RESIDUAL INTERSECTIONS AND LINEAR POWERS

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ABSTRACT. If I is an ideal in a Gorenstein ring S, and S/I is Cohen-Macaulay, then the same is true for any linked ideal I'; but such statements hold for residual intersections of higher codimension only under restrictive hypotheses, not satisfied even by ideals as simple as the ideal L_n of minors of a generic $2 \times n$ matrix when $n > 3$.

In this paper we initiate the study of a different sort of Cohen-Macaulay property that holds for certain general residual intersections of the maximal (interesting) codimension, one less than the analytic spread of I. For example, suppose that K is the residual intersection of L_n by $2n-4$ general quadratic forms in L_n. In this situation we analyze S/K and show that $I^{n-3}(S/K)$ is a self-dual maximal Cohen-Macaulay S/K-module with linear free resolution over S.

The technical heart of the paper is a result about ideals of analytic spread 1 whose high powers are linearly presented.

INTRODUCTION

Let S be a Noetherian ring, and let $I \subset S$ be an ideal of codimension g. Let $J = (a_1, \ldots, a_s)$ be an ideal generated by s elements in I, and consider the residual ideal $K := J : I$. If $\operatorname{codim} K \geq s$ then K (or S/K) is said to be the *s-residual intersection of I with respect to J*, and the residual intersection is called *geometric* if, in addition, $\operatorname{codim}(I + K) > s$.

Let $g = \operatorname{codim} I$. Under strong hypotheses the residual intersection $R := S/K$ is Cohen-Macaulay and, up to shifts, the canonical module ω_R of R is isomorphic to $I^{s-g+1}R$ and I^jR is ω_R-dual to $I^{s-g+1-j}R$ for $0 \leq j \leq s-g+1$ (see [EU] for a summary of the situation). For example, these conclusions are true when I is the ideal of maximal minors of a sufficiently general $(n-1) \times n$ matrix.

However, all these things fail even when I is generated by the maximal minors of a generic $2 \times n$ matrix with $n \geq 4$. The main contribution of this paper is to construct a natural rank 1, self-dual, maximal Cohen-Macaulay module over certain residual intersections of such ideals and many others. The following special case of our main result, Theorem 2.1, will convey the flavor:

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Theorem 0.1. *Let S be a standard graded polynomial ring over a field of characteristic 0, let $I \subset S$ be a nonzero homogeneous ideal generated in a single degree δ with analytic spread ℓ, and let $J \subset I$ be generated by $\ell - 1$ general elements of degree δ. Suppose that $J : I$ is an $(\ell - 1)$-residual intersection of I, and set $R = S/(J : I)$. If all sufficiently high powers of I are linearly presented then, for all $\rho \gg 0$, $I^\rho R$ is a maximal Cohen-Macaulay R-module with linear resolution as an S-module, and is, up to shift, ω_R-self-dual.*

The restriction to $(\ell - 1)$-residual intersections is natural because in that case IR has analytic spread 1, so all high powers are isomorphic. Thus it makes sense to speak of their asymptotic structure.

The idea of the proof of Theorem 0.1is to reduce to the case of analytic spread 1, and then use the fact that $I^\rho R$, for large ρ, can be given the structure of a commutative algebra. The key point is to show that the condition of linear presentation is preserved in the reduction.

In Corollary 2.7 we show that if $I \subset S$ is any homogeneous ideal generated in degree $\leq \delta$, and $J \subset I$ is generated by $\dim S - 1$ general elements of degree $t > \delta$, then for $\rho \gg 0$ the S-module $(I_{\geq t})^\rho(S/(J : I))$ is perfect of codimension $\dim S - 1$, and has linear, symmetric minimal free resolution.

Theorem 0.1 applies, in particular, to the ideal of maximal minors of a generic $2 \times n$ matrix, and more can be said in that case, as in Section 3:

Theorem 0.2. *Let $S = k[x_1, \dots, x_n, y_1, \dots, y_n]$ be a polynomial ring over a field k of characteristic 0. Suppose that $I \subset S$ is the ideal of 2×2 minors of the generic matrix*

$$\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix}.$$

The ideal I has analytic spread $\ell = 2n - 3$.

Let J be an ideal generated by $\ell - 1$ general quadrics in I. The ring $R := S/(J : I)$ is unmixed of codimension $2n - 4$, with isolated singularity. If k is algebraically closed, then $J : I$ is the intersection of $\frac{1}{n-1}\binom{2(n-2)}{n-2}$ linear prime ideals.

The canonical module ω_R is isomorphic to $I^{n-3}R(2n - 10)$. Furthermore, for all $\rho \geq n - 3$, the module $I^\rho R$ is isomorphic up to shift to ω_R, and is a maximal Cohen-Macaulay R-module with linear resolution as an S-module.

Examples suggest that part of the conclusion of Theorem 0.1 holds much more generally:

Conjecture 0.1. *Let S be a standard graded polynomial ring over an infinite field, let $I \subset S$ be a nonzero homogeneous ideal generated in a single degree δ with analytic spread ℓ, and let $J \subset I$ be generated by $\ell - 1$ general elements of degree δ. Set $R = S/(J : I)$. If I is unmixed and $J : I$ is a geometric $(\ell - 1)$-residual intersection of I, then $I^\rho R$ is a maximal Cohen-Macaulay R-module for all $\rho \gg 0$.*

Further examples suggest that Conjecture 0.1 might hold without the assumption that R is obtained as a residual intersection. In addition, the self-duality seems to hold in more general circumstances:

Conjecture 0.2. *Let R be a standard graded algebra over a field. Assume that R is reduced and equidimensional, and that ω_R is generated in a single degree. If ω_R has analytic spread 1 (in the sense that a homogeneous ideal isomorphic to a shift of ω_R has analytic spread 1), then some power of ω_R (as a fractional ideal) is ω_R-self-dual up to a shift.*

Examples we have seen also support another version:

Let R be a standard graded ring over a field, and let $I \subset R$ be a homogeneous ideal of positive codimension. Suppose that:

(1) R is reduced and the truncation I' of I in the degree of the highest generator of I has analytic spread 1;
(2) Some power of I is isomorphic to a shift of a power of ω_R.

Are the high powers of I' always ω_R-self-dual up to a shift?

We shall see in Section 6 that none of the hypotheses can be dropped. Conjecture 0.1 has surprising consequences:

Proposition 0.3. *Suppose that $S = k[x_1, \ldots, x_d]$ is a standard graded polynomial ring over an infinite field, and $I \subset S$ is an ideal generated by forms of a single degree. If Conjecture 0.1 is true then:*

(1) *Suppose that I is generically a complete intersection of codimension g generated by $g + 1$ elements. If I is unmixed, then S/I is Cohen-Macaulay.*
(2) *If $I \subset S$ is an unmixed ideal of linear type (that is, the Rees algebra $\mathcal{R}(I)$ of I is equal to the symmetric algebra of I), then*

$$\mathcal{R}(I)_{(x_1,\ldots,x_d)\mathcal{R}(I)}$$

is Cohen-Macaulay.

The assertion of item (1) is true (independent of Conjecture 0.3) both when I has codimension 2 [EG, Theorem 2.1], and also when S/I is equidimensional and locally of depth $\geq \frac{1}{2}\dim(S/I)_P$ at every prime P containing I (Theorem 5.1 below). In Section 5 we prove Proposition 0.3, and note two special cases where the conjectures are verified.

1. The module of interest

The analytic spread of an ideal plays an important role in this theory. If R is a positively graded algebra over a field k, with maximal homogeneous ideal $\mathfrak{m}$, and $I \subset R$ is an ideal generated by forms of a single degree δ, then the *analytic spread* may be defined as:

$$\ell(I) := \dim k[I_\delta] = \dim(k \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \cdots).$$

Assuming that k is infinite, $\ell(I)$ is the smallest number of generators of a homogeneous ideal over which I is integral, and such an ideal may be taken to be the ideal generated by $\ell(I)$ general forms in I of degree δ. The reduction number $r(I)$ of I is the smallest integer $r \geq 0$ so that $I^{r+1} = \mathfrak{a}I^r$ for some homogeneous $\ell(I)$-generated ideal $\mathfrak{a}$ over which I is integral.

If $s < \ell(I)$ and $K := I : J$ is an s-residual intersection of I with respect to a homogeneous ideal J generated by s elements, so that $\operatorname{codim} K \geq s$, then I cannot be integral over J, so if R is equidimensional then, by [M, Theorem 4.1], $\operatorname{codim} K = s$. However, this may not be the case when $s \geq \ell(I)$.

Now assume that k is infinite. When $s < \ell(I)$ and J is generated by s general forms of degree δ in I, Proposition 1.2 implies that the modules $I^\rho(S/K)$ are nonzero. These, for $\rho \gg 0$ and $s = \ell(I) - 1$, are the modules that are of interest to us.

Proposition-Definition 1.1. Suppose that R is a positively graded algebra over an infinite field k, that $I \subset R$ is generated by forms of a single degree δ, and that I has analytic spread ≤ 1. Let $\overline{R} := R/(0 :_R I^\infty)$ and $\overline{I} := I\overline{R}$.

(1) For $\rho \gg 0$ the module $I^\rho(\rho\delta)$ is, up to homogeneous isomorphism, independent of ρ and I^ρ maps isomorphically to $\overline{I}^\rho$. If I contains a non-zerodivisor of R, the latter is true for all $\rho \geq 0$.

Let $M := M(I)$ be the stable value of $I^\rho(\rho\delta)$.

(2) $E := \operatorname{End}_R(M) \cong M$ as graded R-modules.

