GAUSS–MANIN SYSTEMS OF FAMILIES OF LAURENT POLYNOMIALS AND A-HYPERGEOMETRIC SYSTEMS

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Abstract. In this note we study families of Gauß–Manin systems arising from Laurent polynomials with parametric coefficients under projection to the parameter space. For suitable matrices of exponent vectors, we exhibit a natural four-term exact sequence for which we then give an interpretation via generalized A-hypergeometric systems. We determine the extension groups from the parameter sheaf to the middle term of this sequence and show that the four-term sequence does not split. Auxiliary results include the computation of Ext and Tor groups of A-hypergeometric systems against the parameter sheaf.

Contents

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1. Introduction

During the 1980s Gel′fand, Graev, Kapranov and Zelevinskiĭ introduced a class of systems of complex partial differential equations which are a vast generalization of the Gauß hypergeometric equation and which are nowadays known as *A-hypergeometric* (or GKZ) *systems* (*cf.* [GGZ87, GZK89] and a string of other articles of that period). Such A-hypergeometric system has a hybrid combinatorial and algebraic flavor, its initial datum being an integer matrix A and a parameter vector β in the column space of A. This determines a left ideal $H_A(\beta)$ in the Weyl algebra D and the A-hypergeometric system with respect to A and β is then the cyclic left D-module $M_A^\beta := D/H_A(\beta)$. From this definition it is far from clear that these systems have any geometric interpretation.

The analytic behavior of M_A^β (as a system of PDEs) is highly dependent on the parameter vector β. A technique to study this dependence, the *Euler–Koszul functor*, was developed by Matusevich, Miller and the second author in [MMW05]. This is a functor from the category of toric modules, which are a mild generalization of $\mathbb{Z}A$-graded $\mathbb{C}[\mathbb{N}A]$ modules, to the category of complexes of D-modules. The construction of this functor generalizes the Euler–Koszul complex on the semigroup ring $\mathbb{C}[\mathbb{N}A]$ (already known to Gel′fand, Kapranov and Zelevinskiĭ, [GZK89]) and was inspired by it.

The Euler Koszul complex provides a D-resolution of the corresponding A-hypergeometric system provided β does not lie in the *A-exceptional locus*, defined via the local cohomology of $\mathbb{C}[\mathbb{N}A]$. An important step in the geometric interpretation of A-hypergeometric systems was achieved by Schulze and the second author in [SW09], generalizing work of Gel′fand et al. There they showed, using the Euler–Koszul complex, that the Fourier–Laplace transform of M_A^β can be identifed with the direct image of a twisted structure sheaf on a torus under a monomial map (depending on A) to affine space whenever β is outside the set of *strongly resonant parameters*.

If A is homogeneous, *i.e.* if (1,...,1) is in its row span, then this embedding descends to an embedding of a torus of dimension one less into projective space. It was realized by Brylinski [Bry86] that the Fourier–Laplace transform of a D-module on affine space which is constant on all punctured lines through the origin can be expressed by a Radon transform of the corresponding D-module on the projective space.

Using this Radon transform the first author showed in [Rei14] that homogeneous A-hypergeometric systems with not strongly resonant, but integer parameter vector β carry the structure of a mixed Hodge module. Furthermore, there exists a morphism to M_A^β from the Gauß–Manin system of the maximal family of Laurent polynomials with Newton polytope equal to the convex hull of the columns of A. This map has $\mathcal{O}$-free kernel and cokernel, and is compatible with the natural mixed Hodge module structure on the Gauß–Manin system and on M_A^β respectively.

Since M_A^β is the terminal Euler–Koszul homology of the semigroup ring $\mathbb{C}[\mathbb{N}A]$ one wonders whether the Euler–Koszul homology of other toric modules (for example, A-graded ideals of $\mathbb{C}[\mathbb{N}A]$) carry a natural mixed Hodge module structure as well. In this paper we consider the maximal graded ideal of $\mathbb{C}[\mathbb{N}A]$ and prove that its terminal Euler–Koszul homology is isomorphic to the Gauß–Manin system of a map whose fibers are the complement of the fibers of the Laurent polynomial alluded to above.

