# DIOPHANTINE INEQUALITIES AND QUASI-ALGEBRAICALLY CLOSED FIELDS

CRAIG V. SPENCER AND TREVOR D. WOOLEY*

Abstract. Consider a form $g(x_1, \ldots, x_s)$ of degree $d$, having coefficients in the completion $\mathbb{F}_q((1/t))$ of the field of fractions $\mathbb{F}_q(t)$ associated to the finite field $\mathbb{F}_q$. We establish that whenever $s > d^2$, then the form $g$ takes arbitrarily small values for non-zero arguments $\mathbf{x} \in \mathbb{F}_q[t]^s$. We provide related results for problems involving distribution modulo $\mathbb{F}_q[t]$, and analogous conclusions for quasi-algebraically closed fields in general.

## 1. Introduction

A homogeneous polynomial of odd degree, with real coefficients, assumes arbitrarily small values at non-zero integral arguments provided only that it possesses a number of variables sufficiently large compared to its degree. This conclusion of Schmidt [20] was established by means of an argument remarkable both for its ingenuity and its sophistication. With a similar assumption on the number of variables, the analogous problem of showing that a form of odd degree, with integral coefficients, necessarily vanishes, while plainly no harder, turns out to be considerably more straightforward (see Birch [4]). Motivated by familiar correspondence philosophies, one anticipates that similar conclusions should be accessible in which the role of the integers $\mathbb{Z}$ is replaced by the polynomial ring $\mathbb{F}_q[t]$, and that of the real numbers $\mathbb{R}$ is replaced by the Laurent series $\mathbb{F}_q((1/t))$. In this paper we show not only that such may be achieved, but that in addition much sharper conclusions may be attained with considerable ease. It is our hope that the quantitative results recorded herein may shed light on what is to be expected in the above classical situation.

We begin by introducing some notation. Let $k$ be a field. We say that a zero of a polynomial in several variables is *non-trivial* when it has a non-zero coordinate. We refer to a polynomial having zero constant term as a *Chevalley polynomial*, and call a homogeneous polynomial a *form*. Associated to $k$ is the polynomial ring $k[t]$ and the field of fractions $\mathbb{K} = k(t)$. Write $\mathbb{K}_\infty = k((1/t))$ for the completion of $k(t)$ at $\infty$. Each element $\alpha$ in $\mathbb{K}_\infty$ may be written in the shape $\alpha = \sum_{j \leqslant n} a_j t^j$ for some $n \in \mathbb{Z}$ and coefficients $a_j = a_j(\alpha)$ in $k$ $(j \leqslant n)$. We define ord $\alpha$ by taking $-$ord $\alpha$ to be the largest integer $j$ for which $a_j(\alpha) \neq 0$. Fixing a real number $\gamma$ with $\gamma > 1$, we then write

1991 *Mathematics Subject Classification.* 11D75, 11J25, 12F20, 11T55.

*Key words and phrases.* Diophantine inequalities, function fields, quasi-algebraic closure.

*CVS was supported in part by NSF Grant DMS-0635607 and NSA Young Investigators Grant H98230-10-1-0155, and TDW first by an NSF grant, and subsequently by a Royal Society Wolfson Research Merit Award.

$\langle\alpha\rangle$ for $\gamma^{-\operatorname{ord}\alpha}$, and refer to $\langle\alpha\rangle$ as the *norm* or *magnitude* of $\alpha$. In this context we adopt the convention that $\operatorname{ord} 0 = +\infty$ and $\langle 0\rangle = 0$. Finally, when $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{K}_\infty^n$, we define $\langle\boldsymbol{\beta}\rangle = \max_{1\leqslant i\leqslant n} \langle\beta_n\rangle$.

In this section we concentrate on the situation in which $k$ is a finite field $\mathbb{F}_q$, deferring to later sections a more general discussion of quasi-algebraically closed fields. Our first result is a consequence of Theorem 3.1 below.

**Theorem 1.1.** *Let $k = \mathbb{F}_q$, and let $d$ and $s$ be natural numbers with $s > d^2$. Suppose that $F(\mathbf{x}) \in \mathbb{K}_\infty[x_1, \ldots, x_s]$ is a Chevalley polynomial of degree $d$, whose coefficients have magnitude not exceeding the positive number $H$. Then, whenever $0 < \varepsilon \leqslant \gamma^{-d}H$, the inequality $\langle F(\mathbf{x})\rangle < \varepsilon$ admits a solution $\mathbf{x} \in \mathbb{F}_q[t]^s$ with $0 < \langle\mathbf{x}\rangle \leqslant (H/\varepsilon)^{d/(s-d^2)}$.*

The conclusion of Theorem 1.1 may be compared with work of the first author [21], where a variant of the Davenport-Heilbronn method is applied to investigate the solubility of diagonal diophantine inequalities in the function field setting. Let $F(\mathbf{x}) \in \mathbb{K}_\infty[x_1, \ldots, x_s]$ be a diagonal form of degree $d$ whose coefficients are not all in $\mathbb{F}_q(t)$-rational ratio, which is to say that for no $\lambda \in \mathbb{K}_\infty^\times$ does one have $\lambda F(\mathbf{x}) \in \mathbb{K}[x_1, \ldots, x_s]$. Suppose also that the characteristic of $\mathbb{F}_q$ does not divide $d$, and that the corresponding equation $F(\mathbf{x}) = 0$ has a non-trivial solution over $\mathbb{K}_\infty^s$ (a local solubility condition). Then as a consequence of Theorem 1.1 of [21], when $d$ is large and $s \geqslant (4/3+o(1))d\log d$, it follows that for each $\varepsilon > 0$, the inequality $\langle F(\mathbf{x})\rangle < \varepsilon$ possesses infinitely many primitive solutions $\mathbf{x} \in \mathbb{F}_q[t]^s$. Our theorem requires a larger number of variables in order to be applicable, but in compensation it addresses general homogeneous polynomials, and also supplies an upper bound for the smallest non-trivial solution. We note that Hsu [10], [11] has examined diagonal diophantine inequalities for polynomial rings in which the variables are restricted to be irreducible elements of $\mathbb{F}_q[t]$. The conclusions available in this situation resemble those of [21], save that the number of variables employed is rather larger.

As we have already noted, the classical analogue of Theorem 1.1, in which $\mathbb{R}$ replaces $\mathbb{K}_\infty$ and $\mathbb{Z}$ replaces $\mathbb{F}_q[t]$, is far more difficult to analyse. The results of Schmidt [20] are explicit neither in the number of variables required to guarantee the existence of a solution, nor in terms of the size of the solutions delivered. Freeman [9] has shown that for a given system of $r$ cubic diophantine inequalities in the classical setting, the existence of solutions is assured whenever $s > (10r)^{(10r)^5}$, but apparently no explicit conclusions are available for general forms of higher degree.