(3) Let $a \in I$ be a general homogeneous element of degree δ. Write $\overline{I}$ and $\overline{a}$ for the images of I and a in $\overline{R}$. The element $\overline{a}$ is a non-zerodivisor on $\overline{R}$ and as a graded ring,

$$E \cong \overline{R}[\overline{a}^{-1}\overline{I}] = \overline{a}^{-\rho}\overline{I}^\rho,$$

which is the coordinate ring of the blowup of $\overline{R}$ along $\overline{I}$.

(4) If R is reduced away from $V(I)$, then

$$\overline{R} = R_{\mathrm{red}}/(0 : (IR_{\mathrm{red}})^\infty) = R/(\sqrt{0} : I^\infty)$$

is reduced.

Proof. First note that for a as in item (3), the element $\overline{a}$ is a non-zerodivisor on $\overline{R}$ as $\overline{I}$ has positive grade.

(1) Let ϵ be an integer so large that $0 : I^\infty = 0 : I^\epsilon$. By the Artin-Rees Lemma, $I^\rho \cap (0 :_R I^\epsilon) \subset I^\epsilon\,(0 :_R I^\epsilon) = 0$ for large ρ. Thus I^ρ maps isomorphically to $\overline{I}^\rho$.

Because I has analytic spread ≤ 1, the element $\overline{a}$ generates a reduction of $\overline{I}$. Thus, since a is general, the multiplication map

$$\overline{a} : \overline{I}^\rho(\rho\delta) \to \overline{I}^{\rho+1}((\rho+1)\delta)$$

is surjective for $\rho \geq r(I)$, and since $\overline{a}$ is a non-zerodivisor on $\overline{R}$ the map is injective as well. This proves that M is independent of $\rho \gg 0$.

If a is a non-zerodivisor on R, then multiplication by a on R is a monomorphism, and decreasing induction on ρ shows that I^ρ maps isomorphically to $\overline{I}^\rho$ for $\rho \geq 0$.

(2) We have $\operatorname{End}_R(M) = \operatorname{End}_{\overline{R}}(M)$. Writing Q for the total ring of quotients of $\overline{R}$ and taking $\rho \gg 0$, we have

$$\overline{I}^\rho = \overline{a}^\rho \overline{I}^\rho :_Q \overline{a}^\rho \overline{R} \supset \overline{a}^\rho \overline{I}^\rho :_Q \overline{I}^\rho \supset \overline{I}^\rho$$

because $\overline{a}\overline{R}$ is a reduction of $\overline{I}$. Thus

$$\overline{I}^\rho = \overline{a}^\rho \overline{I}^\rho :_Q \overline{I}^\rho.$$

Further $\overline{a}^\rho \overline{I}^\rho :_Q \overline{I}^\rho \cong \operatorname{End}_{\overline{R}}(\overline{I}^\rho)(-\rho\delta)$, proving the assertion.

(3) Because $\overline{a}\overline{R}$ is a principal reduction of $\overline{I}$, the blowup has only one affine chart, $\overline{R}[\overline{a}^{-1}\overline{I}]$. Further $\overline{a}^{-1}\overline{I} \subset \overline{a}^{-2}\overline{I}^2 \subset \cdots$, and this sequence of fractional ideals stabilizes when $\overline{a}\overline{I}^r = \overline{I}^{r+1}$. Thus for $\rho \gg 0$ we have

$$\overline{R}[\overline{a}^{-1}\overline{I}] = \overline{a}^{-\rho}\overline{I}^\rho = \overline{I}^\rho :_Q \overline{I}^\rho \cong E.$$

(4) If R is reduced away from $V(I)$, then $\overline{R}$ is reduced locally at each of its associated primes, hence reduced. □

The next result provides a different description of $I^\rho \overline{R}$ in a general setting:

Proposition 1.2. *Let S be a Noetherian algebra over an infinite field k, and let I be an ideal of S. Let J be an ideal generated by a sequence of general k-linear combinations of generators of I. Let $R = S/(J : I)$ and $\overline{R} = S/(J : I^\infty)$.*

The natural surjection

$$I^\rho / JI^{\rho-1} \to I^\rho \overline{R}$$

is an isomorphism for $\rho \gg 0$.

Proof. Let ϵ be an integer so large that $H := J : I^\infty = J : I^\epsilon$. For $\rho \gg 0$ the Artin-Rees Lemma gives $I^\rho \cap H \subset I^\epsilon H \subset J$. Furthermore, the generators of J are a superficial sequence for I, so [SH, Lemma 8.5.11] gives $I^\rho \cap J = JI^{\rho-1}$. Thus $I^\rho \cap H = JI^{\rho-1}$ as required. □

Suppose that $S = k[x_1, \dots, x_d]$ is a positively graded polynomial ring over an infinite field k, and let I be an ideal generated by forms of a single degree δ. If the analytic spread of I is ℓ and J is generated by $\ell - 1$ general forms of degree δ then, setting $R = S/(J : I)$, the ideal IR has analytic spread ≤ 1, so we may apply Proposition-Definition 1.1. In this case the module $M = M(IR)$ can be expressed as $(I^\rho / JI^{\rho-1})(\rho\delta)$, as we see from Proposition 1.2.

2. IDEALS WHOSE POWERS HAVE LINEAR PRESENTATION

The following is our main general result.

Theorem 2.1. *Let S be a standard graded polynomial ring over an infinite field k. Let $I \subset S$ be a nonzero ideal generated by forms of a single degree δ. Let $\ell := \ell(I)$ be the analytic spread of I. Let $J \subset I$ be generated by $\ell - 1$ general homogeneous elements of degree δ in I. Set*

$$R := S/(J : I),\ \overline{R} := S/(J : I^\infty),\ \overline{I} := I\overline{R}\,.$$

Let $\overline{a} \in \overline{I}$ be a general form of degree δ and let $M = M(IR)$ be as in Proposition-Definition 1.1.

If R is reduced away from $V(I)$ and all sufficiently high powers of I are linearly presented, then

1. *$\overline{R}$ is equidimensional of dimension $\dim S - \ell + 1$, M is a maximal Cohen-Macaulay $\overline{R}$-module and an Ulrich module, and M is $\omega_{\overline{R}}$-self-dual up to a shift.*
2. *$\dim \overline{R} = \dim R$ if and only if $J : I$ is an $(\ell - 1)$-residual intersection of I, that is, $\operatorname{codim}(J : I) \geq \ell - 1$. In this case, M is a maximal Cohen-Macaulay R-module and an Ulrich module, and M is ω_R-self-dual up to a shift.*
3. *As a graded R-module, $\operatorname{End}_R(M) \cong M = (IR)^\rho(\rho\delta) \cong \overline{I}^\rho(\rho\delta)$ for $\rho \gg 0$. As a graded ring, $\operatorname{End}_R(M)$ is isomorphic to the blow-up*

$$\widetilde{R} := \overline{R}[\overline{a}^{-1}\overline{I}]$$

of $\overline{R}$ along $\overline{I}$ and is regular. In particular $\widetilde{R}$ is the normalization of $\overline{R}$, and the conductor of $\overline{R}$ is $\overline{R}\overline{a}^\rho :_{\overline{R}} \overline{I}^\rho$.

We postpone the proof of Theorem 2.1 until the end of this section.

If the characteristic of k is zero and the forms generating J are sequentially general elements of I, then R is automatically reduced away from $V(I)$ by Bertini's Theorem ([F, (4.8) Korollar]). We provide a direct proof of a slightly stronger result that does not require sequentially general elements:

Theorem 2.2. *Let S be a finitely generated algebra over a field k of characteristic 0, let $I \subset S$ be an ideal, and assume that S is regular away from $V(I)$. Let $J \subset I$ be generated by s general k-linear combinations of a set of generators of I. Then away from $V(I)$, the ring S/J is regular of codimension s.*

Proof. Write $I = (f_1, \ldots, f_n)$. Replacing S by any of its localizations S_{f_i} we may assume that S is regular, and passing to any of the connected components of $\operatorname{Spec} S$ we further suppose that S is a domain, say of dimension d.

Let T be the polynomial ring in sn new variables, and let $\widetilde{J} \subset \widetilde{S} := T \otimes_k S$ be the ideal generated by the s generic linear combinations of the f_i using the new variables as coefficients. Set $\widetilde{R} := \widetilde{S}/\widetilde{J}$. If λ is a rational closed point in $\mathbb{A}_k^{sn}$ with coordinate ring $k(\lambda)$ we set $R_\lambda = k(\lambda) \otimes_T \widetilde{R}$. We must show that for general λ, the ring R_λ is regular of codimension s away from $V(IR_\lambda)$.

Let K be the quotient field of T and let $R_K := K \otimes_T \widetilde{R}$. It is easy to see that away from $V(IR_K)$ the ring R_K is regular of codimension s. Thus, since K is a field of characteristic 0, the module $\Omega_K(R_K) = K \otimes_T \Omega_T(\widetilde{R})$ is free of rank $d - s$ away from $V(IR_K)$. It follows that for some t the ideal $(IR_K)^t$ is contained in $\mathrm{Fitt}_{d-s}(K \otimes_T \Omega_T(\widetilde{R})$. Hence for general λ we have

$$(IR_\lambda)^t \subset \mathrm{Fitt}_{d-s}\big(k(\lambda) \otimes_T \Omega_T(\widetilde{R})\big) = \mathrm{Fitt}_{d-s}\big(\Omega_k(R_\lambda)\big).$$

This implies that locally away from $V(IR_\lambda)$ the module $\Omega_k(R_\lambda)$ is generated by $d - s$ elements. Since $\dim R_\lambda \geq d - s$, we see that R_λ is regular of dimension exactly $d - s$ away from $V(IR_\lambda)$. □

We say that an ideal I satisfies G_s if I_P is generated by at most $\operatorname{codim} P$ elements for all primes P of codimension $< s$ containing I.

