The Loewner Equation with complex drivers

Joan Lind, Jeffrey Utley

Outline

- Properties of the LE
- Hulls with complex-valued driving functions $c\sqrt{1-t}, c\sqrt{t}$
- Phase transitions for $c\sqrt{1-t}$ at time 1
- Further questions

Review of the LE

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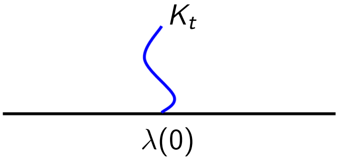

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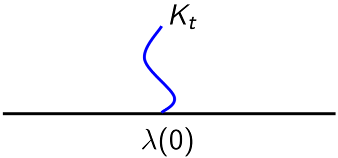

Time t solution is a conformal map $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$.

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Duality Property

$$L_{t,\lambda} = iR_{t,-i\lambda(t-\cdot)}, \quad R_{t,\lambda} = iL_{t,-i\lambda(t-\cdot)}.$$

LE with real drivers

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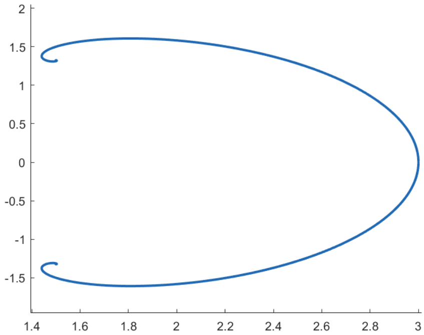
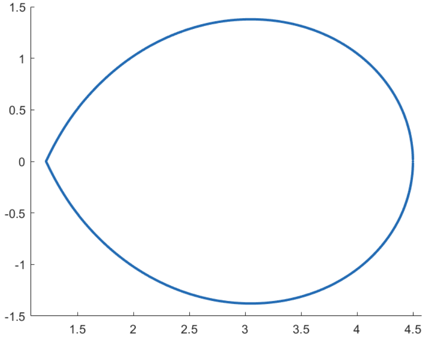


Simple hull L_1 for $c = 3$



Non-simple hull L_1 for $c = 4.5$

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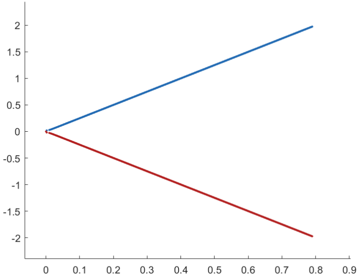

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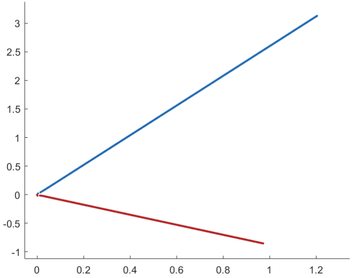

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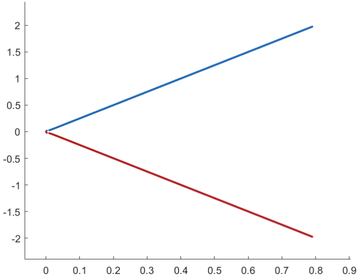

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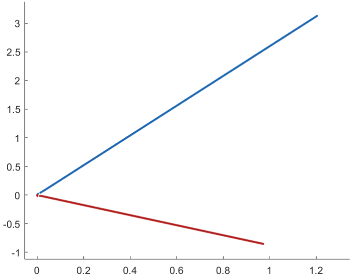

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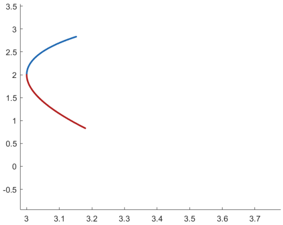
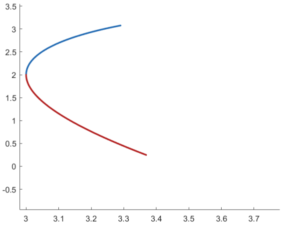
We call the blue curve the upper hull and the red curve the lower hull.