We now give a short overview of the content of this article. In the first section we review the definition of A-hypergeometric systems, of the Euler–Koszul complex, and several functors on D-modules. In the following section we compute the restriction of M_A^β to the origin, its de Rham cohomology, and the groups $\operatorname{Ext}^\bullet_D(\mathcal{M}_A^\beta, \mathcal{O})$. A novel feature of this article is that we work throughout over any field of characteristic zero, rather than specifically over $\mathbb{C}$. In this more general setting we (re)prove that for not strongly resonant parameter the Fourier–Laplace transformed A-hypergeometric system can be viewed as the direct image of a twisted structure sheaf under a torus embedding. In the third section we show that the long exact Euler–Koszul homology sequences induced by the inclusion of the maximal graded ideal in $\mathbb{C}[\mathbb{N}A]$ is isomorphic to certain Gauß–Manin systems coming from a family of Laurent polynomials and compare this sequence with the sequence obtained in [Rei14].

1.1. ***A*-hypergeometric systems.** We introduce here the main notation and review some basis facts on A-hypergeometric systems and the Euler–Koszul functor. We refer to [MMW05, SW09] for more details.

Notation 1.1. Throughout, we work over the field $\Bbbk$ of characteristic zero.

In general we adopt the convention that we denote a sheaf by a calligraphic letter such as $\mathcal{M}$, a module by an Italic letter such as M, and categories and functors by Roman letters such as M.

Notation 1.2. Throughout, A will be an integer matrix that we assume to be pointed: there should be a $\mathbb{Z}$-linear functional on the column space of A that evaluates positively on each column of A.

For any integer matrix A, let $\Bbbk^A$ be a vector space with basis corresponding to the columns $\{\mathbf{a}_j\}_j$ of A. Let R_A (resp. O_A) be the polynomial ring over $\Bbbk$ generated by the variables $\partial_A = \{\partial_j\}_j$ (resp. $x_A = \{x_j\}_j$) corresponding to $\{\mathbf{a}_j\}_j$; we read R_A as coordinate ring on the variety $X_A := \Bbbk^A$. Further, let D_A be the ring of $\Bbbk$-linear differential operators on O_A, where we identify $\frac{\partial}{\partial x_j}$ with ∂_j so that both R_A and O_A are subrings of D_A.

For any (semi)ring of coefficients C we write CA for the set of C-linear combinations of the columns of A. In particular, $\Bbbk A$ is a vector space.

Definition 1.3. Let A be an integer matrix with independent rows whose $\mathbb{Z}$-ideal of maximal minors equals $\mathbb{Z}$.

For the parameter $\beta \in \Bbbk A$ let $H_A(\beta)$ be the D_A-ideal generated by the *homogeneity equations*

$$\{E_i \bullet \phi \quad = \quad \beta_i \cdot \phi\}_i$$

together with the *toric* partial differential *equations*

$$\{(\partial_A^{\mathbf{v}_+} - \partial_A^{\mathbf{v}_-}) \bullet \phi = 0 \quad | \quad A \cdot \mathbf{v} = 0\},$$

using (throughout) multi-index notation. Here, with $\mathbf{0}_A = (0, \ldots, 0)$ in $\Bbbk^A$, we write $E_i := \sum_j a_{ij} x_j \partial_j$ and $\mathbf{v}_+ = \max(\mathbf{v}, \mathbf{0}_A)$, $\mathbf{v}_- = -\min(\mathbf{v}, \mathbf{0}_A)$. We put

$$M_A^\beta := D_A / H_A(\beta).$$

◇

We have

$$\begin{aligned} x^{\mathbf{u}} E_i - E_i x^{\mathbf{u}} &= -(A\cdot \mathbf{u})_i x^{\mathbf{u}}, \\ \partial^{\mathbf{u}} E_i - E_i \partial^{\mathbf{u}} &= (A\cdot \mathbf{u})_i \partial^{\mathbf{u}}. \end{aligned}$$

The *A-degree function* (with values in $\mathbb{Z}A$) on R_A and D_A is:

$$-\deg_A(x_j) := \mathbf{a}_j =: \deg_A(\partial_j).$$

We denote $\deg_{A,i}(-)$ the degree function associated to the weight given by the i-th row of A. Then $E_i P = P(E_i - \deg_{A,i}(P))P$ for A-graded P. Let

$$\varepsilon_A := \sum_j \deg_A(\partial_j) = \sum_j \mathbf{a}_j.$$

Let M be an A-graded D_A-module. There are commuting D_A-linear endomorphisms E_i via

$$E_i \circ m := (E_i + \deg_i(m)) \cdot m.$$

for A-graded $m \in M$. In particular, if N is an A-graded R_A-module one obtains commuting sets of D_A-endomorphisms on the left D_A-module $D_A \otimes_{R_A} N$ by

