ON THE WARING–GOLDBACH PROBLEM FOR EIGHTH AND HIGHER POWERS

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Abstract. Recent progress on Vinogradov's mean value theorem has resulted in improved estimates for exponential sums of Weyl type. We apply these new estimates to obtain sharper bounds for the function $H(k)$ in the Waring–Goldbach problem. We obtain new results for all exponents $k \geq 8$, and in particular establish that $H(k) \leq (4k-2)\log k + k - 7$ when k is large, giving the first improvement on the classical result of Hua from the 1940s.

1. Introduction

A formal application of the Hardy–Littlewood method suggests that whenever s and k are natural numbers with $s \geq k+1$, then all large integers n satisfying appropriate local conditions should be represented as the sum of s kth powers of prime numbers. With this expectation in mind, consider a natural number k and prime number p, and define $\theta = \theta(k,p)$ to be the integer with $p^\theta | k$ but $p^{\theta+1} \nmid k$, and $\gamma = \gamma(k,p)$ by

$$\gamma(k,p) = \begin{cases} \theta+2, & \text{when } p=2 \text{ and } \theta>0, \\ \theta+1, & \text{otherwise.} \end{cases}$$

We then put

$$K(k) = \prod_{(p-1)|k} p^\gamma,$$

and denote by $H(k)$ the least integer s such that every sufficiently large positive integer congruent to s modulo $K(k)$ may be written in the shape

$$p_1^k + p_2^k + \ldots + p_s^k = n, \tag{1.1}$$

with $p_1, \ldots, p_s$ prime numbers. We note that the local conditions introduced in the definition of $H(k)$ are designed to exclude the degenerate situations in which one or more variables might otherwise be forced to be prime divisors of $K(k)$. In such circumstances, the representation problem at hand reduces to a similar problem of Waring–Goldbach type in fewer variables. Thus, for example, since every representation of an even integer n as the sum of three primes reduces to a representation of $n-2$ as the sum of two primes, we investigate the equation (1.1) with $k=1$ and $s=3$ only when n is odd. We direct the reader to recent work [1, 2] for more on the Waring–Goldbach problem in the absence of such restrictions.

The first general bound for $H(k)$ was obtained by Hua [3], who showed that

$$H(k) \leq 2^k + 1 \qquad (k \geq 1). \tag{1.2}$$

This result, which generalizes I. M. Vinogradov's celebrated three primes theorem [17], remains the best known bound on $H(k)$ for $k = 1$, 2 and 3. When $k \geq 4$, on the other hand,

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the bound (1.2) has been sharpened considerably. These improvements may be grouped into three chronological phases: (i) work of Davenport and Hua from the 1940s and 1950s (see Hua [5]); (ii) refinements of the diminishing ranges method in Waring's problem developed in the mid-1980s by Thanigasalam [12, 13] and Vaughan [14]; and (iii) more recent refinements of Zhao [22], the first author [7], and of Kawada and the second author [6]. Thus, for intermediate values of k, the current state of play may be summarised with the bounds

$$H(4) \le 13, \quad H(5) \le 21, \quad H(6) \le 32, \quad H(7) \le 46,$$
$$H(8) \le 63, \quad H(9) \le 83, \quad H(10) \le 103.$$

Here we note that although Thanigasalam [13] claims only the bound $H(10) \le 107$, it is clear that his methods establish the stronger bound recorded above. For larger values of k, Hua [4, 5] adapted ideas from Vinogradov's work on Waring's problem to show that

$$H(k) \le k(4 \log k + 2 \log\log k + O(1)), \quad \text{as } k \to \infty. \tag{1.3}$$

In this paper, we make use of the new estimates for Weyl sums that result from recent work of the second author [18, 21] concerning Vinogradov's mean-value theorem to improve on the above results for $k \ge 8$. In particular, we obtain the following theorem, which represents the first improvement on Hua's bound (1.3) in more than half a century.

Theorem 1. *When k is large, one has $H(k) \le (4k-2)\log k + k - 7$.*

Our computations suggest strongly that the bound recorded in this theorem holds as soon as k exceeds 64. We also obtain the following bounds for $H(k)$ when $8 \le k \le 20$.

Theorem 2. *Let $8 \le k \le 20$. Then $H(k) \le s(k)$, where $s(k)$ is defined by Table 1.*

k	8	9	10	11	12	13	14	15	16	17	18	19	20
$s(k)$	61	75	89	103	117	131	147	163	178	194	211	227	244

Table 1. Upper bounds for $H(k)$ when $8 \le k \le 20$

We remark that we have an alternate approach to bounding $H(k)$ for larger k that yields a bound in which, for all $k \ge 20$, at most three extra variables are required relative to the conclusion of Theorem 1. In combination with Theorem 2 and the above-cited conclusions of earlier scholars, therefore, it follows that for every exponent k with $k \ge 3$, one has

$$H(k) \le (4k-2)\log k + k - 4.$$

Following some discussion of basic generating functions in §2, we adapt Vaughan's variant of the diminishing ranges argument in §3 so as to accommodate recent progress on Vinogradov's mean value theorem. In §4 we apply these ideas to derive the mean value estimates underpinning the proofs of Theorems 1 and 2. We complete the proof of the latter in §5.

Throughout this paper, the letter ε denotes a sufficiently small positive number. Whenever ε occurs in a statement, we assert that the statement holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. The letter p, with or without subscripts, is reserved for prime numbers, and c denotes an absolute constant, not necessarily the same in all occurrences. We also write $e(x)$ for $\exp(2\pi i x)$, and (a, b) for the greatest common divisor of a and b. Finally, for real numbers θ, we denote by $\lfloor \theta \rfloor$ the largest integer not exceeding θ, and by $\lceil \theta \rceil$ the least integer no smaller than θ.