LE with complex drivers: $\lambda(t) = (3+2i) + (1-i)t$



$t = 0.25$



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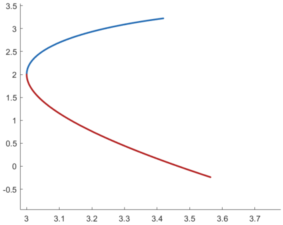
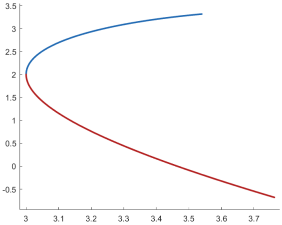


$t = 0.75$



$t = 1$

LE with complex drivers

Theorem (Tran)

There exists $\sigma > 0$ *so that when* $\lambda : [0, T] \to \mathbb{C}$ *has* $Lip(1/2)$ *norm less than* σ, *then* $L_t = \gamma[-t, t]$ *for a simple curve* $\gamma : [-t, t] \to \mathbb{C}$. *Moreover,*

$$\gamma(t) = \lim_{y \to 0^+} g_t^{-1}(\lambda(t) + iy) \quad \textit{and} \quad \gamma(-t) = \lim_{y \to 0^-} g_t^{-1}(\lambda(t) + iy).$$

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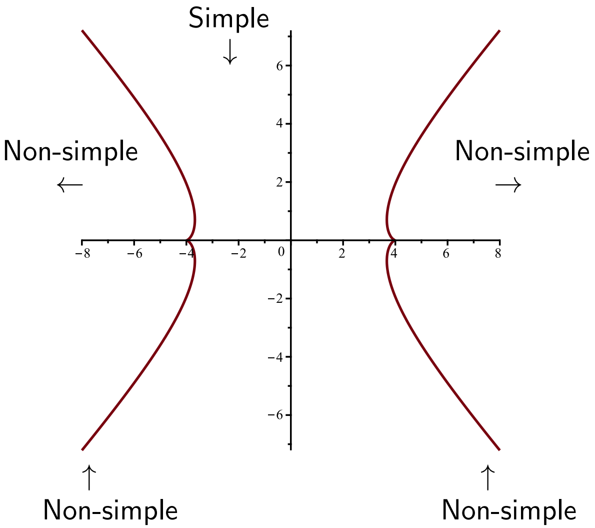
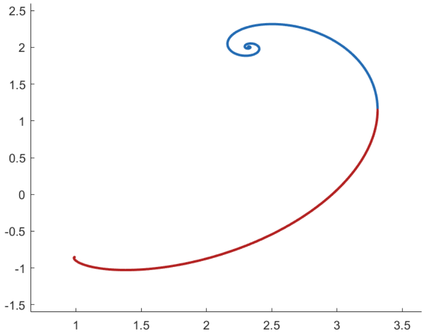
No, we have $\sigma < 3.723$ due to phase transitions for $\mathbb{C}$-valued $c\sqrt{1-t}$.

Complex-valued $c\sqrt{1-t}$: phase transition image

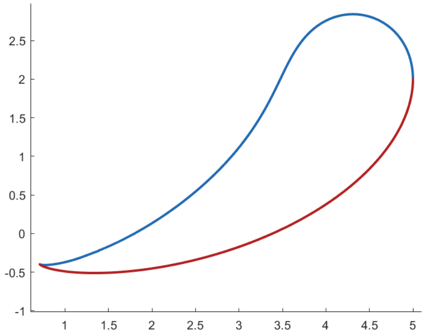


Complex-valued $c\sqrt{1-t}$: simple and non-simple hulls

Examples of simple and non-simple hulls at time 1:



Simple curve $c = 3.31 + 1.15i$



Non-simple curve $c = 5 + 2i$

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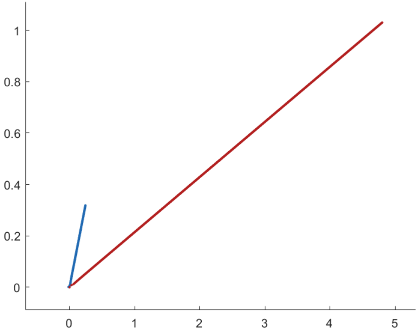
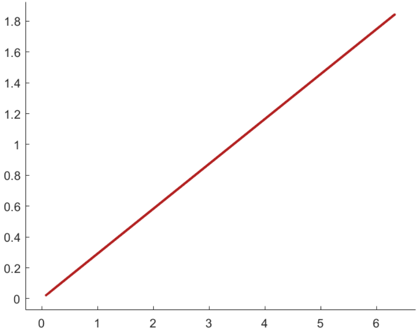


