

ON WARING'S PROBLEM: TWO SQUARES, TWO CUBES AND TWO SIXTH POWERS

TREVOR D. WOOLEY

ABSTRACT. We investigate the number of representations of a large positive integer as the sum of two squares, two positive integral cubes, and two sixth powers, showing that the anticipated asymptotic formula fails for at most $O((\log X)^{3+\varepsilon})$ positive integers not exceeding X .

1. INTRODUCTION

The Hardy-Littlewood (circle) method fails to establish the solubility of problems of Waring-type when the sum of the reciprocals of the exponents does not exceed 2. This well-known consequence of the convexity barrier has been circumvented in very few cases by other devices. A problem that fails to be accessible to the circle method by the narrowest of margins is the notorious one of representing integers as sums of two squares, two positive integral cubes, and two sixth powers. The author has recently applied Golubeva's method [5] to show, subject to the truth of the Generalised Riemann Hypothesis (GRH), that all large integers are thus represented (see [20, Theorem 1.1]). However, this work employs representations of special type, and fails to deliver the anticipated asymptotic formula for their total number. The purpose of this note is to show that, although the expected asymptotic formula may occasionally fail to hold, the set of such exceptional instances is extraordinarily sparse.

Let $R(n)$ denote the number of representations of the integer n in the shape

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^6 + x_6^6 = n, \quad (1.1)$$

with $x_i \in \mathbb{N}$ ($1 \leq i \leq 6$). Also, let

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^q q^{-6} S_2(q, a)^2 S_3(q, a)^2 S_6(q, a)^2 e(-na/q), \quad (1.2)$$

in which we write

$$S_k(q, a) = \sum_{r=1}^q e(ar^k/q),$$

and as usual $e(z)$ denotes $e^{2\pi iz}$. We refer to a function $\psi(t)$ as being a *sedately increasing function* when $\psi(t)$ is a function of a positive variable t , increasing monotonically to infinity, and satisfying the condition that when t is large, one has $\psi(t) = O(t^\delta)$ for a positive number δ sufficiently small in the ambient

2010 *Mathematics Subject Classification.* 11P05, 11D85, 11P55.

Key words and phrases. Waring's problem, Hardy-Littlewood method.

context. Finally, write $E(X; \psi)$ for the number of integers n with $1 \leq n \leq X$ such that

$$\left| R(n) - \frac{27}{32} \sqrt[3]{2} \Gamma\left(\frac{4}{3}\right)^6 \mathfrak{S}(n)n \right| > n\psi(n)^{-1}. \quad (1.3)$$

It is worth remarking here that for every integer n , one has $1 \ll \mathfrak{S}(n) \ll 1$.

Theorem 1.1. *When $\psi(t)$ is a sedately increasing function one has*

$$E(X; \psi) \ll \psi(X)^2 (\log X)^3.$$

It follows that, for each $\varepsilon > 0$, the asymptotic formula

$$R(n) \sim \frac{27}{32} \sqrt[3]{2} \Gamma\left(\frac{4}{3}\right)^6 \mathfrak{S}(n)n \quad (1.4)$$

fails for at most $O((\log X)^{3+\varepsilon})$ of the integers n with $1 \leq n \leq X$, an extremely slim exceptional set indeed. It follows from the final paragraph of §4, in fact, that if for a large integer n the asymptotic formula (1.4) fails, then this formula instead holds for a modification of n in which at most 3 of its digits are altered.

We note that Hooley [9] has confirmed the anticipated asymptotic formula for the number of representations in the analogous problem in which an additional h th power is present in (1.1). Also, the methods of Vaughan [13] apply if one is prepared instead to substitute a fifth power for one of the sixth powers. Both of these situations, of course, are not inhibited by the convexity barrier. There are very few instances in which Waring's problem has been resolved in subconvex situations. Subject to natural conditions of congruential type, Gauss [4] addressed sums of three squares, and Hooley [10] successfully considered sums of two squares and three cubes. Also, work of Golubeva [5] addresses the mixed problem

$$n = x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^4 + x_6^{16} + x_7^{4k+1}.$$

Most recently of all, subject to the validity of the Elliott-Halberstam conjecture together with GRH, work of Friedlander and the author [3] addresses sums of two squares and three biquadrates.

We finish by recording a convention concerning the use of the number ε . Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Note that the "value" of ε may consequently change from statement to statement.