Remark 2.3. If I satisfies $G_{\ell-1}$, then the ideal $J : I$ in Theorem 2.1 is an $(\ell - 1)$-residual intersection.

Theorem 2.4 (Examples)**.** *The following classes of ideals in a polynomial ring in d variables over a field of characteristic 0 all satisfy the hypotheses and conclusions of parts (1)-(3) of Theorem 2.1:*

(1) *ideals of $m \times m$ minors of $m \times n$ matrices A of linear forms such that either* $\operatorname{codim} I_m(A) = d$ *or*

$$\operatorname{codim} I_k(A) \geq \min\{(m - k + 1)(n - m) + 1, d\} \text{ for } 2 \leq k \leq m;$$

(2) *strongly stable monomial ideals generated in one degree;*
(3) *products of ideals of linear forms;*
(4) *polymatroidal ideals;*
(5) *monomial ideals generated in degree 2 and having linear resolution;*
(6) *linearly presented ideals of dimension 0, and ideals of dimension 1 that have linear resolutions for the first $\lceil (d-1)/2 \rceil$ steps.*
(7) *truncations $I_{\geq t}$ of homogeneous ideals I at degree t if I is generated in degrees $\leq t$ and I has a homogeneous reduction generated in degrees $\leq t - 1$.*
(8) *linearly presented ideals of fiber type, such as linearly presented ideals satisfying G_d that are perfect of codimension 2 or Gorenstein of codimension 3.*

More precisely, in the first 5 cases every power of the ideal in question actually has a linear resolution; in cases 6 and 7 all large powers have linear resolution; and in case 8 all powers are linearly presented.

Proof. First notice that R is reduced away from $V(I)$ according to Theorem 2.2.
(1): See Theorem 4.1, the second case of which is [BCV, Theorem 3.7].
(2)–(4): These assertions are copied from the list in [BCV, p. 42], and were proven in [CH].
(5): See [HHZ, Theorem 3.2].
(6): See [EHU, Theorem 7.1 and Corollary 7.7].
(7): See Proposition 2.5.
(8): Symmetric powers of linearly presented ideals are always linearly presented; and fiber type implies that the additional relations on the generators of the powers all have degree 0. The given classes of ideals are of this type by [MU, Theorem 1.3] and [KPU, Theorem 9.1], respectively. In the case of perfect codimension 2 ideals, all the powers have linear resolution by item (1) and the Hilbert-Burch Theorem. ☐

Proposition 2.5. *Suppose that I is a homogeneous ideal in a standard graded polynomial ring over a field. Suppose further that I is generated in degrees $\leq \delta$ and that I has a homogeneous reduction generated in degrees $\leq a$.*

If $t \geq \max\{\delta, a+1\}$, then the high powers of the truncated ideal $I' := I_{\geq t}$ all have linear resolution.

Proof. If $t \geq \delta$ then $(I_{\geq t})^\rho = (I^\rho)_{\geq t\rho}$. Moreover, for large ρ the regularity of I^ρ grows as a linear function bounded by $a\rho + b$, see [K, Theorem 5].

Thus if $t \geq \max\{\delta, a+1\}$ then $I'^\rho = (I^\rho)_{\geq t\rho}$ and, for $\rho \gg 0$, I'^ρ has regularity $\leq t\rho$ since $t \geq a+1$. ☐

Remark 2.6. We do not know of an ideal whose high powers have linear presentation but not linear resolution. Linearly presented ideals of fiber type may provide such examples.

The following consequence of Theorem 2.1 gives some evidence for a positive answer to Question .

Corollary 2.7. *Suppose that I is a nonzero homogeneous ideal in a standard graded polynomial ring S in d variables over an infinite field k. Let t be such that I is generated in degrees $< t$. Let J be an ideal generated by $d-1$ sequentially general forms of degree t in I, write $R = S/(J : I)$, and set $I' = I_{\geq t}$.*

The conclusions of Theorem 2.1 hold if we replace δ by t, ℓ by d, IR by $I'R$, $\overline{I} = I\overline{R}$ by $I'\overline{R}$, and $M(IR)$ by $M(I'R)$.

In particular, as an S-module, $M(I'R)$ is perfect of grade $d-1$, and its minimal graded free resolution is linear and symmetric.

Proof. We wish to apply Theorem 2.1 to the ideal I' and the ring $R' := S/(J : I')$. First, since I in generated in degrees $< t$, it follows that $\ell(I') = d$ and that all sufficiently high powers of I' are linearly presented, see Proposition 2.5.

Next, we show that R' is reduced away from $V(I')$. If the characteristic is zero, this follows from Theorem 2.2. Otherwise, let $\{x_u\}$ be the variables of S and $\{f_v\}$ be a spanning set of I_{t-1}, and apply [FOV, Theorem 3.4.13] to the scheme $\operatorname{Proj} S \setminus V(I')$ and the generating set $\{x_u f_v\}$ of I'. It follows that if a_1 is a general k-linear combination of these generators, then $S/(a_1)$ is regular away from $V(I')$. Iterating, we see that $R' = S/\left((a_1, \ldots, a_{d-1}) : I'\right)$ is regular away from $V(I')$.

Note that R maps onto R', whereas $J : I^\infty = J : I'^\infty$, so $\overline{R} = \overline{R'} := S/(J : I'^\infty)$. Thus, applying Theorem 2.1 to the ideal I' and the ring R' yields the result. ☐

To prove Theorem 2.1 we reduce to the case $\ell = 1$ by factoring out a general $(\ell - 2)$-residual intersection and proving that the hypothesis of linearly presented powers is preserved. We then use the following more general result:

Theorem 2.8. *Let S be a standard graded polynomial ring over an infinite field, let R be a homogeneous factor ring of S, and let $I \subset R$ be an ideal.*

Suppose that:

- *I is generated by forms of a single degree δ and has analytic spread 1.*
- *R is reduced away from $V(I)$.*
- *All sufficiently high powers of I are linearly presented as S-modules.*

Set $\overline{R} := R/(0 : I^\infty)$ and $\overline{I} := I\overline{R}$. Let $\overline{a} \in \overline{I}$ be a general element of degree δ, and let

$$\widetilde{R} := \overline{R}[\overline{a}^{-1}\overline{I}]$$

be the blowup of $\overline{R}$ along $\overline{I}$. Let $M := M(I)$ as in Proposition-Definition 1.1.

We have:

1. *M is a direct sum of maximal Cohen-Macaulay modules on the components of $\widetilde{R}$ and M has a linear resolution as an S-module.*
2. *$\overline{R}$ is equidimensional if and only if M is a maximal Cohen-Macaulay module over $\overline{R}$. In this case M is $\omega_{\overline{R}}$-self-dual up to a shift, and an Ulrich module over $\overline{R}$. Furthermore, M is also ω_R-self-dual if and only if $\dim R = \dim \overline{R}$.*
3. *As a graded R-module, $\operatorname{End}_R(M) \cong M = I^\rho(\rho\delta) \cong \overline{I}^\rho(\rho\delta) \cong \widetilde{R}$ for $\rho \gg 0$. As a graded ring, $\operatorname{End}_R(M) \cong \widetilde{R}$, and this ring is regular. In particular $\widetilde{R}$ is the normalization of $\overline{R}$, and the conductor of $\overline{R}$ is $\overline{R}\overline{a}^\rho :_{\overline{R}} \overline{I}^\rho$ for $\rho \gg 0$.*

Part (3) of the theorem includes the assertion that M is a commutative algebra. The following lemma enables us to exploit this fact:

Lemma 2.9. *Let S be a standard graded polynomial ring over a field k. Suppose that T is a non-negatively graded S-algebra that is finite, generated in degree 0, and linearly presented as a module over S. If T_0 is reduced, then T is a product of standard graded polynomial rings over fields containing k.*

Proof. The ring T_0 is finite over k and hence Artinian. Thus T_0 decomposes as a product of fields e_iT_0, where the e_i are orthogonal idempotents. Since it is a summand of T, the S-module e_iT is also linearly presented, so it suffices to consider the case when $T_0 = K$ is a field. Let $\widetilde{S} = K \otimes_k S$, and note that T is naturally a cyclic $\widetilde{S}$-module, generated in degree 0.

We next show that T is linearly presented as an $\widetilde{S}$-module. Let $\widetilde{F}$ be a minimal homogeneous $\widetilde{S}$-free resolution for T. Using an S-module splitting $\widetilde{S} = S^m$ we see that $\widetilde{F}$ is also an S-free resolution, which we call F. Choosing bases in $\widetilde{F}$ and corresponding bases in F, we see that the entries in the matrix representations of the maps in F have the same positive degrees as occur in $\widetilde{F}$, and thus F is minimal. Since the initial map of F is linear by hypothesis, the initial map of $\widetilde{F}$ is linear.

Thus T is a cyclic, linearly presented $\widetilde{S}$-module. Therefore T may be written as $\widetilde{S}/m$, where m is an ideal generated by linear forms, so T is a polynomial ring over K. □

Proof of Theorem 2.8. Since $\ell(I) = 1$ the ideals I and $\overline{I}$ are not nilpotent, so $\ell(\overline{I}) = 1$ and the ring $\overline{R}$ is nonzero. This ring is reduced by Proposition-Definition 1.1(4).