We use several decompositions of the unit interval into sets of major and minor arcs. In order to facilitate discussion, we introduce some standard notation with which to describe such Hardy-Littlewood dissections. When $1 \le Y \le X$, we define the set of major arcs $\mathfrak{M}(Y, X)$ as the union of the intervals

$$\mathfrak{M}(q, a; Y, X) = \left\{ \alpha \in [0, 1) : \ |q\alpha - a| \le X^{-1} \right\}$$

with $0 \le a \le q \le Y$ and $(a, q) = 1$. We define the corresponding set of minor arcs by putting $\mathfrak{m}(Y, X) = [0, 1) \setminus \mathfrak{M}(Y, X)$.

2. Bounds on exponential sums

Recent progress on Vinogradov's mean value theorem obtained by the second author [18, 19, 20, 21] permits improvements to be made in bounds on exponential sums of Weyl type. In this section, we collect together several such improved estimates for later use. Recall the classical Weyl sum

$$f_k(\alpha; X) = \sum_{X < x \le 2X} e\left(\alpha x^k\right),$$

in which we suppose that $k \ge 2$ is an integer and α is real. We define the exponent $\Sigma(k)$ as in Table 2 when $7 \le k \le 20$, and otherwise by putting

$$\Sigma(k) = 2k^2 - 6k + 4.$$

Note that whenever $k \ge 3$, one has $\Sigma(k) \le 2k^2 - 6k + 4$. Then, when $k \ge 3$ is an integer, we define σ_k by means of the relation

$$\sigma_k^{-1} = \min\left\{2^{k-1}, \Sigma(k)\right\}. \tag{2.1}$$

k	7	8	9	10	11	12	13
$\Sigma(k)$	58.093	80.867	107.396	137.763	172.027	210.222	252.370
k	14	15	16	17	18	19	20
$\Sigma(k)$	298.487	348.580	402.655	460.718	522.771	588.815	658.854

Table 2. Definition of $\Sigma(k)$ for $7 \le k \le 20$

For $k \ge 3$, we define the multiplicative function $w_k(q)$ by taking

$$w_k(p^{uk+v}) = \begin{cases} kp^{-u-1/2}, & \text{when } u \ge 0 \text{ and } v = 1, \\ p^{-u-1}, & \text{when } u \ge 0 \text{ and } 2 \le v \le k, \end{cases}$$

and we note that $q^{-1/2} \le w_k(q) \ll q^{-1/k}$. We are now equipped to announce our main tool in the shape of the following lemma.

Lemma 2.1. *Suppose that $k \ge 3$. Then either one has*

$$f_k(\alpha; X) \ll X^{1-\sigma_k+\varepsilon}, \tag{2.2}$$

or there exist integers a and q such that

$$1 \leq q \leq X^{k\sigma_k}, \quad (a,q)=1 \quad \textit{and} \quad |q\alpha - a| \leq X^{-k+k\sigma_k},$$

in which case

$$f_k(\alpha;X) \ll \frac{w_k(q)X}{1+X^k|\alpha - a/q|} + X^{1/2+\varepsilon}.$$

Proof. Suppose first that $\alpha \in \mathfrak{m}(X, X^{k-1})$. Then the estimate (2.2) follows at once from [20, Theorem 11.5], the refinement of [20, Theorem 11.1] that follows by employing the bounds recorded in [21, Theorem 1.2], and Weyl's inequality (see [15, Lemma 2.4]). Meanwhile, when $\alpha \in \mathfrak{M}(X, X^{k-1})$, the desired conclusion follows by applying the argument of the proof of [6, Lemma 2.1]. The required details will be readily surmised from the special case $y = x$ of the proof of [9, Lemma 2.2]. □

We also require upper bounds for the corresponding Weyl sum over prime numbers,

$$g_k(\alpha;X) = \sum_{X<p\leq 2X} e\left(\alpha p^k\right),$$

and these we summarise in the next lemma.

Lemma 2.2. *Suppose that $k \geq 4$ and $X^{2\sigma_k/3} \leq P \leq X^{9/20}$. Then either one has*

$$g_k(\alpha;X) \ll X^{1-\sigma_k/3+\varepsilon}, \tag{2.3}$$

or there exist integers a and q such that

$$1 \leq q \leq P, \quad (a,q)=1 \quad \textit{and} \quad |q\alpha - a| \leq PX^{-k}, \tag{2.4}$$

in which case

$$g_k(\alpha;X) \ll \frac{X^{1+\varepsilon}}{(q+X^k|q\alpha - a|)^{1/2}}. \tag{2.5}$$

Proof. First, when $\alpha \in \mathfrak{m}(P, P^{-1}X^k)$, the bound (2.3) follows from the special case $\theta = 1$ of [9, Theorem 1.2]. Here, one replaces the exponent σ_k of that paper with the refinement made available in (2.1) by virtue of Lemma 2.1. We note in this context that the present absence of von Mangoldt weights is easily accommodated by applying the usual routine: one eliminates the prime powers p^h $(h \geq 2)$, with an acceptable error term, and then applies partial summation to remove the remaining logarithmic weight. On the other hand, when instead $\alpha \in \mathfrak{M}(P, P^{-1}X^k)$, the hypotheses (2.4) are in play, and the inequality (2.5) is a direct consequence of [8, Theorem 2]. □

Finally, we have need of a variant of a lemma of Vaughan (see [14, Lemma 1]) dealing with an exponential sum over a difference polynomial, namely

$$F_k(\alpha;X,H) = \sum_{1\leq h\leq H} \sum_{X<x\leq 2X} e\left(\alpha\left((x+h)^k - x^k\right)\right),$$

where $1 \leq H \leq X$.