2. PREPARATORY MANOEUVRES

The method that we employ to prove Theorem 1.1 is motivated by our earlier work on sums of one square and five cubes [18]. Suppose that X is a large positive number, and let $\psi(t)$ be a sedately increasing function. We denote by $\mathcal{Z}(X)$ the set of integers n with $X/2 < n \leq X$ for which the lower bound (1.3) holds, and we abbreviate $\text{card}(\mathcal{Z}(X))$ to Z . Write P_k for $[X^{1/k}]$, and define the exponential sum $f_k(\alpha)$ by

$$f_k(\alpha) = \sum_{1 \leq x \leq P_k} e(\alpha x^k).$$

Also, when Q is a positive integer, let $\mathfrak{M}(Q)$ denote the union of the intervals

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \leq QX^{-1}\},$$

with $0 \leq a \leq q \leq Q$ and $(a, q) = 1$. Then, when $1 \leq Q \leq 2X^{1/2}$, we put $\mathfrak{N}(Q) = \mathfrak{M}(Q) \setminus \mathfrak{M}(Q/2)$. In order to ensure that each $\alpha \in [0, 1)$ is associated with a uniquely defined arc $\mathfrak{M}(q, a)$, we adopt the convention that when α lies in more than one arc $\mathfrak{M}(q, a) \subseteq \mathfrak{M}(Q)$, then it is declared to lie in the arc for which q is least. Finally, let ν be a sufficiently small positive number, and write $W = X^\nu$. We then take \mathfrak{P} to be the union of the intervals

$$\mathfrak{P}(q, a) = \{\alpha \in [0, 1) : |\alpha - a/q| \leq WX^{-1}\},$$

with $0 \leq a \leq q \leq W$ and $(a, q) = 1$, and we set $\mathfrak{p} = [0, 1) \setminus \mathfrak{P}$.

We initiate proceedings by outlining how the methods of [15, Chapter 4] lead to the asymptotic formula

$$\int_{\mathfrak{P}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha = \frac{\Gamma(\frac{3}{2})^2 \Gamma(\frac{4}{3})^2 \Gamma(\frac{7}{6})^2}{\Gamma(2)} \mathfrak{S}(n)n + O(n^{1-\tau}), \quad (2.1)$$

for a suitably small positive number τ . Here, the singular series $\mathfrak{S}(n)$ is the one defined in (1.2). Recall the identities

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \quad \text{and} \quad \Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z} \sqrt{\pi} \Gamma(2z).$$

Then one finds that

$$3\Gamma(\frac{4}{3})\Gamma(\frac{2}{3}) = \pi/\sin(\pi/3) \quad \text{and} \quad \Gamma(\frac{2}{3})\Gamma(\frac{7}{6}) = 2^{1-4/3} \sqrt{\pi} \Gamma(\frac{4}{3}),$$

and hence

$$\frac{\Gamma(\frac{3}{2})^2 \Gamma(\frac{4}{3})^2 \Gamma(\frac{7}{6})^2}{\Gamma(2)} = \frac{\frac{1}{4} \pi \Gamma(\frac{4}{3})^2 (2^{-1/3} \sqrt{\pi} \Gamma(\frac{4}{3}))^2}{\Gamma(\frac{2}{3})^2} = \frac{2^{-8/3} \pi^2 \Gamma(\frac{4}{3})^6}{(2\pi/3\sqrt{3})^2} = \frac{27}{32} \sqrt{2} \Gamma(\frac{4}{3})^6.$$

Write

$$v_k(\beta) = \int_0^{P_k} e(\beta\gamma^k) d\gamma,$$

and for the time being define $f_k^*(\alpha)$ for $\alpha \in \mathfrak{P}(q, a) \subseteq \mathfrak{P}$ by putting

$$f_k^*(\alpha) = q^{-1} S_k(q, a) v_k(\alpha - a/q). \quad (2.2)$$

Then it follows from [15, Theorem 4.1] that whenever $\alpha \in \mathfrak{P}(q, a) \subseteq \mathfrak{P}$, one has

$$f_k(\alpha) - f_k^*(\alpha) \ll q^{1/2+\varepsilon} \ll W^{1/2+\varepsilon}.$$

The measure of \mathfrak{P} is $O(W^3 X^{-1})$, and hence we deduce that

$$\begin{aligned} \int_{\mathfrak{P}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha - \int_{\mathfrak{P}} f_2^*(\alpha)^2 f_3^*(\alpha)^2 f_6^*(\alpha)^2 e(-n\alpha) d\alpha \\ \ll W^4 X^{5/6}. \end{aligned}$$