$$E_i \circ (P \otimes Q) := (E_i + \deg_i(P) + \deg_i(Q)) P \otimes Q.$$

The *Euler–Koszul complex* $K_\bullet(N; E-\beta)$ of the A-graded module N is the homological Koszul complex induced by $E - \beta := \{(E_i - \beta_i)\circ\}_i$ on $D_A \otimes_{R_A} N$. In particular, the terminal module $D_A \otimes_{R_A} N$ sits in cohomological degree zero. We denote $\mathcal{K}_\bullet(N; E-\beta)$ the corresponding complex of quasi-coherent sheaves. The cohomology objects are $H_\bullet(N; E-\beta)$ and $\mathcal{H}_\bullet(N; E-\beta)$ respectively. If $N(\alpha)$ denotes the usual shift-of-degree functor on the category of graded R_A-modules, then $K_\bullet(N; E-\beta)(\alpha)$ and $K_\bullet(N(\alpha); E-\beta+\alpha)$ are identical.

Identifying $\mathbb{Z}A$ with $\mathbb{Z}^{\mathrm{rk}(A)}$ we get coordinates $\{t_i\}_i$ on $T_A = \operatorname{Spec}(\Bbbk[\mathbb{Z}A]) = \operatorname{Spec}(\Bbbk[\{t_i^{\pm}\}_i])$ and then an embedding

$$h_A \colon T_A \longrightarrow \operatorname{Spec}(\mathbb{C}[\{\partial_j\}_j]) = \Bbbk^A \tag{1.1.1}$$

induced by the monomial morphism

$$t := \{t_i\}_i \longrightarrow \{\prod_i t_i^{\mathbf{a}_{ij}}\}_j =: t^A \tag{1.1.2}$$

The closure of the image of h_A in X_A becomes a toric variety via h_A and is defined by the R_A-ideal I_A given as the kernel of (1.1.2) and generated by all binomials $\partial_A^{\mathbf{v}_+} - \partial_A^{\mathbf{v}_-}$ where $A\mathbf{v} = 0$. We denote the semigroup ring

$$S_A := R_A / I_A \simeq \Bbbk[\mathbb{N}A].$$

We denote $\widetilde{\mathbb{N}A}$ the saturation of $\mathbb{N}A$ and by $\tilde{S}_A$ the associated semigroup ring, identical with the normalization of S_A.

The *faces* τ of the rational polyhedral cone $\mathbb{R}_+ A$, *i.e.* the subsets of (the columns of) A that minimize (over A) some linear functional $\mathbb{Z}A \longrightarrow \mathbb{Z}$, correspond to A-graded prime ideals I_A^τ of R_A with $I_A^\tau = I_A + R_A(\{\partial_j\}_{j\notin\tau})$. We let $R_A\partial_A$ be the unique A-graded maximal R_A-ideal.

An R_A-module N is *toric* if it is A-graded, and if it has a (finite) A-graded composition chain

$$0 = N_0 \subsetneq N_1 \subseteq N_2 \cdots \subsetneq N_k = N$$

such that each composition factor N_i/N_{i-1} is isomorphic as A-graded R_A-module to a face ring R_A/I_A^τ or one of its shifts by an element of $\mathbb{Z}A$.

For a finitely generated A-graded R_A-module $N = \bigoplus_{\alpha\in\mathbb{Z}A} N_\alpha$, let

$$\begin{aligned} \deg_A(N) &= \{\alpha \in \mathbb{Z}A \mid N_\alpha \neq 0\}, \\ \operatorname{qdeg}_A(N) &= \overline{\deg_A(N)}^{Zar}, \end{aligned}$$

the latter being the Zariski closure of the former in $\Bbbk A = \Bbbk \otimes_{\mathbb{Z}} \mathbb{Z}A$. For unions of such modules, degrees as well as quasi-degrees are defined to be the corresponding unions, compare [SW09].

Let $N = \Bbbk(-\alpha)$ be the graded R_A-module whose module structure is that of $R_A/I_A^\emptyset = R_A/R_A\partial_A \simeq \Bbbk$, and which lives entirely inside degree $\alpha \in \mathbb{Z}^d$. Then $K_\bullet(N; E-\beta)$ is an exact complex if $\beta \neq \alpha$, and its differentials are zero otherwise.