We turn next to consider the extent to which the bounds on solutions presented in Theorem 1.1 can be considered sharp.

**Theorem 1.2.** *Let $k = \mathbb{F}_q$, and let $d$ and $s$ be natural numbers with $s > d^2$. Then there exist arbitrarily large numbers $H$, and forms $F(\mathbf{x}; H) \in \mathbb{K}_\infty[\mathbf{x}]$, of degree $d$ in $s$ variables, satisfying the following properties:*

*(a) the coefficients of $F$ each have magnitude not exceeding $H$, and*

(b) *the smallest non-trivial solution* $\mathbf{x} \in \mathbb{F}_q[t]^s$ *of the inequality* $\langle F(\mathbf{x}; H)\rangle < 1$ *satisfies the bound* $\langle \mathbf{x}\rangle \geqslant (\gamma^{1-d}H)^{d/(s-d^2)}$.

This result, which is a consequence of the more general result recorded in Theorem 4.2 below, shows that the conclusion of Theorem 1.1 is essentially best possible in circumstances wherein $\varepsilon = 1$. More general values of $\varepsilon$ may also be addressed via Theorem 1.2 by simply rescaling the coefficients of the polynomial $F$. Further remarks on such lower bounds are offered in section 4.

We now turn our attention to problems analogous to those in the classical literature concerned with the distribution of polynomial sequences modulo 1. Given $\alpha \in \mathbb{K}_\infty$, we define $\langle\langle \alpha \rangle\rangle = \min_{x \in k[t]} \langle \alpha - x\rangle$. As a special case of Theorem 6.1, we derive the following conclusion.

**Theorem 1.3.** *Let* $k = \mathbb{F}_q$, *and suppose that* $f(x) \in \mathbb{K}_\infty[x]$ *is a Chevalley polynomial of degree* $d$. *Then for each positive number* $N$, *there exists a non-zero polynomial* $x \in k[t]$, *with* $\langle x\rangle \leqslant N$, *for which* $\langle\langle f(x)\rangle\rangle < N^{-1/d}$.

Define $\|\alpha\|$ for $\alpha \in \mathbb{R}$ by putting $\|\alpha\| = \min_{y\in\mathbb{Z}} |\alpha - y|$, in which $|\cdot|$ denotes the ordinary absolute value, so that $\|\cdot\|$ is the classical analogue of $\langle\langle \cdot \rangle\rangle$. Also, let $f(t) \in \mathbb{R}[t]$ be a Chevalley polynomial of degree $d$. Then, beginning with work of Vinogradov [22] in the special case $f(t) = \alpha t^d$, a host of authors have established estimates of the type

$$\min_{1\leqslant n\leqslant N} \|f(n)\| \ll_{d,\varepsilon} N^{\varepsilon-\sigma(d)},$$

valid for each positive number $\varepsilon$, in which $\sigma(d)$ is a suitable positive exponent. The current state of the art is given by the permissible exponents $\sigma(d) = 2^{1-d}$ (Schmidt [19] for $d = 2$, and R. C. Baker [1], [3] for $d \geqslant 3$), and $\sigma(d) = S(d)^{-1}$ for a certain exponent $S(d)$ with $S(d) \sim 4d^2 \log d$ (see Corollary 1.3 of Wooley [24]). The conclusion of Theorem 1.3 is therefore rather sharper than conclusions available in the analogous classical situation whenever $d > 2$. In Theorem 6.1 below, we offer more general conclusions. These may be compared with results in Chapter 10 of [2] that address situations in which the polynomials $F_j$ are either quadratic or diagonal forms.

The authors are grateful to the referee for numerous suggestions concerning the exposition of this paper.

## 2. Quasi-algebraically closed fields

Our conclusions extend to cover function fields in which the field of constants is any quasi-algebraically closed field. We recall that a field $k$ is called *quasi-algebraically closed* if every non-constant homogeneous polynomial over $k$, having a number of variables exceeding its degree, possesses a non-trivial zero. In this context we recall the language of Lang [13], and introduce some of our own. We say that $k$ is a *strongly* $C_i$*-field*, or more briefly a $C_i^*$*-field*, when any Chevalley polynomial of positive degree $d$ lying in $k[\mathbf{x}]$, having more than $d^i$ variables, necessarily possesses a non-trivial $k$-rational zero. When such a conclusion holds only for forms, we say instead that $k$ is a $C_i$*-field*. In this terminology, algebraically closed fields such as $\mathbb{C}$ are $C_0^*$-fields, and from the Chevalley-Warning theorem (see [8] and [23]) it follows that the finite field $\mathbb{F}_q$ having $q$ elements is a $C_1^*$-field. Work of Lang [13] and Nagata [16], moreover, shows that algebraic extensions of $C_i^*$-fields are $C_i^*$, and that a transcendental extension, of transcendence degree $j$, over a $C_i^*$-field is $C_{i+j}^*$. The same conclusions hold in the absence of asterisk decorations.

In this section we recall elements of $C_i$-theory relevant to our subsequent arguments.

**Lemma 2.1.** *Let $k$ be a $C_i^*$-field, and suppose that for $1 \leqslant j \leqslant r$, the polynomial $g_j(\mathbf{x}) \in k[x_1, \dots, x_s]$ is Chevalley of degree at most $d$. Suppose also that $s > rd^i$. Then the system of equations $g_j(\mathbf{x}) = 0$ $(1 \leqslant j \leqslant r)$ possesses a non-trivial $k$-rational solution. When $k$ is merely a $C_i$-field, the same conclusion holds provided that the polynomials $g_j$ are forms.*

*Proof.* This is Theorem 1b of Nagata [16]. □

Note that when $k$ is a $C_i$-field, then it is a consequence of Lemma 2.1 that $k$ is a $C_{i+1}^*$-field. For if $g(\mathbf{x}) \in k[x_1, \dots, x_s]$ is a Chevalley polynomial of degree $d$, then one may write $g$ in the shape $g(\mathbf{x}) = g_1(\mathbf{x}) + \dots + g_d(\mathbf{x})$, where each $g_j$ is homogeneous of degree $j$. In particular, the equation $g(\mathbf{x}) = 0$ has a non-trivial $k$-rational solution provided only that the system $g_j(\mathbf{x}) = 0$ $(1 \leqslant j \leqslant d)$ has such a solution. But the latter is a system of $d$ simultaneous homogeneous equations of degree at most $d$, and by Lemma 2.1 this system has a non-trivial $k$-rational solution whenever $s > d^{i+1}$, thereby confirming our earlier claim.