A routine computation leads from (2.2) to the formula

$$\int_{\mathfrak{P}} f_2^*(\alpha)^2 f_3^*(\alpha)^2 f_6^*(\alpha)^2 e(-n\alpha) d\alpha = \mathfrak{S}(n; W) J(n; W),$$

where

$$\mathfrak{S}(n; W) = \sum_{1 \leq q \leq W} A(q; n), \quad (2.3)$$

in which we have written

$$A(q; n) = \sum_{\substack{a=1 \\ (a,q)=1}}^q q^{-6} S_2(q, a)^2 S_3(q, a)^2 S_6(q, a)^2 e(-na/q), \quad (2.4)$$

and

$$J(n; W) = \int_{-WX^{-1}}^{WX^{-1}} v_2(\beta)^2 v_3(\beta)^2 v_6(\beta)^2 e(-\beta n) d\beta.$$

Thus we may conclude at this stage of the argument that

$$\int_{\mathfrak{P}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha = \mathfrak{S}(n; W) J(n; W) + O(X^{1-\tau}), \quad (2.5)$$

for a suitable positive number τ .

We next recall from [15, Theorem 7.3] that

$$v_k(\beta) \ll P_k(1 + |\beta|X)^{-1/k},$$

whence the singular integral $J(n; W)$ converges absolutely as $W \rightarrow \infty$. The discussion concluding [2, Chapter 4] consequently delivers the relation

$$J(n; W) = \frac{\Gamma(\frac{3}{2})^2 \Gamma(\frac{4}{3})^2 \Gamma(\frac{7}{6})^2}{\Gamma(2)} n + O(n^{1-\tau}). \quad (2.6)$$

In order to analyse the truncated singular series $\mathfrak{S}(n; W)$, we begin by defining the multiplicative function $w_k(q)$ by taking

$$w_k(p^{uk+v}) = \begin{cases} kp^{-u-1/2}, & \text{when } u \geq 0 \text{ and } v = 1, \\ p^{-u-1}, & \text{when } u \geq 0 \text{ and } 2 \leq v \leq k. \end{cases}$$

Then according to [14, Lemma 3], whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$, one has $q^{-1} S_k(q, a) \ll w_k(q)$. On employing this estimate within (2.4), we see that when $q = p^h$, then

$$q^{1/2} A(q; n) \ll q^{3/2} w_2(q)^2 w_3(q)^2 w_6(q)^2 \ll p^{-\max\{3/2, h/2\}},$$

and thus

$$\sum_{h=1}^{\infty} p^{h/2} |A(p^h; n)| \ll p^{-3/2} + \sum_{h=4}^{\infty} p^{-h/2} \ll p^{-3/2}.$$

By applying the multiplicativity of the function $A(q; n)$, we consequently deduce that for a suitable positive number B , one has

$$\begin{aligned} \sum_{q=1}^{\infty} (q/W)^{1/2} |A(q; n)| &= W^{-1/2} \prod_p \left(1 + \sum_{h=1}^{\infty} p^{h/2} |A(p^h; n)| \right) \\ &\ll W^{-1/2} \prod_p (1 + Bp^{-3/2}) \ll W^{-1/2}. \end{aligned}$$

In this way, we find that

$$\sum_{q>W} |A(q; n)| \ll W^{-1/2},$$

so that the singular series $\mathfrak{S}(n)$ defined in (1.2) converges absolutely. In particular, we find from (2.3) that one has

$$\mathfrak{S}(n) - \mathfrak{S}(n; W) \ll W^{-1/2}.$$

On recalling (2.5) and (2.6), we confirm that the formula (2.1) does indeed hold.

We pause here to note that the standard theory of singular series in Waring's problem (see [15, Chapter 4]) shows that $\mathfrak{S}(n) \gg 1$ provided only that the congruence

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^6 + x_6^6 \equiv n \pmod{q}$$

is soluble for every natural number q . That such is indeed the case follows directly by applying the Cauchy-Davenport Lemma (see [15, Lemma 2.14]). This confirms the remark preceding the statement of Theorem 1.1 above to the effect that $1 \ll \mathfrak{S}(n) \ll 1$ for every integer n .

