EXPERIMENTS WITH THE RESTRICTION FUNCTOR

ULI WALTHER

Abstract. We consider holonomic modules arising in singularity theory and hypergeometric systems. We study their restrictions in the D-module sense. This gives rise to interesting problems and conjectures arising from the computations.

Let D_n be the n-th Weyl algebra,

$$D_n = \frac{\mathbb{C}\{x_1, \partial_1, \dots, x_n, \partial_n\}}{\langle \partial_i\partial_j - \partial_j\partial_i, x_ix_j - x_jx_i, \partial_ix_i - x_i\partial_i - 1, \partial_ix_j - x_j\partial_i : 1 \leq i < j \leq n\rangle}.$$

In [17] an algorithm is explained that has as input a holonomic D_n-module M and returns a stratification of $\mathbb{C}^n$ such that on each stratum the i-th restriction $\rho^i_{X,\mathbb{C}^n}(M)$ of M to X is a connection of rank $r_i(X, M)$.

This paper deals with applications of two types of this algorithm. We first consider the Malgrange module M_f associated to a polynomial $f \in R_n = \mathbb{C}[x_1, \dots, x_n]$ and its endomorphism s with minimal Bernstein-Sato polynomial $b_f(s)$. The second type of modules that we study are hypergeometric systems of the type introduced by Gelfand, Graev, Kapranov and Zelevinsky.

In both cases the interest derives from the change of the stratification (or certain multiplicities of the strata) under the variation of an external quantity. For the singularity examples, this external quantity is the choice of a factor of the Bernstein-Sato polynomial. In the GKZ-case we vary the parameter vector β. The explicit computations give rise to speculations and questions that are listed together with comments and progress known to the author.

1. Introductions

In this section we briefly explain some background on $\mathcal{D}$-modules. We refer to [17] for more details and references.

1.1. **$\mathcal{D}$-basics.** The general references for $\mathcal{D}$-module theory are [1, 2].

Let X^{an} be an analytic manifold over $\mathbb{C}$. We always use the superscript to indicate analytic spaces. Symbols without the superscript refer to algebraic objects. The sheaf of differential operators $\mathcal{D}_{X^{an}}$ acts via $\bullet$ on the sheaf of holomorphic functions $\mathcal{O}_{X^{an}}$. If X is algebraic, let $\mathcal{D}_X, \mathcal{O}_X$ be the algebraic versions of these sheaves. Abusing notation, $\mathcal{O}_X \subseteq \mathcal{O}_{X^{an}}$ and $\mathcal{D}_{X^{an}} = \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} \mathcal{D}_X$.

Let $\mathcal{M} \in \mathcal{D}_X$-mods. The filtration on $\mathcal{D}_{X^{an}}$ by order of differential operators induces a graded object $\mathrm{gr}_{(0,1)}(\mathcal{M})$. The *singular locus* $\mathrm{sing}(\mathcal{M})$ is the variety of the intersection of the annihilator of $\mathrm{gr}_{(0,1)}(\mathcal{M})$ with the subring $\mathcal{O}_X$ of $\mathrm{gr}_{(0,1)}(\mathcal{D})$.

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1.2. **Holonomic modules and the $\mathcal{S}ol$-functor.** The category of *holonomic* modules $\operatorname{Hol}(X) = \{\mathcal{M}^\bullet \in \mathcal{D}_X\text{-mods} : \dim \operatorname{gr}_{(0,1)}(H^i(\mathcal{M}^\bullet)) = \dim(X) \forall i\}$ is a full subcategory of $\mathcal{D}_X$-mods, closed under subquotients and extensions. Near $p \notin \operatorname{sing}(\mathcal{M})$, $\mathcal{M}_{an} \in \operatorname{Hol}(X^{an})$ is a connection and $\dim_{\mathbb{C}}(\operatorname{Hom}_{\mathcal{D}_{an,p}}(\mathcal{M}_{an,p}, \mathcal{O}_{an,p}))$ is the *rank* $\operatorname{rk}(\mathcal{M})$.

Let $\mathcal{M}^\bullet \in \mathcal{D}_{X^{an}}$-mods. Then $\mathcal{S}ol(\mathcal{M}^\bullet) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_{X^{an}}}(\mathcal{M}^\bullet, \mathcal{O}_{X^{an}})$ is its *solution complex*. By [6] if $\mathcal{M}^\bullet, \mathcal{L}^\bullet \in \operatorname{Hol}(X)$ then the sheaves $\mathcal{E}xt^i_{\mathcal{D}_{X^{an}}}(\mathcal{M}^\bullet, \mathcal{L}^\bullet)$ are constructible and $\dim_{\mathbb{C}}(\mathcal{E}xt^i_{\mathcal{D}^{an}}(\mathcal{M}^\bullet, \mathcal{L}^\bullet))_p < \infty \forall p \in X$.