Lemma 2.3. *Suppose that $k \geq 4$. Then either one has*

$$F_k(\alpha;X,H) \ll HX^{1-\sigma_{k-1}+\varepsilon}, \tag{2.6}$$

or there exist integers a and q such that

$$1 \leq q \leq X^{(k-2)\sigma_{k-1}}, \quad (a,q)=1 \quad \text{and} \quad |q\alpha - a| \leq X^{(k-2)\sigma_{k-1}}(HX^{k-1})^{-1}, \tag{2.7}$$

in which case

$$F_k(\alpha; X, H) \ll \frac{q^{-1/(k-2)}HX^{1+\varepsilon}}{1+HX^{k-1}|\alpha - a/q|} + HX^{1/3+\varepsilon}. \tag{2.8}$$

Proof. Put $C = k^{3k}$ and let $Q = CHX^{k-2}$. Define $\mathfrak{n}$ to be the set of points $\alpha \in [0,1)$ with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q)=1$ and $|q\alpha - a| \leq Q^{-1}$, then $q > X$. Suppose in the first instance that $k \geq 8$. We bound $|F_k(\alpha; X, H)|$ for $\alpha \in \mathfrak{n}$ by applying the method of proof of [16, Lemma 10.3], in which we formally take $M = \frac{1}{2}$ and $R = 2$. By substituting the conclusion of [20, Theorem 11.5], and the refinement of [20, Theorem 11.1] utilising [21, Theorem 1.2], for [16, Lemma 10.2], one obtains the bound

$$\sup_{\alpha \in \mathfrak{n}} |F_k(\alpha; X, H)| \ll X^{1-\sigma_{k-1}+\varepsilon}H. \tag{2.9}$$

When $4 \leq k \leq 7$, meanwhile, the bound (2.9) with $\sigma_{k-1} = 2^{2-k}$ follows from [14, Lemma 1]. This establishes (2.6) when $\alpha \in \mathfrak{n}$.

Suppose next that $\alpha \notin \mathfrak{n}$. By Dirichlet's theorem on Diophantine approximation, there exist integers a and q with

$$1 \leq q \leq Q, \quad (a,q)=1 \quad \text{and} \quad |q\alpha - a| \leq Q^{-1}.$$

Since $\alpha \notin \mathfrak{n}$, it follows that $q \leq X$. Then [14, Lemma 2] yields the bound

$$F_k(\alpha; X, H) \ll \frac{q^{-1/(k-2)}HX^{1+\varepsilon}}{1+HX^{k-1}|\alpha - a/q|} + Hq^{(k-2)/(k-1)+\varepsilon}. \tag{2.10}$$

If either inequality in (2.7) fails, then this implies (2.6) once more. Finally, when (2.7) holds, we have $q^{(k-2)/(k-1)} \leq X^{(k-2)\sigma_{k-1}} \leq X^{1/3}$, and (2.8) follows from (2.10). This completes the proof of the lemma. □

3. Mean-values for kth powers

We describe in this section an enhanced diminishing ranges argument of Vaughan [14] of use for mean values of intermediate and larger orders. Our first lemma is a variant of [14, Theorem 3] that makes use of Lemma 2.3.

Lemma 3.1. *Let $k \geq 4$ be an integer, and define σ_k by means of the relation* (2.1)*. Also, let $s = \lfloor \frac{1}{2}(k+3) \rfloor$, consider real numbers $\lambda_1, \ldots, \lambda_s$ with*

$$\lambda_1 = 1, \quad 1 \geq \lambda_2 \geq 1 - 1/k, \quad \lambda_2 \geq \lambda_i > 1/2 \quad (2 \leq i \leq s),$$

and set $\nu = k\lambda_2 - k + 1$. Consider a large real number N, and put $N_i = N^{\lambda_i}$ $(1 \leq i \leq s)$. Let $R(m)$ be a non-negative arithmetic function. Finally, with $C = C(k, \varepsilon) \geq 2$, define

$$\mathcal{G}(\alpha) = \sum_{1 \leq m \leq CN_2^k} R(m)e(\alpha m)$$

and

$$\mathcal{F}_j(\alpha) = \mathcal{G}(\alpha) \prod_{i=j}^{s} f_k(\alpha; N_i) \quad (j = 1, 2).$$

Then

$$\int_0^1 |\mathcal{F}_1(\alpha)|^2\,\mathrm{d}\alpha \ll \left(N + N^{1+\nu-\sigma_{k-1}+\varepsilon}\right)\int_0^1 |\mathcal{F}_2(\alpha)|^2\,\mathrm{d}\alpha + \mathcal{F}_1(0)^2 N^{\varepsilon-k}. \tag{3.1}$$

Proof. Write

$$R_1(n) = \sum_{N_1<x_1\le 2N_1} \dots \sum_{N_s<x_s\le 2N_s} \sum_{1\le m\le (2N_2)^k} R(m),$$

in which the summation is subject to the condition $n = m + x_1^k + \ldots + x_s^k$. Then it follows via orthogonality that the mean value on the left hand side of (3.1) is bounded above by $\sum_n R_1(n)^2$. We may therefore follow the argument of the proof of [14, Theorem 3] leading to formula (2.8) of the latter source. That formula defines the integral

$$M = \int_0^1 F_k(\alpha; N, H)|\mathcal{F}_2(\alpha)|^2\,\mathrm{d}\alpha,$$

where $H = \min\{CN^\nu, N\}$.

Define the set of major arcs $\mathfrak{N}$ to be the union of the intervals

$$\mathfrak{N}(q,a) = \mathfrak{M}(q, a; N^{(k-2)\sigma_{k-1}}, HN^{k-1-(k-2)\sigma_{k-1}}),$$

over integers a and q satisfying (2.7). Also, define the function $G_1(\alpha)$ on $[0,1)$ by taking

$$G_1(\alpha) = \frac{q^{-1/(k-2)}HN}{1 + HN^{k-1}|\alpha - a/q|},$$

when $\alpha \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}$, and otherwise by putting $G_1(\alpha) = 0$. Then by applying Lemma 2.3, we obtain the estimate

$$M \ll N^\varepsilon \int_{\mathfrak{N}} G_1(\alpha)|\mathcal{F}_2(\alpha)|^2\,\mathrm{d}\alpha + N^{1+\nu-\sigma_{k-1}+\varepsilon}\int_0^1 |\mathcal{F}_2(\alpha)|^2\,\mathrm{d}\alpha. \tag{3.2}$$

This inequality replaces [14, equations (2.17)–(2.21)]. To complete the proof of the lemma, we follow the remainder of Vaughan's argument on [14, pages 451–452], noting that the first integral on the right hand side of (3.2) is the quantity K defined in [14, equation (2.21)]. We remark that although our set of major arcs $\mathfrak{N}$ is somewhat larger than the respective set in Vaughan's paper, this does not pose any problems, since we nonetheless have $(k-2)\sigma_{k-1} \le \frac{1}{2}$, which serves as a satisfactory substitute for the relevant bound $(k-2)2^{2-k} \le \frac{1}{2}$ at the top of [14, page 452]). □

