DEGREE BOUNDS FOR LOCAL COHOMOLOGY

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ABSTRACT. It has long been known how to read information about the socle degrees of the local cohomology $\mathrm{H}^0_{\mathfrak{m}}(M)$ of a graded R-module from the twists in position $d = \dim R$, in a resolution of M by free R-modules. It has also long been known how to use local cohomology to read valuable information from complexes which approximate resolutions in the sense that they have positive homology of small Krull dimension. The present paper reads information about the maximal generator degree (rather than the socle degree) of $\mathrm{H}^0_{\mathfrak{m}}(M)$ from the twists in position $d-1$ (rather than position d) in an approximate resolution of M.

We apply the local cohomology results to draw conclusions about the maximum generator degree of the second symbolic power of the prime ideal defining a monomial curve and the second symbolic power of the ideal defining a finite set of points in projective space. There is also an application to hyperplane sections of subschemes of projective space and to partial Castelnuovo-Mumford regularity. Perhaps, the most important application is to the study of blow-up algebras and their defining equations. The techniques of the present paper are the main tool used in [10] to bound the degrees of these equations and thus to identify them in some cases.

1. INTRODUCTION

For the time being, let R be a non-negatively graded polynomial ring in d variables over a field and M be a finitely generated graded R-module of depth zero. It is well known how to

(1.0.1) read the socle degrees of M from the twists at the end of a minimal homogeneous finite free resolution of M.

Of course, the socle of M is the socle of the local cohomology module $\mathrm{H}^0_{\mathfrak{m}}(M)$, where $\mathfrak{m}$ is the maximal homogeneous ideal of R. In this paper we find bounds on the degrees of interesting elements of $\mathrm{H}^i_{\mathfrak{m}}(M)$ in terms of information about the ring R and information that can be read from a homogeneous complex of finitely generated R-modules $C_\bullet : \cdots \to C_2 \to C_1 \to C_0 \to 0$ with $\mathrm{H}_0(C_\bullet) = M$. The ring R need not be a polynomial ring, the complex $C_\bullet$ need not be finite, need not be acyclic, and need not consist of free modules, and the parameter i need not be zero. Instead, we impose hypotheses on the Krull dimension of $\mathrm{H}_j(C_\bullet)$ and the depth of C_j in order make various local cohomology modules $\mathrm{H}^\ell_{\mathfrak{m}}(\mathrm{H}_j(C_\bullet))$ and $\mathrm{H}^\ell_{\mathfrak{m}}(C_j)$ vanish.

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The crucial technical result is Proposition 4.1. In Theorem 4.3, our main theorem, we bound the maximal generator degree of $\mathrm{H}^i_{\mathfrak{m}}(M)$ in terms of the maximal generator degree of C_j for appropriately related i and j. In particular, in Corollary 4.8, we bound the maximal generator degree of $\mathrm{H}^0_{\mathfrak{m}}(M)$ in terms of the maximal generator degree of C_{d-1}. The hypotheses of Corollary 4.8 hold if $C_\bullet$ is a free resolution of M; consequently, this result is completely analogous to (1.0.1) where $\max\{r \mid [\mathrm{H}^0_{\mathfrak{m}}(M)]_r \neq 0\}$ is read from the generator degrees of C_d. Corollary 4.7 is an intriguing generalization of the well-known fact that a maximal Cohen-Macaulay module over a polynomial ring is free.

Corollary 4.8 is precisely the result that we use in [10] to identify the torsion submodule of the symmetric powers $\mathrm{Sym}_\ell(I)$ where I is a grade three Gorenstein ideal in an even-dimensional polynomial ring. A more elementary result (Proposition 3.6) may be used to identify the torsion submodule of $\mathrm{Sym}_\ell(I)$ when I is a grade three Gorenstein ideal in an odd-dimensional polynomial ring. We view Proposition 3.6 as a model for the main results in the present paper.

In Section 3, using a spectral sequence argument, we relate the cohomology of $\mathrm{Hom}(C_\bullet, N)$ to $\mathrm{Ext}^\bullet(M,N)$, where $C_\bullet$ is a complex with $\mathrm{H}_0(C_\bullet) = M$ and M and N are arbitrary modules. In spite of the a priori lack of hypotheses we obtain a significant, multi-faceted, result which is the basis for Section 4; and hence the rest of the paper.

In Section 5, we apply the local cohomology techniques of Section 4 to draw conclusions about geometric situations. Corollary 5.2 shows that if $\mathfrak{p}$ is the prime ideal which defines the monomial curve associated to a numerical semigroup H, then the maximal generator degree of the second symbolic power of $\mathfrak{p}$ satisfies

$$b_0(\mathfrak{p}^{(2)}) \leq \sup\{b_0(\mathfrak{p}) + \text{ the maximal generator of } H + \text{ the Frobenius number of } H, 2b_0(\mathfrak{p})\}.$$

Corollary 5.4 shows that if I is the ideal which defines a finite set of points in projective space, then

$$b_0(I^{(2)}) \leq b_0(I) + p(P/I) + 2,$$

where $p(P/I)$ is the postulation number of the homogeneous coordinate ring of the set of points. Corollary 5.7 is about hyperplane sections of subschemes of projective space. Let V be the subscheme of $\mathbb{P}^{d-1}_k$ defined by the homogeneous ideal I in $R = k[x_1, \dots, x_d]$ and H be a linear subspace of $\mathbb{P}^{d-1}_k$ defined by general linear forms in $k[x_1, \dots, x_d]$. We produce an upper bound for the maximal generator degree of the saturated ideal defining the subscheme $V \cap H$ of H, in terms of information that can be read from a single shift in the minimal homogeneous resolution of R/I.

Acknolwedgment. We are indebted to the referee of the first version of this paper. Owing to the referee's suggestions, the proof of Lemma 3.2 was simplified substantially and the assumptions of Theorem 4.3 were weakened.

2. Conventions, notation, and preliminary results

2.1. By a graded ring or module we mean a $\mathbb{Z}$-graded ring or module, unless otherwise specified. Notice that every ring and module is graded, namely trivially graded.

2.2. If M and N are graded modules over a graded ring R, then N is a homogeneous *subquotient* of M if N is isomorphic to a homogeneous submodule of a graded homomorphic image of M; that is, if there exists a graded R-module P with

$$\begin{tikzcd} & M \arrow[d, two heads] \\ N \arrow[r, hook] & P. \end{tikzcd}$$

Of course, N is also a subquotient of M if N is isomorphic to a graded homomorphic image of a homogeneous submodule of M. The property of being a subquotient is transitive in the sense that if M_1 is a subquotient of M_2 and M_2 is a subquotient of M_3, then M_1 is a subquotient of M_3.

2.3. Let M and N be graded modules over a graded ring R. By ${}^*\operatorname{Hom}_R(M,N)$ and ${}^*\operatorname{Ext}^i_R(M,N)$ we denoted the graded Hom and Ext modules. They coincide with the usual Hom and Ext modules if R is Noetherian and M is finitely generated or if R, M, and N are trivially graded.

2.4. Let M and N be graded modules over a graded ring R. In order to simplify our formulas we set $\operatorname{Ext}^i_R(M,N) = 0$ and ${}^*\operatorname{Ext}^i_R(M,N) = 0$ for $i < 0$. If R is a non-negatively graded Noetherian ring, R_0 is local, and $\mathfrak{m}$ is the maximal homogeneous ideal of R, we also set $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i < 0$.