Definition 1.4. If the row span of A contains $\mathbf{1}_A$ we call A *homogeneous.* Homogeneity is equivalent to I_A defining a projective variety, and to the system $H_A(\beta)$ having only regular singularities [SW08]. ⋄

1.2. **$\mathcal{D}$-module functors.** Let X be a smooth algebraic $\Bbbk$-variety of dimension d_X. We denote by $\mathcal{D}_X$ the sheaf of algebraic differential operators and by D_X its ring of global sections. For $X = \mathbb{A}^n$ we sometimes write D_n . We denote by $\operatorname{Mod}(\mathcal{D}_X)$ the Abelian category of left $\mathcal{D}_X$-modules. The full triangulated subcategories of the derived category $\mathrm{D}^b(\mathcal{D}_X) := \mathrm{D}^b(\operatorname{Mod}(\mathcal{D}_X))$ consisting of objects with $\mathcal{O}_X$-quasi-coherent (resp. holonomic) cohomology are denoted by $\mathrm{D}^b_{qc}(\mathcal{D}_X)$ (resp. $\mathrm{D}^b_h(\mathcal{D}_X)$).

We recall the notation for cohomological shifting a complex $C^\bullet$: $C^\bullet[1]$ is the complex $C^\bullet$ shifted one step left, $(C^\bullet[1])^i = C^{i+1}$, with corresponding shift of the morphisms.

Let $f : X \to Y$ be a map between smooth algebraic varieties. Let $\mathcal{M} \in \mathrm{D}^b_{qc}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathrm{D}^b_{qc}(\mathcal{D}_Y)$, then we denote by

$$f_+\mathcal{M} := \mathrm{R}f_*(\mathcal{D}_{Y\leftarrow X} \overset{L}{\otimes} \mathcal{M}) \quad \text{and} \quad f^+\mathcal{N} := \mathcal{D}_{X\to Y} \overset{L}{\otimes} f^{-1}\mathcal{N}[d_X - d_Y]$$

the direct and inverse image functors for $\mathcal{D}$-modules; both preserve holonomicity and if f is non-characteristic with respect to $\mathcal{N}$ then f^+ is exact (up to a shift), (see e.g. [HTT08, Def. 2.4.2 & Thm 2.4.6]). We denote by

$$\begin{aligned} \mathbb{D} : \mathrm{D}^b_h(\mathcal{D}_X) &\longrightarrow (\mathrm{D}^b_h(\mathcal{D}_X))^{opp} \\ \mathcal{M} &\mapsto \mathrm{R}\,\mathcal{H}om(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X] \end{aligned}$$

the duality functor, which also preserves holonomicity. We additionally define the functors

$$f_\dagger := \mathbb{D} \circ f_+ \circ \mathbb{D} \quad \text{and} \quad f^\dagger := \mathbb{D} \circ f^+ \circ \mathbb{D}.$$

If X is an affine variety, we have an equivalence of categories

$$\begin{aligned} \operatorname{Mod}(\mathcal{D}_X) &\longrightarrow \operatorname{Mod}(D_X) \\ \mathcal{M} &\mapsto M := \Gamma(X, \mathcal{M}) \end{aligned} \tag{1.2.1}$$

where $\operatorname{Mod}(D_X)$ is the category of left D_X-modules.

Definition 1.5. Let

$$\langle -, - \rangle : \mathbb{A}^\ell \times \hat{\mathbb{A}}^\ell \to \mathbb{A}^1, \qquad (\lambda_1, \dots, \lambda_\ell, \mu_1, \dots \mu_\ell) \mapsto \sum_{i=1}^{\ell} \lambda_i \mu_i.$$

(Here, and elsewhere, $\hat{\mathbb{A}}^\ell$ denotes an affine space of dimension ℓ; we use the "hat" to keep apart source and range of the two functors defined in (1.2.2) below). Now define two $\mathcal{D}_{\mathbb{A}^\ell\times\hat{\mathbb{A}}^\ell}$-modules by

$$\mathcal{L} := \mathcal{O}_{\mathbb{A}^\ell\times\hat{\mathbb{A}}^\ell} e^{\langle\cdot,\cdot\rangle}, \qquad \overline{\mathcal{L}} := \mathcal{O}_{\mathbb{A}^\ell\times\hat{\mathbb{A}}^\ell} e^{-\langle\cdot,\cdot\rangle}.$$