Before introducing our basic mean-value estimate, we define a set of admissible exponents for kth powers as follows. Let $t = t_k$ and $u = u_k$ be positive integral parameters to be chosen in due course. Then, with $\theta = 1 - 1/k$, we first set

$$\lambda_i = (\theta + \sigma_{k-1}/k)^{i-1} \quad (1 \le i \le u+1). \tag{3.3}$$

Finally, we define $\lambda_{u+2}, \dots, \lambda_{u+t}$ by putting

$$\lambda_{u+2} = \frac{k^2 - \theta^{t-3}}{k^2 + k - k\theta^{t-3}}\lambda_{u+1}, \tag{3.4}$$

$$\lambda_{u+j} = \frac{k^2 - k - 1}{k^2 + k - k\theta^{t-3}}\theta^{j-3}\lambda_{u+1} \quad (3 \le j \le t), \tag{3.5}$$

and then set

$$\Lambda = \lambda_1 + \ldots + \lambda_{t+u}. \tag{3.6}$$

Lemma 3.2. *Let k, t and u be positive integers with $k \geq 3$ and $t \geq \lfloor \frac{1}{2}(k+3) \rfloor$, and let v be a non-negative real number. Define the exponents λ_j and Λ by means of (3.3)-(3.6). Then, when N is large,*

$$\int_0^1 |f_k(\alpha; N)|^v \prod_{j=1}^{t+u} \left| g_k(\alpha; N^{\lambda_j}) \right|^2 \, \mathrm{d}\alpha \ll N^{2\Lambda+v-k+\varepsilon} \left(1 + N^{k-\Lambda-v\sigma_k}\right). \tag{3.7}$$

Proof. We establish the bound (3.7) through an application of the Hardy-Littlewood method. We begin by examining an auxiliary mean value, establishing the bound

$$\int_0^1 \prod_{j=i}^{t+u} |f_k(\alpha; N^{\lambda_j})|^2 \, \mathrm{d}\alpha \ll N^{2\Lambda_i-k+\varepsilon} \left(1 + N^{k-\Lambda_i}\right), \tag{3.8}$$

for $1 \leq i \leq u+1$, where $\Lambda_i = \lambda_i + \ldots + \lambda_{t+u}$. Observe first that, by orthogonality, the bound

$$\int_0^1 \prod_{j=u+1}^{u+t} \left| f_k\left(\alpha; N^{\lambda_j}\right) \right|^2 \, \mathrm{d}\alpha \ll N^{\Lambda_{u+1}+\varepsilon}$$

follows as a direct consequence of Vaughan [15, Theorem 6.1]. When $1 \leq i \leq u$, meanwhile, we apply Lemma 3.1 with $\lambda_2 = \theta + \sigma_{k-1}/k$ and $R(m)$ equal to the number of integral representations of the integer m in the form

$$m = x_{s+1}^k + \ldots + x_{t+u}^k,$$

with $N^{\lambda_j} < x_j \leq 2N^{\lambda_j}$ $(s+1 \leq j \leq t+u)$. Since one then has $k\lambda_2 - k + 1 = \sigma_{k-1}$, we deduce that whenever the estimate (3.8) holds for $i = I+1$, then it holds also for $i = I$. The desired bound (3.8) therefore follows for $1 \leq i \leq u+1$ by backwards induction, starting from the base case $i = u+1$.

Now, with $P = N^{k\sigma_k}$, put $\mathfrak{M} = \mathfrak{M}(P, N^k P^{-1})$ and $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. Then by Lemma 2.1,

$$\sup_{\alpha \in \mathfrak{m}} |f_k(\alpha; N)| \ll N^{1-\sigma_k+\varepsilon}.$$

Furthermore, by comparing the underlying Diophantine equations, we have

$$\int_0^1 \prod_{j=1}^{t+u} |g_k(\alpha; N^{\lambda_j})|^2 \, \mathrm{d}\alpha \leq \int_0^1 \prod_{j=1}^{t+u} |f_k(\alpha; N^{\lambda_j})|^2 \, \mathrm{d}\alpha.$$

We therefore deduce from (3.8) that

$$\begin{aligned} \int_{\mathfrak{m}} |f_k(\alpha; N)|^v \prod_{j=1}^{t+u} \left| g_k(\alpha; N^{\lambda_j}) \right|^2 \, \mathrm{d}\alpha &\ll \left(\sup_{\alpha \in \mathfrak{m}} |f_k(\alpha; N)|\right)^v \int_{\mathfrak{m}} \prod_{j=1}^{t+u} \left| g_k(\alpha; N^{\lambda_j}) \right|^2 \, \mathrm{d}\alpha \\ &\ll N^{2\Lambda+v-k+\varepsilon}(N^{-v\sigma_k} + N^{k-\Lambda-v\sigma_k}). \end{aligned} \tag{3.9}$$

In order to estimate the contribution of the major arcs $\mathfrak{M}$ to the left side of (3.7), we note that when $\alpha \in \mathfrak{M}(q, a; P, N^k P^{-1}) \subseteq \mathfrak{M}$, it follows from Lemmata 2.1 and 2.2 that

$$f_k(\alpha; N) \ll \frac{q^{-1/k} N}{1 + N^k |\alpha - a/q|},$$

and

$$g_k(\alpha;N_j) \ll \frac{q^{-1/2}N_j^{1+\varepsilon}}{(1+N_j^k|\alpha-a/q|)^{1/2}} \quad (j=1,2).$$

Applying these two inequalities in combination with a trivial bound for $g_k(\alpha;N_j)$ $(j\geq 3)$, we find that

$$\int_{\mathfrak{M}} |f_k(\alpha;N)|^v \prod_{j=1}^{t+u} \left|g_k(\alpha;N^{\lambda_j})\right|^2 \,\mathrm{d}\alpha \ll \sum_{\substack{0\leq a\leq q\leq P\\(a,q)=1}} \int_{\mathfrak{M}(q,a)} \frac{q^{-2-v/k}N^{2\Lambda+v+\varepsilon}}{(1+N^k|\alpha-a/q|)^{1+v}}.$$

Thus we arrive at the estimate

$$\int_{\mathfrak{M}} |f_k(\alpha;N)|^v \prod_{j=1}^{t+u} \left|g_k(\alpha;N^{\lambda_j})\right|^2 \,\mathrm{d}\alpha \ll N^{2\Lambda+v-k+\varepsilon},$$

which, in combination with (3.9), confirms the desired bound (3.7). □

An immediate consequence of Lemma 3.2 supplies useful mean value estimates in the Waring–Goldbach problem.