We say that a form $\Psi(\mathbf{x}) \in k[x_1, \dots, x_s]$ is *normic* when it satisfies the property that the equation $\Psi(\mathbf{x}) = 0$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. When such is the case, and the form $\Psi(\mathbf{x})$ has degree $d$ and contains $d^i$ variables, then we say that $\Psi$ is *normic of order $i$*. Plainly, when $k$ is a $C_i$-field, any normic form $\Psi(\mathbf{x})$ of degree $d$ can have at most $d^i$ variables. We note also that when $k = \mathbb{F}_q$, then for each natural number $d$ there exist normic forms of degree $d$ possessing precisely $d$ variables. In order to exhibit such a form, consider a field extension $L$ of $\mathbb{F}_q$ of degree $d$, and examine the norm form $\Psi(\mathbf{x})$ defined by considering the norm map from $L$ to $\mathbb{F}_q$ with respect to a coordinate basis for the field extension of $L$ over $\mathbb{F}_q$.

When $m$ is a non-negative integer, and $F_1, \dots, F_r \in \mathbb{K}_\infty[x_1, \dots, x_s]$, it is convenient to define $D_m(\mathbf{F}) = D_m(F_1, \dots, F_r)$ by putting

$$D_m(F_1, \dots, F_r) = (\deg F_1)^m + \dots + (\deg F_r)^m.$$

**Lemma 2.2.** *Let $k$ be a $C_i^*$-field, and suppose that for $1 \leqslant j \leqslant r$, the polynomial $g_j(\mathbf{x}) \in k[x_1, \dots, x_s]$ is Chevalley. Suppose also that there are normic forms over $k$ of order $i$ of each positive degree. Then whenever $s > D_i(\mathbf{g})$, the system of equations $g_j(\mathbf{x}) = 0$ $(1 \leqslant j \leqslant r)$ possesses a non-trivial $k$-rational solution. When $k$ is merely a $C_i$-field, the same conclusion holds provided that the polynomials $g_j$ are forms.*

*Proof.* This is Theorem 4 of Lang [13] when $k$ is a $C_i$-field, whilst the argument of the proof of this theorem delivers the desired conclusion also when $k$ is $C_i^*$. □

## 3. Solving inequalities via $C_i$-theory

We now apply the theory of $C_i$-fields, due to Lang [13] and Nagata [16], so as to bound the solutions of diophantine inequalities over function fields $k(t)$.

**Theorem 3.1.** *Let $k$ be a $C_i^*$-field. Suppose that $F_j(\mathbf{x}) \in \mathbb{K}_\infty[x_1, \ldots, x_s]$ $(1 \leqslant j \leqslant r)$ are Chevalley polynomials of degree at most $d$, whose coefficients have magnitude not exceeding the positive number $H$. Put $\Delta = \deg F_1 + \ldots + \deg F_r$, and suppose that $s > \Delta d^i$. Then whenever $0 < \varepsilon \leqslant \gamma^{-d} H$, the system of inequalities*

$$\langle F_j(\mathbf{x}) \rangle < \varepsilon \quad (1 \leqslant j \leqslant r) \tag{3.1}$$

*admits a solution $\mathbf{x} \in k[t]^s$ satisfying $0 < \langle \mathbf{x} \rangle \leqslant (H/\varepsilon)^{rd^i/(s-\Delta d^i)}$. The same conclusion holds for $C_i$-fields $k$ when the polynomials $F_j$ are forms.*

*Proof.* We suppose that $k$ is a $C_i^*$-field, and that for $1 \leqslant j \leqslant r$, the polynomial $F_j(\mathbf{x}) \in \mathbb{K}_\infty[x_1, \ldots, x_s]$ is Chevalley of degree $d_j \leqslant d$. Let $H$ be an upper bound for the magnitude of the non-zero coefficients occurring in $F_j(\mathbf{x})$ $(1 \leqslant j \leqslant r)$, and write $h$ for the largest integer for which $\gamma^h \leqslant H$. It follows that for $1 \leqslant j \leqslant r$, the coefficients of $F_j(\mathbf{x})$ each have degree at most $h$. We take $B$ to be a non-negative integer to be chosen later, and consider an $s$-tuple $(x_1, \ldots, x_s) \in k[t]^s$ wherein each coordinate $x_n$ has $t$-degree $B$. Put

$$x_n = y_{n0} + y_{n1} t + \ldots + y_{nB} t^B \quad (1 \leqslant n \leqslant s), \tag{3.2}$$

with $y_{n0}, \ldots, y_{nB} \in k$ $(1 \leqslant n \leqslant s)$, and consider the polynomial obtained by substituting this choice for $\mathbf{x}$ into $F_j(\mathbf{x})$ $(1 \leqslant j \leqslant r)$. Thus, for $1 \leqslant j \leqslant r$, we obtain

$$F_j(\mathbf{x}) = \sum_{m \leqslant d_j B + h} G_{jm}(\mathbf{y}) t^m, \tag{3.3}$$

where each polynomial $G_{jm}(\mathbf{y}) \in k[y_{10}, \ldots, y_{sB}]$ is Chevalley of degree at most $d_j$ for $1 \leqslant j \leqslant r$. Let $M$ be the least integer for which $\gamma^M > 1/\varepsilon$, so that $\gamma^{M-1} \leqslant 1/\varepsilon$. We seek a non-trivial solution $\mathbf{y} \in k^{s(B+1)}$ to the system of equations

$$G_{jm}(\mathbf{y}) = 0 \quad (-M < m \leqslant d_j B + h,\ 1 \leqslant j \leqslant r). \tag{3.4}$$

In view of (3.3), the $s$-tuple $\mathbf{x} \in k[t]^s$, associated to $\mathbf{y}$ via (3.2), provides a non-trivial solution to the system of inequalities

$$\langle F_j(\mathbf{x}) \rangle \leqslant \gamma^{-M} \quad (1 \leqslant j \leqslant r),$$

and hence to the system (3.1).