R_1 for $c = 3.31 + 1.15i$



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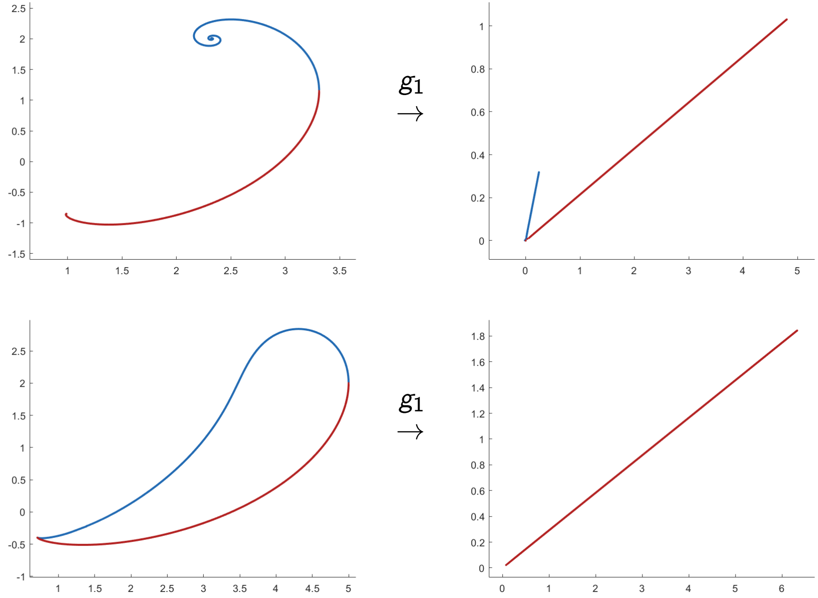

The concatenation property

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In fact,

$$g_t(K_{t+s,\lambda} \setminus K_{t,\lambda}) = K_{s,\lambda(t+\cdot)} \setminus \{\lambda(t)\},$$

$$K_{t+s,\lambda} = K_{t,\lambda} \cup g_t^{-1}(K_{s,\lambda(t+\cdot)} \setminus \{\lambda(t)\}).$$

The concatenation property for $\mathbb{C}$-valued hulls

For complex-valued driving functions, the property is not so nice:

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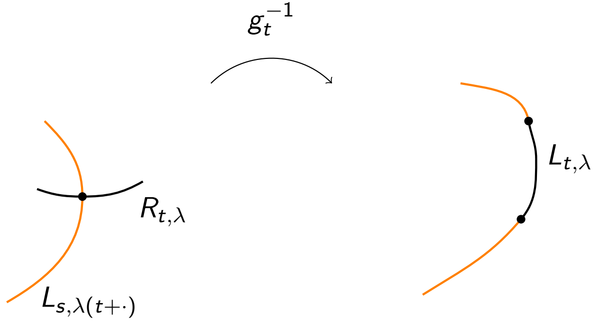

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On an interval $[s, t] \subset [0, 1)$, $\lambda(t) = c\sqrt{1-t}$ has Lip$(\frac{1}{2})$ norm $|c|\sqrt{t-s}$, so the theorem applies on small enough intervals.

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We can build up the simple curve L_t (for any $t < 1$) by attaching simple curves using the concatenation property.

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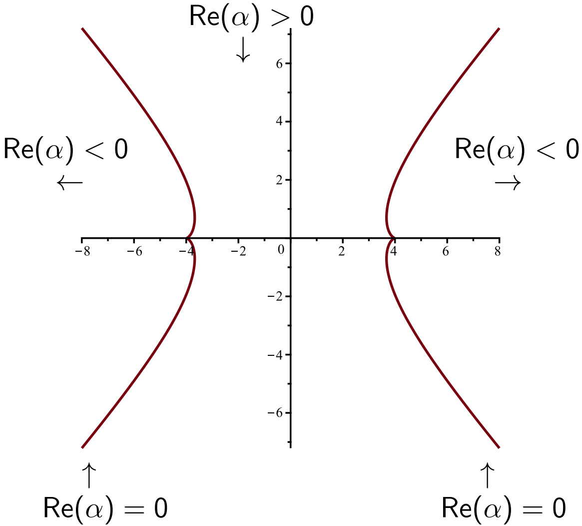

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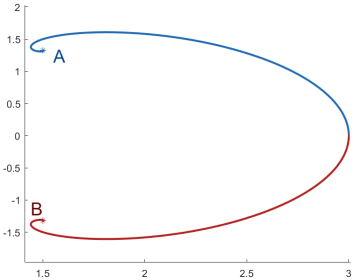

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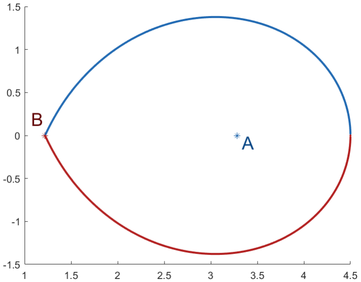

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This allows us to extend the results

$$Re(\alpha) > 0 \implies \gamma(1) = A, \gamma(-1) = B,$$
$$Re(\alpha) < 0 \implies \gamma(1) = \gamma(-1) = B$$

to $\mathbb{C}$, proving our criteria for simple and non-simple hulls.