(3) Note that M is generated in degree 0 as an S-module and is linearly presented. Further, by Proposition-Definition 1.1, $M = \widetilde{R}$ is an S-algebra. Since $\widetilde{R}$ is reduced, Lemma 2.9 shows that $\widetilde{R}$ is regular. The rest of part (3) now follows from Proposition-Definition 1.1.

(1) This follows from (3): We have $M \cong \widetilde{R}$. Any regular ring is Cohen-Macaulay. By Lemma 2.9, $\widetilde{R}$ is a product of rings, each of which is obtained from S by extending the ground field and factoring out linear forms. Such a ring has a linear resolution as an S-module.

(2) The ring $\widetilde{R}$ is a finitely generated R-module by (3). The ring $\overline{R}$ is equidimensional if and only if each component of $\widetilde{R}$ has maximal dimension as an $\overline{R}$-module. By (1), this condition is equivalent to the condition that M is a maximal Cohen-Macaulay module over $\overline{R}$.

Set $d := \dim \overline{R}$. If each component of $\widetilde{R}$ has maximal dimension as an $\overline{R}$-module, then each such component is a polynomial ring in d variables over a field containing k by Lemma 2.9. Thus, writing $\omega_{\widetilde{R}}$ for the direct sum of the canonical modules of these components, we have $\omega_{\widetilde{R}} \cong \widetilde{R}(-d)$.

On the other hand, $\omega_{\widetilde{R}} \cong \mathrm{Hom}_{\overline{R}}(\widetilde{R}, \omega_{\overline{R}})$. Thus as graded $\overline{R}$-modules,

$$\mathrm{Hom}_{\overline{R}}(M, \omega_{\overline{R}}) \cong \mathrm{Hom}_{\widetilde{R}}(M, \omega_{\widetilde{R}}) \cong \mathrm{Hom}_{\widetilde{R}}(\widetilde{R}, \omega_{\widetilde{R}}) \cong \widetilde{R}(-d) \cong M(-d).$$

If $\dim R = \dim \overline{R}$, these statements follow for R in place of $\overline{R}$, and the converse is obvious.

The fact that M is Ulrich follows from the linearity statement in (1). ☐

The following lemma shows that the linearity hypothesis in Theorem 2.1 is preserved in the reduction to Theorem 2.8.

Lemma 2.10. *Let S be a standard graded polynomial ring over an infinite field, and let $I \subset S$ be an ideal generated by forms of degree δ. Let $a_1, \ldots, a_n$ be homogeneous elements of I of degree $\leq \delta+1$. For $0 \leq \nu \leq n$, set $J_\nu = (a_1, \ldots, a_\nu) \subset S$.*

If the high powers of I have linear presentation, then for all $0 \leq \nu \leq n$ and all sufficiently large ρ the module $I^\rho / J_\nu I^{\rho-1}$ has linear presentation as an S-module.

Remark 2.11. If $a_1, \ldots, a_n$ are chosen sequentially generally, the same proof, together with Proposition 1.2, shows that if the high powers of I have linear resolution then for all $0 \leq \nu \leq n$ and all sufficiently large ρ the module $I^\rho / J_\nu I^{\rho-1}$ has linear resolution as an S-module.

Proof. We do induction on ν, the case $\nu = 0$ being the hypothesis. We assume the result for ν. For $\rho \gg 0$ and $0 \leq \nu \leq n-1$ the sequences

$$(I^{\rho-1}/J_\nu I^{\rho-2})(-\delta') \xrightarrow{a_{\nu+1}} I^\rho / J_\nu I^{\rho-1} \to I^\rho / J_{\nu+1} I^{\rho-1} \to 0$$

are exact, where $\delta \leq \delta' := \deg a_{\nu+1} \leq \delta + 1$. Using the minimal homogeneous generators of the middle term $I^\rho / J_\nu I^{\rho-1}$, we get a possibly non-minimal set of generators of $I^\rho / J_{\nu+1} I^{\rho-1}$ having degree $\rho\delta$, with relations of degrees $\rho\delta + 1$ and $(\rho-1)\delta + \delta' \leq \rho\delta + 1$. This implies that $I^\rho / J_{\nu+1} I^{\rho-1}$ has linear presentation. ☐

Proof of Theorem 2.1. Write $d = \dim S$ and let $a_1, \ldots, a_{\ell-1}$ be the $\ell - 1$ general elements of I that generate J. We apply Theorem 2.8 to the ideal $IR \subset R$, and verify the hypotheses as follows:

- $\ell(IR) = 1$: In fact $\ell(IR) \leq 1$ because the general elements $a_1, \ldots, a_{\ell-1}$ are part of a minimal generating set for a minimal reduction of I. If $\ell(IR) = 0$ then $(IR)^\rho = 0$ for $\rho \gg 0$, and hence Proposition 1.2 shows that $I^\rho = JI^{\rho-1}$, which is impossible since J is generated by fewer than $\ell(I)$ elements of I (see also [S, Theorem 4]).
- M is linearly presented as an S-module: By Proposition 1.2, $M \cong (I^\rho / JI^{\rho-1})(\rho\delta)$ for $\rho \gg 0$. By Lemma 2.10, this is linearly presented as an S-module.

We begin by showing that $\overline{R}$ is equidimensional of dimension $d - \ell + 1$:

Since IR is not nilpotent it follows that $\overline{R} \neq 0$. Any minimal prime of the ring $\overline{R}$ arises from a minimal prime Q of the ideal $K := J : I$ that does not contain I, and we need to show that $\operatorname{codim} Q = \ell - 1$ or, equivalently, $\operatorname{codim} K_Q = \ell - 1$. But $K_Q = J_Q$ since $I \not\subset Q$. Finally, the generators $\frac{a_1}{1}, \ldots, \frac{a_{\ell-1}}{1}$ of J_Q form a regular sequence on S_Q, because the general elements $a_1, \dots, a_{\ell-1}$ of I are a filter regular sequence with respect to I and $Q \not\supset I$.

Thus $\operatorname{codim} K_Q = \operatorname{codim} J_Q = \ell - 1$. Together with item (2) of Theorem 2.8, this gives the assertion of item **(1)**.

We now prove item **(2)**. We have seen in the previous step that $d - \ell + 1 = \dim \overline{R} \leq \dim R$. Thus $\dim \overline{R} = \dim R$ if and only if $\operatorname{codim} K \geq \ell - 1$.

Finally, item (3) of Theorem 2.8 now implies **(3)**. □

3. The minors of a $2 \times n$ matrix of linear forms

Throughout this section we assume $n \geq 3$.

We say that an ideal I of a Noetherian ring S is *s-residually S_2* if, for every $i \leq s$ and every i-residual intersection K of I, the ring S/K satisfies Serre's condition S_2; see [CEU] for more information.

Theorem 3.1. *Let T be a local Gorenstein ring containing a field of characteristic 0. Suppose that $I \subset S = T[[x_{1,1}, \dots, x_{2,n}]]$ is the ideal of 2×2 minors of the generic matrix*

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ x_{2,1} & \dots & x_{2,n} \end{pmatrix}.$$

Let $\ell := \ell(I)$, which is equal to $2n - 3$ by Proposition 4.2. The ideal I is $(\ell - 2)$-residually S_2. In particular, if $s \leq \ell - 1$ and K is an s-residual intersection of I, then K is unmixed of codimension exactly s. If, in addition, the residual intersection is geometric, then the image of I in S/K contains a non-zerodivisor.

Proof. Note that I satisfies G_{2n}. By [RWW, Theorem 4.3], $\operatorname{Ext}_S^{n+j-1}(S/I^j, S) = 0$ for $2 \leq j \leq n - 3 = (\ell - 2) - \operatorname{codim} I + 1$ (this is where we require characteristic 0). The same vanishing holds trivially for $j = 1$. By [CEU, Corollary 4.2], this implies that I is $(\ell - 2)$-residually S_2.

From [CEU, Proposition 3.1] we know that I is $(\ell - 1)$-parsimonious. Note that K is a proper ideal because s is less than the minimal number of generators of I. Thus we may apply [CEU, Proposition 3.3(a)] and conclude that K is unmixed of codimension exactly s. If, in addition, K is a geometric residual intersection, then $\operatorname{codim}(I + K) \geq s + 1$, so I is not in any associated prime of K. □

Corollary 3.2. *Suppose that I is the ideal of 2×2 minors of a $2 \times n$ matrix A over a local Gorenstein ring T containing a field of characteristic 0, and assume that $\operatorname{codim} I = n - 1$. If $s \leq 2n - 4$ and K is an s-residual intersection of I, then every minimal prime of K has codimension exactly s. If, in addition, the residual intersection is geometric, then I is in no minimal prime of K.*

Proof. We may assume that the entries of A are in the maximal ideal of T, since otherwise I is a complete intersection and the result follows, for instance, from [HU1, Theorem 5.1].

Let $J \subset I$ be an ideal with s generators such that $K = J : I$. Let $\widetilde{T} = T[\![x_{1,1}, \dots, x_{2,n}]\!]$, and let $\pi : \widetilde{T} \to T$ be the T-algebra map sending $x_{i,j}$ to the (i,j) entry $A_{i,j}$ of A. Note that the kernel of π is generated by the regular sequence $\alpha_{i,j} = x_{i,j} - A_{i,j}$. Let $\widetilde{I}$ be the ideal of 2×2 minors of the generic $2 \times n$ matrix

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ x_{2,1} & \dots & x_{2,n} \end{pmatrix},$$

so that $\pi(\widetilde{I}) = I$. Let $\widetilde{J} \subset \widetilde{I}$ be an ideal with s generators such that $\pi(\widetilde{J}) = J$, and let $\widetilde{K} = \widetilde{J} : \widetilde{I}$.