2.5. We collect names for some of the invariants associated to a graded module. Let R be a graded ring and M be a graded R-module. Define

$$\begin{aligned} \operatorname{topdeg} M &= \sup\{j \mid M_j \neq 0\}, \\ \operatorname{indeg} M &= \inf\{j \mid M_j \neq 0\}, \\ b_0(M) &= \inf\left\{b \,\middle|\, R\left(\textstyle\bigoplus_{j \le b} M_j\right) = M\right\}. \end{aligned}$$

If R is a non-negatively graded Noetherian ring, R_0 is local, $\mathfrak{m}$ is the maximal homogeneous ideal of R, and M is finitely generated, then also define

$$\begin{aligned} a_i(M) &= \operatorname{topdeg} \mathrm{H}^i_{\mathfrak{m}}(M) \ \text{ and} \\ b_i(M) &= \operatorname{topdeg} \operatorname{Tor}^R_i(M, R/\mathfrak{m}). \end{aligned}$$

Observe that both definitions of the maximal generator degree $b_0(M)$ give the same value. The expressions "topdeg", "indeg", and "b_i" are read "top degree", "initial degree", and "maximal i-th shift in a minimal homogeneous free resolution", respectively. If M is the zero module, then

$$\operatorname{topdeg}(M) = b_0(M) = -\infty \quad \text{and} \quad \operatorname{indeg} M = \infty.$$

In general one has

$$a_i(M) < \infty \quad \text{and} \quad b_i(M) < \infty.$$

We often use the data of 2.6.

2.6. Let R be a non-negatively graded Noetherian ring with R_0 local, let $\mathfrak{m}$ be the maximal homogeneous ideal of R, and write $k = R/\mathfrak{m}$ for the residue field.

Lemma 2.7. *Use the data of 2.6 and let M be a (not necessarily finitely generated) graded R-module. Assume that M_j is a finitely generated R_0-module for every j. Then*

$$b_0(M) = b_0(k \otimes_R M)\,.$$

In particular, $b_0(M) = -\infty$ if and only if $M = \mathfrak{m}M$.

Proof. We show that $b_0(M) \le b_0(k \otimes_R M)$, for which it suffices to prove that if $k \otimes_R M$ can be generated by homogeneous elements of degrees at most b then so does M. Notice that $\mathfrak{m}_0$ is the maximal ideal of the local ring R_0. Write $R_+ = \oplus_{j>0} R_j$, so that $\mathfrak{m} = \mathfrak{m}_0 + R_+$.

Let N be the submodule of M generated by the homogeneous elements of degrees at most b and set $K = M/(N + \mathfrak{m}_0 M)$. Our assumption on the generation of $k \otimes_R M = M/(\mathfrak{m}_0 M + R_+ M)$ implies that $K/R_+K = 0$ or, equivalently, $K = R_+K$. Since $K_j = 0$ for every $j \le b$, it follows that $K = 0$. Thus $M/N = 0$ by Nakayama's Lemma because every graded component $(M/N)_j$ is a finitely generated module over the local ring $(R_0, \mathfrak{m}_0)$. □

2.8. Take $R, R_0, \mathfrak{m}$ as described in 2.6. For a minimal homogeneous generating set $y_1, \dots, y_n$ of $\mathfrak{m}$, with $\deg y_1 \ge \deg y_2 \ge \cdots \ge \deg y_n$, let $\Theta_t = \sum_{j=1}^{t} \deg y_j$ if $t \le n$ and $\Theta_t = -\infty$ otherwise.

2.9. Take $R, R_0, \mathfrak{m}$ as described in 2.6. Assume further that R_0 is a factor ring of a local Gorenstein ring T. Let $S = T[x_1, \dots, x_n]$ be a graded polynomial ring which maps homogeneously onto R. If g is the codimension of R as an S-module, then the graded canonical module of R is

$$\omega_R = \operatorname{Ext}_S^g(R, S)(-\textstyle\sum \deg x_i).$$

2.10. Take $R, R_0, \mathfrak{m}$ as described in 2.6 with $\dim R = d$. Recall the numerical functions of 2.5. The a-invariant of R is

$$a(R) = a_d(R).$$

Furthermore, if ω_R is the graded canonical module of R (see 2.9), then

$$a(R) = -\operatorname{indeg} \omega_R.$$

2.11. The graded ring $R = \bigoplus_{i \ge 0} R_i$ is a *standard graded R_0-algebra* if R is generated as an R_0-algebra by R_1 and R_1 is finitely generated as an R_0-module.

2.12. Let R be a standard graded polynomial ring over a field k, $\mathfrak{m}$ be the maximal homogeneous ideal of R, and M be a finitely generated graded R-module. Recall the numerical functions of 2.5. The Castelnuovo-Mumford *regularity* of M is

$$\operatorname{reg} M = \sup\{a_i(M) + i\} = \sup\{b_i(M) - i\}.$$

2.13. Let q be an integer and R be a ring. A complex of finitely generated free R-modules

$$\dots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

is called *q-linear* if $C_i = R(-q-i)^{\beta_i}$ for some β_i, for all i with $0 \le i$.

3. First bounds on local cohomology modules

The main result of this paper is Theorem 4.3. Lemma 3.2 is the first step in the proof of Theorem 4.3. In Lemma 3.2 we relate the cohomology of $\mathrm{Hom}(C_\bullet, N)$ to $\mathrm{Ext}^\bullet(M,N)$, where $C_\bullet$ is a complex with $\mathrm{H}_0(C_\bullet) = M$ and N is an arbitrary module.

Setup 3.1. Let R be a graded ring, M and N be graded R-modules, and

$$C_\bullet : \quad \cdots \longrightarrow C_1 \longrightarrow C_0 \to 0$$

be a homogeneous complex of graded R-modules with $\mathrm{H}_0(C_\bullet) = M$. Fix an integer i. We consider two hypotheses which can be imposed on the above data:

- The data satisfies U_i if ${}^*\mathrm{Ext}^{i-j}(\mathrm{H}_j(C_\bullet), N) = 0$ for all integers j with $1 \le j \le i$.
- The data satisfies V_i if ${}^*\mathrm{Ext}^{i-j}(C_j, N) = 0$ for all integers j with $0 \le j \le i-1$.

Lemma 3.2. *In the setup of* 3.1*, the following statements hold.*

(a) *If the hypotheses U_{i-1}, U_i, V_{i-1}, and V_i all are in effect, then there is a natural homogeneous isomorphism* $\mathrm{H}^i({}^*\mathrm{Hom}(C_\bullet, N)) \simeq {}^*\mathrm{Ext}^i(M,N)$.

(b) *If the hypotheses U_i, V_{i-1}, and V_i are in effect, then there is a natural homogeneous surjection*

$${}^*\mathrm{Ext}^i(M,N) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{H}^i({}^*\mathrm{Hom}(C_\bullet, N)) \ .$$

(c) *If the hypotheses U_{i-1}, U_i, and V_i are in effect, then there is a natural homogeneous surjection*

$$\mathrm{H}^i({}^*\mathrm{Hom}(C_\bullet, N)) \longrightarrow\!\!\!\!\!\rightarrow {}^*\mathrm{Ext}^i(M,N) \ .$$

(d) *If the hypotheses U_{i-1}, U_i, and V_{i-1} are in effect, then there is a natural homogeneous injection*

$$\mathrm{H}^i({}^*\mathrm{Hom}(C_\bullet, N)) \hookrightarrow {}^*\mathrm{Ext}^i(M,N) \ .$$

(e) *If the hypotheses U_{i-1}, V_{i-1}, and V_i are in effect, then there is a natural homogeneous injection*

$${}^*\mathrm{Ext}^i(M,N) \hookrightarrow \mathrm{H}^i({}^*\mathrm{Hom}(C_\bullet, N)) \ .$$

(f) *If the hypotheses U_i and V_{i-1} are in effect, then* $\mathrm{H}^i({}^*\mathrm{Hom}(C_\bullet, N))$ *is a natural homogeneous subquotient of* ${}^*\mathrm{Ext}^i(M,N)$.

(g) *If the hypotheses U_{i-1} and V_i are in effect, then* ${}^*\mathrm{Ext}^i(M,N)$ *is a natural homogeneous subquotient of* $\mathrm{H}^i({}^*\mathrm{Hom}(C_\bullet, N))$.