1.3. **Restriction.** Let $i : Y \hookrightarrow X$ and suppose that $Y = \operatorname{Var}(\mathcal{I}(Y))$ is smooth of codimension $d_{X/Y} := \dim(X) - \dim(Y)$. For $\mathcal{M}^\bullet \in \mathcal{D}_X$-mods set $\rho_{Y,X}(\mathcal{M}^\bullet) = i^{-1}\left(\mathcal{D}_X/\mathcal{I}(Y)\cdot\mathcal{D}_X \otimes^L_{\mathcal{D}_X} (\mathcal{M}^\bullet)\right)[d_{X/Y}]$. If $\mathcal{M}$ is a connection on X then $\rho_{Y,X}(\mathcal{M})$ is one on Y, in cohomological degree $d_{X/Y}$. For all $\mathcal{M}^\bullet \in \operatorname{Hol}(X)$, the complexes $\rho_{x,X}(\mathcal{M})$ are smooth deformations of each other for $x \in X \setminus \operatorname{sing}(\mathcal{M}^\bullet)$.

1.4. **Regular singular modules.** If $\mathcal{M} \in \operatorname{Hol}(X^{an})$ it is called *regular at* $p \in X$ if and only if the natural map

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{M}_p, \mathcal{O}_{X^{an},p}) \quad\to\quad \mathbb{R}\operatorname{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{M}_p, \hat{\mathcal{O}}_{X^{an},p}) \tag{1.1}$$

is a quasi-isomorphism, and simply *regular* if for all $i : X^{an} \to \overline{X^{an}}$ with compact $\overline{X^{an}}$, the direct image $i_+(\mathcal{M})$ is regular at all $p \in \overline{X^{an}}$. We write $\operatorname{Reg}(X)$ for the category of complexes with regular cohomology.

Let $p \in X$, $i : p \hookrightarrow X$ the embedding. There is a natural isomorphism

$$\mathbb{R}\operatorname{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{M}, \hat{\mathcal{O}}_{X,p}) = \mathbb{R}\operatorname{Hom}_{\mathbb{C}}(\rho_{p,\mathbb{C}^n}(\mathcal{M}), \mathbb{C}).$$

induced by $\operatorname{Hom}_{\mathbb{C}}(i^{-1}(\mathcal{O}_X/\mathcal{I}(p) \otimes M), \mathbb{C}) \ni f \mapsto \sum_{\alpha\in\mathbb{N}^n} x^\alpha \cdot \frac{f(\partial^\alpha)}{\alpha!}$. So for $\mathcal{M} \in \operatorname{Reg}(X)$, $\rho_{p,X}(\mathcal{M})$ and $\mathcal{S}ol(\mathcal{M})_p$ are equivalent.

It remains an important open question to find an algorithm that determines regularity of a $\mathcal{D}$-module near a given point. However, $\operatorname{Reg}(X)$ contains $\mathcal{O}_X$, is closed under formation of direct and inverse images, and exterior tensor product (i.e., tensors over $\mathbb{C}$ in different sets of variables), as well as subquotients and extensions.

The assignment $\mathcal{M}^\bullet \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_{X_{an}}}(\mathcal{M}^\bullet_{an}, \mathcal{O}_{X^{an}})$ sets up an equivalence from $\operatorname{Hol}(X)$ to the category of constructible sheaves.

In [6, 7] it was proved that if $X = \mathbb{C}^n$ and $M^\bullet, L^\bullet \in \operatorname{Hol}(\mathbb{C}^n)$ are algebraic then there exists an algebraic stratification S of $\mathbb{C}^n$ such that on the strata $\rho^i_{p,X}(M^\bullet)$ is locally constant as a sheaf of vector spaces. In [17] an algorithm was given to find such a stratification, and to determine the vector space dimensions of these restrictions $\forall p \in X, \forall i$. We call such a stratification a *Kashiwara stratification of* $\mathcal{M}^\bullet, L^\bullet$.

Example 1.1. Consider $P = x_1^2\partial_1 + 1$ on $\mathbb{C}^1$ and $M = D_1/D_1P$. It has singular locus equal to the origin. Outside the singular locus, it is a connection of rank 1. At $x_1 = 0$ we have a restriction $D_1/x_1D_1 \otimes^L_{D_1} M$ equal to zero. This is clear for the 1-st restriction as $\exp(1/x_1)$ is singular there. For the 0-th restriction one may use a method of Oaku [12] as implemented in [5], or note that M is x_1-torsion free.

On the other hand, let p be the origin. The solution complex $\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})_p$ turns out to have one-dimensional cohomology in degree 1, because $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M}, \mathcal{O})_p$ equals $\mathcal{O}_p$ modulo the convergent series expressions in the $\mathbb{C}$-linear relations $x_1^n + nx_1^{n+1}$, which are the images under P of x_1^n. So the $\mathcal{E}xt^1$-group is generated, for example, by x_1. That this is not the zero class follows from the fact that if one tries to reduce x_1 modulo $P\mathcal{D}$ one has

$$\begin{aligned} x_1 &= (x_1+x_1^2)-x_1^2 \\ &= (x_1+x_1^2)-(x_1^2+2x_1^3)+2x_1^3 \\ &= (x_1+x_1^2)-(x_1^2+2x_1^3)+2(x_1^3+3x_1^4)-6x_1^4 \\ &= (x_1+x_1^2)-(x_1^2+2x_1^3)+2(x_1^3+3x_1^4)-\ldots \pm (n-1)!(x_1^n+nx_1^{n+1}) \mp n!x_1^{n+1} \end{aligned}$$

and the series $f=-\sum_{n=1}^{\infty}(n-1)!(-x_1)^n$ is not convergent.