We refer to [KS97, Section 5] for details on these sheaves. Denote by $p_1 : \mathbb{A}^\ell\times\hat{\mathbb{A}}^\ell \to \mathbb{A}^\ell$ and $p_2 : \mathbb{A}^\ell\times\hat{\mathbb{A}}^\ell \to \hat{\mathbb{A}}^\ell$ the projection to the first and second factors respectively. The *Fourier–Laplace transform* is defined by

$$\begin{aligned} \mathrm{FL} : \mathrm{D}^b_{qc}(\mathcal{D}_{\mathbb{A}^\ell}) &\longrightarrow \mathrm{D}^b_{qc}(\mathcal{D}_{\hat{\mathbb{A}}^\ell}) \\ M &\mapsto p_{2+}(p_1^+ M \overset{L}{\otimes} \mathcal{L}) \end{aligned} \tag{1.2.2}$$

and

$$\begin{aligned} \mathrm{FL}^{-1} : \mathrm{D}^b_{qc}(\mathcal{D}_{\hat{\mathbb{A}}^\ell}) &\longrightarrow \mathrm{D}^b_{qc}(\mathcal{D}_{\mathbb{A}^\ell}) \\ M &\mapsto p_{1+}(p_2^+ M \overset{L}{\otimes} \overline{\mathcal{L}}) \end{aligned} \tag{1.2.3}$$

Then $\mathrm{FL}^{-1}\circ\mathrm{FL}(M) \simeq \iota^+ M$ where ι is given by $\lambda \mapsto -\lambda$, and we set

$$\hat{\mathcal{M}}^\beta_A := \mathrm{FL}^{-1}(\mathcal{M}^\beta_A)$$

with global sections $\hat{M}^\beta_A$. ⋄

Notation 1.6. If $\mathbb{A}^\ell$ and $\hat{\mathbb{A}}^\ell$ are an FL-pair with $\mathbb{A}^\ell = \Bbbk^A$ for some matrix A, we shall denote by $\hat{R}_A, \hat{O}_A, \hat{S}_A, \ldots$ the A-graded objects on $\hat{\mathbb{A}}^\ell$ corresponding to the A-graded objects $R_A, O_A, S_A, \ldots$ on $\mathbb{A}^\ell$. ⋄

2. Restriction and de Rham functors of Euler–Koszul complexes

In this section we make some computations considering certain functors on the class of (generalized) hypergeometric systems.

2.1. **Local cohomology.** Relevant in several ways are the local cohomology functors $H^\bullet_{\partial_A}(-)$ given as the higher derived functors of the ∂_A-torsion functor

$$\Gamma_{\partial_A}(M) := \{m \in M \mid \partial_i^k \cdot m = 0\, \forall k \gg 0, \forall i\},$$

a subfunctor of the identity functor on the category of R_A-modules. If M is A-graded, so are all $H^i_{\partial_A}(M)$ since the support ideal is A-graded. See [ILL+07] for details and background.

Lemma 2.1. *For any R_A-module N there is a functorial isomorphism*

$$\mathrm{R}\Gamma_{\partial_A}(N)[\dim(X_A)] = (D_A/\partial_A D_A) \otimes^L_{R_A} N$$

so that $H^\bullet_{\partial_A}(N) = \mathrm{Tor}^{R_A}_{\dim(X)-\bullet}(D_A/\partial_A D_A, N)$. Any R_A-grading $\deg(-)$ on N makes this isomorphism graded if the right side is shifted by $\sum_j \deg(\partial_j)$.

Proof. One representative for $\mathrm{R}\Gamma_{\partial_A}(-)$ is the Čech (*i.e.*, stable Koszul) complex $\check{C}^\bullet_A(-) = (-)\otimes_{R_A} \bigotimes_j (R_A \longrightarrow R_A[1/\partial_j])$. On R_A, this returns a D_A-complex with unique cohomology group, in cohomological degree $\dim X_A$, given by $\bigoplus_{\mathbf{v}<0} \Bbbk\cdot\partial^{\mathbf{v}}$ where $\mathbf{v}$ is componentwise negative. The D_A-isomorphism of this module with $D_A/D_A\partial_A$ that identifies the coset of $1/\prod_j \partial_j$ in the former with the coset of 1 in the latter (is A-graded of degree ε_A and) shows that (up to this shift in degree) this is the injective hull of $R_A/R_A\cdot\partial_A$ over R_A. The anti-automorphism induced by $x^{\mathbf{u}}\partial^{\mathbf{v}} \longrightarrow \partial^{\mathbf{v}}(-x)^{\mathbf{u}}$ allows to view $\check{C}^\bullet_A$ as complex of right D_A-modules without affecting the R_A-structure. Then $H^{\dim X_A}_{\partial_A}(R_A) = D_A/\partial_A D_A$ is the canonical module of R_A with its natural right D_A-structure.