Proof. For $i \le 0$ the assertions are obvious, hence we may assume that i is a positive integer. Let $I^\bullet$ be a homogenous resolution of N by graded injective modules. Consider the double complex $D^{pq} := {}^*\mathrm{Hom}(C_q, I^p)$ and write $T^\bullet$ for its total complex. The horizontal and vertical filtration of the double complex yield first quadrant spectral sequences whose E_2 terms are

$${}^{\prime}E_2^{p,q} \simeq {}^*\mathrm{Ext}^p(\mathrm{H}_q(C_\bullet), N) \qquad \text{and} \qquad {}^{\prime\prime}E_2^{p,q} \simeq \mathrm{H}^q({}^*\mathrm{Ext}^p(C_\bullet, N)),$$

respectively.

The infinity term ${}'E_\infty^{p,q}$ is a homogeneous subquotient of ${}'E_2^{p,q}$. It is a homogeneous epimorphic image if $q=0$,

$$
{}'E_2^{p,0} \twoheadrightarrow {}'E_\infty^{p,0}\,. \tag{3.2.1}
$$

On the other hand there is a homogeneous embedding

$$
{}'E_\infty^{p,0} \hookrightarrow \mathrm{H}^p(T^\bullet)\,, \tag{3.2.2}
$$

whose cokernel has a filtration with factors ${}'E_\infty^{p-q,q}$ for $1 \le q \le p$.

The condition U_{p-1} is equivalent to ${}'E_2^{p-1-j,j} = 0$ for $1 \le j \le p-1$, which implies that ${}'E_{j+1}^{p-1-j,j} = 0$ for $1 \le j \le p-1$ and hence ${}'E_{j+1}^{p-1-j,j} = 0$ for $1 \le j$. It follows that on the $(j+1)$-st page of the spectral sequence the natural map ${}'E_{j+1}^{p-1-j,j} \to {}'E_{j+1}^{p,0}$ is the zero map for $1 \le j$. Hence the epimorphism of (3.2.1) is an isomorphism.

The condition U_p means that ${}'E_2^{p-q,q} = 0$ for $1 \le q \le p$, which gives ${}'E_\infty^{p-q,q} = 0$ for $1 \le q \le p$. Hence the inclusion of (3.2.2) is an isomorphism.

Likewise, ${}''E_\infty^{p,q}$ is a homogeneous subquotient of ${}''E_2^{p,q}$ and a homogeneous epimorphic image if $p=0$. Hence

$$
{}''E_2^{0,q} \twoheadrightarrow {}''E_\infty^{0,q}\,. \tag{3.2.3}
$$

Also there is a homogeneous embedding

$$
{}''E_\infty^{0,q} \hookrightarrow \mathrm{H}^q(T^\bullet)\,, \tag{3.2.4}
$$

and the cokernel of this embedding has a filtration whose factors are ${}''E_\infty^{p,q-p}$ for $1 \le p \le q$.

Now V_{q-1} implies ${}''E_2^{j-1,q-j} = 0$ for $2 \le j \le q$, which gives ${}''E_j^{j-1,q-j} = 0$ for $2 \le j$. Therefore the natural map ${}''E_j^{j-1,q-j} \to {}''E_j^{0,q}$ is the zero map for $2 \le j$, and hence the epimorphism of (3.2.3) is an isomorphism.

Finally, V_q gives ${}''E_2^{p,q-p} = 0$ for $1 \le p \le q$. Therefore ${}''E_\infty^{p,q-p} = 0$, which means that the embedding of (3.2.4) is an isomorphism.

The lemma now follows from the homogenous epimorphisms and embeddings (3.2.1), (3.2.2), (3.2.4), (3.2.3) for $p=q=i$ and the various conditions for when they are isomorphisms. $\square$

Remark 3.3. The above proof shows that in Lemma 3.2 the condition V_i can be replaced by the weaker assumption that $\mathrm{H}^j(\mathrm{Ext}^{i-j}(C_\bullet, N)) = 0$ for all j with $0 \le j \le i-1$, and likewise for V_{i-1}.

Observation 3.5 shows that the hypotheses of Lemma 3.2 are implied by some natural assumptions on a complex.

Setup 3.4. Let R be a non-negatively graded Cohen-Macaulay ring with R_0 a factor ring of a local Gorenstein ring. Let d be the dimension of R and assume that $1 \le d$. Let $\mathfrak{m}$ be the maximal homogeneous ideal of R, $k = R/\mathfrak{m}$ its residue field, and $\omega = \omega_R$ its graded canonical module; see 2.9. Let

$$
(C_\bullet, \partial_\bullet): \quad \dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0
$$

be a homogeneous complex of finitely generated graded R-modules. Write $M = \mathrm{H}_0(C_\bullet)$ and $\mathrm{H}_\bullet = \mathrm{H}_\bullet(C_\bullet)$. Let $(-)^\vee$ denote the functor $\mathrm{Hom}_R(-,\omega)$.

Observation 3.5. *Adopt the setup of 3.4 and use the hypotheses U_i, V_i of 3.1 with $N = \omega$.*

(a) *Fix an integer i. If $\dim \mathrm{H}_j \le d - i + j$ for every j with $1 \le j \le i-1$, then the data satisfies condition U_ℓ for all ℓ with $\ell \le i-1$.*

(b) *Fix integers r and s. If $\min\{d, d-r+j+1\} \le \operatorname{depth} C_j$ for every j with $0 \le j \le s-1$, then the data satisfies condition V_ℓ for all ℓ with $r \le \ell \le s$.*

Proof. We may assume that the local ring R_0 is complete. In the setting of (a), we have $\mathrm{H}^k_{\mathfrak{m}}(H_j) = 0$ for every k with $d-i+j+1 \le k$; that is, $d-(i-j-1) \le k$. By graded duality, this gives $\mathrm{Ext}^h_R(H_j,\omega) = 0$ for every h with $h \le i-j-1$. In (b), we have $\mathrm{H}^k_{\mathfrak{m}}(C_j) = 0$ whenever

$$k \le \min\{d, d-r+j+1\} - 1 = d - \max\{1, r-j\},$$

which gives $\mathrm{Ext}^h_R(C_j,\omega) = 0$ for every h with $\max\{1, r-j\} \le h$. □

Our first application of Lemma 3.2 is the next result, Proposition 3.6, which relates local cohomology modules along complexes and yields bounds on the top degree of such modules. Proposition 3.6 is essentially known; the idea goes back to Gruson, Lazarsfeld, and Peskine [7, 1.6], at least. Proposition 3.6 has found applications in [10], where we determine the implicit equations defining Rees rings of linearly presented grade three Gorenstein ideals.

Recall the numerical functions of 2.5 and 2.10.

Proposition 3.6. *Let R be a non-negatively graded Noetherian algebra over a local ring R_0 with $\dim R = d$, $\mathfrak{m}$ be the maximal homogeneous ideal of R, M be a graded R-module, and*

$$C_\bullet : \qquad \dots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

be a homogeneous complex of finitely generated graded R-modules with $\mathrm{H}_0(C_\bullet) = M$. Fix an integer i. Assume that

(1) $\dim \mathrm{H}_j(C_\bullet) \le j + i$ *for all j with $1 \le j \le d-i-1$, and*

(2) $j + i + 1 \le \operatorname{depth} C_j$ *for all j with $0 \le j \le d-i-1$.*

Then

(a) $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is a graded subquotient of* $\mathrm{H}^d_{\mathfrak{m}}(C_{d-i})$, *and*

(b) $a_i(M) \le b_0(C_{d-i}) + a(R)$.

Remark 3.7. If

(3.7.1) $\quad \mathrm{H}_j(C_\bullet)_{\mathfrak{p}} = 0$ for all j and $\mathfrak{p}$ with $1 \le j \le d-i-1$, $\mathfrak{p} \in \mathrm{Spec}(R)$, and $i+2 \le \dim R/\mathfrak{p}$,

then hypothesis (1) is satisfied. Typically, one applies Proposition 3.6 when the modules C_j are maximal Cohen-Macaulay modules (for example, free modules over a Cohen-Macaulay ring), because, in this case, hypothesis (2) about $\operatorname{depth} C_j$ is automatically satisfied.