Lemma 3.3. *Let k, t and u be positive integers with $k\geq 3$ and $t\geq \left\lfloor \frac{1}{2}(k+3)\right\rfloor$, and let w be a non-negative integer. Define the exponents λ_j and Λ by means of* (3.3)-(3.6), *and put $\eta=\max\{0,k-\Lambda-2w\sigma_k\}$. Then when N is sufficiently large, one has*

$$\int_0^1 |g_k(\alpha;N)|^{2w} \prod_{j=1}^{t+u} \left|g_k(\alpha;N^{\lambda_j})\right|^2 \,\mathrm{d}\alpha \ll N^{2\Lambda+2w-k+\eta+\varepsilon}.$$

Proof. By considering the underlying Diophantine equation, it follows via orthogonality that the mean value in question is bounded above by

$$\int_0^1 |f_k(\alpha;N)|^{2w} \prod_{j=1}^{t+u} \left|g_k(\alpha;N^{\lambda_j})\right|^2 \,\mathrm{d}\alpha.$$

Then we deduce from Lemma 3.2 that

$$\int_0^1 |g_k(\alpha;N)|^{2w} \prod_{j=1}^{t+u} \left|g_k(\alpha;N^{\lambda_j})\right|^2 \,\mathrm{d}\alpha \ll N^{2\Lambda+2w-k+\varepsilon}\left(1+N^{k-\Lambda-2w\sigma_k}\right),$$

and the proof of the lemma is complete. □

4. An upper bound for $H(k)$

An upper bound for $H(k)$ follows by combining the mean value estimate supplied by Lemma 3.3 with the Weyl-type estimate stemming from Lemma 2.2. In certain circumstances an extra variable can be saved by employing the device of Zhao [22, equation (3.10)]. In this section we prepare a general lemma that captures both the results stemming from the basic strategy, and those reflecting the refinement stemming from the argument of Zhao.

Lemma 4.1. *Let k, t and u be positive integers with $k\geq 3$ and $t\geq \left\lfloor \frac{1}{2}(k+3)\right\rfloor$. Define the exponent Λ by means of* (3.6), *and put*

$$v=\lfloor (k-\Lambda)/(2\sigma_k)\rfloor \quad \textit{and} \quad \eta^*=k-\Lambda-2v\sigma_k.$$

Finally, define

$$h = \begin{cases} 1, & \text{when } 0 \le \eta^* < \tfrac{1}{2}\sigma_k, \\ 2, & \text{when } \tfrac{1}{2}\sigma_k \le \eta^* < \sigma_k, \\ 3, & \text{when } \sigma_k \le \eta^* < 2\sigma_k. \end{cases}$$

Suppose in addition that $2(t+u+v)+h \ge 3k+1$ *and, when* $h \in \{1,2\}$, *that either* $v \ge 3$ *or* $\eta^* < h\sigma_k/3$. *Then*

$$H(k) \le 2(t+u+v)+h.$$

Proof. Let $s = 2(t+u+v)+h$, and note that we are permitted to assume that $s \ge 3k+1$. Consider a large integer n satisfying the congruence condition $n \equiv s \pmod{K(k)}$. Set $\lambda_j = 1$ for $j > u+t$, and write $N = \frac{1}{2}n^{1/k}$. Denote by $R_{k,s}(n)$ the number of representations of n in the form

$$n = p_1^k + p_2^k + \cdots + p_s^k,$$

subject to $N^{\lambda_{j+1}} < p_{2j+\omega} \le 2N^{\lambda_{j+1}}$ for $0 \le j \le \frac{1}{2}(s-\omega)$ and $\omega \in \{1,2\}$. Finally, write

$$\mathcal{G}(\alpha) = \prod_{j=1}^{u+t+v} g_k(\alpha; N^{\lambda_j}).$$

Then

$$R_{k,s}(n) = \int_0^1 g_k(\alpha;N)^h \mathcal{G}(\alpha)^2 e(-\alpha n)\,\mathrm{d}\alpha. \tag{4.1}$$

We now dissect the unit interval into sets of major and minor arcs. Let

$$P = N^{1/3}, \quad Q = N^k P^{-1}, \quad L = \log N, \quad X = N^{2\Lambda+2v+h-k}L^{-s},$$

in which Λ is defined via (3.6). We choose the set of major arcs to be $\mathfrak{M} = \mathfrak{M}(P,Q)$ and write $\mathfrak{m} = [0,1)\backslash\mathfrak{M}$. The major arc contribution to the integral in (4.1) can be approximated using the methods in Chapters 6 and 9 of a forthcoming monograph by Liu and Zhan [11]. Alternatively, we may refer to the main theorem in Liu [10], which establishes that for any fixed $A > 0$, one has the asymptotic formula

$$\int_{\mathfrak{M}} g_k(\alpha;N)^h \mathcal{G}(\alpha)^2 e(-\alpha n)\,\mathrm{d}\alpha = \mathfrak{S}_{k,s}(n) J_{k,s}(n) + O\left(XL^{-A}\right), \tag{4.2}$$

where

$$\mathfrak{S}_{k,s}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \phi(q)^{-s} \Bigg(\sum_{\substack{r=1 \\ (r,q)=1}}^{q} e(ar^k/q) \Bigg)^s e(-an/q)$$

and

$$J_{k,s}(n) = \int_{-\infty}^{\infty} V(\beta;N)^h \Bigg(\prod_{i=1}^{u+t+v} V(\beta; N^{\lambda_j}) \Bigg)^2 e(-\beta n)\,\mathrm{d}\beta,$$

in which

$$V(\beta;Z) = \int_Z^{2Z} \frac{e(\beta\gamma^k)}{\log \gamma}\,\mathrm{d}\gamma.$$