This means also that the natural morphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O})_x = \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_x,\mathcal{O}_x) \to \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}_x,\hat{\mathcal{O}}_x) \tag{1.2}$$

is not an isomorphism. Namely, we just established that the coset of $x_1 = -P \bullet \sum_{n=1}^{\infty}(n-1)!(-x_1)^n$ is a boundary with formal power series, while it is not so for convergent ones. Consequently, $\mathrm{Hom}_{D_1}(M_x,\hat{\mathcal{O}}_x/\mathcal{O}_x)$ is nonzero. Indeed, the power series f given above has the property that $Pf \in \mathcal{O}_x$ so that it represents a homomorphism from $\mathcal{D}$ to $\hat{\mathcal{O}}_x/\mathcal{O}_x$ that is zero on $\mathcal{D}P$.

2. Examples

In this section we study explicit examples with the help of computer algebra programs.

2.1. **Singularities.** Let $f \in R_n$ and pick a new indeterminate s. Differentiating f^s formally with respect to $x_1,\ldots,x_n$, one obtains the (finitely generated) D_n-module

$$M_f = \frac{D[s]\bullet f^s}{D[s]\cdot f \bullet f^s}.$$

The action of s on M_f has a minimal polynomial, $b_f(s)$, called the *Bernstein-Sato polynomial* of f. The module M_f is regular holonomic supported in $\mathrm{Var}(f)$ (see [1]) and was introduced by Malgrange [9]. Locally near $p \in \mathbb{C}^n$, the part of M_f that is annihilated by $s+1$ agrees with M_f if and only if f is smooth near p.

Example 2.1 (Whitney umbrella)**.** Let $n=3$, $f(x)=x^2-y^2z$ and $X=\mathbb{C}^3$. With *Macaulay2* [5] one computes a presentation for M_f:

```
{-2} | yDy-2zDz                        |
{-3} | yzDx+xDy                        |
{-3} | y2Dx+2xDz                       |
{-3} | x2Dx+2xzDz+2x                   |
{-3} | y2z-x2                          |
{-4} | z2DxDz+1/2xDy^2+1/2zDx          |
{-4} | yz2Dz-1/2x2Dy+yz                |
{-5} | z3Dz^2-1/4x2Dy^2+5/2z2Dz+1/2z   |
```

The characteristic variety has three components. Their projections under $\pi_X : T^*(X) \to X$ are to

$$\{\mathrm{Var}(x,y,z)=o, \mathrm{Var}(x,y)=l, \mathrm{Var}(f)=V\}.$$

These are the strata of a Kashiwara stratification of M_f. For $p \in X$, write $\rho(M_f,p)$ for the *restriction diagram* $(\dim_{\mathbb{C}}\rho^0_{p,X}(M_f),\ldots,\dim_{\mathbb{C}}\rho^3_{p,X}(M_f))$. By definition, a Kashiwara stratification on X makes the $\rho(M_f,p)$ constant along strata, since on all strata the restrictions are fiber bundles. To compute these diagrams along all strata one picks a point on each stratum and restricts M_f to that point. We obtain

p	stratum	$\rho(M_f, p)$
$(0,0,0)$	o	$(0,1,0,1)$
$(0,0,1)$	l	$(0,1,1,0)$
$(1,1,1)$	V	$(0,1,0,0)$

where (here and in all further examples below) all points $p \in X$ that are not listed in the table have $\rho(M_f, p)$ equal to zero.

Consider now the singular part $M_{f,sing} = (s+1)M_f$ of M_f with presentation

```
{-1} | x          |
{-2} | yDy-2zDz   |
{-2} | yz         |
{-2} | y2         |
{-3} | z2Dz+1/2z  |
```

Now we obtain the following nonzero restriction data:

p	stratum	$\rho(M_{f,sing}, p)$
$(0,0,0)$	o	$(0,0,0,1)$
$(0,0,1)$	l	$(0,0,1,0)$

In fact every factor of the Bernstein-Sato polynomial $b_f(s) = (s+1)^2(s+3/2)$ may be used to study M_f. The following table lists $\rho(q(s)M_f, p)$ for all factors:

stratum $\setminus q$	1	$(s+1)$	$(s+1)^2$	$(s+3/2)$	$(s+1)(s+3/2)$
o	$(0,1,0,1)$	$(0,0,0,1)$	$(0,0,0,1)$	$(0,1,0,0)$	$(0,0,0,0)$
l	$(0,1,1,0)$	$(0,0,1,0)$	$(0,0,0,0)$	$(0,1,1,0)$	$(0,0,1,0)$
V	$(0,1,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(0,1,0,0)$	$(0,0,0,0)$

One can see that multiplication by $(s+1)$ kills $\rho^1(M_f, p)$ for all p, but has no other effect. Although $(s+1)$ will always kill a "one" in $\rho^1(M_f, o)$, it may have other effects as well.