Phase Transitions: 2-segment hulls

The two points added to a hull at time t are given by the formulas

$$\gamma(t) = \lim_{y \to 0^+} g_t^{-1}(\lambda(t) + iy) \quad \text{and} \quad \gamma(-t) = \lim_{y \to 0^-} g_t^{-1}(\lambda(t) + iy).$$

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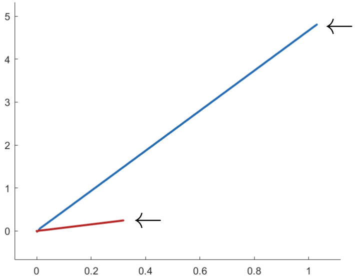
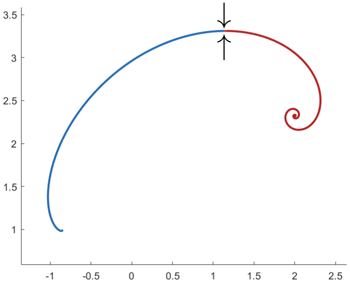
Driving function $\lambda(t) = (1.15 + 3.31i)\sqrt{t}$



L_1



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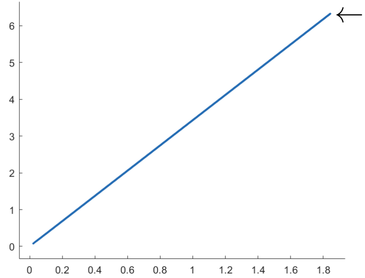
Driving function $\lambda(t) = (2 + 5i)\sqrt{t}$



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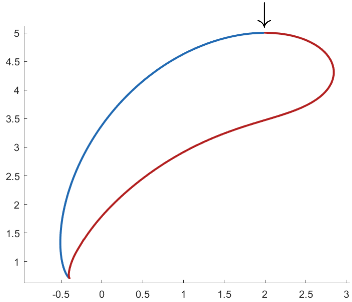

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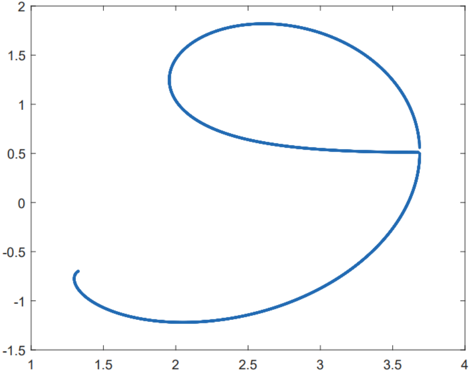

$L_{1,\lambda}$ for $\lambda(t) = c\sqrt{1-t}$, $c \approx 3.687 + 0.511i$

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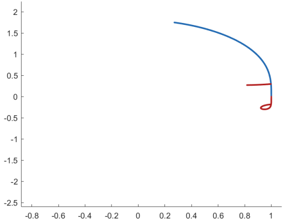
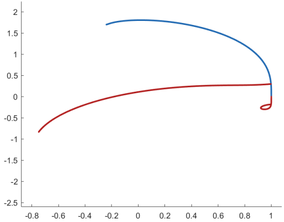
- Do all hulls grow continuously?

  Driving functions such as $\lambda(t) = e^{it}, t \in [0, 1]$ which intersect themselves may lead to "jumps" in growth.

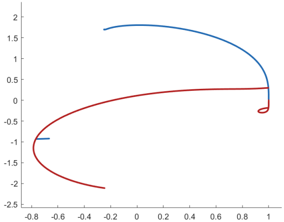
Further Questions: driving function $\lambda(t) = e^{it}$



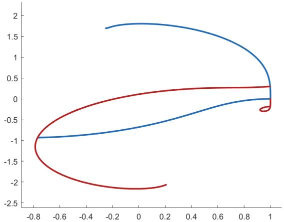
$t = 0.25$



$t = 0.5$



$t = 0.75$



$t = 1$

Thank you