Since $\widetilde{T}/\widetilde{I}$ is Cohen-Macaulay, and the codimension of $\widetilde{I}$ is equal to that of I, the $2n$ elements $\alpha_{1,1}, \dots, \alpha_{2,n}$ form a regular sequence on $\widetilde{T}/\widetilde{I}$. It now follows from [HU1, Lemma 4.1] that $\sqrt{\pi(\widetilde{K})} = \sqrt{K}$. Thus

$$\operatorname{codim} \widetilde{K} \geq \operatorname{codim} \pi(\widetilde{K}) = \operatorname{codim} K \geq s,$$

so $\widetilde{K}$ is an s-residual intersection of $\widetilde{I}$. As $s \leq 2n-4$, Theorem 3.1 implies that $\widetilde{K}$ is unmixed of codimension exactly s.

Since $\operatorname{codim} \pi(\widetilde{K}) \geq s$, it follows that the sequence $\alpha_{1,1}, \dots, \alpha_{2,n}$ is part of a system of parameters of $\widetilde{T}/\widetilde{K}$, and thus all minimal primes of $\pi(\widetilde{K})$ have codimension exactly s.

Using $\sqrt{\pi(\widetilde{K})} = \sqrt{K}$ again, we see that all minimal primes of K have codimension exactly s.

The last statement follows immediately. □

Theorem 3.3. *Suppose that I is the ideal of 2×2 minors of a $2 \times n$ matrix A of linear forms in a polynomial ring S over a field of characteristic 0, and suppose that the entries of A span a vector space of dimension c.*

If I has codimension $\min\{n-1, c\}$, then the hypotheses and conclusions of Theorem 2.1 hold for I, and the ring $\overline{R}$ in Theorem 2.1 is $R_{\text{red}} = R/\sqrt{0}$. In addition, the equivalent conditions of Theorem 2.1(2) are satisfied.

Proof. Theorem 2.4(1) implies that the hypotheses and conclusions of Theorem 2.1 hold for I.

The ideal I satisfies G_c, and $\ell := \ell(I) \leq c$. Thus K is a geometric $(\ell-1)$-residual intersection. Hence the equivalent conditions of Theorem 2.1(2) are satisfied.

If $\operatorname{codim} I = n-1$, then Corollary 3.2 shows that I is not contained in any minimal prime of K. On the other hand, if $\operatorname{codim} I = c$ then $c = \ell$, so K is a complete intersection of codimension $c-1$, and again I is not contained in any minimal prime of K.

Since R is reduced away from $V(I)$, it follows in both cases that

$$\overline{R} = R/(0 : I^\infty) = R/(\sqrt{0} : I^\infty) = R/\sqrt{0}\,.$$ □

In the case of a generic $2 \times n$ matrix, we can be very explicit.

Theorem 3.4. *Let $S = k[x_1, \dots, x_n, y_1, \dots, y_n]$ be a polynomial ring over a field k of characteristic 0. Suppose that $I \subset S$ is the ideal of 2×2 minors of the generic matrix*

$$\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix}.$$

The ideal I has analytic spread $\ell := \ell(I) = 2n - 3$ and reduction number $r := r(I) = n - 3$ by Proposition 4.2.

Let J be an ideal generated by $\ell - 1$ general quadrics $a_1, \dots, a_{\ell-1}$ in I. Set $R = S/(J : I)$, $\overline{R} = R/(0 : I^\infty)$, and $M = M(IR)$. In addition to the assertions of Theorem 2.1 we have:

(1) *R has an isolated singularity, and $\overline{R} = R$.*
(2) *If a is a general quadric in I and $\rho \geq r$ then $(IR)^r(2r) \xrightarrow{a^{\rho-r}} (IR)^\rho(2\rho)$ is an isomorphism, so $M = (IR)^r(2r)$.*
(3) *$M \cong \omega_R(4)$.*
(4) *If k is algebraically closed, then $J : I$ is the intersection of $\frac{1}{n-1}\binom{2(n-2)}{n-2}$ linear prime ideals of codimension $2n-4$. These may be described as follows: the quadratic forms $a_1, \dots, a_{2n-4}$ may be regarded as linear forms in $k[I_2]$, which may be identified with the homogeneous coordinate ring of the Grassmanian $G(2, n)$ of $(n-2)$-dimensional subspaces of $k^n = \langle x_1, \dots, x_n \rangle$. Since the a_i are general, the space cut out by these forms intersects the Grassmanian in $\frac{1}{n-1}\binom{2(n-2)}{n-2}$ reduced points. Each of these points corresponds to a subspace $\langle L_1(x), \dots, L_{n-2}(x) \rangle$, which yields a linear prime ideal*

$$(L_1(x), \dots, L_{n-2}(x), L_1(y), \dots, L_{n-2}(y))$$

that is a minimal prime of $J : I$.

Proof. For $i \leq \ell - 1$ we set $J_i = (a_1, \dots, a_i)$. Since I satisfies G_{2n}, the ideal $J_i : I$ is a geometric residual intersection of I and is unmixed of codimension i by Theorem 3.1.

(1) By Theorem 3.1 the image of I in R contains a non-zerodivisor. Thus $R = \overline{R}$.

We now form the generic $(\ell - 1)$-residual intersection by tensoring with a polynomial ring T in $(\ell - 1)\binom{n}{2}$ new variables, and forming the residual intersection of I with respect to $\ell - 1$ generic linear combinations of the minors. All minimal primes of this residual intersection have codimension $\ell - 1$ according to Theorem 3.1. By [HU2, Theorem 2.4], applied to the punctured spectrum of S, the ring of the generic $(\ell - 1)$-residual intersection is nonsingular in codimension 3. Since the characteristic is 0, one sees as in the second half of the proof of Theorem 2.2 that the fiber over a general rational closed point of $\operatorname{Spec} T$ is also nonsingular in codimension 3. This fiber, which is a domain in codimension 3, surjects onto $R = S/\left((a_1, \ldots, a_{\ell-1}) : I\right)$, where $a_1, \ldots, a_{\ell-1}$ are linear combinations of the minors corresponding to the general rational closed point of $\operatorname{Spec} T$. As the codimensions of these two rings are the same, namely $\ell - 1$, the map is an isomorphism locally in codimension 3. Hence R is also nonsingular in codimension 3, and since R has dimension 4, its singularity is isolated.

(2) Let $a \in I$ be a general quadric. By Theorem 3.1 the element a is a non-zerodivisor on R. The ideal IR has analytic spread at most 1, and as IR contains a non-zerodivisor it has analytic spread exactly 1. Since $a_1, \ldots, a_{\ell-1}$ are general, [SH, Theorem 8.6.6] shows that the reduction number of IR is at most $r = r(I)$. Thus if $\rho \geq r$, then $a^{\rho-r} : (IR)^r(2r) \to (IR)^\rho(2\rho)$ is a surjection by the same reference, hence an isomorphism, and so $M = (IR)^r(2r)$.

(3) Let $M_{j,i} := I^j/J_iI^{j-1}(2j)$, which is generated in degree 0. We will show that $M \cong M_{r,\ell-1} \cong M_{\rho,\ell-1}$ for all $\rho \geq r$. Moreover, we will show that $M_{r+1,\ell-1} \cong \omega_R(4)$. For this we must estimate the depth of $M_{r,\ell-1}$, and for this in turn we first prove that certain syzygies of $M_{j,i}$ have low degree.

Lemma 3.5. *If $1 \leq j \leq r$ and $0 \leq i \leq n + j - 1$, then $\operatorname{depth} M_{j,i} \geq 4$ and the m-th free module in a minimal graded S-free resolution of $M_{j,i}$ is generated in degree m for all $m > i$.*

Proof. Note that if $n \leq 3$ then $r = 0$, so the statement is vacuous. We thus assume $n \geq 4$.

We adopt the notation of the proof above. We first consider the case $j = 1$. Set $g := \operatorname{codim} I = n - 1$. We must treat the cases with $i \leq n = g + 1$. For $i \leq g$ the ideal J_i is a complete intersection of codimension i. Since S/I is Cohen-Macaulay, the link $S/(J_g : I)$ is also Cohen-Macaulay. Thus by [U, Proposition 1.7(b)], the depth of S/J_{g+1} is at least $\dim S - g - 1 = n$. In each of these cases the length of the minimal graded free resolution of $J_i(2)$ is at most $i - 1$. On the other hand, the minimal graded free resolution of $I(2)$ has length $n - 2$ and is linear. Thus the long exact sequence in $\operatorname{Tor}_\bullet^S(k, -)$ proves both statements of the Lemma for $M_{1,i} = (I/J_i)(2)$.

We now do induction on i, assuming that $j \geq 2$.

If $i = 0$, then $M_{j,i} = I^j(2j)$. By [ABW, Theorem 5.4 and the beginning of its proof] the minimal graded free resolution of I^j is linear and of length at most $2n-4$ for every j, as required.

We now suppose $i > 0$. Consider the sequence

$$0 \to M_{j-1,i-1} \xrightarrow{\alpha} M_{j,i-1} \longrightarrow M_{j,i} \to 0\,,$$

where α is multiplication by a_i. It follows from the definitions that the sequence is right exact. We will show that it is exact.

Since I is a complete intersection on the punctured spectrum, [U, Lemma 2.7(a)] with $s := \ell - 1$ shows that the left-hand map in this sequence is a monomorphism locally on the punctured spectrum because $r \leq 2n-4-(n-1)+2$. By induction $M_{j-1,i-1}$ has positive depth, so the sequence is also left exact as claimed.