We are now equipped to examine the exceptional set $\mathcal{Z}(X)$. For each integer $n \in \mathcal{Z}(X)$, it follows from (1.3) via orthogonality that

$$\left| \int_0^1 f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha - \frac{27}{32} \sqrt[3]{2} \Gamma\left(\frac{4}{3}\right)^6 \mathfrak{S}(n)n \right| > \frac{1}{2} X \psi(X)^{-1}.$$

By employing the relation (2.1), therefore, we obtain the lower bound

$$\left| \int_{\mathfrak{p}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha \right| > \frac{1}{4} X \psi(X)^{-1},$$

and from this we see that

$$\sum_{n \in \mathcal{Z}(X)} \left| \int_{\mathfrak{p}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha \right| \gg ZX \psi(X)^{-1}. \quad (2.7)$$

For each integer n with $n \in \mathcal{Z}(X)$, there exists a complex number η_n , with $|\eta_n| = 1$, satisfying the condition that

$$\left| \int_{\mathfrak{p}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha \right| = \eta_n \int_{\mathfrak{p}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha.$$

Define the exponential sums $K(\alpha)$ and $K^*(\alpha)$ by putting

$$K(\alpha) = \sum_{n \in \mathcal{Z}(X)} \eta_n e(-n\alpha) \quad \text{and} \quad K^*(\alpha) = \sum_{n \in \mathcal{Z}(X)} e(-n\alpha).$$

Then it follows from (2.7) that

$$\begin{aligned} ZX \psi(X)^{-1} &\ll \sum_{n \in \mathcal{Z}(X)} \eta_n \int_{\mathfrak{p}} f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha) d\alpha \\ &\leq \int_{\mathfrak{p}} |f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| d\alpha. \end{aligned} \quad (2.8)$$

This relation is the starting point for our estimation of Z , which we achieve by bounding the integral on the right hand side of (2.8).

3. AUXILIARY ESTIMATES

Before making further progress, we pause to derive some auxiliary mean value estimates of use in our subsequent argument, and these we collect together in the present section. In this and the next section, we write $L = L(X)$ for $\log X$. We begin by supplying a sharp version of a mean value involving both $f_3(\alpha)$ and $f_6(\alpha)$.

Lemma 3.1. *One has*

$$\int_0^1 |f_3(\alpha)^2 f_6(\alpha)^4| d\alpha \leq 3X^{2/3} + O(X^{23/36+\varepsilon}). \quad (3.1)$$

Proof. By orthogonality, the integral I_1 on the left hand side of (3.1) counts the number of integral solutions of the equation

$$x_1^3 - x_2^3 = y_1^6 + y_2^6 - y_3^6 - y_4^6, \quad (3.2)$$

with $1 \leq x_1, x_2 \leq P_3$ and $1 \leq y_i \leq P_6$ ($1 \leq i \leq 4$). The number of solutions of (3.2) counted by I_1 with $x_1 = x_2$ is at most $P_3 I_2$, where I_2 denotes the number of integral solutions of the equation

$$y_1^6 + y_2^6 = y_3^6 + y_4^6,$$

with $1 \leq y_i \leq P_6$ ($1 \leq i \leq 4$). It was shown by Greaves [6], Skinner and Wooley [12] and Hooley [11], in close chronological proximity, that $I_2 = 2P_6^2 + O(P_6^{2-\delta})$ for some $\delta > 0$. The last named source obtains such a conclusion for any $\delta < \frac{1}{3}$. The most recent results can be found in Heath-Brown [8, Theorem 11]. On the other hand, the solutions of (3.2) counted by I_1 with $x_1 \neq x_2$ are of two types. There are the solutions \mathbf{x}, \mathbf{y} in which the integer

$$y_1^6 + y_2^6 - y_3^6 - y_4^6 \quad (3.3)$$

has precisely one representation in the form $x_1^3 - x_2^3$, of which there are at most P_6^4 . When the expression (3.3) has more than one representation in the shape $x_1^3 - x_2^3$, meanwhile, we must proceed in a more elaborate fashion.