The system (3.4) consists of $d_j B + h + M$ equations of degree at most $d_j$, for $1 \leqslant j \leqslant r$, in $s(B+1)$ variables. Since $k$ is presently supposed to be a

$C_i^*$-field, we find from the first conclusion of Lemma 2.1 that the system (3.4) possesses a non-trivial solution $\mathbf{y} \in k^{s(B+1)}$ whenever

$$s(B+1) > d^i \sum_{j=1}^{r} (d_j B + h + M). \tag{3.5}$$

Write $\Delta = d_1 + \ldots + d_r$. The hypotheses of the statement of the theorem permit us to assume that $\varepsilon \leqslant \gamma^{-d} H$, which implies that $\gamma^{-M} < \varepsilon \leqslant \gamma^{-d} H < \gamma^{h+1-d}$. We therefore have $h + M \geqslant d$, so that when $s > \Delta d^i$, the condition (3.5) is satisfied for the largest non-negative integral value of $B$ satisfying $(s - \Delta d^i)B \leqslant r d^i (h+M) - \Delta d^i$. On recalling that our definition of $M$ ensures that $1/\varepsilon < \gamma^M \leqslant \gamma/\varepsilon$, it follows in particular that there exists a non-trivial solution $\mathbf{x} \in k[t]^s$ to the system (3.1) with

$$\langle \mathbf{x} \rangle^{s-\Delta d^i} \leqslant \gamma^{-\Delta d^i} (\gamma^{h+M})^{r d^i} \leqslant \gamma^{(r-\Delta) d^i} (H/\varepsilon)^{r d^i}.$$

Since the lower bound $\Delta \geqslant r$ follows from the hypotheses of the statement of the theorem, the first conclusion of Theorem 3.1 now follows. The second follows in like manner by making use of the final assertion of Lemma 2.1. $\square$

Theorem 1.1 is an immediate consequence of the last theorem, since $\mathbb{F}_q$ is a $C_1^*$-field. We remark that when $k$ is a $C_i^*$-field, and there are normic forms of order $i$ for each positive degree, then the conclusions of Theorem 3.1 may be sharpened. If one makes use of Lemma 2.2 in place of Lemma 2.1 in the above argument, then one may replace the constraint (3.5) by the condition

$$s(B+1) > \sum_{j=1}^{r} (d_j B + h + M) d_j^i.$$

From here one finds that whenever $s > D_{i+1}(\mathbf{F})$, a solution of the system (3.1) exists for which $\langle \mathbf{x} \rangle \leqslant (H/\varepsilon)^{D_i(\mathbf{F})/(s - D_{i+1}(\mathbf{F}))}$. The same conclusion holds for $C_i$-fields when the polynomials $F_j$ are forms.

We have already remarked on the paucity of explicit results, in the classical rational case, for general homogeneous forms of higher degree. In the diagonal situation, on the other hand, much more is known, and one even has available reasonable bounds for the size of the smallest non-trivial solutions. Put $\rho(8) = 15/8$ and $\rho(9) = 1$. Also, let $s$ be either 8 or 9, and consider non-zero real numbers $\lambda_1, \ldots, \lambda_s$. Then it follows from work of Brüdern [6] that for each positive number $\varepsilon$, and for any exponent $\rho$ exceeding $\rho(s)$, the inequality

$$|\lambda_1 x_1^3 + \ldots + \lambda_s x_s^3| < \varepsilon$$

possesses an integral solution $\mathbf{x}$ satisfying

$$0 < |\lambda_1 x_1^3| + \ldots + |\lambda_s x_s^3| \ll |\lambda_1 \cdots \lambda_s|^{\rho} (1/\varepsilon)^{s\rho - 1}. \tag{3.6}$$

Sharper conclusions are available when the coefficients $\lambda_i$ are integral. Indeed, Brüdern [5] shows that in such circumstances the exponent $\rho(8) = 15/8$ may be replaced by 5/3. We refer the reader to [17] for earlier work on this topic.

The argument that we employ to establish Theorem 3.1 is easily adapted to provide bounds of the shape (3.6), and leads to the following conclusion.

**Theorem 3.2.** *Let $k$ be a $C_i$-field, and let $s$ and $d$ be natural numbers with $s > d^{i+1}$. Put $\rho = 1/(s - d^{i+1})$. Then whenever $\lambda_j \in \mathbb{K}_\infty^\times$ $(1 \leqslant j \leqslant s)$, and*

$$0 < \varepsilon \leqslant \gamma^{-d} \langle \boldsymbol{\lambda} \rangle^{1-s/d^{i+1}} \langle \lambda_1 \cdots \lambda_s \rangle^{1/d^{i+1}}, \tag{3.7}$$

*the inequality*

$$\langle \lambda_1 x_1^d + \ldots + \lambda_s x_s^d \rangle < \varepsilon \tag{3.8}$$

*possesses a solution $\mathbf{x} \in k[t]^s$ satisfying*

$$0 < \max_{1 \leqslant n \leqslant s} \langle \lambda_n x_n^d \rangle < \gamma^{d-1} \langle \lambda_1 \cdots \lambda_s \rangle^\rho (1/\varepsilon)^{s\rho - 1}.$$

*Proof.* We adopt an approach similar to that employed in our proof of Theorem 3.1. Let $k$ be a $C_i$-field. For $1 \leqslant j \leqslant s$, put $h_j = -\mathrm{ord}\, \lambda_j$, and let $h = \max\{h_1, \ldots, h_s\}$. We take $B$ to be a non-negative integer to be chosen in due course, and on this occasion we consider an $s$-tuple $(x_1, \ldots, x_s) \in k[t]^s$ with the property that for $1 \leqslant j \leqslant s$, the polynomial $x_j$ has $t$-degree $B_j = B + [(h - h_j)/d]$. Here, as usual, we write $[\theta]$ for the largest integer not exceeding $\theta$. Taking

$$x_n = y_{n0} + y_{n1} t + \ldots + y_{nB_n} t^{B_n} \quad (1 \leqslant n \leqslant s), \tag{3.9}$$

with $y_{n0}, y_{n1}, \ldots, y_{nB_n} \in k$ $(1 \leqslant n \leqslant s)$, we obtain the expression

$$\lambda_1 x_1^d + \ldots + \lambda_s x_s^d = \sum_{m \leqslant dB+h} G_m(\mathbf{y}) t^m, \tag{3.10}$$

where each polynomial $G_m(\mathbf{y}) \in k[y_{10}, \ldots, y_{sB_s}]$ is homogeneous of degree $d$. Notice here that our definition of $B_j$ ensures that the summation over $m$ on the right hand side of (3.10) need not extend beyond $dB + h$, for when $1 \leqslant j \leqslant s$, one has

$$-\mathrm{ord}(\lambda_j x_j^d) = dB_j + h_j = dB + d[(h - h_j)/d] + h_j \leqslant dB + h.$$

Let $M$ be the least integer for which $\gamma^M > 1/\varepsilon$, so that $\gamma^{M-1} \leqslant 1/\varepsilon$, and put $D = B_1 + \ldots + B_s$. We seek a non-trivial solution $\mathbf{y} \in k^{D+s}$ to the system

$$G_m(\mathbf{y}) = 0 \quad (-M < d \leqslant dB + h). \tag{3.11}$$

In view of (3.10) and our choice for $M$, the $s$-tuple $\mathbf{x} \in k[t]^s$, associated to $\mathbf{y}$ via the relations (3.9), then provides a non-zero solution of the inequality (3.8).