Proof. (a) Completing R_0 does not change the local cohomology modules in question; hence we may assume that R_0 is complete. We use the notation of 2.9; most notably, $S = T[x_1,\dots,x_n]$ with $x_1,\dots,x_n$ homogeneous variables of positive degree over a local Gorenstein ring T, which we may assume to be complete, and R is obtained from S by factoring out a homogeneous ideal J of height g. The ideal J contains a homogenous regular sequence $\underline{\alpha}$ of length g. (Indeed, after factoring out a maximal T-regular sequence in J_0, we may assume that $J_0 \subset \mathfrak{p}$ for some minimal prime ideal $\mathfrak{p}$ of T. Hence $J_\mathfrak{p} \neq S_\mathfrak{p}$ and $n = \dim S_\mathfrak{p} \geq \operatorname{ht} J_\mathfrak{p} \geq \operatorname{ht} J = g$. It follows that $\operatorname{ht}(x_1,\dots,x_n)J = \min\{n,g\} = g$. Since the ideal $(x_1,\dots,x_n)J$ has grade g and is generated by forms of positive degree, it contains a homogenous regular sequence $\underline{\alpha}$ of length g, hence so does J.) Now $S/(\underline{\alpha})$ and R have the same dimension d, and we may safely replace the latter ring by the former in order to assume that R is Cohen-Macaulay with R_0 complete.

When the hypotheses of Proposition 3.6 are inserted into Observation 3.5, one obtains, in particular, that the conditions U_{d-i-1} and V_{d-i} hold; so Lemma 3.2.g guarantees that $\operatorname{Ext}^{d-i}(M,\omega)$ is a graded subquotient of $\mathrm{H}^{d-i}(\operatorname{Hom}(C_\bullet,\omega))$, which is a graded subquotient of $\operatorname{Hom}(C_{d-i},\omega)$. Graded duality yields $\mathrm{H}^i_\mathfrak{m}(M)$ is a graded subquotient of $\mathrm{H}^d_\mathfrak{m}(C_{d-i})$.

(b) Apply (a) to see that

$$a_i(M) \leq a_d(C_{d-i}).$$

Let F be a finitely generated graded free R-module which maps surjectively onto C_{d-i} so that $b_0(F) = b_0(C_{d-i})$. The long exact sequence of local cohomology gives a surjection

$$\mathrm{H}^d_\mathfrak{m}(F) \twoheadrightarrow \mathrm{H}^d_\mathfrak{m}(C_{d-i}) \ ,$$

which shows that

$$a_d(C_{d-i}) \leq a_d(F) = b_0(F) + a(R) = b_0(C_{d-i}) + a(R).$$

□

Corollary 3.8. *Adopt the hypotheses of Proposition* 3.6*, with $i = 0$. Then*

$$[\mathrm{H}^0_\mathfrak{m}(M)]_\ell = 0 \quad \textit{for all } \ell \textit{ with} \quad b_0(C_d) + a(R) < \ell.$$

Reminder. Keep in mind that the hypotheses of Proposition 3.6 are satisfied if the conditions of Remark 3.7 are in effect.

Corollary 3.9. *Let $R = k[x_1,\dots,x_d]$ be a standard graded polynomial ring over a field, with maximal homogeneous ideal $\mathfrak{m}$,*

$$C_\bullet : \quad \dots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

be a homogeneous complex of finitely generated graded free R-modules, and $M = \mathrm{H}_0(C_\bullet)$. Assume that $\dim \mathrm{H}_j(C_\bullet) \leq j$ *for all* $1 \leq j \leq d-1$ *and that the subcomplex*

$$C_d \longrightarrow \dots \longrightarrow C_0 \longrightarrow 0$$

of $C_\bullet$ is q-linear for some integer q. Then $\mathrm{H}^0_\mathfrak{m}(M)$ is concentrated in degree q; that is, $[\mathrm{H}^0_\mathfrak{m}(M)]_\ell = 0$ for all ℓ with $\ell \neq q$.

Proof. Apply Corollary 3.8. We see that $[\mathrm{H}^0_{\mathfrak{m}}(M)]_\ell = 0$ for ℓ with $b_0(C_d)+a(R)<\ell$. But

$$b_0(C_d)+a(R) \le (q+d)-d = q;$$

so $[\mathrm{H}^0_{\mathfrak{m}}(M)]_\ell = 0$ for $q<\ell$. On the other hand, $\mathrm{H}^0_{\mathfrak{m}}(M)$ is a graded submodule of M and $[M]_\ell = 0$ for all ℓ with $\ell < q$. □

4. Bounds on generator degrees of local cohomology modules

In this section, we use Lemma 3.2 and Proposition 4.1 below to prove our main result, Theorem 4.3. The rest of the section is devoted to applications of this result.

Proposition 4.1. *Adopt the setup of* 3.4. *Fix integers i and t with* $1 \le t$. *Assume*

(1) $\mathrm{H}^\ell_{\mathfrak{m}}(M) = 0$ *for all* ℓ *with* $d-i+1 \le \ell \le d-i+t-1$,
(2) $\dim \mathrm{H}_j(C_\bullet) \le d-i+j$ *for all* j *with* $1 \le j \le i-1$,
(3) $\min\{d, d-i+t+j+1\} \le \operatorname{depth} C_j$ *for all* j *with* $0 \le j \le i-1$, *and*
(4) $i-j+1 \le \operatorname{depth} C_j^\vee$ *for all* j *with* $i-t+1 \le j \le i-1$.

Then there is a natural homogeneous injection

$$\operatorname{socle}(\operatorname{Ext}^i_R(M,\omega_R)) \hookrightarrow \operatorname{Ext}^t_R(k, \operatorname{im}(\partial^\vee_{i-t+1}))\ .$$

Moreover, $t \le \operatorname{depth}(\operatorname{im}(\partial^\vee_{i-t+1}))$, *and equality holds if* $\operatorname{depth}\operatorname{Ext}^i_R(M,\omega_R) = 0$.

Remark 4.2. Hypothesis (1) is always satisfied when $t=1$, or $\operatorname{depth} M/\mathrm{H}^0_{\mathfrak{m}}(M) \ge d-i+t$, or, in particular, $M/\mathrm{H}^0_{\mathfrak{m}}(M) = 0$. A slightly modified proof shows that condition (1) can be replaced by the weaker assumption

$$\operatorname{depth}\operatorname{Ext}^\ell_R(M,\omega) \ge i+2-\ell \quad \text{for all } \ell \text{ with } i-t+1 \le \ell \le i-1\,.$$

Hypothesis (2) is satisfied for $i \le d$ if

$$\mathrm{H}_j(C_\bullet)_{\mathfrak{p}} = 0 \quad \text{for all } j \text{ and } \mathfrak{p} \text{ with } 1 \le j \le i-1 \text{ and } 2 \le \dim R/\mathfrak{p}\,.$$

As observed in Remark 3.7, one typically applies Proposition 4.1 when the modules C_j are maximal Cohen-Macaulay modules, because, in this case, hypotheses (3) and (4) are automatically satisfied for $t \le d$.

Proof. Hypotheses (2) and (3) and Observation 3.5 imply that

(4.2.1) the condition U_h of 3.1 holds when $h \le i-1$

(4.2.2) the condition V_h of 3.1 holds when $i-t \le h \le i$.