Example 2.2. If $f = x^3 + y^3 + z^3$, then the restriction of M_f to the origin gives $(0,1,1,9)$ while the restriction of $(s+1)M_f$ to the origin o results in $(0,0,0,8)$. The Bernstein-Sato polynomial is here $(s+1)(s+31/30)(s+37/30)(s+41/30)(s+43/30)(s+47/30)(s+49/30)(s+53/30)(s+59/30)$, and each factor contributes a one-dimensional space to $\rho^3(M_f, o)$.

Example 2.3. Consider now $f = xy(x+y)(x+2y)$ in $\mathbb{C}^2$. Here $b_f(s) = (s+1)(s+2/4)(s+3/4)(s+4/4)(s+5/4)(s+6/4)$. The origin is the only singular point of f, so we move directly to the table of restrictions to the origin.

$q(s)$	$\rho(q(s) \cdot M_f, (0,0))^*$
1	$(0,1,9)$
$(s+1)$	$(0,0,9)$
$(s+1)(s+1/2)$	$(0,0,8)$
$(s+1)(s+3/4)$	$(0,0,7)$
$(s+1)^2$	$(0,0,6)$
$(s+1)(s+5/4)$	$(0,0,7)$
$(s+1)(s+3/2)$	$(0,0,8)$

One is tempted to associate (for example) to $(s+5/4)$ the diagram $(0,0,2)$ since this is the drop in the restriction diagram caused by considering $(s+5/4)M_f$ instead of M_f. There is a 2-dimensional eigenspace of the de Rham cohomology of the

Milnor fiber $f^{-1}(1)$ of f at the origin corresponding to $s+5/4$. Indeed, one has the following general fact (see, for example, [8, 16]): if f is a homogeneous isolated singularity then $(s+1)M_f$ is isomorphic as a graded module to the quotient of R_n by the Jacobian ideal of f. This space in turn is identified with the top de Rham cohomology of the Milnor fiber of f, and both isomorphisms preserve the s-action. It follows that the restriction diagrams mirror the eigenspaces of the top cohomology of the Milnor fiber under the s-action, which turns out to be multiplication by (shifted) degree of the forms.

Example 2.4. We consider a generic hyperplane arrangement f of five planes in $\mathbb{C}^3$, where by generic we mean that the intersection of any i of the participating five hyperplanes meets in a space of codimension $\min(3, i)$. The most interesting singular point is the origin as in all other points f is a normal crossing divisor where the restriction diagram is always given by binomial coefficients. The Bernstein-Sato polynomial of f is $(s+1)^2(s+\frac{3+0}{5})\cdots(s+\frac{3+5}{5})$ The restriction diagrams for some choices of $q(s)$ are as follows.

$q(s)$	$\rho(q(s)\cdot M_f,(0,0,0))^*$
1	$(0,1,4,18)$
$(s+1)$	$(0,0,4,18)$
$(s+1)^2$	$(0,0,0,18)$
$(s+1)^2(s+8/5)$	$(0,0,0,16)$
$(s+1)^2(s+7/5)$	$(0,0,0,15)$
$(s+1)^2(s+6/5)$	$(0,0,0,15)$
$(s+1)^3$	$(0,0,0,12)$
$(s+1)^2(s+4/5)$	$(0,0,0,15)$
$(s+1)^2(s+3/5)$	$(0,0,0,17)$

It is known that generic arrangements have their Bernstein-Sato polynomial only depend on the dimension of the ambient space and the number of participating hyperplanes. It is not clear that the same holds for the restriction diagrams.

Example 2.5. Let us consider the singularity $f = xyz(x+y)(x+z)$. This is a non-generic central arrangement. The Kashiwara stratification of M_f is (as for all arrangements) the natural one coming from the intersection lattice. There are two types of points that are not normal crossings: the origin, and the lines $x = y = 0$ and $x = z = 0$. Correspondingly, the Bernstein-Sato polynomial is

$$\begin{aligned} b_f(s) \quad = \quad & (s+1) \\ & (s+2/3)(s+3/3)(s+4/3) \\ & (s+3/5)(s+4/5)(s+5/5)(s+6/5)(s+7/5) \end{aligned}$$

At the origin one obtains the following restriction diagrams:

$q(s)$	$\rho(q(s)\cdot M_f,(0,0,0))$
1	$(0,1,4,8)$
$(s+1)$	$(0,0,4,8)$
$(s+1)^2$	$(0,0,0,8)$
$(s+1)^2(s+8/5)$	$(0,0,0,7)$
$(s+1)^2(s+7/5)$	$(0,0,0,7)$
$(s+1)^2(s+6/5)$	$(0,0,0,7)$
$(s+1)^3$	$(0,0,0,4)$
$(s+1)^2(s+4/5)$	$(0,0,0,7)$
$(s+1)^2(s+3/5)$	$(0,0,0,7)$

Again, one would associate "drop diagrams" to certain factors of $b_f(s)$. We compare in this way this f to a generic 5-arrangement

	3/5	4/5	5/5	6/5	7/5	8/5
generic	1	3	6	3	3	2
f	1	1	4	1	1	0

We remark that $(s+2/3)$, $(s+3/3)$ and $(s+4/3)$ give drop diagrams for f at $p \in \operatorname{Var}(x,y) \cup \operatorname{Var}(x,z) \setminus \operatorname{Var}(x,y,z)$ since in these points f is equivalent to a 3 lines in the plane.