The system (3.11) consists of $dB + h + M$ homogeneous equations of degree $d$ in $D + s$ variables. Since $k$ is a $C_i$-field, we find from the second conclusion of Lemma 2.1 that the system (3.11) possesses a non-trivial solution $\mathbf{y} \in k^{D+s}$ whenever $D + s > (dB + h + M)d^i$. This condition is equivalent to the constraint

$$s(B+1) + \sum_{j=1}^s [(h - h_j)/d] > d^{i+1} B + d^i (h + M),$$

which is to say

$$(s - d^{i+1})B > d^i(h + M) - s - \sum_{j=1}^s [(h - h_j)/d]. \tag{3.12}$$

Since $\varepsilon > \gamma^{-M}$, the hypothesis (3.7) permits us to assume that

$$h + M > d + \sum_{j=1}^{s} (h - h_j)/d^{i+1}.$$

It follows that the condition (3.12) is satisfied for a non-negative integral value of $B$ with

$$(s - d^{i+1})B \leqslant d^i(h + M) - d^{i+1} - \sum_{j=1}^{s} [(h - h_j)/d].$$

The last condition is satisfied with a value of $B$ satisfying the condition

$$\begin{aligned}(s - d^{i+1})(dB + h) &\leqslant d^{i+1}M - d^{i+2} + sh - \sum_{j=1}^{s} (h - h_j) + s(d - 1)\\ &\leqslant d^{i+1}(M - 1) + \sum_{j=1}^{s} h_j + (s - d^{i+1})(d - 1).\end{aligned}$$

In particular, when $s > d^{i+1}$, there exists a solution $\mathbf{x} \in k[t]^s$ to the inequality (3.8) with

$$0 < \max_{1 \leqslant n \leqslant s} \langle \lambda_n x_n^d \rangle^{s-d^{i+1}} \leqslant \langle \lambda_1 \cdots \lambda_s \rangle (\gamma^{d-1})^{s-d^{i+1}} (\gamma^{M-1})^{d^{i+1}},$$

and the conclusion of the theorem is now immediate. □

## 4. Lower bounds

By adapting an argument employed by Cassels [7] in his work on solutions of rational quadratic forms, we are able to derive lower bounds for the magnitude of non-trivial solutions of certain diophantine equations over $\mathbb{F}_q[t]$. Such lower bounds apply also, of course, to the solutions of corresponding diophantine inequalities. We begin with a simple lemma.

**Lemma 4.1.** *Let $k$ be a field with the property that there exists a normic form $\Psi(\mathbf{x}) \in k[x_1, \ldots, x_D]$ of degree $d$. When $r \geqslant 1$, put $\Delta = rdD$, and define the polynomials $\Phi_m(\mathbf{x}) \in \mathbb{K}[x_1, \ldots, x_\Delta]$ by putting*

$$\Phi_m(\mathbf{x}) = \sum_{j=0}^{d-1} t^j \Psi(x_{mdD+jD+1}, \ldots, x_{mdD+jD+D}) \quad (0 \leqslant m < r). \tag{4.1}$$

*Then, whenever $\mathbf{x} \in k[t]^\Delta$, one has $\Phi_m(\mathbf{x}) = 0$ $(0 \leqslant m < r)$ if and only if $\mathbf{x} = \mathbf{0}$.*

*Proof.* Observe first that since the variables occurring in the respective polynomials of the system (4.1) are disjoint, it suffices to prove the conclusion of the lemma when $r = 1$. Consider then the polynomial

$$\Phi(\mathbf{x}) = \sum_{j=0}^{d-1} t^j \Psi(x_{jD+1}, \ldots, x_{jD+D}).$$

Suppose, if possible, that the equation $\Phi(\mathbf{x}) = 0$ possesses a non-trivial solution $\mathbf{x} \in k[t]^{dD}$. There is no loss of generality in supposing that for some index $j$ with $1 \leqslant j \leqslant dD$, the polynomial $x_j$ is not divisible by $t$, for the homogeneity of $\Psi$ ensures that we may divide every coordinate through by any power of $t$ without impacting solubility. Given such a primitive solution, we first reduce modulo $t$ to obtain the congruence

$$\Phi(\mathbf{x}) \equiv \Psi(x_1, \dots, x_D) \equiv 0 \pmod{t}. \tag{4.2}$$

Let $x_{j0}$ denote the constant term of $x_j$ for $1 \leqslant j \leqslant D$. Then the congruence (4.2) implies that $\Psi(x_{10}, \dots, x_{D0}) = 0$. Since $\Psi$ is normic, it follows that $x_{j0} = 0$ $(1 \leqslant j \leqslant D)$, and hence that $t$ divides $x_j$ for $1 \leqslant j \leqslant D$. Write $y_j = t^{-1}x_j$ $(1 \leqslant j \leqslant D)$, and substitute into the equation $\Phi(\mathbf{x}) = 0$. On dividing by $t$, we obtain the new equation

$$\sum_{j=0}^{d-2} t^j \Psi(x_{(j+1)D+1}, \dots, x_{(j+1)D+D}) + t^{d-1}\Psi(y_1, \dots, y_D) = 0.$$

A comparison with our original equation $\Phi(\mathbf{x}) = 0$ leads to the conclusion that $t$ divides $x_{D+j}$ $(1 \leqslant j \leqslant D)$, and so we may repeat the previous manipulation. Proceeding inductively, we find that $t|x_j$ for $1 \leqslant j \leqslant dD$, contradicting the assumption that $x_j$ is not divisible by $t$ for some index $j$. We are therefore forced to conclude that when $\mathbf{x} \in k[t]^{dD}$ and $\Phi(\mathbf{x}) = 0$, then necessarily $\mathbf{x} = \mathbf{0}$. In view of our earlier discussion, this completes the proof of the lemma. $\square$