It follows from the work of Heath-Brown [7], as explained following equation (9.9) of Wooley [17], that there are $O(P_3^{4/3+\varepsilon})$ integers m having two or more representations in the shape $m = x_1^3 - x_2^3$, with $1 \leq x_1, x_2 \leq P_3$. Write \mathfrak{X} for the set of such integers m , and $\rho(n)$ for the number of representations of an integer n in the shape $n = y_1^6 + y_2^6 - y_3^6 - y_4^6$, with $1 \leq y_i \leq P_6$ ($1 \leq i \leq 4$). Then by Cauchy's inequality, the number of solutions of (3.2), in which (3.3) has more than one representation in the shape $x_1^3 - x_2^3$, is

$$\sum_{m \in \mathfrak{X}} \rho(m) \leq \left(\sum_{m \in \mathfrak{X}} 1 \right)^{1/2} \left(\sum_{m \in \mathfrak{X}} \rho(m)^2 \right)^{1/2} \ll (P_3^{4/3+\varepsilon})^{1/2} \left(\int_0^1 |f_6(\alpha)|^8 d\alpha \right)^{1/2}.$$

By Hua's Lemma (see [15, Lemma 2.5]), one has

$$\int_0^1 |f_6(\alpha)|^8 d\alpha \ll P_6^{5+\varepsilon},$$

and thus we conclude that

$$\sum_{m \in \mathfrak{X}} \rho(m) \ll (P_3^{4/3+\varepsilon})^{1/2} (P_6^{5+\varepsilon})^{1/2} \ll X^{23/36+\varepsilon}.$$

Combining all of the above contributions, therefore, we conclude that

$$I_1 \leq 2P_3P_6^2 + P_6^4 + O(X^{23/36+\varepsilon}) = 3X^{2/3} + O(X^{23/36+\varepsilon}).$$

This completes the proof of the lemma. \square

Lemma 3.2. *One has*

$$\int_0^1 |f_3(\alpha)^2 K(\alpha)^2| d\alpha \ll P_3 Z + P_3^\varepsilon Z^2.$$

Proof. The desired conclusion is immediate from [19, Lemma 2.1] and its proof. \square

We finish this section by recording a variant of Brüdern's pruning lemma suitable for our purposes.

Lemma 3.3. *Let N, Q be real numbers with $Q \leq N$. For $0 \leq a \leq q \leq Q$ and $(a, q) = 1$, let $\mathcal{M}(q, a)$ denote an interval contained in $[a/q - \frac{1}{2}, a/q + \frac{1}{2}]$. Write \mathcal{M} for the union of all $\mathcal{M}(q, a)$. Let $\gamma < 1$ be a positive number, and let $G : \mathcal{M} \rightarrow \mathbb{C}$ be a function which for $\alpha \in \mathcal{M}(q, a)$ satisfies*

$$G(\alpha) \ll (q + N|q\alpha - a|)^{-1-\gamma}.$$

Furthermore, let $\Psi : \mathbb{R} \rightarrow [0, \infty)$ be a function with a Fourier expansion

$$\Psi(\alpha) = \sum_{|h| \leq H} \psi_h e(\alpha h)$$

such that $\log H \ll \log N$. Then

$$\int_{\mathcal{M}} G(\alpha) \Psi(\alpha) d\alpha \ll_{\gamma} N^{-1} \left(Q^{1-\gamma} \psi_0 + N^{\varepsilon} \sum_{h \neq 0} |\psi_h| \right).$$

Proof. Following the argument of the proof of [1, Lemma 2], mutatis mutandis, we see that

$$\int_{\mathcal{M}} G(\alpha) \Psi(\alpha) d\alpha \ll \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \sum_{|h| \leq H} q^{-1-\gamma} \psi_h \rho_h e(ah/q),$$

where

$$\rho_h = \int_{-1/2}^{1/2} (1 + N|\alpha|)^{-1-\gamma} e(\alpha h) d\alpha \ll_{\gamma} N^{-1}. \quad (3.4)$$

Hence, just as in the analogous argument of [1, Lemma 2], we obtain

$$\begin{aligned} \int_{\mathcal{M}} G(\alpha)\Psi(\alpha) \, d\alpha &\ll \sum_{q \leq Q} q^{-\gamma} \psi_0 \rho_0 + \sum_{h \neq 0} |\psi_h \rho_h| \sum_{q \leq Q} \sum_{d|(q,h)} dq^{-1-\gamma} \\ &\ll_{\gamma} Q^{1-\gamma} \psi_0 \rho_0 + N^{\varepsilon} \sum_{h \neq 0} |\psi_h \rho_h|. \end{aligned}$$