The drop diagrams we computed for the generic arrangement are in exact agreement with the graded components of the top cohomology of the Milnor fiber. These graded components can be computed as the integral of a suitable power on a decone of the arrangement:

For generic arrangements, Orlik and Randell have conjectured a formula for these numbers [13].

We now list a few questions that arose from our computations.

(1) In [16] we introduced $\mathfrak{a}_{q(s)} = \{r \in R | q(s) \cdot r = 0 \in M_f\}$. Can one relate them to strata of $\rho_{0,\mathbb{C}^n}(M_f)$? Are the restriction dimensions determined by the multiplicities of these schemes?
(2) Can one give a formula for the eigenvalues of the top de Rham cohomology of the Milnor fiber in terms of restrictions?
(3) Is there an explicit formula for the dimensions of the graded pieces of the Milnor fiber cohomology for arrangements via the intersection lattice?
(4) Considering $f, f^2, f^3, \ldots$, is there an asymptotic formula for $\rho(M_{f^k}, 0)$?
(5) The Bernstein-Sato polynomial is related to jump loci of multiplier ideals [3]. The ideals $\mathfrak{a}_{q(s)}$ were used in [16] to compute the Bernstein-Sato polynomial. Are the ideals $\mathfrak{a}_{q(s)}$ related to multiplier ideals?
(6) In [16] the Bernstein-Sato polynomials of generic arrangements are calculated by providing a sequence of polynomials $q_i(s)$ such that $\operatorname{codim}(q_i(s) \cdot M_f) = i$. Is there a way of "peeling off" higher dimensional components of M_f one at a time while preserving the interesting features of the s-action?

Example 2.6. If f is the generic arrangement of five planes in 3-space, $q_1(s) = s+1$, $q_2(s) = (s+1)^2$ and $q_3(s) = \prod_{i=0}^{5}(5s+i+3)$. M_f itself is supported on the entire arrangement, $q_1(s) \cdot M_f$ only in the ten lines of intersection, and $q_2(s) \cdot M_f$ only at the origin. Both $q_1(s)$ and $q_2(s)$ are minimal with respect to the support property.

The question is really asking whether the structure of the singular locus, the restrictions of M_f and the Bernstein-Satop polynomial can be put into one statement.

2.2. **GKZ-systems.** Let $A \in \mathbb{Z}^{d,n}$ of rank d with columns $a_1, \ldots, a_n$, and $\beta \in \mathbb{C}^d$. Following [4] we set $I_A = \langle \partial^u - \partial^v : u, v \in \mathbb{N}^n, Au = Av \rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]$ and $E_i = \sum_j A_{i,j} x_j \partial_j$ for $1 \le i \le d$. The A-hypergeometric system to β is $H_A(\beta) = D_n \cdot (I_A, \{E_i - \beta_i\}_1^n)$, [4, 15].

The matrix A induces a torus action of $(\mathbb{C}^\times)^d$ on $\mathbb{C}^n$ via the action of the j-th column on the j-th component of $\mathbb{C}^n$: $(t_1, \ldots, t_d) \bullet y_i = y_i \cdot \prod_{i=1}^d (t_i)^{a_{i,j}}$. Reading $\mathbb{C}[\partial_1, \ldots, \partial_n]$ as cootdinate ring of $\mathbb{C}^n$, the orbit under this action of $(1, \ldots, 1)$ is isomorphic to T and its closure in $\mathbb{C}^n$ is an algebraic variety with defining ideal I_A.

It is proved in [10] that there is $r \in \mathbb{N}$, called the rank of $H_A(\beta)$, such that in a generic point $p \in \mathbb{C}^n$ the number of local holomorphic solutions to $H_A(\beta)$ equals r as long as β is outside a certain subspace arrangement, while if β is in this arrangement then number of solutions strictly exceeds r. Near a generic point $p \in \mathbb{C}^n$ holomorphic modules are regular, so the rank of $H_A(\beta)$ is equal to $\dim \rho^0_{p,\mathbb{C}^n}(D/H_A(\beta))$. In fact, $H_A(\beta)$ is regular for all matrices A for which I_A is homogenous in the usual sense – this happens if and only if all columns of A lie in a hyperplane of $\mathbb{C}^n$ not crossing the origin.

We now consider the behaviour of the Kashiwara stratification of $H_A(\beta)$ under variation of β.

Example 2.7. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$. The associated GKZ-system

$$\begin{array}{rcl} x_1\partial_1 + \ x_2\partial_2 + \ x_3\partial_3 + \ x_4\partial_4 & - & \beta_1 \\ x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 & - & \beta_2 \\ \partial_2\partial_3 - \partial_1\partial_4,\ \partial_3^3 - \partial_2\partial_4^2,\ \partial_1\partial_3^2 - \partial_2^2\partial_4,\ \partial_2^3 - \partial_1^2\partial_3 & & \end{array}$$

has rank 4 unless $\beta = (1,2)^T$. If $\beta = (1,2)^T$ then the rank is 5.