Use (4.2.1), (4.2.2), and Lemma 3.2.e in order to conclude that there is a natural homogeneous injection

$$\operatorname{Ext}^i(M,\omega) \hookrightarrow \mathrm{H}^i(C^\vee_\bullet)\ . \tag{4.2.3}$$

Combine (4.2.3) and the natural inclusion $\mathrm{H}^i(C_\bullet^\vee) \hookrightarrow \operatorname{coker}(\partial_i^\vee)$ to see that

$$\operatorname{socle}(\operatorname{Ext}_R^i(M,\omega)) \hookrightarrow \operatorname{socle}(\operatorname{coker}(\partial_i^\vee)) .$$

Assumption (1) implies that

$$\operatorname{Ext}_R^h(M,\omega) = 0 \text{ for } i-t+1 \le h \le i-1.$$

Use (4.2.1), (4.2.2), and Lemma 3.2.a to conclude that $\mathrm{H}^h(C_\bullet^\vee) \simeq \operatorname{Ext}^h(M,\omega)$ for $i-t+1 \le h \le i-1$; and therefore

$$\mathrm{H}^h(C_\bullet^\vee) = 0 \text{ for } i-t+1 \le h \le i-1.$$

It follows that the complex

$$C_{i-t}^\vee \xrightarrow{\partial_{i-t+1}^\vee} C_{i-t+1}^\vee \xrightarrow{\partial_{i-t+2}^\vee} \cdots \xrightarrow{\partial_{i-1}^\vee} C_{i-1}^\vee \xrightarrow{\partial_i^\vee} C_i^\vee \to \operatorname{coker}(\partial_i^\vee) \to 0 \tag{4.2.4}$$

is exact. Notice that $1 \le \operatorname{depth} C_i^\vee$ since $1 \le \operatorname{depth} \omega_R$. Hence the inequality in Assumption (4) holds for j with $i-t+1 \le j \le i$. It follows that $\operatorname{Ext}_R^h(k, C_{i-h}^\vee) = 0$ for $0 \le h \le t-1$. Long exact sequences associated to $\operatorname{Ext}_R^\bullet(k,-)$ then show that

$$\operatorname{socle}(\operatorname{coker}(\partial_i^\vee)) \simeq \operatorname{Hom}_R(k, \operatorname{coker}(\partial_i^\vee)) \hookrightarrow \operatorname{Ext}_R^t(k, \operatorname{im}(\partial_{i-t+1}^\vee)).$$

Moreover, $t \le \operatorname{depth}(\operatorname{im}(\partial_{i-t+1}^\vee))$ by the exact complex (4.2.4). Equality holds unless

$$\operatorname{Ext}_R^t(k, \operatorname{im}(\partial_{i-t+1}^\vee)) = 0,$$

which means $\operatorname{socle}(\operatorname{Ext}_R^i(M,\omega)) = 0$, hence $0 < \operatorname{depth} \operatorname{Ext}_R^i(M,\omega)$. □

Theorem 4.3. *Adopt the setup of* 3.4 *and the hypotheses of* 4.1. *Let $a(R)$ denote the a-invariant of R and let Θ_t be the invariant of R from* 2.8. *If the graded components of $\mathrm{H}_{\mathfrak{m}}^{d-i}(M)$ are finitely generated as an R_0-module, then*

$$b_0(\mathrm{H}_{\mathfrak{m}}^{d-i}(M)) \le b_0(C_{i-t}) + \Theta_t + a(R).$$

Reminder. Keep in mind that the hypotheses of Proposition 4.1 are satisfied if the conditions of Remark 4.2 are in effect.

Remark 4.4. The assumption in Theorem 4.3 that the graded components of $\mathrm{H}_{\mathfrak{m}}^{d-i}(M)$ are finitely generated as an R_0-module is satisfied if R_0 is Artinian or if $\mathrm{H}_{\mathfrak{m}}^{d-i}(M)$ is finitely generated as an R-module. The latter condition holds if $i = d$ because $\mathrm{H}_{\mathfrak{m}}^0(M)$ is a graded submodule of M. From graded duality it follows that the same condition also holds if $\dim R_{\mathfrak{p}} - \operatorname{depth} M_{\mathfrak{p}} < i$ for all $\mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$. The last assumption is satisfied, for instance, if R_0 is universally catenary, $\operatorname{Supp}_R(M)$ is equidimensional of dimension $\neq d-i$, and M is Cohen-Macaulay locally on the punctured spectrum of R.

Proof. We may assume that R_0 is complete. Applying graded duality and Hom-tensor adjointness twice one obtains isomorphisms of graded k-vector spaces

$$\begin{array}{rcl} \operatorname{Hom}_R(k,\operatorname{Ext}^i_R(M,\omega)) & \simeq & {}^*\operatorname{Hom}_{R_0}(k\otimes_R \mathrm{H}^{d-i}_{\mathfrak{m}}(M),E_{R_0}(k)) \\ & \simeq & {}^*\operatorname{Hom}_R(\mathrm{H}^{d-i}_{\mathfrak{m}}(M),\operatorname{Hom}_{R_0}(k,E_{R_0}(k))) \\ & \simeq & {}^*\operatorname{Hom}_R(\mathrm{H}^{d-i}_{\mathfrak{m}}(M),k). \end{array}$$

It follows that

$$(4.4.1) \qquad {}^*\operatorname{Hom}_k(k\otimes_R \mathrm{H}^{d-i}_{\mathfrak{m}}(M),k)\simeq \operatorname{socle}(\operatorname{Ext}^i_R(M,\omega)).$$

Since the graded components of $\mathrm{H}^{d-i}_{\mathfrak{m}}(M)$ are finitely generated R_0-modules Lemma 2.7 gives

$$b_0(\mathrm{H}^{d-i}_{\mathfrak{m}}(M)) = b_0(k\otimes_R \mathrm{H}^{d-i}_{\mathfrak{m}}(M))\,.$$

Now from (4.4.1) one attains

$$b_0(\mathrm{H}^{d-i}_{\mathfrak{m}}(M)) = -\operatorname{indeg}(\operatorname{socle}(\operatorname{Ext}^i_R(M,\omega))).$$

Thus, it remains to prove that

$$\operatorname{indeg}(\operatorname{socle}(\operatorname{Ext}^i_R(M,\omega)) \geq -b_0(C_{i-t}) - \Theta_t - a(R).$$

From Proposition 4.1 we have a homogeneous embedding

$$\operatorname{socle}(\operatorname{Ext}^i_R(M,\omega)) \hookrightarrow \operatorname{Ext}^t_R(k,N),$$

where $N = \operatorname{im}(\partial^\vee_{i-t+1})$. By the same proposition, $\operatorname{depth} N \geq t$. Hence conditions U_{t-1} and U_t of 3.1 are both satisfied for $C_\bullet := K_\bullet$, the Koszul complex of a minimal homogenous generating set of $\mathfrak{m}$ as in (2.8). The conditions V_{t-1} and V_t are trivially satisfied as $K_\bullet$ is a complex of free modules. Therefore Lemma 3.2.a gives the well-known identification

$$\operatorname{Ext}^t_R(k,N) \simeq \mathrm{H}^t(\operatorname{Hom}_R(K_\bullet,N))\,.$$

The latter homology module is a homogenous subquotient of $\operatorname{Hom}_R(K_t,N)$, which in turn is an epimorphic image of $\operatorname{Hom}_R(K_t,C^\vee_{i-t})$. Let F be a finitely generated graded free R-module, with $b_0(F) = b_0(C_{i-t})$, that maps homogeneously onto C_{i-t}. We obtain a homogeneous inclusion $C^\vee_{i-t} \hookrightarrow F^\vee$. Thus:

$$\begin{tikzcd} \operatorname{Hom}_R(K_t,C^\vee_{i-t}) \arrow[r, hook] \arrow[d, two heads] & \operatorname{Hom}_R(K_t,F^\vee) \\ \operatorname{Hom}_R(K_t,N). & \end{tikzcd}$$

We conclude that $\operatorname{socle}(\operatorname{Ext}^i_R(M,\omega))$ is a homogeneous subquotient of $\operatorname{Hom}_R(K_t,F^\vee)$. Therefore,

$$\begin{array}{rcl} \operatorname{indeg}(\operatorname{socle}(\operatorname{Ext}^i_R(M,\omega))) & \geq & \operatorname{indeg}\operatorname{Hom}_R(K_t,F^\vee) \\ & = & \operatorname{indeg} F^\vee - \Theta_t \\ & = & -b_0(F) + \operatorname{indeg}\omega - \Theta_t \\ & = & -b_0(C_{i-t}) - a(R) - \Theta_t. \end{array}$$

□

Corollary 4.5 is a self-contained reformulation of Theorem 4.3. The purpose of this reformulation is to obtain one simultaneous bound for $\operatorname{topdeg} \mathrm{H}^r_{\mathfrak{m}}(M)$ (which is the subject of Proposition 3.6) and $b_0(\mathrm{H}^s_{\mathfrak{m}}(M))$ (which is the subject of Theorem 4.3) for appropriately related r and s. We resume this theme in Corollary 4.6.