Recalling the upper bound (3.4), the desired conclusion follows at once. \square

4. THE CORE ANALYSIS

We initiate the main thrust of our work on the integral on the right hand side of (2.8) by introducing some auxiliary mean values that include major arc majorants to $f_k(\alpha)$ ($k = 2, 3$). Let δ be a sufficiently small positive number. Define the function $f_2(\alpha)$ for $\alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}(Q)$ by taking

$$\tilde{f}_2(\alpha) = P_2(q + P_2^2|q\alpha - a|)^{-1/2}, \quad (4.1)$$

and then put

$$T_1(Q) = \int_{\mathfrak{M}(Q)} \tilde{f}_2(\alpha)^2 |f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha. \quad (4.2)$$

In addition, define the function $f_3^*(\alpha)$ for $\alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}(Q)$ by taking

$$f_3^*(\alpha) = q^{-1} S_3(q, a) v_3(\alpha - a/q), \quad (4.3)$$

and then put

$$T_2(Q) = \int_{\mathfrak{M}(Q)} \tilde{f}_2(\alpha)^2 |f_3^*(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha. \quad (4.4)$$

Lemma 4.1. *When $1 \leq Q \leq X^{1/2}$, one has*

$$T_1(Q) \ll XZ^{1/2} + X^{7/6+\varepsilon} Q^{\delta-1/2} Z.$$

Proof. Let γ be a small positive number. Then by applying Hölder's inequality to (4.2), one obtains the bound

$$\begin{aligned} T_1(Q) &\leq \left(\sup_{\alpha \in \mathfrak{M}(Q)} \tilde{f}_2(\alpha) \right)^{1-\gamma} \left(\int_0^1 |f_3(\alpha)^2 f_6(\alpha)^4| \, d\alpha \right)^{1/2} \\ &\quad \times \left(\int_{\mathfrak{M}(Q)} \tilde{f}_2(\alpha)^{2+2\gamma} |f_3(\alpha)^2 K(\alpha)^2| \, d\alpha \right)^{1/2}. \end{aligned} \quad (4.5)$$

Making use of (4.1) and applying Lemma 3.3, we see that

$$\begin{aligned} &\int_{\mathfrak{M}(Q)} \tilde{f}_2(\alpha)^{2+2\gamma} |f_3(\alpha)^2 K(\alpha)^2| \, d\alpha \\ &\ll P_2^{2+2\gamma} X^{-1} \left(Q^{1-\gamma} \int_0^1 |f_3(\alpha)^2 K(\alpha)^2| \, d\alpha + X^{\varepsilon} f_3(0)^2 K^*(0)^2 \right). \end{aligned}$$

Then as a consequence of Lemma 3.2, one finds that

$$\begin{aligned} \int_{\mathfrak{M}(Q)} \tilde{f}_2(\alpha)^{2+2\gamma} |f_3(\alpha)^2 K(\alpha)^2| d\alpha \\ \ll P_2^{2+2\gamma} X^{-1} (Q^{1-\gamma} (P_3 Z + P_3^\varepsilon Z^2) + P_3^{2+\varepsilon} Z^2) \\ \ll P_2^{2\gamma} (P_3 Q^{1-\gamma} Z + P_3^{2+\varepsilon} Z^2). \end{aligned}$$

On recalling (4.1) and Lemma 3.1, we therefore conclude from (4.5) that

$$\begin{aligned} T_1(Q) &\ll (P_2 Q^{-1/2})^{1-\gamma} (X^{2/3})^{1/2} P_2^\gamma (P_3 Q^{1-\gamma} Z + P_3^{2+\varepsilon} Z^2)^{1/2} \\ &\ll P_2 X^{1/3} P_3^{1/2} Z^{1/2} + P_2 X^{1/3+\varepsilon} P_3 Q^{(\gamma-1)/2} Z. \end{aligned}$$