One of the features of A-hpergeometric systems that lend them to explicit study is the fact that their characteristic variety can only take finitely many values for fixed A but varying β. In fact, for this particular A independently of the parameter vector, $\mathrm{char}(H_A(\beta))$ is the union of $\mathrm{Var}(\langle D_1, D_2, D_3, x_4\rangle)$, $\mathrm{Var}(\langle D_2, D_3, D_4, x_1\rangle)$, $\mathrm{Var}(\langle D_1, D_2, D_3, D_4\rangle)$ and the closure of the conormal bundle of the big torus orbit $T \bullet (1,1,1,1)$ which is the variety of I_A without $\mathrm{Var}(x_1x_4)$. For all β then the Kashiwara stratification is the weakest Whitney stratification that contains the four varieties above.

Denote

$$F_1 = \langle x_2x_3 + 8x_1x_4, x_3^3 + 8x_2x_4^2, x_1x_3^2 - x_2^2x_4, x_2^3 + 8x_1^2x_3\rangle$$

and

$$F_2 = \langle x_2x_3 - 4x_1x_4, x_3^3 + 4x_2x_4^2, x_1x_3^2 + x_2^2x_4, x_2^3 + 4x_1^2x_3\rangle.$$

These two ideals show up in the primary decomposition of the singular locus of the discriminant. Write also

$$G_1 = \langle x_1, x_3^3 + 27/4x_2x_4^2\rangle, \qquad G_2 = \langle x_4, x_2^3 + 27/4x_1^2x_3\rangle.$$

These arise as components of intersections of $(f = 0)$ with $(x_1 = 0)$ and $(x_4 = 0)$. In the following table we have in each column the restriction diagrams at the given value β to the various strata of $\mathfrak{S}(H_A(\beta))$. We only give the first four entries of

$\rho(p, H_A(\beta))$, since $\rho^4_{p,\mathbb{C}^4}(H_A(\beta)) = \rho^5_{p,\mathbb{C}^4}(H_A(\beta)) = 0$ in all cases.

variety$\backslash\beta =$	$(1,2)$	$(2,-1)$	$(2,4)$	$(-1,-1)$	$(0,3)$	$(*,*)$
$\langle\emptyset\rangle$	$(5,0,0)$	$(4,0,0)$	$(4,0,0)$	$(4,0,0)$	$(4,0,0)$	$(4,0,0)$
$\langle x_1\rangle$	$(4,0,0)$	$(4,1,0)$	$(3,0,0)$	$(3,0,0)$	$(3,0,0)$	$(3,0,0)$
$\langle x_4\rangle$	$(4,0,0)$	$(3,0,0)$	$(3,0,0)$	$(3,0,0)$	$(4,1,0)$	$(3,0,0)$
$\langle f\rangle$	$(4,0,0)$	$(3,0,0)$	$(3,0,0)$	$(3,0,0)$	$(3,0,0)$	$(3,0,0)$
F_1	$(3,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$
F_2	$(3,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$
$\langle x_3, x_4\rangle$	$(2,0,0)$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(2,1,0)$	$(1,0,0)$
$\langle x_1, x_2\rangle$	$(2,0,0)$	$(2,1,0)$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$
G_1	$(3,0,0)$	$(3,1,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$
G_2	$(3,0,0)$	$(2,0,0)$	$(2,0,0)$	$(2,0,0)$	$(3,1,0)$	$(2,0,0)$
$\langle x_1, x_4\rangle$	$(3,0,0)$	$(3,1,0)$	$(2,0,0)$	$(3,1,0)$	$(3,1,0)$	$(2,0,0)$
$\langle x_1, x_2, x_3\rangle$	$(1,0,0)$	$(1,1,0)$	$(1,1,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
$\langle x_1, x_2, x_4\rangle$	$(1,0,0)$	$(1,1,0)$	$(1,1,0)$	$(1,1,0)$	$(1,1,0)$	$(0,0,0)$
$\langle x_1, x_3, x_4\rangle$	$(1,0,0)$	$(1,1,0)$	$(1,1,0)$	$(1,1,0)$	$(1,1,0)$	$(0,0,0)$
$\langle x_2, x_3, x_4\rangle$	$(1,0,0)$	$(0,0,0)$	$(1,1,0)$	$(0,0,0)$	$(1,1,0)$	$(0,0,0)$
$\langle x_1, ..., x_4\rangle$	$(0,0,1)$	$(0,0,0)$	$(1,2,1)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$
	$H^1_{\mathfrak{m}}(R/I_A)$	$H^1_{\langle a_4\rangle}(R/I_A)$	$\mathbb{N}A$	$H^2_{\mathfrak{m}}(R/I_A)$	$H^1_{\langle a_1\rangle}(R/I_A)$	

It is very interesting to contemplate the "A-partition" of the parameter space $\mathbb{C}^2$ into subsets where the restriction diagrams are the same for all $p \in \mathbb{C}^4$. Since the Kashiwara stratification $\mathfrak{S}(H_A(\beta))$ is always the same we suspect this to be a finite partition. To be more precise, we believe that the A-partition of $\mathbb{C}^2$ has six components, and our six parameter choices are a set of representatives for the cosets of this partition. In fact, the partition seems to correspond to the natural partition of multidegrees induced by the Čech complex of $\{\partial_1, \partial_4\}$ on the toric ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.