Corollary 4.5. *Adopt the setup of* 3.4. *Fix integers i and t with $1 \le t \le \mu(\mathfrak{m})$. Let $a(R)$ denote the a-invariant of R and let Θ_t be the invariant of R from* 2.8. *Assume*

(1) $\mathrm{H}^\ell_{\mathfrak{m}}(M) = 0$ *for all ℓ with $i-t+1 \le \ell \le i-1$,*
(2) $\dim \mathrm{H}_j(C_\bullet) \le j+i-t$ *for all j with $1 \le j \le d-1-i+t$, and*
(3) *C_j is a maximal Cohen-Macaulay module for all j with $0 \le j \le d-1-i+t$.*

If the graded components of $\mathrm{H}^{i-t}_{\mathfrak{m}}(M)$ are finitely generated as an R_0-module, then

$$\sup\{b_0(\mathrm{H}^{i-t}_{\mathfrak{m}}(M)) - \Theta_t, a_i(M)\} \le b_0(C_{d-i}) + a(R).$$

Proof. Apply Theorem 4.3 (with i replaced by $d-i+t$) and Proposition 3.6.b. □

Various forms of partial regularity appear in the literature; see, for example, [8, 3, 11]. Recall from 2.12 that

$$\operatorname{reg} M = \sup\{a_i(M) + i \mid 0 \le i\}.$$

The number $\operatorname{reg}(M/\mathrm{H}^0_{\mathfrak{m}}(M))$ which appears on the left side of (4.6.1) is equal to the *partial regularity*

$$\sup\{a_i(M) + i \mid 1 \le i\} \tag{4.5.1}$$

of M. We obtain the right side of (4.6.1) as an upper bound for the partial regularity (4.5.1) of M. Then we show that the maximal generator degree of the submodule $\mathrm{H}^0_{\mathfrak{m}}(M)$ of M that is ignored in the calculation of (4.5.1) satisfies the same bound.

We also offer the following interpretation of the right side of (4.6.1). If $C_\bullet$ had been a minimal homogeneous resolution of M by free R-modules, then $b_0(C_i)$ would equal $b_i(M)$. Of course,

$$\sup\{b_i(M) - i \mid 0 \le i \le d-t\}$$

is another partial regularity of M.

Corollary 4.6. *Adopt the setup of* 3.4 *and assume in addition that R is a standard graded ring with R_0 a field. Fix an integer t with $1 \le t \le \mu(\mathfrak{m})$. Assume*

(1) $t \le \operatorname{depth} M/\mathrm{H}^0_{\mathfrak{m}}(M)$,
(2) $\dim \mathrm{H}_j(C_\bullet) \le j$ *for all j with $1 \le j \le d-1$, and*
(3) *C_j is a maximal Cohen Macaulay module for all j with $0 \le j \le d-1$.*

Then

$$\sup\{b_0(\mathrm{H}^0_{\mathfrak{m}}(M)), \operatorname{reg}(M/\mathrm{H}^0_{\mathfrak{m}}(M))\} \le \sup\{b_0(C_i) - i \mid 0 \le i \le d-t\} + \operatorname{reg}(R). \tag{4.6.1}$$

Proof. The assumption on the depth of $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ implies that

$$\mathrm{H}^i_{\mathfrak{m}}\left(M/\mathrm{H}^0_{\mathfrak{m}}(M)\right) = 0 \quad \text{for } 0 \le i \le t-1.$$

Therefore

$$\begin{aligned} \operatorname{reg}(M/\mathrm{H}^0_{\mathfrak{m}}(M)) &= \sup\{a_i(M) + i \mid t \le i \le d\} \\ &\le \sup\{b_0(C_{d-i}) + a(R) + i \mid t \le i \le d\} && \text{by Proposition 3.6.b} \\ &= \sup\{b_0(C_i) + a(R) + d - i \mid 0 \le i \le d-t\}. \end{aligned}$$

Apply Theorem 4.3 with $i = d$ to obtain

$$b_0(\mathrm{H}^0_{\mathfrak{m}}(M)) \le b_0(C_{d-t}) + t + a(R).$$

The invariant Θ_t from 4.3 and 2.8 is t because R is standard graded, and $a(R) + d = \operatorname{reg}(R)$ because R is standard graded and Cohen-Macaulay. □

Corollary 4.7 is well known and easy to prove if R is a standard graded polynomial ring over a field because, in this case, the sum $\Theta_d + a(R)$ is zero, the maximal Cohen-Macaulay module $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ is free, $\mathrm{H}^0_{\mathfrak{m}}(M)$ is a direct summand of M, and it is true and clear that

$$b_0(\mathrm{H}^0_{\mathfrak{m}}(M)) \le b_0(M).$$

On the other hand, the result in the stated generality is new and intriguing.

Corollary 4.7. *Let R be a non-negatively graded Cohen-Macaulay ring with R_0 a local ring. Denote the maximal homogeneous ideal of R by $\mathfrak{m}$ and $\dim R$ by d. Let M be a finitely generated graded R-module and assume that $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ is a maximal Cohen-Macaulay R-module. Then*

$$b_0(\mathrm{H}^0_{\mathfrak{m}}(M)) \le b_0(M) + \Theta_d + a(R),$$

where Θ_d is defined in 2.8 *and $a(R)$ is the a-invariant of R.*

Proof. Again we may assume that $1 \le d$. Apply Theorem 4.3 with $i = t = d$ and $C_\bullet$ a free resolution of M. □

Corollary 4.8 is the numerical consequence of Theorem 4.3 that we apply most often.

Corollary 4.8. *Adopt the setup of* 3.4 *and assume in addition that $R = k[x_1, \dots, x_d]$ be a standard graded polynomial ring over a field. Assume that $\dim \mathrm{H}_j(C_\bullet) \le j$ whenever $1 \le j \le d-1$ and that $\min\{d, j+2\} \le \operatorname{depth} C_j$ whenever $0 \le j \le d-1$. Then*

$$b_0(\mathrm{H}^0_{\mathfrak{m}}(M)) \le b_0(C_{d-1}) - d + 1.$$

Proof. Apply Theorem 4.3 with $i = d$ and $t = 1$. □

The next result is analogous to Corollary 3.9. The hypothesis is weaker than the hypothesis in Corollary 3.9 because we do not require that the complex $C_\bullet$ be linear quite as far in the present result. Alas, the conclusion is also weaker. We conclude that the generators of $\mathrm{H}^0_{\mathfrak{m}}(M)$ are concentrated in one degree rather than learning that all of $\mathrm{H}^0_{\mathfrak{m}}(M)$ is concentrated in one degree.

Corollary 4.9. *Let $R = k[x_1, \dots, x_d]$ be a standard graded polynomial ring over a field, with maximal homogeneous ideal $\mathfrak{m}$,*

$$C_\bullet : \quad \dots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

be a homogeneous complex of finitely generated graded free R-modules, and $M = \mathrm{H}_0(C_\bullet)$. Assume that $\dim \mathrm{H}_j(C_\bullet) \le j$ for all $1 \le j \le d-1$ and that the subcomplex

$$C_{d-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow 0$$

of $C_\bullet$ is q-linear for some integer q. Then every minimal homogeneous generator of $\mathrm{H}^0_{\mathfrak{m}}(M)$ has degree q.