The conclusion of the lemma follows on recalling that $P_k = X^{1/k}$. \square

Lemma 4.2. *When $1 \leq Q \leq X^{1/2}$, one has*

$$T_2(Q) \ll X^{5/6+\varepsilon} Q^{1/3} Z + X^{1+\varepsilon} Q^{-2/3} Z.$$

Proof. On making use of the definition (4.3), we find from [15, Theorems 4.2 and 7.3] that when $\alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}(Q)$ one has

$$f_3^*(\alpha) \ll P_3 (q + P_3^3 |q\alpha - a|)^{-1/3},$$

whence

$$\sup_{\alpha \in \mathfrak{M}(Q)} |f_3^*(\alpha)| \ll P_3 Q^{-1/3}.$$

On recalling (4.1), we therefore deduce from [1, Lemma 2] and (4.4) that whenever γ is a sufficiently small positive number, one has

$$\begin{aligned} T_2(Q) &\ll \left(\sup_{\alpha \in \mathfrak{M}(Q)} |f_3^*(\alpha)| \right)^2 K^*(0) \int_{\mathfrak{M}(Q)} \tilde{f}_2(\alpha)^2 |f_6(\alpha)|^2 d\alpha \\ &\ll X^\varepsilon P_3^2 Q^{-2/3} Z \left(Q \int_0^1 |f_6(\alpha)|^2 d\alpha + P_6^2 \right) \\ &\ll X^\varepsilon P_3^2 Q^{-2/3} Z (Q P_6 + P_6^2). \end{aligned}$$

The conclusion of the lemma now follows. \square

Write

$$T_0(Q) = \int_{\mathfrak{M}(Q)} |f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| d\alpha. \quad (4.6)$$

According to [16, Theorem 4], when $\alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}(Q)$ and $1 \leq Q \leq 2P_2$, one has

$$\begin{aligned} f_2(\alpha) &\ll P_2 (q + P_2^2 |q\alpha - a|)^{-1/2} + (q + P_2^2 |q\alpha - a|)^{1/2} \\ &\ll P_2 (q + P_2^2 |q\alpha - a|)^{-1/2}. \end{aligned}$$

Consequently, when $\alpha \in \mathfrak{M}(Q)$ and $1 \leq Q \leq X^{1/2}$, one has

$$f_2(\alpha) \ll \tilde{f}_2(\alpha). \quad (4.7)$$

It therefore follows from (4.6) and Lemma 4.1 that whenever $X^{1/3+\delta} \leq Q \leq X^{1/2}$, one has

$$\begin{aligned} T_0(Q) &\ll \int_{\mathfrak{N}(Q)} \tilde{f}_2(\alpha)^2 |f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha \\ &\ll XZ^{1/2} + X^{7/6+\varepsilon} Q^{\delta-1/2} Z \\ &\ll XZ^{1/2} + X^{1-\delta^2} Z. \end{aligned} \quad (4.8)$$

Next, from [15, Theorem 4.1], one finds that when $\alpha \in \mathfrak{N}(Q)$ one has

$$f_3(\alpha) - f_3^*(\alpha) \ll Q^{1/2+\varepsilon},$$

whence

$$\begin{aligned} \int_{\mathfrak{N}(Q)} \tilde{f}_2(\alpha)^2 |f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha &\ll \int_{\mathfrak{N}(Q)} \tilde{f}_2(\alpha)^2 |f_3^*(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha \\ &\quad + Q^{1+\varepsilon} K^*(0) \int_{\mathfrak{N}(Q)} \tilde{f}_2(\alpha)^2 |f_6(\alpha)|^2 \, d\alpha. \end{aligned}$$

On recalling (4.1), (4.4) and [1, Lemma 2], therefore, we see as in the proof of Lemma 4.2 that

$$\int_{\mathfrak{N}(Q)} \tilde{f}_2(\alpha)^2 |f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha \ll T_2(Q) + X^\varepsilon QZ(QP_6 + P_6^2).$$

Then on making use of Lemma 4.2, we conclude from (4.7) that whenever $1 \leq Q \leq X^{1/2}$, one has

$$\begin{aligned} T_0(Q) &\ll \int_{\mathfrak{N}(Q)} \tilde{f}_2(\alpha)^2 |f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha \\ &\ll X^\varepsilon Z(Q^2 X^{1/6} + QX^{1/3} + Q^{1/3} X^{5/6} + Q^{-2/3} X). \end{aligned}$$