Suppose one only considers the rank $\rho^0_{p,\mathbb{C}^4}(H_A(\beta))$ in a generic point p. As I_A is homogeneous in the usual sense, $\mathrm{rk}(H_A(\beta)) = \mathrm{vol}(A)$ for generic β by [15]. A theorem of Matusevich, Miller and the author [10] states that $E(A) = \{\beta : \mathrm{rk}(H_A(\beta)) > \mathrm{vol}(A)\}$ agrees with the Zariski closure $\mathcal{E}(A)$ in $\mathbb{C}^d$ of $\{\deg(\gamma) : 0 \neq \gamma \in H^i_{\mathfrak{m}}(\mathbb{C}[\{\partial_i\}_1^n]/I_A), i < d\}$, the set of multidegrees of local cohomology elements witnessing the failure of R/I_A to be Cohen-Macaulay. This is a subspace arrangement.

The local cohomology partition induced on the parameter space is not fine enough to stratify the entire restriction diagram, because in our example $(1,2)$ is the only "unusual" degree predicted by $\mathcal{E}(A)$. As stated above, we think that the six sets $E_1 = \mathbb{N}A$, $E_2 = (1,2)$, $E_3 = (\mathbb{N}A + \mathbb{Z}a_1) \setminus (\mathbb{N}A \cup (1,2))$, $E_4 = (\mathbb{N}A + \mathbb{Z}a_4) \setminus (\mathbb{N}A \cup (1,2))$, $E_5 = \mathbb{Z}^2 \setminus ((\mathbb{N}A + \mathbb{Z}a_1) \cup (\mathbb{N}A + \mathbb{Z}a_4))$ and $\mathbb{C}^2 \setminus \bigcup_{i=1}^5 E_i$ form the A-partition of $\mathbb{C}^2$.

This suggests in particular a combinatorial, but non-algebraic, A-partition. In general, the A-partition of $\mathbb{C}^d$ cannot be algebraic. For example, if $A = (1)$ then $\beta \in \mathbb{N}$ and $\beta \notin \mathbb{N}$ have different restriction at the origin due to the existence of the polynomial solution x^β (or the failure of its existence). Again, this stratification is the one naturally induced by the Čech complex.

The next example gives more evidence to this pattern but shows that the situation is more complicated than one may expect.

can i make laura's 5-point example? (1,0,0),(1,1,0),(1,0,1),(1,1,1),(1,0,-2)

Example 2.8. Even within $\mathbb{N}A$ with Cohen-Macaulay I_A not all β are created equal. The example is due to M. Saito who used it to illustrate his sets $E_\tau(\beta)$, [14]. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Then the parameters $\beta = a_1$, $\beta = a_3$ give different $\rho^0_{p,\mathbb{C}^4}(H_A(\beta)$ on some strata:

variety$\backslash\beta$	a_3	a_1	$(1,1,0) \in$ $\mathbb{Q}(a_1,a_4) \setminus \mathbb{Z}(a_1,a_4)$	$(1,1,-1) \in$ $(\mathbb{N}A + \mathbb{Z}a_2) \cap (\mathbb{N}A + \mathbb{Z}a_3) \setminus \mathbb{N}A$
$\langle\emptyset\rangle$	$(3,0,0,0,0)$	$(3,0,0,0,0)$	$(3,0,0,0,0)$	$(3,0,0,0,0)$
$\langle x_1\rangle$	$(1,0,0,0,0)$	$(1,0,0,0,0)$	$(2,1,0,0,0)$	$(2,1,0,0,0)$
$\langle x_2\rangle$	$(2,0,0,0,0)$	$(2,0,0,0,0)$	$(2,0,0,0,0)$	$(2,0,0,0,0)$
$\langle x_3\rangle$	$(2,0,0,0,0)$	$(2,0,0,0,0)$	$(2,0,0,0,0)$	$(2,0,0,0,0)$
$\langle x_4\rangle$	$(1,0,0,0,0)$	$(1,0,0,0,0)$	$(2,1,0,0,0)$	$(2,1,0,0,0)$
$\langle x_1,x_2\rangle$	$(1,1,0,0,0)$	$(1,1,0,0,0)$	$(1,1,0,0,0)$	$(1,1,0,0,0)$
$\boxed{\langle x_2,x_3\rangle}$	$(2,2,0,0,0)$	$(1,1,0,0,0)$	$(1,1,0,0,0)$	$(0,0,0,0,0)$
$\langle x_1,x_4\rangle$	$(1,1,0,0,0)$	$(1,1,0,0,0)$	$(1,1,0,0,0)$	$(2,3,1,0,0)$
$\langle x_3,x_4\rangle$	$(1,1,0,0,0)$	$(1,1,0,0,0)$	$(1,1,0,0,0)$	$(1,1,0,0,0)$
$\boxed{\langle x_1x_3^2+x_2^2x_4\rangle}$	$(3,1,0,0,0)$	$(2,0,0,0,0)$	$(3,1,0,0,0)$	$(2,0,0,0,0)$
$\langle x_1,x_2,x_3,x_4\rangle$	$(1,3,3,1,0)$	$(1,3,3,1,0)$	$(0,0,0,0,0)$	$(0,0,0,0,0)$
$\langle x_1,x_2,x_3\rangle$	$(1,2,1,0,0)$	$(1,2,1,0,0)$	$(0,0,0,0,0)$	$(0,0,0,0,0)$
$\langle x_1,x_2,x_4\rangle$	$(1,2,1,0,0)$	$(1,2,1,0,0)$	$(1,2,1,0,0)$	$(1,2,1,0,0)$
$\langle x_1,x_3,x_4\rangle$	$(1,2,1,0,0)$	$(1,2,1,0,0)$	$(1,2,1,0,0)$	$(1,2,1,0,0)$
$\langle x_2,x_3,x_4\rangle$	$(1,2,1,0,0)$	$(1,2,1,0,0)$	$(0,0,0,0,0)$	$(0,0,0,0,0)$