Here, the expression $\mathfrak{S}_{k,s}(n)$ is the singular series associated with sums of s kth powers of primes and $J_{k,s}(n)$ is the singular integral associated with the representations counted by $R_{k,s}(n)$ (see Liu and Zhan [11, equations (4.50) and (9.16)]). We remark that while the main result in Liu [10] states (4.2) for a different set of major arcs, corresponding to a larger choice of P than that given above, Liu's argument applies also to $\mathfrak{M}$. Moreover, with our choice of N, standard estimates for the singular series and the singular integral (see Liu and Zhan [11, §§4.6 and 6.2]) confirm that for all sufficiently large integers n with $n \equiv s \pmod{K(k)}$,

$$X \ll \mathfrak{S}_{k,s}(n)J_{k,s}(n) \ll X. \tag{4.3}$$

We note in this context that the conditions $s \geq 3k+1$ and $n \equiv s \pmod{K(k)}$ ensure that the singular series is positive (see [5, Theorem 12]).

We turn next to the contribution of the minor arcs, first considering the situation with $\sigma_k \leq \eta^* < 2\sigma_k$, in which case $h = 3$, together with that in which $h \in \{1, 2\}$ and $\eta^* < h\sigma_k/3$. By Lemma 2.2, one has

$$\sup_{\alpha \in \mathfrak{m}} |g_k(\alpha; N)| \ll N^{1-\sigma_k/3+\varepsilon}. \tag{4.4}$$

Write

$$\Theta = \int_0^1 |g_k(\alpha; N)\mathcal{G}(\alpha)|^2 \, \mathrm{d}\alpha. \tag{4.5}$$

Then, since the definition of v ensures that $k - \Lambda < (2v+2)\sigma_k$, we find from Lemma 3.3 that

$$\Theta \ll N^{2\Lambda+2v+2-k+\varepsilon}. \tag{4.6}$$

Thus, when $h = 3$, we obtain the bound

$$\int_{\mathfrak{m}} |g_k(\alpha; N)^h \mathcal{G}(\alpha)^2| \, d\alpha \ll \Big(\sup_{\alpha \in \mathfrak{m}} |g_k(\alpha; N)|\Big)\Theta \ll XN^{-\sigma_k/4}. \tag{4.7}$$

On the other hand, when $h \in \{1, 2\}$ and $\eta^* < h\sigma_k/3$, we find instead that

$$\int_{\mathfrak{m}} |g_k(\alpha; N)^h \mathcal{G}(\alpha)^2| \, d\alpha \ll \Big(\sup_{\alpha \in \mathfrak{m}} |g_k(\alpha; N)|\Big)^h \int_0^1 |\mathcal{G}(\alpha)|^2 \, \mathrm{d}\alpha \ll XN^{\eta^*-h\sigma_k/3+\varepsilon}.$$

By combining these estimates with (4.1)-(4.3), we conclude in these cases that

$$R_{k,s}(n) = \mathfrak{S}_{k,s}(n)J_{k,s}(n) + O(XL^{-A}) \gg X. \tag{4.8}$$

Thus $H(k) \leq s$, completing the proof of the lemma when $\sigma_k \leq \eta^* < 2\sigma_k$.

The final case to consider is that in which $h \in \{1, 2\}$ and $0 \leq \eta^* < \sigma_k$. Here, we employ the method of Zhao [22, equation (3.10)]. For simplicity of exposition, we provide a detailed account of the situation with $h = 2$, although it will be clear how to adjust the argument to handle the case $h = 1$. We begin with the observation that

$$\int_{\mathfrak{m}} g_k(\alpha; N)^h \mathcal{G}(\alpha)^2 e(-n\alpha) \, \mathrm{d}\alpha = \sum_{N<p_1,p_2\leq 2N} \int_{\mathfrak{m}} \mathcal{G}(\alpha)^2 e((p_1^k + p_2^k - n)\alpha) \, \mathrm{d}\alpha,$$

whence, by Cauchy's inequality,

$$\left| \int_{\mathfrak{m}} g_k(\alpha; N)^h \mathcal{G}(\alpha)^2 e(-n\alpha) \, \mathrm{d}\alpha \right| \leq N^{h/2}\Upsilon^{1/2}, \tag{4.9}$$

where

$$\Upsilon = \sum_{N<x_1,x_2\le 2N} \left| \int_{\mathfrak{m}} \mathcal{G}(\alpha)^2 e((x_1^k + x_2^k - n)\alpha)\, \mathrm{d}\alpha \right|^2$$
$$= \int_{\mathfrak{m}} \int_{\mathfrak{m}} \mathcal{G}(\alpha)^2 \mathcal{G}(-\beta)^2 f_k(\alpha-\beta;N)^h e(-n(\alpha-\beta))\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

Put $\mathfrak{N}(q,a) = \mathfrak{M}(q,a;N^{k\sigma_k},N^{k-k\sigma_k})$ and $\mathfrak{N} = \mathfrak{M}(N^{k\sigma_k},N^{k-k\sigma_k})$. Also, write $\mathfrak{n} = [0,1)\setminus\mathfrak{N}$. Next, denote by $\mathfrak{B}$ the set of ordered pairs $(\alpha,\beta)\in\mathfrak{m}^2$ for which $\alpha-\beta\in\mathfrak{N}$ (mod 1), and put $\mathfrak{b} = \mathfrak{m}^2\setminus\mathfrak{B}$. Let $\Psi:[0,1)\to[0,\infty)$ denote the function defined by taking

$$\Psi(\alpha) = w_k(q)N(1+N^k|\alpha-a/q|)^{-1},$$

when $\alpha\in\mathfrak{N}(q,a)\subseteq\mathfrak{N}$, and otherwise by taking $\Psi(\alpha)=0$. Then it follows from an application of the triangle inequality that

$$\Upsilon \le \iint_{\mathfrak{b}} |f_k(\alpha-\beta;N)^h \mathcal{G}(\alpha)^2\mathcal{G}(\beta)^2|\, \mathrm{d}\alpha\, \mathrm{d}\beta + \iint_{\mathfrak{B}} |f_k(\alpha-\beta;N)^h \mathcal{G}(\alpha)^2\mathcal{G}(\beta)^2|\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