Consequently, when $X^\nu \leq Q \leq X^{1/3+\delta}$ one deduces that

$$T_0(Q) \ll X^\varepsilon Z(X^{17/18+\delta} + X^{1-2\nu/3}). \quad (4.9)$$

We now cover \mathfrak{p} by dyadic sets of arcs $\mathfrak{N}(Q)$ with $Q = X^\nu, 2X^\nu, \dots$ and $Q \leq X^{1/2}$, as is possible as a consequence of Dirichlet's theorem on Diophantine approximation. Thus we see from (4.8) and (4.9) that

$$\int_{\mathfrak{p}} |f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 K(\alpha)| \, d\alpha \ll L(XZ^{1/2} + X^{1-\delta^2} Z + X^{1-\nu/2} Z).$$

On substituting into (2.8), we therefore conclude that

$$ZX\psi(X)^{-1} \ll LXZ^{1/2} + X^{1-\delta^3} Z,$$

whence $Z \ll L^2\psi(X)^2$. Summing over dyadic intervals to cover the set of integers $[1, X] \cap \mathbb{Z}$, we therefore arrive at the estimate

$$E(X; \psi) \leq \sum_{2^j \leq X} \text{card}(\mathcal{Z}(2^{j+1})) \ll L^3\psi(X)^2.$$

This completes the proof of Theorem 1.1.

REFERENCES

- [1] J. Brüdern, *A problem in additive number theory*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 27–33.
- [2] H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Second edition, Cambridge University Press, Cambridge, 2005.
- [3] J. B. Friedlander and T. D. Wooley, *On Waring's problem: two squares and three biquadrates*, submitted, 13pp, preprint available as arxiv:1211.1823.
- [4] C. F. Gauss, *Disquisitiones arithmeticae*, Leipzig, 1801.
- [5] E. P. Golubeva, *A bound for the representability of large numbers by ternary forms, and nonhomogeneous Waring equations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **357** (2008), Analiticheskaya Teoriya Chisel i Teoriya Funktsii. **23**, 5–21, 224; translation in J. Math. Sci. (New York) **157** (2009), 543–552.
- [6] G. Greaves, *Representation of a number by the sum of two fourth powers*, Mat. Zametki **55** (1994), 47–58.
- [7] D. R. Heath-Brown, *The density of rational points on cubic surfaces*, Acta Arith. **79** (1997), 17–30.
- [8] D. R. Heath-Brown, *The density of rational points on curves and surfaces*, Ann. of Math. (2) **155** (2002), 553–595.
- [9] C. Hooley, *On a new approach to various problems of Waring's type*, Recent progress in analytic number theory, Vol. 1 (Durham, 1979), pp. 127–191, Academic Press, London-New York, 1981.
- [10] C. Hooley, *On Waring's problem for two squares and three cubes*, J. Reine Angew. Math. **328** (1981), 161–207.
- [11] C. Hooley, *On another sieve method and the numbers that are a sum of two h th powers, II*, J. Reine Angew. Math. **475** (1996), 55–75.
- [12] C. M. Skinner and T. D. Wooley, *Sums of two k th powers*, J. Reine Angew. Math. **462** (1995), 57–68.
- [13] R. C. Vaughan, *A ternary additive problem*, Proc. London Math. Soc. (3) **41** (1980), 516–532.
- [14] R. C. Vaughan, *On Waring's problem for smaller exponents*, Proc. London Math. Soc. (3) **52** (1986), 445–463.
- [15] R. C. Vaughan, *The Hardy-Littlewood method*, 2nd edition, Cambridge University Press, Cambridge, 1997.
- [16] R. C. Vaughan, *On generating functions in additive number theory, I*, Analytic number theory, pp. 436–448, Cambridge Univ. Press, Cambridge, 2009.
- [17] T. D. Wooley, *Slim exceptional sets for sums of cubes*, Canad. J. Math. **54** (2002), 417–448.
- [18] T. D. Wooley, *Slim exceptional sets in Waring's problem: one square and five cubes*, Quart. J. Math. Oxford **53** (2002), 111–118.
- [19] T. D. Wooley, *Slim exceptional sets and the asymptotic formula in Waring's problem*, Math. Proc. Cambridge Philos. Soc. **134** (2003), 193–206.
- [20] T. D. Wooley, *On Waring's problem: some consequences of Golubeva's method*, submitted.

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, CLIFTON,
BRISTOL BS8 1TW, UNITED KINGDOM

E-mail address: matdw@bristol.ac.uk