The modules in $\check{C}^\bullet_A$ are flat, so $\check{C}^\bullet_A \otimes_{R_A} N[\dim X_A] = D_A/\partial_A D_A \otimes^L_{R_A} N$. If N is graded, then—since ∂_A is monomial—$D_A/\partial_A D_A$ and its flat resolution $\check{C}^\bullet_A$ are also graded. Hence $\check{C}^\bullet_A \otimes_{R_A} N$ has graded cohomology. The identification $H^{\dim X_A}(\check{C}^\bullet_A)[\dim X_A] \simeq D_A/\partial_A D_A$ shifts the grading by the degree of the socle element $1/\prod_j \partial_j$ of the left hand side. □

2.2. **Strongly resonant parameters.** We recall from [MMW05, SW09] the following important sets. The *exceptional locus* $\mathcal{E}_A$ is

$$\mathcal{E}_A := \operatorname{qdeg}_A\left(\bigoplus_{k>\dim X_A-\dim T_A} \operatorname{Ext}^k_{R_A}(S_A, R_A)\right) = \bigcup_{k<\dim T_A}\left(\overline{\deg_A H^k_{\partial_A}(S_A)}^{Zar}\right).$$

A larger interesting set is

$$\operatorname{sRes}(A) := \bigcup_j \operatorname{qdeg}_A(H^1_{\partial_j}(S_A)),$$

the *strongly resonant* parameters of A.

For $\Bbbk = \mathbb{C}$ the following results were shown in [MMW05, SW09]. A parameter is in $\mathcal{E}_A$ if and only if the complex $\mathcal{K}_\bullet(S_A; E-\beta)$ fails to be a resolution of $\mathcal{M}^\beta_A$; it is in $\operatorname{sRes}(A)$ if and only if $\mathcal{K}_\bullet(S_A; E-\beta)$ fails to resolve the Fourier–Laplace transform of $h_{A+}(\mathcal{O}^\beta_{T_A})$ where

$$\mathcal{O}^\beta_{T_A} = \mathcal{D}_{T_A}/\mathcal{D}_{T_A}(\{\partial_{t_i} t_i + \beta_i\}_i),$$

or alternatively if and only if $h_{A+}(\mathcal{O}^\beta_{T_A})$ disagrees with $\hat{\mathcal{M}}^\beta_A$. We are interested in these results over $\Bbbk$:

Theorem 2.2. *Let $\Bbbk$ be an arbitrary field of characteristic zero. For each j, the following are equivalent:*

(1) $\beta \notin \operatorname{sRes}_j(A) := \operatorname{qdeg}_A(H^1_{\partial_j}(S_A))$;
(2) left-multiplication by ∂_{x_j} is a quasi-isomorphism on $K_\bullet(E-\beta; S_A)$.

Corollary 2.3. *Over any coefficient field $\Bbbk$ of characteristic zero, the following are equivalent:*

(1) $\beta \notin \operatorname{sRes}(A)$;
(2) $\mathcal{K}_\bullet(E-\beta; S_A)$ represents the Fourier–Laplace transform of $h_{A+}\mathcal{O}^\beta_{T_A}$
(3) $\mathcal{M}^\beta_A$ is naturally isomorphic to the Fourier–Laplace transform of $\mathcal{H}^0 h_{A+}\mathcal{O}^\beta_{T_A}$

Inspection shows that, apart from formal computations that do not depend on $\Bbbk$, there are the following logical dependencies in [SW09].

- [SW09, Cor. 3.7] needs [SW09, Thm. 3.5, Cor. 3.1,Prop. 2.1] and [MMW05, Prop. 5.3], and the fact that higher Euler–Koszul homology is $(\prod_j \partial_j)$-torsion.
- [SW09, Thm. 3.5] needs [SW09, Lem. 3.2] and [MMW05, Prop. 5.3].
- [SW09, Lem. 3.2] is completely formal and independent of the field $\Bbbk$.
- [SW09, Cor. 3.1] needs the left and right Øre properties of D_A, and [SW09, Prop. 2.1].

