1$. Then by [24, Lemma 1.6] (or the argument of the proof of [32, Theorem 7.1]),

$$S(p^t, a_1, a_2) \ll p^{2t/3}.$$

Furthermore, $S(p^t, a_1, a_2) = 0$ if $p|a_1$ but $p \nmid a_2$, so by the argument given in the proof of [22, Lemma 7], we have

$$S(p^t, a_1, a_2) \ll p^{t/2}(p^t, a_2)^{1/4}.$$

Therefore

$$\begin{aligned} F(p^t) &\ll p^{-2t} \sum_{a_2=1}^{p^t} \min\left\{p^t, (p^t, a_2)^{3/2}\right\} \\ &\leq p^{-2t} \sum_{m=0}^{t} p^{t-m} \min\{p^t, p^{3m/2}\}, \end{aligned}$$

and the lemma follows.

We are now in a position to estimate $\Delta(P^{3/2}, P^{1/2})$.

Theorem A.4. *There are positive absolute constants D_1 and D_2 such that*

$$D_1(\log P)^5 \ll \Delta(P^{3/2}, P^{1/2}) \ll D_2(\log P)^5.$$

Proof. For the upper bound, we observe that by positivity, and a rearrangement of terms,

$$\Delta(P^{3/2}, P^{1/2}) \leq J \sum_{1 \leq q_1 \leq P^{3/2}} \sum_{1 \leq q_2 \leq P^{1/2}} v(\mathbf{q}) \leq J \sum_{1 \leq q \leq P^2} F(q).$$

A standard argument shows that $F(q)$ is multiplicative, and hence by Lemmata A.2 and A.3,

$$\Delta(P^{3/2}, P^{1/2}) \leq J \prod_{p \leq P^2} \sum_{h=0}^{\infty} F(p^h) \leq J \prod_{p \leq P^2} \left(1 + 5p^{-1} + O(p^{-4/3})\right),$$

and the upper bound follows easily.

For the lower bound, we observe that by positivity,

$$\Delta(P^{3/2}, P^{1/2}) \geq \sum_{[q_1, q_2] \leq P^{1/2}} J(\mathbf{q}) v(\mathbf{q}).$$

Further, by Lemma A.1, the latter expression is

$$\geq JT_0(P^{1/2}) + O\left(T_1(P^{1/2})\right),$$

where

$$T_r(X) = X^{-r} \sum_{1\leq q\leq X} q^r F(q) \quad (r=0,1).$$

In order to estimate the sums T_r, we let $H^{\pm}$ be the multiplicative function satisfying $H^{\pm}(p^t) = p^{-t}\binom{\pm 5}{t}$ for prime powers p^t, in which we have made the usual extension of the definition of the binomial coefficient. Thus

$$F(n) = \sum_{d|n} H^+(d)G(n/d),$$

where

$$G(n) = \sum_{d|n} H^-(d)F(n/d).$$

But by Lemmata A.2 and A.3, we have $G(p^t) \ll p^{-2t/3}$, and when $p>3$ we have $G(p) = (5-10p)p^{-3}$. Thus

$$\sum_{t=0}^{m} G(p^t) = \sum_{u=0}^{m} F(p^u)(1+1/p)^{-5} + O(p^{-2m/3}).$$

Then on recalling that $F(q)$ is positive, and applying Lemmata A.2 and A.3, we deduce that when m is large,

$$\sum_{t=0}^{m} G(p^t) > \frac{1}{2}(1+1/p)^{-5} \sum_{u=0}^{m} F(p^u) > 0.$$

Thus $\sum_{t=0}^{\infty} G(p^t)$ converges absolutely and is nonzero. Moreover,

$$\sum_{t=0}^{\infty} |G(p^t)| = 1 + O(p^{-4/3}).$$

Thus $\prod_p \sum_{t=0}^{\infty} G(p^t)$ converges absolutely, and hence so does $\sum_{n=1}^{\infty} G(n)$, say to Γ (a positive number). Therefore, on applying a standard argument,

$$\begin{aligned}\sum_{q\leq X} F(q) &= \sum_{d\leq X} H^+(d) \sum_{r\leq X/d} G(r)\\ &= \sum_{d\leq X} H^+(d)\left(\Gamma + O(d/X)\right)\\ &= \frac{1}{5!}\Gamma(\log X)^5 + O\left((\log X)^4\right).\end{aligned}$$

Thus $T_0(X) \sim \frac{1}{5!}\Gamma(\log X)^5$, and a similar argument shows that $T_1(X) \ll (\log X)^4$. This completes the proof of the theorem.

The hypothesis that the system (1.2) is QHL now leads to the conclusion that when δ is a sufficiently small positive number,

$$\tilde{N}(P, P^\delta) \asymp P^2(\log P)^5.$$

Here $\tilde{N}(P, P^\delta)$ denotes the number of solutions of the system (1.2) with (1.3), and satisfying the additional condition that none of these points lie on a proper subvariety of integral degree at most 4 of height at most P^δ. A simple check reveals that the number of solutions lying on the latter subvarieties is equal to the number lying on the subvarieties corresponding to systems of equations in which the x_i are permutations of the y_j, together with $O(P^2)$ other such. Thus we expect that

$$S_3(P; 1, 3) - 6P^3 \asymp P^2(\log P)^5,$$

which is (1.6). Notice that the system (1.2) is not a Hardy-Littlewood system, but, as is evident from Theorem 1.2, shows every likelihood of being QHL.