Let $\widetilde{\alpha} : F_\bullet \to G_\bullet$ be the map of minimal graded free resolutions induced by α. The minimal graded free resolution $H_\bullet$ of $M_{j,i}$ is a direct summand of the mapping cone of $\widetilde{\alpha}$. Hence it follows by induction that H_m is generated in degree m for all $m > i$.

Finally, we must show that the length of $H_\bullet$ is at most $2n-4$. By the induction hypothesis, the length is at most $2n-3$. Further, H_{2n-3} is a direct summand of F_{2n-4}. Moreover, F_{2n-4} is generated in degree $2n-4$ because $2n-4 = n+r-1 \geq n+j-1 > i-1$. Thus H_{2n-3} is generated in degree $2n-4$. Since $H_\bullet$ is the minimal graded free resolution of a module generated in degree zero, it follows that H_{2n-3} is in fact 0, as required. □

Continuing with the proof of part (3), we have a natural surjection of R-modules $\pi : M_{r,\ell-1} = (I^r/JI^{r-1})(2r) \twoheadrightarrow (IR)^r(2r) = M$. Recall that $J : I$ is a geometric $(\ell-1)$-residual intersection. Moreover, on the punctured spectrum I is a complete intersection, hence by [U, Lemma 2.7(c)] with $s := \ell - 1$, the kernel of π is 0 locally on the punctured spectrum, again because $r \leq 2n-4-(n-1)+2$. On the other hand, $M_{r,\ell-1}$ has depth ≥ 1 by Lemma 3.5, so the kernel is 0 and we see that π is an isomorphism.

Let $a \in I$ be a general quadric, and consider the diagram

$$\begin{array}{ccc} M_{r,\ell-1} & \xrightarrow[\pi]{\cong} & (IR)^r(2r) \\ \Big\downarrow{\scriptstyle a^{\rho-r}} & & \Big\downarrow{\scriptstyle a^{\rho-r}} \\ M_{\rho,\ell-1} & \longrightarrow & (IR)^\rho(2\rho) \end{array}$$

with $\rho \geq r$. By item (2) the right-hand vertical map is an isomorphism. It follows that the left-hand vertical map is a monomorphism. For $\rho \geq r = r(I)$ we have again by [SH, Theorem 8.6.6] that

$$I^\rho = (J,a)^{\rho-r}I^r = JI^{\rho-1} + a^{\rho-r}I^r,$$

so the left-hand vertical map is also a surjection. Thus all the maps in the square are isomorphisms, so $M \cong M_{r,\ell-1} \cong M_{\rho,\ell-1}$.

By Theorem 3.1 none of the associated primes of $J_i : I$ contains I for $i \leq \ell - 1$. It follows that the inclusion

$$J_i : I \subset (J_{i-1} : I, a_i) : I$$

is an equality, since, after localizing at any associated prime P of $J_i : I$, both ideals become equal to $(J_i)_P$.

Again by Theorem 3.1 the left-hand side, and thus also the right-hand side, has codimension exactly i. This verifies the hypothesis of [EU, Theorem 4.1], and thus there is a natural homogeneous map

$$\mu : \big(I^{(\ell-1)-g+1}/J_{\ell-1}I^{(\ell-1)-g}\big)(2(\ell-1)-2n) \longrightarrow \omega_{S/(J_{\ell-1}:I)} = \omega_R$$

that is an isomorphism on the punctured spectrum since I is locally a complete intersection there. (Though [EU, Theorem 4.1] was proven in the local case, the twists can be recovered from the proof.)

We have $\ell-1-g+1 = n-2 = r+1$ and $2(\ell-1)-2n = 2(r+1)-4$, so the source of μ is $M_{r+1,\ell-1}(-4) \cong M_{r,\ell-1}(-4) \cong M(-4)$. Since this module has depth ≥ 2 by Lemma 3.5, it follows that μ is an isomorphism, proving (3).

(4) We next prove that the linear prime ideals described in (4) contain $J : I$. A point z on the Grassmannian corresponds to a $2 \times n$ matrix of rank 2, which, after coordinate transformation, may be taken to be

$$\begin{pmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

The Plücker coordinates of z are then

$$p_{\mu,\nu} = \begin{cases} 1 & \text{if } (\mu,\nu) = (n-1,n) \\ 0 & \text{otherwise .} \end{cases}$$

We may write the a_i in the form $\sum \lambda^i_{\mu,\nu} p_{\mu,\nu}$. To say the point z is on the linear section defined by the a_i means that the coefficients $\lambda^i_{n-1,n}$ are all 0. Thus the a_i are in the ideal $L := (x_1, \dots, x_{n-2}, y_1, \dots, y_{n-2})$, which is the prime corresponding to z. Finally, this implies that $J : I \subset L$ because $I \not\subset L$. In particular this shows that the multiplicity of R is at least the degree of the Grassmanian, which is $\frac{1}{n-1}\binom{2(n-2)}{n-2}$.

The degree $2r$ component of the graded module $(IR)^r$ is a homomorphic image of the degree r component of $k[I_2]/(a_1, \dots, a_{2n-4})$. The a_i are general linear forms in $k[I_2]$, the coordinate ring of the Grassmannian in the Plücker embedding. Because this ring is Cohen-Macaulay of dimension $2n-3$, the ring $k[I_2]/(a_1, \dots, a_{2n-4})$ is a one-dimensional Cohen-Macaulay ring of multiplicity equal to the degree of the Grassmannian, and thus the minimal number of generators of $(IR)^r$ is bounded by the degree of the Grassmannian. On the other hand, by Theorem 2.1(2), the R-module $M = (IR)^r(2r)$ is an Ulrich module of rank 1, which shows that the minimal number of generators of $(IR)^r$ is equal to the multiplicity of R.

We deduce that the multiplicity of R is equal to the number of linear minimal primes as above. Since R is unmixed of codimension $2n-4$, this shows that $J : I$ is the intersection of these linear primes, proving (4). ☐

4. Determinantal ideals

Theorem 3.7 of Bruns, Conca and Varbaro [BCV] gives a large family of determinantal ideals whose powers have linear resolutions, reproduced in part (2) of the following theorem:

Theorem 4.1. *Suppose that A is an $m \times n$ matrix of linear forms in a polynomial ring over a field, with $m \leq n$, and suppose that the entries of A generate a vector space of dimension c. Let I be the ideal of $m \times m$ minors of A. If either*

(1) $\operatorname{codim} I = c$, *or*
(2) $\operatorname{codim} I = n - m + 1$ *and for $2 \leq k \leq m-1$ the ideal of $k \times k$ minors of A has codimension $\geq \min\{(m-k+1)(n-m)+1, c\}$,*

then every power of I has a linear resolution.

Since the powers of the ideal of the Veronese surface also have linear resolutions ([BCV, Proposition 3.12]), the powers of the ideal of every geometrically integral scheme of minimal degree have linear resolutions.

It seems plausible that if I is the ideal of maximal minors of a matrix of linear forms and I itself has linear presentation (respectively, linear resolution), then all its powers do too. In the case $m = 2$, the condition for I itself to have linear presentation or resolution is known in terms of the Kronecker classification of linear $2 \times n$ matrices; see [CJ] and [ZN]. In fact, the condition that high powers have linear resolution appears to be more general still: for example, let I be the ideal of 2×2 minors of the matrix

$$\begin{pmatrix} 0 & x_1 & \cdots & x_5 & \mid & y_0 & y_1 & y_2 \\ x_1 & \cdots & x_5 & 0 & \mid & y_1 & y_2 & y_3 \end{pmatrix}.$$

According to Macaulay2 [M2], the Betti tables of the first 3 powers of I (in characteristic 101) are:

```
2: 36 169 383 514 430 221 64   8 .
3: .  .       3  17  40  50 35 13 2


4: 414 2542 7124 11752 12385 8494 3688 924 102
5: .   .    .    .     2     10   20   20  10   2


6: 2544 17028 50967 88676 97776 69804 31458 8172 936
```

and the 4th power also has linear resolution, suggesting that higher powers will too.

Proof of Theorem 4.1. Suppose first that $\operatorname{codim} I = c$, so that in particular $c \leq n - m + 1$. We may harmlessly assume that the entries of A span the space of all linear forms and that the ground field is infinite. We may write the ambient polynomial ring S as T/J where T is a polynomial ring in mn variables in such a way that A is the specialization of a generic matrix B. For a generic choice of intermediate specialization T' of dimension $n-m+1$ with

$$T \twoheadrightarrow T' \twoheadrightarrow S\,,$$

the ideal of $m \times m$ minors I' of the specialization B' of B to T' will have codimension $n-m+1$. It follows that the minimal resolution of I' is the Eagon-Northcott complex, and thus the $\binom{n}{m}$ minors of B' are linearly independent. Since the vector space dimension of the degree m component of T' is also $\binom{n}{m}$, the ideal I' is the m-th power of the maximal homogeneous ideal of T'. Specializing further to S we see that I is the m-th power of the maximal homogeneous ideal.

The sufficiency of (2) is [BCV, Theorem 3.7]. □

Generic matrices. The analytic spread and reduction number of an ideal of maximal minors of a generic matrix are known; for the reader's convenience we reproduce the result.

Proposition 4.2. *Let $\mathcal{X}$ be the generic $m \times n$ matrix of variables of the ring $S = k[x_{1,1}, \dots, x_{m,n}]$, with $m \le n$, and let $I = I_m(\mathcal{X})$ be the ideal of $m \times m$ minors. The analytic spread of I is $\ell(I) = m(n-m)+1$ and, when the ground field k is infinite and $m < n$, the reduction number of I is $r(I) = \ell(I) - n$.*

Proof. Let $\mathfrak{m} \subset S$ be the ideal generated by the entries $x_{i,j}$ of $\mathcal{X}$. The special fiber ring $\mathcal{F}(I) := S/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \cdots$ of I is the homogeneous coordinate ring of the Grassmannian $G(m,n)$ in its Plücker embedding. Since $G(m,n)$ is a variety of dimension $m(n-m)$, the analytic spread of I is $\ell(I) = \dim \mathcal{F}(I) = m(n-m)+1$.