- [SW09, Prop. 2.1] needs that direct images over $\Bbbk$ are formally the same for all $\Bbbk$ (which they are), plus $\mathcal{D}$-affinity of tori, plus various formal computations contained in [BGK+87], namely an identification of a direct image module in VI.7.3, the chain rule in VI.4.1, exactness of direct images for affine closed embeddings in VI.8.1, and equality of direct images under open embeddings in the $\mathcal{D}$- and $\mathcal{O}$-category in VI.5.2.

Tori are $\mathcal{D}$-affine since they are $\mathcal{O}$-affine. Higher Euler–Koszul homology is $(\prod_j \partial_j)$-torsion since localizing every ∂_j leads to Euler–Koszul homology of the quasi-toric Cohen–Macaulay module $\Bbbk[\mathbb{Z}A]$ (compare [SW09] for quasi-toricity). The Øre properties of D_A rely on the Leibniz rule and are unaffected by $\Bbbk$. Any closed embedding over $\Bbbk$ can be base-changed to a closed embedding (and hence to an affine faithful map) over $\mathbb{C}$, by viewing $\Bbbk$ (algebraically) as a subfield of $\mathbb{C}$. Since $\mathbb{C}$ is fully faithful over $\Bbbk$ and affine faithful maps over $\mathbb{C}$ yield exact direct image functors, so do they over $\Bbbk$. Direct images for open embeddings agree over $\mathcal{D}$ and $\mathcal{O}$ more or less by definition. It therefore remains to inspect [MMW05, Prop. 5.3] and exactness of the Euler–Koszul complex on maximal Cohen–Macaulay input over $\Bbbk$.

Using superscripts to indicate base fields, $K_A^{\Bbbk}(D_A^{\Bbbk}; E-\beta) \otimes_{\Bbbk} \mathbb{C} = K_A^{\mathbb{C}}(D_A^{\mathbb{C}}; E-\beta)$ as long as $\beta \in \Bbbk A$. The notion of a toric module is formally independent of $\Bbbk$, and so the categories of toric modules and their Euler–Koszul complexes embed into one another for containments of fields. In particular, the formal mechanisms are identical and scale from one field to another faithfully.

The required part of [MMW05, Prop. 5.3] is the equivalence (3)⇔(4). The proof passes through the equivalences (2)⇔(3) and (2)⇔(4). For both we need, modulo formal computations involving toric composition chains, only to check the equivalence of conditions (1) and (3) in [MMW05, Lem. 4.9]. The implication (1)⇒(3) is linear algebra over any field. The reverse follows by contradiction from base change to $\mathbb{C}$.

Finally, if M is a maximal Cohen–Macaulay toric module over $\Bbbk$ then vanishing of higher Euler–Koszul homology follows like over $\mathbb{C}$ from the spectral sequence [MMW05, Thm. 6.3] since the existence of the spectral sequence is abstract homological nonsense. However, this use of the spectral sequence requires the concept of holonomicity: one would like to use that Euler–Koszul homology modules are holonomic and that therefore their duals are modules.

The Euler–Koszul homology modules induce $\mathcal{D}_A$-modules on affine space. On that class, (dimension, and hence) holonomicity can be defined over all fields, via the theory of good filtrations. That holonomic modules have holonomic modules as their duals was proved by Roos, see the Bernstein notes [Ber, Thm. 3.15].

2.3. **Restriction to the origin.** Let ρ be the restriction functor to $\mathbf{0}_A \in \Bbbk^A$,

$$\rho(-) := (D_A/x_A D_A) \otimes^L_{R_A} (-)$$

from the category of (A-graded) $\mathcal{D}_A$- or D_A-modules to the category of (A-graded) $\Bbbk$-vector spaces. Denote $\rho_k(-)$ its k-th homology.

We start with a topological observation derived from [SW09]. By $H^\bullet_{dR}(-;\Bbbk)$ we mean the algebraic de Rham cohomology in the sense of Grothendieck [Gro66].