**Theorem 4.2.** *Let $k$ be a $C_i$-field, and suppose that a normic form of degree $d$ exists in $k[\mathbf{x}]$ with $D$ variables. Suppose also that $r$ is a natural number and $s > rdD$. Then there exist arbitrarily large numbers $H$, and systems of forms $F_j(\mathbf{x}; H) \in \mathbb{K}_\infty[x_1, \dots, x_s]$ $(1 \leqslant j \leqslant r)$ of degree $d$, satisfying the following properties:*

*(a) the coefficients of $F_1, \dots, F_r$ each have magnitude not exceeding $H$, and*
*(b) the smallest non-trivial solution $\mathbf{x} \in k[t]^s$ of the simultaneous inequalities $\langle F_j(\mathbf{x}; H)\rangle < 1$ $(1 \leqslant j \leqslant r)$ satisfies the bound $\langle \mathbf{x} \rangle \geqslant (\gamma^{1-d}H)^{rD/(s-rdD)}$.*

*Proof.* We seek polynomials $F_j(\mathbf{x}; H)$ $(1 \leqslant j \leqslant r)$ having coefficients lying in $k[t]$. Of course, whenever $\mathbf{x} \in k[t]^s$, the polynomials $F_j(\mathbf{x}; H)$ $(1 \leqslant j \leqslant r)$ all lie in $k[t]$. Consequently, given such polynomials, the system of inequalities $\langle F_j(\mathbf{x}; H)\rangle < 1$ $(1 \leqslant j \leqslant r)$ has a non-trivial solution $\mathbf{x} \in k[t]^s$ if and only if the system of equations $F_j(\mathbf{x}; H) = 0$ $(1 \leqslant j \leqslant r)$ has a non-trivial solution $\mathbf{x} \in k[t]^s$. Let $\Psi(x_1, \dots, x_D)$ be a normic form of degree $d$, the existence of which is assured by the hypotheses of the statement of the theorem. For the sake of convenience, write $\Delta = rdD$. We define the polynomials $\Phi_m(\mathbf{x}) \in \mathbb{K}[x_1, \dots, x_\Delta]$ as in (4.1), and observe that the polynomials $\Phi_m(\mathbf{x})$ $(0 \leqslant m < r)$ have coefficients lying in $k[t]$. In view of Lemma 4.1, these polynomials have the property that, when $\mathbf{x} \in k[t]^\Delta$, one has $\Phi_m(\mathbf{x}) = 0$ $(0 \leqslant m < r)$ if and only if $\mathbf{x} = \mathbf{0}$.

Now let $s$ be an integer with $s > \Delta$, and let $h \in \mathbb{N}$. We claim that there exists a positive integer $\delta$ having the property that there exist at least $h\Delta(s -$

$\Delta + 1)$ distinct monic irreducible polynomials in $k[t]$ of degree $\delta$. When $k$ has infinitely many elements, our claim follows with $\delta = 1$ by considering polynomials of the shape $t + \lambda$, with $\lambda \in k$. When $k$ is a finite field, on the other hand, then $k$ is isomorphic to $\mathbb{F}_q$ for some prime power $q$, and so it suffices to consider polynomials of degree sufficiently large in terms of $h$, $\Delta$ and $s$. Here we make use of the analogue of the prime number theorem for the ring $\mathbb{F}_q[t]$, so that the number of monic irreducible polynomials of degree $\delta$ in $\mathbb{F}_q[t]$ is asymptotic to $q^\delta/\delta$, as $\delta \to \infty$, a quantity which tends to infinity with $\delta$ (see [18, Corollary to Proposition 2.1]). We may therefore take distinct monic irreducible polynomials

$$\pi_{uwl} \in k[t] \quad (1 \leqslant u \leqslant \Delta,\, 0 \leqslant w \leqslant s - \Delta,\, 1 \leqslant l \leqslant h),$$

each of degree $\delta$. When $1 \leqslant u \leqslant \Delta$ and $0 \leqslant w \leqslant s - \Delta$, write

$$\varpi_{uw} = \prod_{1 \leqslant l \leqslant h} \pi_{uwl},$$

put

$$a_{uv} = \prod_{\substack{0 \leqslant w \leqslant s-\Delta \\ w \neq v}} \varpi_{uw} \quad (1 \leqslant u \leqslant \Delta,\, 0 \leqslant v \leqslant s - \Delta), \tag{4.3}$$

and consider the linear forms

$$L_u(\mathbf{x}) = a_{u0}x_{s-\Delta+u} + \sum_{v=1}^{s-\Delta} a_{uv}x_v \quad (1 \leqslant u \leqslant \Delta). \tag{4.4}$$

An examination of the definitions (4.3) and (4.4) reveals that whenever $\mathbf{x} \in k[t]^s$ and $L_u(\mathbf{x}) = 0$, then necessarily $\varpi_{u0}|x_{s-\Delta+u}$ and $\varpi_{uv}|x_v$ $(1 \leqslant v \leqslant s - \Delta)$.

We now seek a non-trivial solution $\mathbf{x} \in k[t]^s$ of the system of equations

$$\Phi_m(L_1(\mathbf{x}), \dots, L_\Delta(\mathbf{x})) = 0 \quad (0 \leqslant m < r). \tag{4.5}$$

From the discussion in the opening paragraph of this proof, we find that the system (4.5) has a non-trivial solution $\mathbf{x} \in k[t]^s$ if and only if the same holds for the system

$$L_1(\mathbf{x}) = \ldots = L_\Delta(\mathbf{x}) = 0.$$

This is a system of $\Delta$ homogeneous linear equations in the variables $x_1, \dots, x_s$. Since, by hypothesis, we have $s > \Delta$, this system of equations has a non-trivial solution $\mathbf{x} \in k[t]^s$. If one were to have $x_1 = \ldots = x_{s-\Delta} = 0$, then it would follow from (4.4) that $x_{s-\Delta+u} = 0$ for $1 \leqslant u \leqslant \Delta$. The latter implies that $\mathbf{x} = \mathbf{0}$, contradicting the non-triviality of $\mathbf{x}$. Consequently, there exists an integer $v$, with $1 \leqslant v \leqslant s - \Delta$, for which $x_v \neq 0$. But the conclusion of the previous paragraph then implies that $\varpi_{uv}|x_v$ $(1 \leqslant u \leqslant \Delta)$. Since the monic irreducibles $\pi_{uwl}$ are distinct, it therefore follows that $x_v$ is divisible by the polynomial $\varpi_{1v} \cdots \varpi_{\Delta v}$. We thus deduce that any non-trivial solution of the system (4.5) satisfies

$$\langle \mathbf{x} \rangle \geqslant \langle \varpi_{1v} \cdots \varpi_{\Delta v} \rangle = (\gamma^{\delta h})^\Delta. \tag{4.6}$$