Proof. We may assume that $1 \le d$ since otherwise $\mathrm{H}^0_{\mathfrak{m}}(M) = M$. Apply Corollary 4.8 to conclude

$$b_0(\mathrm{H}^0_{\mathfrak{m}}(M)) \le b_0(C_{d-1}) - d + 1 = q.$$

On the other hand, $\mathrm{H}^0_{\mathfrak{m}}(M)$ is a submodule of M and every minimal homogeneous generator of M has degree q. □

5. Geometric Applications.

We apply the local cohomology techniques of Section 4 to draw conclusions about the generator degrees of the second symbolic power of the prime ideal which defines a monomial curve in affine space; of the second symbolic power of the ideal which defines a finite set of points in projective space; and of the saturated ideal defining the intersection of a projective scheme with a general linear subspace.

Recall that if I is an ideal in a Noetherian ring R, then the t-th symbolic power of I is $I^t R_W \cap R$, where W is the complement of the union of the associated primes of I and R_W is the localization of R at the multiplicative system W; see [9]. The first two applications in this section make use of local cohomology by way of the following lemma.

Lemma 5.1. *Let P be a non-negatively graded Noetherian ring with P_0 an Artinian local ring. Let I be a homogeneous ideal in P with P/I a Cohen-Macaulay ring of dimension one. Write $R = P/I$ and denote the maximal homogeneous ideal of R by $\mathfrak{m}$. Then*

$$b_0(I^{(2)}) \le \sup\{b_0(I) + b_0(\mathfrak{m}) + a(R), 2b_0(I)\}.$$

Proof. Let M be the R-module I/I^2. Notice that $\mathrm{H}^0_{\mathfrak{m}}(M) = I^{(2)}/I^2$. If $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ is the zero module, then $I^{(2)} = I$ and the degree bounds hold automatically. Otherwise, $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ has positive depth and is a maximal Cohen-Macaulay R-module. Apply Corollary 4.7 to the R-module M to conclude that

$$b_0(\mathrm{H}^0_{\mathfrak{m}}(I/I^2)) \le b_0(I/I^2) + \Theta_1 + a(R). \tag{5.1.1}$$

Nakayama's Lemma guarantees that $b_0(I/I^2) = b_0(I)$. In the present situation, Θ_1, which is defined in (2.8), is equal to $b_0(\mathfrak{m})$. Thus,

$$\begin{aligned} b_0(I^{(2)}) \le \sup\left\{b_0\left(I^{(2)}/I^2\right), b_0(I^2)\right\} &= \sup\left\{b_0\left(\mathrm{H}^0_{\mathfrak{m}}(I/I^2)\right), b_0(I^2)\right\} \\ &\le \sup\{b_0(I)+b_0(\mathfrak{m})+a(R), 2b_0(I)\}. \end{aligned}$$

□

Our first application of Lemma 5.1 is to monomial curves. The hypothesis in Corollary 5.2 that H is a numerical semigroup includes the requirement that all large positive integers are in H. The Frobenius number of H, denoted $F(H)$, is the largest integer b with $b \notin H$. One way to see the connection between $F(H)$ and the language of Lemma 5.1 is described below.

Let R be a non-negatively graded ring over a field. Denote the maximal homogeneous ideal of R by $\mathfrak{m}$ and the dimension of R by d. A well known theorem of Serre, see for example [2, 4.3.5], shows that

$$\begin{aligned} &\max\{n \,|\, \text{Hilbert Function}_R(n) \neq \text{Hilbert Quasi-Polynomial}_R(n)\} \\ &= \max\left\{ n \;\middle|\; \sum_{i=0}^{d} (-1)^i \dim \mathrm{H}^i_{\mathfrak{m}}(R)_n \neq 0 \right\}. \end{aligned} \tag{5.1.2}$$

If R is Cohen-Macaulay, then the number on the right side of (5.1.2) is equal to the a-invariant of R. If $R \subset k[t]$ is the coordinate ring of a monomial curve over an infinite field, then the number on the left side of (5.1.2) represents the largest exponent n with $t^n \notin R$. If R is a standard graded ring, then the number on the left is often called the postulation number of R.

Corollary 5.2. *Let k be an infinite field, H be a numerical semigroup minimally generated by the positive integers $h_1 < h_2 < \cdots < h_\ell$, P be the polynomial ring $k[x_1,\ldots,x_\ell]$, and $\mathfrak{p} \subset P$ be the prime ideal which defines the monomial curve*

$$\{(\tau^{h_1},\ldots,\tau^{h_\ell}) \subset \mathbb{A}^\ell_k \mid \tau \in k\}.$$

Then the maximal generator degree of the second symbolic power of $\mathfrak{p}$ satisfies

$$b_0(\mathfrak{p}^{(2)}) \le \sup\{b_0(\mathfrak{p}) + \text{ the maximal generator of } H + \text{ the Frobenius number of } H, 2b_0(\mathfrak{p})\}.$$

Proof. View P as a graded ring with $\deg x_i = h_i$. Then $\mathfrak{p}$ is the kernel of the homogeneous ring homomorphism $P \to k[t]$ with $x_i \mapsto t^{h_i}$. Let $R = P/\mathfrak{p}$ and $\mathfrak{m}$ denote the maximal homogeneous ideal of R. We see that R is a one-dimensional Cohen-Macaulay domain, the a-invariant of R is the Frobenius number of H, and $b_0(\mathfrak{m}) = h_\ell$ is the maximal generator of H. The assertion follows from Lemma 5.1. □

Example 5.3. In the language of Corollary 5.2, if $H = \langle 3,4,5 \rangle$, then

$$b_0(\mathfrak{p}) + b_0(\mathfrak{m}) + F(H) = 10+5+2 = 17, \quad 2b_0(\mathfrak{p}) = 20, \quad \text{and} \quad b_0(\mathfrak{p}^{(2)}) = 18;$$

so there are situations where some minimal generator of $\mathfrak{p}^2$ of degree more than

$$b_0(\mathfrak{p}) + b_0(\mathfrak{m}) + F(H)$$

is also a minimal generator of $\mathfrak{p}^{(2)}$. The calculation of $b_0(\mathfrak{p}^{(2)})$ was made in Macaulay2 [6] over the field of rational numbers.

Our second application of Lemma 5.1 concerns the ideal of a finite set of points in projective space and to other similar ideals. In the situation of Corollary 5.4,

$$\operatorname{reg}(R) = a(R) + 1 = \text{ the postulation number of } P/I \text{ plus one;}$$

see the discussion surrounding (5.1.2).

Corollary 5.4. *Let P be a standard graded polynomial ring over a field and I be a homogeneous ideal in P with $R = P/I$ Cohen-Macaulay of dimension one. Then*

$$b_0(I^{(2)}) \le b_0(I) + \operatorname{reg}(R) + 1.$$

Furthermore, if, in addition to the above hypotheses, the minimal homogeneous resolution of I by free P-modules is not linear, then

$$b_0(I^{(2)}) \le b_0(I) + \operatorname{reg}(R). \tag{5.4.1}$$

Proof. The ring R is a standard graded ring over a field; so every minimal generator of the maximal homogeneous ideal $\mathfrak{m}$ of R has degree one; furthermore $1 + a(R) = \operatorname{reg}(R)$. Apply Lemma 5.1 to obtain

$$\begin{aligned} b_0(I^{(2)}) \le \sup\{b_0(I) + b_0(\mathfrak{m}) + a(R), 2b_0(I)\} &\le \sup\{b_0(I) + \operatorname{reg}(R), 2b_0(I)\} \\ &= b_0(I) + \max\{\operatorname{reg}(R), b_0(I)\}. \end{aligned}$$

In the general case,

$$b_0(I) \le \operatorname{reg}(I) \le \operatorname{reg}(R) + 1.$$

On the other hand, if the minimal homogeneous resolution of I by free P-modules is not linear, then

$$b_0(I) < \operatorname{reg}(I) \le \operatorname{reg}(R) + 1;$$

hence $b_0(I) \le \operatorname{reg}(R)$. □

Example 5.5. If $P = \mathbb{Q}[x,y,z]$ and I is the ideal of P generated by the 2×2 minors of

$$\begin{bmatrix} x & y & z \\ y & z & x \end{bmatrix},$$

then $R = P/I$ is a one dimensional Cohen-Macaulay ring, the minimal homogeneous resolution of I by free P-modules is linear,

$$b_0(I) + \operatorname{reg}(R) = 2 + 1 = 3, \quad \text{and} \quad b_0(I^{(2)}) = 4;$$

so the inequality (5.4.1) does not hold in the general case. Again, $b_0(I^{(2)})$ was computed using Macaulay2 [6].