Now assume that the ground field is infinite. The reduction number $r(I)$ of $\mathcal{F}(I)$ is the maximal degree of a socle element after reducing $\mathcal{F}(I)$ modulo a general linear system of parameters [SH, Theorem 8.6.6]. Because the homogeneous coordinate ring of the Grassmannian is Cohen-Macaulay, we can relate this to the degree of the generators of the canonical module. The canonical module of the Grassmannian $G(m,n)$ is $\mathcal{O}_G(-n)$ in the Plücker embedding (see for example [EH, Proposition 5.25]). Thus modulo a general sequence of $\ell(I) = m(n-m)+1$ linear forms, the socle is in degree $\ell(I) - n$, and the reduction number is thus $r(I) = \ell(I) - n$. □

It is interesting to ask when an ideal of maximal minors has an $(\ell - 1)$-residual intersection, so that part (2) of Theorem 2.1 applies. We thank Monte Cooper and Edward Price for pointing out an error in a previous version of the next Proposition, and providing a correction.

Proposition 4.3. *Let $\mathcal{X}$ be the generic $m \times n$ matrix of variables of the ring $k[x_{1,1}, \dots, x_{m,n}]$, with $m \le n$, and let $I = I_m(\mathcal{X})$ be the ideal of $m \times m$ minors. Let $\ell := \ell(I)$, which is $m(n-m)+1$ by Proposition 4.2.*

(1) *The ideal I satisfies G_ℓ if and only if one of the following holds:*

- $m \leq 2$;
- $n \leq m+2$;
- $n = m+3$ *and* $m \leq 5$.

(2) *The ideal* I *satisfies* $G_{\ell-1}$ *if and only if it satisfies* G_ℓ *or*
- $n = 7$ *and* $m = 3$;
- $n = m+3$ *and* $m \leq 6$.

(3) I *does not have any* $(\ell-1)$*-residual intersection if one of the following holds:*
- $n = m+3$ *and* $m = 10$ *or* 11 *or* $m \geq 14$;
- $n = m+4$ *and* $m \geq 6$;
- $n \geq m+5$ *and* $m \geq 3$.

Proof. For every prime $P \in V(I)$ one has $P \in V(I_{t+1}(\mathcal{X})) \setminus V(I_t(\mathcal{X}))$ for some t with $0 \leq t \leq m-1$, and the minimal number of generators of I_P is exactly $\binom{n-t}{m-t}$.

Thus the condition G_s holds for I if and only if

$$\binom{n-t}{m-t} \leq \operatorname{codim} I_{t+1}(\mathcal{X}) = (m-t)(n-t)$$

whenever $\operatorname{codim} I_{t+1}(\mathcal{X}) \leq s-1$. Given this, the verification of items (1) and (2) is not difficult.

If I admits an $(\ell-1)$-residual intersection, then locally in codimension $\ell-2$, the ideal I can be generated by $\ell-1$ elements. In other words,

$$\binom{n-t}{m-t} \leq \ell - 1 = m(n-m)$$

whenever $\operatorname{codim} I_{t+1}(\mathcal{X}) = (m-t)(n-t) \leq \ell-2$. Again, part (3) follows easily from this. ☐

5. Implications and special cases of the conjectures

5.1. Implications of Conjecture 0.1.

Proof of Proposition 0.3. We may assume that k is infinite.
(1) The result is trivial if I is a complete intersection, so we assume that it is not. In this case, $\ell > g$ by [CN]. Thus $\ell = g+1$. It follows that the ideal $J : I$ of Conjecture 0.1 is a link, hence unmixed, and the ideal $IR \subset R := S/(J : I)$ is principal. As I is generically a complete intersection, the link is geometric and IR is generated by a single non-zerodivisor. If $I^\rho R$ were a maximal Cohen-Macaulay R-module for some $\rho > 0$, then $R = S/(J : I)$ is Cohen-Macaulay, hence so is S/I because the unmixed ideal I is also a link of $J : I$.
(2) We may assume that $I \neq 0$. Because I is of linear type, ℓ is the minimal number of generators of I. Let ϕ be a homogeneous presentation matrix of I with respect to a general choice of homogeneous generators $f_1, \ldots, f_\ell$ of I. The ideal P defining

the symmetric algebra of I as a quotient of $S' := k[T_1, \dots, T_\ell] \otimes_k S$ is generated by the entries of the row vector $(T_1, \dots, T_\ell) \circ \phi$.

Let $S'' = k(T_1, \dots, T_\ell) \otimes_k S$. Over S'', the row vector $(T_1, \dots, T_\ell) \circ \phi$ is the last row of a presentation matrix of IS'' with respect to some homogeneous generators $g_1, \dots, g_\ell$. Thus PS'' has the form $(g_1, \dots, g_{\ell-1}) : IS''$. Since $f_1, \dots, f_\ell$ were chosen generally over k, they are general over $k(T_1, \dots, T_\ell)$, and it follows that $g_1, \dots, g_\ell$ are general over $k(T_1, \dots, T_\ell)$.

By hypothesis, $\mathrm{Sym}(I) = \mathcal{R}(I)$, a domain of dimension $d + 1$. Thus PS'' is a geometric $(\ell - 1)$-residual intersection of IS'', and $I(S''/PS'')$ is generated by a non-zerodivisor. By Conjecture 0.1, $I^\rho(S''/PS'')$ is a maximal Cohen-Macaulay module over S''/PS'' for some $\rho > 0$. Since this is a principal ideal generated by a non-zerodivisor, $S''/PS'' = \mathrm{Sym}(I) \otimes_{S'} S'' = \mathcal{R}(I) \otimes_{S'} S''$ is Cohen-Macaulay, and it follows that $\mathcal{R}(I)_{(x_1,\dots,x_d)\mathcal{R}(I)}$ is too. □

5.2. **Special cases of the conjectures.** The next result has been proven with an additional hypothesis in [H1, Theorem 2.6].

Theorem 5.1. *Let S be a local Gorenstein ring and let $I \subset S$ be an almost complete intersection ideal such that S/I is equidimensional. If $\operatorname{depth}(S/I)_P \geq \frac{1}{2}\dim(S/I)_P$ for every $P \in V(I)$, then S/I is Cohen-Macaulay.*

Proof. Let $J \subset I$ be a complete intersection of the same codimension as I such that I/J is cyclic, and consider $K = J : I$. Our assumptions imply that I is unmixed. Therefore $I = J : K$ and it suffices to prove the Cohen-Macaulayness of S/K.

Notice that $\omega_{S/K} \cong I/J \cong S/K$. Thus by [HO, Theorem 1.6] or [H2, Lemma 5.8] it suffices to show that

$$\operatorname{depth}(S/K)_P \geq 1 + \frac{1}{2}\dim(S/K)_P$$

for every $P \in V(K)$ with $\dim(S/K)_P \geq 2$. We may assume that $P \in V(I)$ since otherwise $(S/K)_P = (S/J)_P$ is Cohen-Macaulay. But then $\operatorname{depth}(S/I)_P \geq \frac{1}{2}\dim(S/I)_P$ and $\dim(S/I)_P = \dim(S/K)_P$. Now the exact sequence

$$0 \to S/K \cong I/J \longrightarrow S/J \longrightarrow S/I \to 0$$

shows that $\operatorname{depth}(S/K)_P \geq 1 + \frac{1}{2}\dim(S/K)_P$, as required. □

Notice that an almost complete intersection ideal $I \subset S$ satisfies the assumptions of Theorem 5.1 if I is unmixed and S/I is almost Cohen-Macaulay, which means that $\operatorname{depth} S/I \geq \dim S/I - 1$.

Corollary 5.2. *If $I \subset S = k[x_1, \dots, x_d]$ is an unmixed monomial almost complete intersection, then S/I is Cohen-Macaulay.*

Proof. The Taylor resolution shows that the projective dimension of the polynomial ring modulo a monomial ideal is bounded by the minimal number of generators of the ideal; thus any monomial almost complete intersection is almost Cohen-Macaulay. □

Corollary 5.3. *With hypotheses as in Theorem 5.1, suppose in addition that the residue field of S is infinite and that I is generically a complete intersection. Let J be an ideal generated by $g := \operatorname{codim} I$ general elements of I, and let $K = J : I$. For all ρ the module $I^\rho(S/K)$ is an ω-self-dual Cohen-Macaulay S/K-module. In particular, Conjecture 0.1 is true under these additional hypotheses.*

Proof. We may assume that I is not a complete intersection. Thus by [CN] the analytic spread of I is $g+1$, and K is a geometric link of I. By Theorem 5.1, the ring S/K is Gorenstein, and $I(S/K)$ is generated by a non-zerodivisor. The conclusion is now immediate. □

When $J : I$ is a $(g+1)$-residual intersection, I/J itself has good properties:

Proposition 5.4. *Let S be a local Gorenstein ring with infinite residue field and let $I \subset S$ be generically a complete intersection of codimension g such that S/I is Cohen-Macaulay. Let $J \subsetneq I$ be generated by $g+1$ general elements of I and set $K = J : I$. The module I/J is $\omega_{S/K}$-self-dual and is a Cohen-Macaulay module of dimension $\dim S - g - 1 = \dim S/K$.*

Proof. We note that the ideal $K = J : I$ has codimension $\geq g+1$, hence is a $(g+1)$-residual intersection of I. Since S/I is Cohen-Macaulay and I is generically a complete intersection, K has codimension exactly $g+1$ ([U, Proposition 1.7(a)]). A result of van Straten and Warmt implies that I/J is $\omega_{S/K}$-self-dual; see Theorem 2.1 of [EU] where Huneke's simplified proof is given.