As for M_f we offer some problems and questions that arose from our computations.

- Can one give a ρ-classification of β? As a fist step towards this question one should perhaps determine whether the cosets are countable unions of algebraic sets.

  Consider the example $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ but this time we view β as an indeterminate. If one restricts $H_A(\beta)$ to a generic point $x_1 - a_1 = x_2 - a_2 = x_3 - a_3 = x_4 - a_4 = 0$, one obtains a module over $\mathbb{C}[\beta_1, \beta_2]$ with five generators and the following syzygy:

$$
\begin{aligned}
&[(\beta_2-0)(\beta_2-1)(\beta_2-2)3a_1^3a_3,\\
&(\beta_1-1)(\beta_1(4a_1a_2^3a_4+2a_2^4a_3)+(4a_1a_2^3a_4+18a_1^2a_2a_3^2+2a_2^4a_3))\\
&\quad+(\beta_2-2)(-\beta_1(5a_1a_2^3a_4+18a_1^2a_2a_3^2+4a_2^4a_3)-\beta_2(24a_1^3a_3a_4-a_1a_2^3a_4-12a_1^2a_2a_3^2-2a_2^4a_3)\\
&\quad+(60a_1^3a_3a_4+a_1a_2^3a_4-30a_1^2a_2a_3^2-4a_2^4a_3)),\\
&(\beta_1-1)(12a_1a_2^3a_4^2+36a_1^2a_2a_3^2a_4+6a_2^4a_3a_4)+(\beta_2-2)(48a_1^3a_3a_4^2-3a_1a_2^3a_4^2-39a_1^2a_2a_3^2a_4-6a_2^4a_3a_4),\\
&(\beta_1-1)(-\beta_1 3a_1a_2^3a_3+4a_1^2a_2^2a_4+2a_1a_2^3a_3)\\
&\quad+(\beta_2-2)(\beta_1(-4a_1^2a_2^2a_4+4a_1a_2^3a_3)+\beta_2(a_1^2a_2^2a_4-9a_1^3a_3^2-a_1a_2^3a_3)+(a_1^2a_2^2a_4+9a_1^3a_3^2-a_1a_2^3a_3)),\\
&(\beta_1-1)(16a_1^2a_2^2a_4^2+7a_1a_2^3a_3a_4+27a_1^2a_2a_3^3+4a_2^4a_3^2)\\
&\quad+(\beta_2-2)(-4a_1^2a_2^2a_4^2+36a_1^3a_3^2a_4-a_1a_2^3a_3a_4-27a_1^2a_2a_3^3-4a_2^4a_3^2)]
\end{aligned}
$$

  In particular, the syzygy vanishes at $\mathcal{E}(A)$. Since the restrictions of $H_A(\beta)$ to a generic point can be completed to formal solutions of the hypergeometric ideal, and since in a generic point formal and convergent solutions agree, this syzygy "explains" the rank jump at $\mathcal{E}(A)$.

A natural question is whether there always are algebraic syzygies for the solutions at indeterminate points. This is not obvious since the functions in question are not algebraic most of the time.

- Do the restriction diagrams determine the isomorphism class of $H_A(\beta)$? Saito has shown that isomorphy of A-hypergeometric systems is measured by his sets $E_\tau(\beta)$. The question hence becomes whether the A-partition of $\mathbb{C}^d$ agrees with the partition induced by the $E_\tau(\beta)$.
- Can one determine the restriction diagrams combinatorially? For the rank, this is in part answered by [10]. Before, for simplicial A, Saito had done it [14].
- Can one give a good bound for the restriction numbers? There is one for the rank [15], but it seems not very sharp. It would be nice to get a bound through understanding the restrictions step by step.

It is a natural to ask whether one can make rank jumps arbitrarily large. Recently Matusevich and the author [11] have shown that rank jumps can be arbitrarily high. Indeed, a family of systems is given where the difference between the rank and the volume grows linearly with $n = 2d$.

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Purdue University
E-mail address: walther@math.purdue.edu