The polynomial $\Psi(y_1, \dots, y_D)$ has coefficients from $k$, and the magnitude of each of the non-zero coefficients of the linear forms $L_u(\mathbf{x})$ is precisely $(\gamma^{\delta h})^{s-\Delta}$. Thus, considered as a polynomial in $\mathbb{K}[\mathbf{x}]$ with coefficients lying in $k[t]$, the size of the coefficient of greatest magnitude within the system of polynomials

$$\Psi(L_{mdD+jD+1}(\mathbf{x}), \dots, L_{mdD+jD+D}(\mathbf{x})) \quad (0 \leqslant m < r,\ 0 \leqslant j < d)$$

is at most $(\gamma^{\delta h})^{d(s-\Delta)}$. From (4.1), it therefore follows that the size of the coefficient of greatest magnitude within the polynomials $\Phi_m(\mathbf{x})$ $(0 \leqslant m < r)$ is at most $H = \gamma^{d-1}(\gamma^{\delta h})^{d(s-\Delta)}$. On applying the latter expression to rewrite $\gamma^{\delta h}$ in terms of $H$ in (4.6), and recalling that $\Delta = rdD$, one deduces that any non-trivial solution of the system (4.5) satisfies

$$\langle \mathbf{x} \rangle \geqslant (\gamma^{1-d}H)^{\Delta/(d(s-\Delta))} = (\gamma^{1-d}H)^{rD/(s-rdD)}.$$

This completes the proof of the theorem. □

In the finite field $\mathbb{F}_q$, there exists a normic form of degree $d$, in $d$ variables, for every positive integer $d$. The conclusion of Theorem 1.2 therefore follows at once. We note that Cassels [7] has established analogous lower bounds in the classical situation for rational zeros of a quadratic form. One should observe, however, that in Cassels' work, the integer $D$ may be taken arbitrarily large, owing to the existence of definite forms in any given number of variables. There is also related work of Masser [15] concerning integral zeros of quadratic polynomials.

## 5. Oddly $C_i$-fields

We refer to a polynomial having no monomials of even degree as an *odd Chevalley polynomial*. Motivated by work of Lang concerning the theory of real places (see §3 of [14]), we say that $k$ is an *oddly* $C_i^*$-field when any odd Chevalley polynomial lying in $k[\mathbf{x}]$, having more than $d^i$ variables, necessarily possesses a non-trivial $k$-rational zero. When such holds only for *forms* of odd degree, we say instead that $k$ is *oddly* $C_i$. A field $k$ is called *real* if $-1$ cannot be expressed as a sum of squares in $k$. The field $k$ is described as *real closed* when it is maximal with respect to this property in its algebraic closure. Thus, the field of real numbers $\mathbb{R}$ is both real closed and oddly $C_0^*$. Also, a generalisation of the Corollary to Theorem 15 of Lang [14][1] shows that the function field $\mathbb{R}(t_1, \dots, t_n)$ is oddly $C_n^*$. We briefly sketch below how to establish an odd version of Theorem 3.1.

**Theorem 5.1.** *Modify Theorem 3.1 so that when $i \geqslant 1$, the assumption that $k$ be a $C_i^*$-field is replaced by the hypothesis that it be oddly $C_i^*$, and likewise in the absence of asterisk decorations. In addition, replace the assumption that $k$ be a $C_0^*$-field by the hypothesis that it be real closed. Also, let $F_j(\mathbf{x}) \in \mathbb{K}_\infty[x_1, \dots, x_s]$ be odd Chevalley polynomials of degree at most $d$. Then, under the remaining hypotheses of Theorem 3.1, one has the same conclusions.*

[1] Here we have noted the transparent typographic error in the statement of this corollary.

In our proof of Theorem 5.1, we can afford to be relatively informal, the hard work having already been accomplished. The proof of the second conclusion of Theorem 5.1 follows in precisely the same manner as that of Theorem 3.1, substituting Theorems 12 and 15 of Lang [14] in place of Lemma 2.1. In order to avoid hypotheses concerning the existence of normic forms in such an argument, one should modify the proof of Theorem 12 of [14] along the lines of the proof of Theorem 1a of Nagata [16]. For the corresponding conclusion on oddly $C_i^*$-fields with $i \geqslant 1$, one may proceed in like manner. When $k$ is real closed, it remains to verify that any system of $r$ odd Chevalley polynomials, in more than $r$ variables, possesses a non-trivial zero. This we achieve by means of an application of an algebraic version of the Borsuk-Ulam theorem. Let $F_j(\mathbf{x}) \in k[x_1, \dots, x_s]$ $(1 \leqslant j \leqslant r)$ be odd Chevalley polynomials, and suppose that $s > r$. By setting $x_j = 0$ for $r+1 < j \leqslant s$, we may suppose without loss that $s = r+1$. The map $f : k^s \to k^r$, defined by taking $f(\mathbf{x}) = (F_1(\mathbf{x}), \dots, F_r(\mathbf{x}))$, maps the $r$-dimensional sphere, defined by the equation $x_1^2 + \cdots + x_s^2 = 1$, into $k^r$. Then by the algebraic version of the Borsuk-Ulam theorem (see Knebusch [12]), there exists a point $\mathbf{a} \in k^s$ with $a_1^2 + \cdots + a_s^2 = 1$ for which $F_j(\mathbf{a}) = 0$ $(1 \leqslant j \leqslant r)$. This is achieved, in fact, by finding such a point with $F_j(\mathbf{a}) = F_j(-\mathbf{a})$ $(1 \leqslant j \leqslant r)$. Not only does this confirm our earlier assertion, but by utilising the discussion surrounding the Corollary to Theorem 15 of Lang [14], one finds also that a function field in $n$ variables over a real closed field is oddly $C_n^*$.

The refinements to Theorem 3.1 described in its sequel apply, mutatis mutandis, to the conclusions of Theorem 5.1.

## 6. Distribution modulo $k[t]$

A simple modification of the argument employed in the proof of Theorem 3.1 delivers a result on the distribution of polynomials modulo $k[t]$.