Thus we deduce from Lemma 2.1 that

$$\Upsilon \ll \Upsilon_1 + N^{h-1}\Upsilon_2, \tag{4.10}$$

where

$$\Upsilon_1 = N^{h-h\sigma_k+\varepsilon} \int_0^1 \int_0^1 |\mathcal{G}(\alpha)\mathcal{G}(\beta)|^2\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

and

$$\Upsilon_2 = \iint_{\mathfrak{B}} \Psi(\alpha-\beta)|\mathcal{G}(\alpha)\mathcal{G}(\beta)|^2\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

It follows from Lemma 3.3 that

$$\Upsilon_1 \ll N^{h-h\sigma_k+\varepsilon}\left(N^{2\Lambda+2v-k+\eta^*+\varepsilon}\right)^2 = X^2N^{-h+2\eta^*-h\sigma_k+3\varepsilon}. \tag{4.11}$$

In order to bound Υ_2, we begin by using the estimate

$$|g_k(\alpha;N)g_k(\beta;N)|^2 \ll |g_k(\alpha;N)|^4 + |g_k(\beta;N)|^4,$$

in combination with trivial estimates and symmetry, to obtain

$$\Upsilon_2 \ll N^{\Lambda-1-\lambda_2} \int_{\mathfrak{m}} \int_{\mathfrak{m}} \Psi(\alpha-\beta)|g_k(\alpha;N)^{v-1}g_k(\alpha;N^{\lambda_2})g_k(\beta;N)^2\mathcal{G}(\alpha)\mathcal{G}(\beta)^2|\, \mathrm{d}\alpha\, \mathrm{d}\beta.$$

Write

$$\Phi = \sup_{\beta\in[0,1)} \int_0^1 \Psi(\alpha-\beta)^2|g_k(\alpha;N^{\lambda_2})|^2\, \mathrm{d}\alpha.$$

Then since [22, Lemma 2.2] supplies the bound $\Phi \ll N^{2+2\lambda_2-k+\varepsilon}$, it follows from Cauchy's inequality in combination with (4.4)-(4.6) that

$$\int_{\mathfrak{m}} \Psi(\alpha-\beta)|g_k(\alpha;N)^{v-1}g_k(\alpha;N^{\lambda_2})\mathcal{G}(\alpha)|\, \mathrm{d}\alpha \ll \left(\sup_{\alpha\in\mathfrak{m}} |g_k(\alpha;N)|\right)^{v-2}\Theta^{1/2}\Phi^{1/2}$$
$$\ll (N^{1-\sigma_k/3})^{v-2}N^{\Lambda+\lambda_2+v+2-k+\varepsilon}.$$

We therefore conclude by means of (4.6) that

$$\begin{aligned}\Upsilon_2 &\ll N^{2\Lambda+2v-k-1-\sigma_k/4}\int_0^1 |g_k(\beta;N)\mathcal{G}(\beta)|^2\,\mathrm{d}\beta\\ &\ll N^{4\Lambda+4v+1-2k-\sigma_k/5}.\end{aligned}$$

On substituting this estimate together with (4.11) into (4.10), and noting that, by hypothesis, we have $h\sigma_k > 2\eta^*$, we deduce that for some positive number ν, one has

$$\Upsilon \ll X^2 N^{-h-2\nu}.$$

Inserting this bound into (4.9), we arrive at the estimate

$$\int_{\mathfrak{m}} g_k(\alpha;N)^h \mathcal{G}(\alpha)^2 e(-n\alpha)\,\mathrm{d}\alpha \ll XN^{-\nu},$$

an estimate that may be employed as a viable substitute for (4.7) in the argument leading to (4.8). Thus the conclusion of the lemma follows also in these final cases, and so the proof of the lemma is complete. □

5. Proof of Theorems 1 and 2

Theorems 1 and 2 are direct consequences of Lemma 4.1. Recall (3.3)-(3.6), and write $\sigma = \sigma_{k-1}$ and $\phi = \theta + \sigma/k$. Then one has

$$\sum_{i=1}^{u+1} \lambda_i = \frac{1-\phi^{u+1}}{1-\phi} = \frac{k}{1-\sigma}(1-\phi^{u+1})$$

and

$$\begin{aligned}\sum_{j=2}^{t} \lambda_{u+j} &= \left(\frac{k^2-\theta^{t-3}}{k^2+k-k\theta^{t-3}}\right)\lambda_{u+1} + \left(\frac{k^2-k-1}{k^2+k-k\theta^{t-3}}\right)\left(\frac{1-\theta^{t-2}}{1-\theta}\right)\lambda_{u+1}\\ &= \left(\frac{k^3-k-(k^3-2k^2+2)\theta^{t-3}}{k^2+k-k\theta^{t-3}}\right)\phi^u.\end{aligned}$$

Thus

$$\Lambda = \frac{k}{1-\sigma} + \left(\frac{(k^3-k-(k^3-2k^2+2)\theta^{t-3})(1-\sigma)-(k-1+\sigma)(k^2+k-k\theta^{t-3})}{(k^2+k-k\theta^{t-3})(1-\sigma)}\right)\phi^u,$$

and hence

$$k-\Lambda = -\frac{k\sigma}{1-\sigma} + \left(\frac{k^2(k+1)\sigma+\theta^{t-3}((k^3-3k^2+k+2)-\sigma(k^3-2k^2+k+2))}{(k^2+k-k\theta^{t-3})(1-\sigma)}\right)\phi^u.$$

This formula provides the key input into our application of Lemma 4.1.