Remark 5.6. If one re-does the calculation of Corollary 5.4 starting at (5.1.1), then one can read the conclusion of Corollary 5.4 as

$$b_0(I^{(2)}/I^2) \le b_0(I) + \operatorname{reg}(R). \tag{5.6.1}$$

Indeed,

$$b_0(I^{(2)}/I^2) = b_0(\mathrm{H}^0_{\mathfrak{m}}(I/I^2)) \le b_0(I/I^2) + \Theta_1 + a(R) = b_0(I) + 1 + a(R) = b_0(I) + \operatorname{reg}(R).$$

The formulation (5.6.1) affords a direct comparison with the relevant part of [4, Cor 7.8]:

$$\operatorname{topdeg}(I^{(2)}/I^2) \le b_1(I) - 1 + \operatorname{reg}(R).$$

Observe that $b_0(I^{(2)}/I^2) \le \operatorname{topdeg}(I^{(2)}/I^2)$ and $b_0(I) \le b_1(I) - 1$. (Recall the meaning of b_i from (2.5).)

Corollary 5.7 is about hyperplane sections of subschemes of projective space. For instance, let V be the subscheme of $\mathbb{P}^{d-1}_k$ defined by the homogeneous ideal I in $R = k[x_1, \dots, x_d]$ and H be a linear subspace of $\mathbb{P}^{d-1}_k$ defined by general linear forms in $k[x_1, \dots, x_d]$. We produce an upper bound for the maximal generator degree of the saturated ideal defining $V \cap H$, in terms of information that can be read from a single shift in the minimal homogeneous resolution of R/I. The analogous bound for the highest degree of a form that is in the saturated ideal of $V \cap H$ but not in the image of I was proved in [4, 5.1]. Notice that the saturated ideal of $V \cap H$ is the ideal of polynomials vanishing on $V \cap H$ if k is algebraically closed and I is radical [5, 5.2].

Corollary 5.7. *Let $R = k[x_1, \dots, x_d]$ be a standard graded polynomial ring over a field k, I be a homogeneous ideal of R, and L be an ideal minimally generated by c linear forms in R. Assume that* $\dim \operatorname{Tor}_1^R(R/I, R/L) \le 1$. *Let $\bar{I}$ be the image of I in $\overline{R} = R/L$ and J be the saturation $J = \bar{I}^{\mathrm{sat}}$ of $\bar{I}$. Then*

$$b_0(J) \le \max\{b_0(I), b_{d-c-2}(I) - d + c + 1\}.$$

Furthermore, if $c = \dim(R/I) - 1$, then

$$b_0(J) \le b_{d-c-2}(I) - d + c + 2.$$

Remark 5.8. The inequality $\dim \operatorname{Tor}_1^R(R/I, R/L) \le 1$ is satisfied if $\dim(R/I) \le 1$. It also holds if k is infinite and L is generated by general linear forms, because such forms are a filtered regular sequence on R/I, see [12, 2.3] for a proof.

Proof. If $d - 1 \le c$, then $\overline{R}$ is a principal ideal domain and hence $J = \bar{I}$. Thus we may assume that $c \le d - 2$. Denote the maximal homogeneous ideal of R by $\mathfrak{m}$ and the maximal homogeneous ideal of $\overline{R}$ by $\overline{\mathfrak{m}}$. The ideal J of $\overline{R}$ is equal to

$$J = \bigcup_i (\bar{I} :_{\overline{R}} \overline{\mathfrak{m}}^i);$$

therefore

$$J/\bar{I} = \mathrm{H}^0_{\overline{\mathfrak{m}}}(\overline{R}/\bar{I})$$

and

$$b_0(J) \le \sup\{b_0(\bar{I}), b_0(\mathrm{H}^0_{\mathfrak{m}}(\overline{R}/\bar{I}))\} \le \sup\{b_0(I), b_0(\mathrm{H}^0_{\mathfrak{m}}(\overline{R}/\bar{I}))\}.$$

We now bound $b_0(\mathrm{H}^0_{\mathfrak{m}}(\overline{R}/\bar{I})$. Let $C_\bullet$ be a minimal homogeneous resolution of R/I by free R-modules. Consider the complex $\overline{C}_\bullet = C_\bullet \otimes_R \overline{R}$. Our assumption on Tor_1 and the rigidity of Tor [1, 2.1] imply that dim $\mathrm{Tor}_i^R(R/I, R/L) \le 1$ for all positive i. Keep in mind that $\overline{R}$ is a polynomial ring of dimension $d-c$. Apply Corollary 4.8 to the complex $\overline{C}_\bullet$ to obtain

$$\begin{aligned} b_0(\mathrm{H}^0_{\mathfrak{m}}(\overline{R}/\bar{I})) &\le b_0(C_{d-c-1}) - (d-c) + 1 = b_{d-c-1}(R/I) - d + c + 1 \\ &= b_{d-c-2}(I) - d + c + 1. \end{aligned}$$

This completes the proof of the general case.

If $c = \dim(R/I) - 1$, then $d - c - 1 = \mathrm{ht}\, I = \mathrm{grade}\, I$ and

$$b_0(R/I) < b_1(R/I) < \cdots < b_{d-c-1}(R/I)$$

because

$$0 \longrightarrow C_0^* \longrightarrow \ldots \longrightarrow C_{d-c-1}^*$$

is a minimal resolution. It follows that

$$b_0(I) < \cdots < b_{d-c-2}(I);$$

and therefore, $b_0(I) \le b_{d-c-2}(I) - d + c + 2$. □

REFERENCES

[1] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math **5** (1961), 631–647.
[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.
[3] M. Chardin, *Powers of ideals and the cohomology of stalks and fibers of morphisms*, Algebra Number Theory **7** (2013), 1–18.
[4] D. Eisenbud, C. Huneke, and B. Ulrich, *The regularity of Tor and graded Betti numbers*, Amer. J. Math. **128** (2006), 573–605.
[5] H. Flenner, *Die Sätze von Bertini für lokale Ringe*, Math. Ann. **229** (1977), 97–111.
[6] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/.
[7] L. Gruson, R. Lazarsfeld, and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), 491–506.
[8] L. T. Hoa and T. N. Trung, *Partial Castelnuovo-Mumford regularities of sums and intersections of powers of monomial ideals*, Math. Proc. Camb. Phil. Soc. **149** (2010), 229–246.
[9] M. Hochster and C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147 (2002), 349–369.
[10] A. Kustin, C. Polini, and B. Ulrich, *The equations defining blowup algebras of height three Gorenstein ideals*, Algebra Number Theory **11** (2017), 1489–1525.
[11] W. Niu *Some results on asymptotic regularity of ideal sheaves*, J. Algebra **377** (2013), 157–172.
[12] Y. Xie, *Formulas for the multiplicity of graded algebras*, Trans. Amer. Math. Soc. **364** (2012), 4085–4106.

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