**Theorem 6.1.** *Let $k$ be a $C_i^*$-field, and suppose that for $1 \leqslant j \leqslant r$, the polynomial $F_j(\mathbf{x}) \in \mathbb{K}_\infty[x_1, \dots, x_s]$ is Chevalley of degree at most $d$. Then for each positive number $N$, the simultaneous inequalities*

$$\langle\!\langle F_j(\mathbf{x}) \rangle\!\rangle < N^{-s/(rd^i)} \quad (1 \leqslant j \leqslant r)$$

*possess a non-trivial solution $\mathbf{x} \in k[t]^s$ with $\langle \mathbf{x} \rangle \leqslant N$. When $k$ is a $C_i$-field, the same conclusion holds provided that the polynomials $F_j(\mathbf{x})$ are forms.*

*Proof.* The proof of Theorem 6.1 is swiftly accomplished by means of the argument of the proof of Theorem 3.1. With the same notation as that employed in the latter, we seek a non-trivial solution $\mathbf{y} \in k^{s(B+1)}$ to the system of Chevalley polynomial equations

$$G_{jm}(\mathbf{y}) = 0 \quad (-M < m < 0,\ 1 \leqslant j \leqslant r), \tag{6.1}$$

in place of (3.4). In view of (3.3), the element $\mathbf{x} \in k[t]^s$, associated to $\mathbf{y}$ via the relations (3.2), provides a non-trivial solution of the system of inequalities

$$\langle\!\langle F_j(\mathbf{x}) \rangle\!\rangle \leqslant \gamma^{-M} \quad (1 \leqslant j \leqslant r). \tag{6.2}$$

The system (6.1) consists of $M-1$ equations of degree at most $d$, for $1 \leqslant j \leqslant r$, in $s(B+1)$ variables. Since we may currently suppose $k$ to be a $C_i^*$-field, we find from the first conclusion of Lemma 2.1 that the system (6.1) possesses a non-trivial solution $\mathbf{y} \in k^{s(B+1)}$ whenever

$$s(B+1) > d^i \sum_{j=1}^{r} (M-1) = (M-1)rd^i.$$

This condition is satisfied for the least integral value of $M$ satisfying $rd^iM \geqslant s(B+1)$, and hence the system (6.2) has a non-trivial solution $\mathbf{x} \in k[t]^s$ with $\langle \mathbf{x} \rangle \leqslant \gamma^B$ and $\gamma^{-M} \leqslant (\gamma^{B+1})^{-s/(rd^i)}$. The first conclusion of Theorem 6.1 follows on taking $B$ to be the largest non-negative integer satisfying $\gamma^B \leqslant N$, since then we have $(\gamma^{B+1})^{-1} < N^{-1}$. The second conclusion of the theorem follows in a similar manner. $\square$

Making use, once again, of the fact that $\mathbb{F}_q$ is a $C_1^*$-field, we derive the consequence of Theorem 6.1 recorded in Theorem 1.3. Finally, we note that a conclusion analogous to that of Theorem 6.1 follows for oddly $C_i^*$-fields, and for oddly $C_i$-fields, provided that the polynomials $F_j(\mathbf{x})$ are respectively odd Chevalley polynomials, and forms of odd degree.

## References


[1] R. C. Baker, *Weyl sums and Diophantine approximation*, J. London Math. Soc. (2) **25** (1982), 25–34.

[2] R. C. Baker, *Diophantine inequalities*, London Math. Soc. Monographs, Oxford University Press, Oxford, 1986.

[3] R. C. Baker, *Correction to: "Weyl sums and Diophantine approximation"*, J. London Math. Soc. (2) **46** (1992), 202–204.

[4] B. J. Birch, *Homogeneous forms of odd degree in a large number of variables*, Mathematika **4** (1957), 102–105.

[5] J. Brüdern, *Small solutions of additive cubic equations*, J. London Math. Soc. (2) **50** (1994), 25–42.

[6] J. Brüdern, *Cubic Diophantine inequalities, II*, J. London Math. Soc. (2) **53** (1996), 1–18.

[7] J. W. S. Cassels, *Bounds for the least solutions of homogeneous quadratic equations*, Proc. Cambridge Philos. Soc. **51** (1955) 262–264.

[8] C. Chevalley, *Démonstration d'une hypothèse de M. Artin*, Abh. Math. Sem. Hamburg Univ. **11** (1936), 73–75.

[9] D. E. Freeman, *Systems of cubic Diophantine inequalities*, J. Reine Angew. Math. **570** (2004), 1–46.

[10] C.-N. Hsu, *Diophantine inequalities for polynomial rings*, J. Number Theory **78** (1999), 46–61.

[11] C.-N. Hsu, *Diophantine inequalities for the non-Archimedean line* $\mathbb{F}_q((1/T))$, Acta Arith. **97** (2001), 253–267.

[12] M. Knebusch, *An algebraic proof of the Borsuk-Ulam theorem for polynomial mappings*, Proc. Amer. Math. Soc. **84** (1982), 29–32.

[13] S. Lang, *On quasi-algebraic closure*, Ann. of Math. (2) **55** (1952), 373–390.

[14] S. Lang, *The theory of real places*, Ann. of Math. (2) **57** (1953), 378–391.

[15] D. W. Masser, *How to solve a quadratic equation in rationals*, Bull. London Math. Soc. **30** (1998), 24–28.

[16] M. Nagata, *Note on a paper of Lang concerning quasi-algebraic closure*, Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math. **30** (1957), 237–241.

[17] J. Pitman and D. Ridout, *Diagonal cubic equations and inequalities*, Proc. Roy. Soc. London Ser. A **297** (1967), 476–502.

[18] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics, vol. 210, Springer-Verlag, New York, 2002.

[19] W. M. Schmidt, *On the distribution modulo* 1 *of the sequence* $\alpha n^2 + \beta n$, Canad. J. Math. **29** (1977), 819–826.

[20] W. M. Schmidt, *Diophantine inequalities for forms of odd degree*, Adv. in Math. **38** (1980), 128–151.

[21] C. V. Spencer, *Diophantine inequalities in function fields*, Bull. London Math. Soc. **41** (2009), 341–353.

[22] I. M. Vinogradov, *Analytischer Beweis des Satzes über die Verteilung der Bruchteile eines ganzen Polynoms*, Bull. Acad. Sci. USSR (6) **21** (1927), 567–578.

[23] E. Warning, *Bemerkung zur vorstehenden Arbeit von Herrn Chevalley*, Abh. Math. Sem. Hamburg Univ. **11** (1936), 76–83.

[24] T. D. Wooley, *On Vinogradov's mean value theorem*, Mathematika **39** (1992), 379–399.

CVS: Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, U.S.A.

*E-mail address*: cvs@math.ksu.edu

TDW: School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom

*E-mail address*: matdw@bristol.ac.uk