As a final footnote, we mention that the ideas outlined above have a natural extension to inhomogeneous equations. Such would explain the observation of Szekeres [30], that the equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 1 \tag{A.26}$$

has more solutions than are predicted by the major arcs in the circle method. Szekeres found that the parametric solutions $(x_1, x_2, x_3, x_4) = (1, m, -m, 0)$, and permutations thereof, outnumber the product of local densities. However, these parametric solutions are, in our language, "trivial", and we would predict that the major arc contribution in fact corresponds to the number of non-trivial solutions.

References

1. V. Batyrev and Yu. Manin, *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques*, Math. Ann. **286** (1990), 27–43.
2. B. C. Berndt and S. Bhargava, *A remarkable identity found in Ramanujan's third notebook*, Glasgow Math. J. **34** (1992), 341–345.
3. B. J. Birch, *Forms in many variables*, Proc. Roy. Soc. Ser. A **265** (1961), 245–263.
4. K. D. Boklan, *The asymptotic formula in Waring's problem*, Mathematika **41** (1994), 329–347.
5. A. Bremner and S. Brudno, *A complete determination of the zeros of weight-1 6j coefficients*, J. Math. Phys. **27** (1986), 2613–2615.
6. A. Bremner, *A geometric approach to equal sums of sixth powers*, Proc. London Math. Soc. (3) **43** (1981), 544–581.
7. A. Bremner, *A geometric approach to equal sums of fifth powers*, J. Number Theory **13** (1981), 337-354.
8. J. W. S. Cassels and M. J. T. Guy, *On the Hasse principle for cubic surfaces*, Mathematika **13** (1966), 111–120.
9. A. Choudhry, *Symmetric diophantine systems*, Acta Arith. **59** (1991), 291–307.

10. L. E. Dickson, *History of the Theory of Numbers*, vol. 2, Chelsea, New-York, 1952.
11. W. Duke, Z. Rudnick and P. Sarnak, *Density of integer points on affine homogeneous varieties*, Duke Math. J. **71** (1993), 143–179.
12. J. Franke, Yu. Manin and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), 421–435.
13. A. Gloden, *Mehrgradige Gleichungen*, Noordhoff, Groningen, 1944.
14. G. Greaves, *On the representation of a number as a sum of two fourth powers*, Math. Z. **94** (1966), 223–234.
15. G. Greaves, *Some diophantine equations with almost all solutions trivial*, Mathematika (to appear).
16. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers, fifth edition, fourth reprint*, Clarendon Press, Oxford, 1989.
17. D. R. Heath-Brown, *Weyl's inequality, Hua's inequality and Waring's problem*, J. London Math. Soc. (2) **38** (1988), 216–230.
18. C. Hooley, *On the representations of a number as the sum of two cubes*, Math. Z. **82** (1963), 259–266.
19. C. Hooley, *On the representaion of a number as the sum of two h-th powers*, Math. Z. **84** (1964), 126–136.
20. C. Hooley, *On the numbers that are representable as the sum of two cubes*, J. reine angew. Math. **314** (1980), 146–173.
21. C. Hooley, *On another sieve method and the numbers that are representable as the sum of two h-th powers*, Proc. London Math. Soc. (3) **43** (1981), 73–109.
22. C. Hooley, *On Waring's problem*, Acta Mathematica **157** (1986), 49–97.
23. C. Hooley, *On some topics connected with Waring's problem*, J. reine angew. Math. **369** (1986), 110–153.
24. L.-K. Hua, *Additive Theory of Prime Numbers*, A. M. S., Providence, 1965.
25. Y. Katznelson, *Singular matrices and a uniform bound for congruence groups of* $SL_n(\mathbb{Z})$, Duke Math. J. **69** (1993), 121–136.
26. S. Lang (Ed.), *Number Theory III: Diophantine Geometry*, vol. 60, Springer-Verlag, Berlin, E.M.S., 1991.
27. Yu. I. Manin, *Notes on the arithmetic of Fano threefolds*, Compositio Math. **85** (1993), 37–55.
28. N. N. Rogovskaya, *An asymptotic formula for the number of solutions of a system of equations*, Diophantine approximations, Part II, Moskov. Gos. Univ. Moscow, 1986, pp. 78–84. (Russian)
29. W. Schmidt, *The density of integer points on homogeneous varieties*, Acta Math. **154** (1985), 243–296.
30. G. Szekeres, *Major arcs in the four cubes problem*, J. Austral. Math. Soc. Ser. A **25** (1978), 423–437.
31. E. C. Titchmarsh, *The theory of the Riemann zeta-function, second edition*, Clarendon Press, Oxford, 1986.
32. R. C. Vaughan, *The Hardy-Littlewood Method*, University Press, Cambridge, 1981.

RCV: Mathematics Department, Huxley Building, Imperial College, 180 Queen's Gate, London, SW7 2BZ, U.K.
E-mail address: `rvaughan@ma.ic.ac.uk`

TDW: Mathematics Department, University of Michigan, Ann Arbor, Michigan, 48109-1003, U.S.A.
E-mail address: `wooley@math.lsa.umich.edu`