Let $J_g \subset J$ be the ideal generated by g general elements of J. We obviously have $J_g : J \supset J_g : I \supset (J_g : J)K$. Every associated prime of $J_g : I$ has codimension g, and hence does not contain K. Thus, $J_g : J = J_g : I$. Therefore, $J/J_g \cong S/(J_g : I)$, which has depth $\dim S - g$. It follows that $\operatorname{depth} S/J \geq \dim S - g - 1$, so $\operatorname{depth} I/J \geq \dim S - g - 1$; that is, I/J is a maximal Cohen-Macaulay S/K -module. □

Remark 5.5. If in addition to the hypotheses of Proposition 5.4 the ideal I satisfies G_{g+2}, then the module I/J is naturally isomorphic to $I(S/K)$; this follows because K is a geometric $(g+1)$-residual intersection of I due to the G_{g+2} assumption, and so $J = I \cap K$ by [U, Proposition 1.7(c)].

Remark 5.6. There are certainly further phenomena to explain in these directions. For example, let I be the ideal of 2×2 minors of the generic 3×3 matrix over a field k of characteristic 0, and let S be the polynomial ring in 9 variables over k. We have S/I is Gorenstein, $\operatorname{codim} I = 4$, and $\ell(I) = 9$ according to [CN] (or because I is of linear type by [H3, Theorem 2.4]).

For s with $\operatorname{codim} I = 4 \leq s \leq 8 = \ell(I) - 1$, let $K_s = J_s : I$ and $R_s = S/K_s$, where J_s is generated by s general forms of degree 2 in I.

By Brodmann's inequality [B, (2) Theorem] the rings S/I^ρ have depth 0 for $\rho \gg 0$. They have linear resolution for all $\rho \geq 2$ according to [R, Theorem 5.1]. The modules IR_s are maximal Cohen-Macaulay R_s-modules and:

- $\operatorname{depth} R_4 = 5$; so this ring is a Cohen-Macaulay almost complete intersection;
- $\operatorname{depth} R_5 = 1$ and $\operatorname{depth} R_6 = 0$;
- R_7 and R_8 are Gorenstein rings of dimensions 2 and 1, respectively.

The statement about IR_s and the statements in the last two bullets are the result of Macaulay2 computations [M2], though Theorems 2.1(1) and 2.2 already imply that $R_8/0 : (IR_8)^\infty$ is Gorenstein.

6. Necessity of the hypotheses

We next give examples showing that the hypotheses in Conjecture 0.1 cannot simply be dropped. The following examples were discovered and checked using the program Macaulay2 [M2].

Example 6.1. We first consider the ideal K of a smooth rational quartic in $\mathbb{P}^3_{\mathbb{Q}}$ as a general link: Let $K' \subset \mathbb{Q}[x_1, \dots, x_4]$ be the ideal of the smooth rational quartic in $\mathbb{P}^3_{\mathbb{Q}}$, and let $J' \subset K'$ be the ideal generated by two general cubic forms in K'. Let $I' = J' : K'$, which is the ideal of a smooth genus 1 quintic curve in $\mathbb{P}^3_{\mathbb{Q}}$. It turns out that I' is minimally generated by 5 cubic forms. If a is a general cubic in I' and $I := (J', a)$, then I is minimally generated by 3 forms of degree 3 and is generically a complete intersection, and $\ell(I) = 3$ by [CN]. Finally, let J be the ideal generated by two general cubics in I.

The ideal $K := J : I$ is again the ideal of a smooth rational quartic, and thus neither $R := S/K$ nor any power of the principal ideal IR is Cohen-Macaulay.

Here all the assumptions of Conjecture 0.1 are satisfied except that I is not unmixed. Note that IR is not unmixed either.

Example 6.2. Let k be an infinite field, and let $X \subset \mathbb{P}^{d-1}_k$ be an abelian surface embedded by a complete linear series of high degree. Let $S = k[x_1, \dots, x_d]$ be the homogeneous coordinate ring of $\mathbb{P}^{d-1}_k$, and let I_X be the homogeneous ideal of X. The canonical module ω of S/I_X is isomorphic to S/I_X as a graded module, and S/I_X is not Cohen-Macaulay because $\mathrm{H}^1(\mathcal{O}_X) \neq 0$. Let I be a homogeneous geometric link of I_X, so that I is an unmixed but not Cohen-Macaulay almost complete intersection that is generically a complete intersection.

Let K be any homogeneous link of I with respect to a subset of a system of homogeneous minimal generators, chosen sufficiently generally that K is a geometric link of I. Set $R = S/K$. Since I is an almost complete intersection, the ideal IR is generated by a single non-zerodivisor. The canonical module of R is isomorphic to IR, up to shift – that is, R is a quasi-Gorenstein ring.

Since IR is generated by a non-zerodivisor and R is not Cohen-Macaulay, no power of IR can be a Cohen-Macaulay module (though all powers of IR are ω_R-self-dual up to a shift).

Here all the assumptions of Conjecture 0.1 are satisfied except possibly that I is generated in a single degree.

Now specialize to the case where X is the Segre embedding of the product of two smooth cubic curves in $\mathbb{P}^2_{\mathbb{Q}}$. In fact I is not generated in a single degree.

Example 6.3. Let $S = \mathbb{Q}[x_1, \ldots, x_7]$ and let

$$I = (x_1x_4x_7^4, x_5x_6^2x_7^3, x_1x_4x_5^2x_6^2, x_1^2x_3x_4x_5x_6).$$

The ideal I has codimension 2 and analytic spread 4. If J is generated by 3 general forms of degree 6 in I, then $K := J : I$ is a 3-residual intersection. Set $R = S/K$. Because IR is principal but not nilpotent by Proposition 1.2, the high powers of IR are isomorphic, up to a shift, to $S/(J : I^\infty)$, which is not Cohen-Macaulay. It is interesting to note that R is Cohen-Macaulay.

Here all the assumptions of Conjecture 0.1 are satisfied except that I has embedded components and K is not a geometric residual intersection of I.

The following examples show that none of the hypotheses listed in Conjecture 0.2 and Question can simply be dropped.

Example 6.4. Let $H \subset \mathbb{Q}[x_1, x_2, x_3]$ be the ideal of maximal minors of the matrix

$$\begin{pmatrix} x_1^2 & x_1^2 & x_2^2x_3 \\ x_2^2 & x_2^2 & x_1x_2x_3 \\ x_2^2 & x_3^2 & x_1x_2x_3 \\ x_3^2 & 0 & x_1^3 \end{pmatrix}.$$

Let R be the ring defined by the link of H with respect to the minors deleting the first and second rows. The ring R is Cohen-Macaulay and generically a complete intersection of dimension 1. The canonical ideal I is generated in a single degree, but no power of I is ω_R-self-dual up to a shift. (Note that because R is 1-dimensional only powers up to the reduction number of I need to be checked.)

Here all the assumptions of Conjecture 0.2 are satisfied except that R is not reduced.

Example 6.5. Let R be the homogeneous coordinate ring of 11 points in $\mathbb{P}^2_{\mathbb{Q}}$, 6 of which are general and 5 are on a line. The ring R is reduced and 1-dimensional, but the canonical ideal I has no self-dual power. Here all the assumptions of Conjecture 0.2 are satisfied except that I is not equigenerated.

In this case the fractional ideal I is generated in degrees -3 and -2. If we take I' to be the truncation of I in degree -2, then the square of I' is self-dual, as is every higher power, giving a positive answer for Question in this case.

Example 6.6. Let R be the homogeneous coordinate ring of 5 points in $\mathbb{P}^2_{\mathbb{Q}}$, of which 3 lie on one line and 3 on another line (the point of intersection is one of the 5 points):

$$[1:0:0], [0:1:0], [1:1:0], [1:1:1], [0:0:1].$$

The ideal

$$I = (x_1^2 + x_2^2 + x_3^2, x_2x_3)R$$

is equigenerated, but has no self-dual power.

Here all the assumptions of Question are satisfied except that I and the canonical ideal have no power in common up to a shift.

Curiously, the minimal graded R-free resolutions of I and the canonical ideal of R have the same graded betti numbers for at least 10 steps. However, I and I^2 are both generated by 2 elements, whereas the square and cube of the canonical ideal require 3 generators.

Example 6.7. Let $S = \mathbb{Q}[x_1, \dots, x_7]$ and let

$$I = (x_3x_5x_7^4,\ x_2^2x_6^2x_7^2,\ x_2x_3x_4x_5x_6^2,\ x_1x_2x_3x_4x_5x_6).$$

The codimension of I is 2, its analytic spread is 4, and I satisfies G_4.

If J is generated by 3 general forms of degree 6 in I, then $K := J : I$ is a geometric 3-residual intersection, necessarily of codimension 3 by [M, Theorem 4.1] as $3 < 4 = \ell(I)$. The ring $R := S/K$ is Cohen-Macaulay, hence reduced by Theorem 2.2, but R is not Gorenstein.

Since IR is principal, generated by a non-zerodivisor, no power of IR can be self-dual.

Here, as in Example 6.6, all the assumptions of Question are satisfied except that IR and the canonical ideal of R have no power in common up to a shift.

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