The proof of Theorem 2. Let k be an integer with $8 \le k \le 20$, and define $t = t_k$, $u = u_k$, $v = v_k$ and $h = h_k$ according to Table 3. Then a straightforward computer program confirms that the hypotheses of Lemma 4.1 hold, and hence that $H(k) \le 2(t+u+v)+h$. Indeed, with h_k^* defined by Table 4, one finds for these values of k that $2\eta^*/\sigma_k < h_k^*$. All entries in this table have been rounded up in the final decimal place recorded. This completes the proof of Theorem 2.

□

k	8	9	10	11	12	13	14	15	16	17	18	19	20
t_k	9	18	13	14	9	20	13	14	26	28	33	23	35
u_k	18	15	27	30	44	41	56	64	56	62	68	81	78
v_k	3	4	3	7	5	4	4	3	6	6	4	9	8
h_k	1	1	3	1	1	1	1	1	2	2	1	1	2

TABLE 3. The values of t_k, u_k, v_k and h_k for $8 \le k \le 20$

k	8	9	10	11	12	13	14
h_k^*	0.56062	0.09534	2.05276	0.01726	0.00008	0.99878	0.01987
k	15	16	17	18	19	20	
h_k^*	0.00055	1.90169	1.99481	0.00497	0.00294	1.10563	

TABLE 4. The values of h_k^* for $8 \le k \le 20$

It is evident that there is substantial non-monotonicity in the values of t_k, u_k and v_k recorded in Table 3. It seems to the authors that since θ and ϕ have values that are rather close together, then there is relatively little sensitivity in the optimisation to the specific values of t_k and u_k, but rather it is the sum $t_k + u_k$ that is important. We note also that the values of h_k^* are extremely small for a number of the exponents k, so that relatively modest improvements to the values $\Sigma(k)$ recorded in Table 2 will lead to improved values of $H(k)$.

The proof of Theorem 1. We may now suppose that k is large. We put $t = t_k$ and $u = u_k$, where

$$t_k = \lceil \tfrac{1}{2} k \log k \rceil \quad \text{and} \quad u_k = \lceil 2k \log k \rceil - t - 4.$$

It is convenient for later use to put $\gamma = \lceil 2k \log k \rceil - 2k \log k$. Also, we write

$$\tau = \frac{1}{2k^2 - 6k + 4} \quad \text{and} \quad \sigma = \frac{1}{2k^2 - 10k + 12},$$

so that $\sigma_k = \tau$ and $\sigma_{k-1} = \sigma$. Our formula for $k - \Lambda$ may now be written in the shape

$$k - \Lambda = -\frac{k\sigma}{1-\sigma} + \left(\frac{k^2(k+1)(k-1)^3\sigma + \theta^t k^3(k^3 - 3k^2 + O(k))}{(k-1)^3(k^2 + k - k\theta^{t-3})(1-\sigma)} \right) \phi^u.$$

Since

$$\log \theta = \log \left(1 - \frac{1}{k} \right) = -\frac{1}{k} - \frac{1}{2k^2} + O\left(\frac{1}{k^3} \right),$$

it follows that

$$t \log \theta = -\frac{t}{k} - \frac{\log k}{4k} + O\left(\frac{\log k}{k^2} \right),$$

and hence

$$\theta^t = e^{-t/k} \left(1 - \frac{\log k}{4k} + O(k^{-3/2}) \right) \asymp k^{-1/2}.$$

Similarly, since

$$\log\phi = \log\left(1 - \frac{1-\sigma}{k}\right) = -\frac{1}{k} - \frac{1}{2k^2} + O\left(\frac{1}{k^3}\right),$$

we have

$$\phi^u = e^{-u/k}\left(1 - \frac{3\log k}{4k} + O(k^{-3/2})\right) \ll k^{-3/2}.$$

In particular, we find that

$$k - \Lambda = -\frac{k\sigma}{1-\sigma} + \left(k - 1 + O(k^{-1/2})\right)\theta^t\phi^u + O(k^{-5/2}),$$

and that

$$\begin{aligned}\theta^t\phi^u &= e^{-(t+u)/k}\left(1 - \frac{\log k}{k} + O(k^{-3/2})\right)\\ &= e^{(4-\gamma)/k}\left(\frac{1}{k^2} - \frac{\log k}{k^3} + O(k^{-7/2})\right).\end{aligned}$$

On noting that

$$\frac{\sigma}{\tau} = \frac{2k^2 - 6k + 4}{2k^2 - 10k + 12} = 1 + \frac{2}{k} + O\left(\frac{1}{k^2}\right),$$

it follows that

$$\begin{aligned}\frac{k-\Lambda}{2\tau} &= -\tfrac{1}{2}(k+2) + \frac{e^{(4-\gamma)/k}(k^2 - 3k + 2)}{k^3}(k-1)(k - \log k) + O(k^{-1/2})\\ &= -\tfrac{1}{2}(k+2) + (k - \log k - 4)\left(1 + \frac{4-\gamma}{k}\right) + O(k^{-1/2})\\ &= \tfrac{1}{2}k - \log k - 1 - \gamma + O(k^{-1/2}).\end{aligned}$$

Let $v = \lfloor (k-\Lambda)/(2\tau)\rfloor$, put $\eta^* = k - \Lambda - 2v\tau$, and define h as in the statement of Lemma 4.1. In particular, one has $0 \le \eta^* < 2\tau$, and no matter what the value of η^* may be, one confirms that

$$2v + h = \frac{k - \Lambda - \eta^*}{\tau} + h \le \frac{k-\Lambda}{\tau} + 2 \le k - 2\log k - 2\gamma + O(k^{-1/2}).$$

Therefore, since

$$2(t+u+v) + h \le 2(2k\log k + \gamma - 4) + k - 2\log k - 2\gamma + O(k^{-1/2}),$$

we conclude from Lemma 4.1 that

$$H(k) \le (4k-2)\log k + k - 8 + O(k^{-1/2}).$$

In view of our assumption that k is sufficiently large, it follows that

$$H(k) \le (4k-2)\log k + k - 7,$$

and the proof of Theorem 1 is complete. □

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