Lemma 2.4. *If $\beta \notin \operatorname{sRes}(A)$ then $\rho(\mathcal{M}_A^\beta)$ is naturally identified with the homology of the local system to $\mathcal{O}_{T_A}^\beta$ on the torus T_A:*

$$H_j(\rho(\mathcal{M}_A^\beta)) = \rho_j(\mathcal{M}_A^\beta) \simeq \begin{cases} H^j_{dR}(T_A; \Bbbk) & \text{if } \beta \in \mathbb{Z}A \smallsetminus \operatorname{sRes}(A); \\ 0 & \text{if } \beta \in \Bbbk A \smallsetminus (\mathbb{Z}A \cup \operatorname{sRes}(A)). \end{cases}$$

Proof. By [SW09] and 2.3, if $\beta \notin \operatorname{sRes}(A)$ then $\mathrm{FL}^{-1}(\mathcal{M}_A^\beta) \simeq h_{A+}(\mathcal{O}_{T_A}^\beta)$. Under Fourier–Laplace, restriction ρ converts to the functor $(D_A/\partial_A D_A) \otimes^L_{D_A} (-)$. On the affine space $X_A = \Bbbk^A$ this is the D-module direct image under the map to a point. Hence with $\beta \notin \operatorname{sRes}(A)$, $\rho(\mathcal{M}_A^\beta)$ represents the direct image of $\mathcal{O}_{T_A}^\beta$ under projection to a point—in other words, the cohomology of the local system. □

We next extend this lemma by identifying algebraically $\rho(M_A^\beta)$ with $\bigwedge(\Bbbk A)$ for non-exceptional β. (We view the exterior algebra as an abstract copy of the cohomology of T_A). Note that in this case the Euler–Koszul complex resolves $\mathcal{M}_A^\beta$ but is not necessarily a representative for $\mathrm{FL}^{-1} h_{A+}(\mathcal{O}_{T_A}^\beta)$. Studying restrictions of Euler–Koszul complexes turns out to be very down to earth.

Lemma 2.5. *If $\phi \colon N \longrightarrow N'$ is an A-graded morphism (of degree zero) of A-graded R_A-modules then*

(1) the restriction $\rho(\mathcal{K}_\bullet(N; E-\beta))$ is naturally $\bigwedge(\Bbbk A) \otimes_\Bbbk N_\beta$, in the sense that

(2) the induced morphism $\rho(\mathcal{K}_\bullet(N; E-\beta)) \longrightarrow \rho(\mathcal{K}_\bullet(N'; E-\beta))$ is identified with the morphism $\bigwedge(\Bbbk A) \otimes_\Bbbk N_\beta \longrightarrow \bigwedge(\Bbbk A) \otimes_\Bbbk N'_\beta$ induced from ϕ_β.

Proof. We extend the domain of the Euler–Koszul functor to modules of the form $Q \otimes_{R_A} N$ where N is an A-graded R_A-module and Q a right A-graded D_A-module by setting $E_i \circ (q \otimes \nu) = q(E_i + \deg_{A,i}(\nu)) \otimes \nu$.

Morally, $E_i \circ (-)$ remains right-multiplication by E_i and (since multiplications on the left and right commute) one easily checks that there is an isomorphism of functors $\rho(\mathcal{K}_\bullet(-; E-\beta)) = \mathcal{K}_\bullet(\rho(D_A \otimes_{R_A} (-)); E-\beta) = \mathcal{K}_\bullet((D_A/x_A D_A) \otimes^L_{R_A} (-); E-\beta)$ from the category of A-graded R_A-modules to the category of A-graded vector spaces.

As right R_A-module, $(D_A/x_A D_A) \otimes_{R_A} N = N$ for any A-graded N. Hence $(D_A/x_A D_A) \otimes^L_{R_A} (-) = (D_A/x_A D_A) \otimes_{R_A} (-)$. The E_i-action is then $E_i \circ (1 \otimes \nu) = (\deg_{A,i}(\nu)) \otimes \nu$. In particular, the Euler–Koszul complex of $E-\beta$ on $(D_A/x_A D_A) \otimes N$ is in degree $\alpha \in \mathbb{Z}A$ the Koszul complex on N_α induced by the numbers $\{\alpha_i - \beta_i\}_i$. If $\alpha = \beta$ then this Koszul complex is $\bigwedge(\Bbbk A) \otimes_\Bbbk N_\alpha$ with zero differential. If $\alpha \neq \beta$ then this Koszul complex is the Koszul complex (over $\mathbb{Z}$) of a set of generators of the unit ideal and hence exact.

The final claim is clear from the construction. □

Corollary 2.6. *If $\beta \notin \mathcal{E}_A$ then*

$$\rho_j(\mathcal{M}_A^\beta) \simeq \begin{cases} H^j_{dR}(T_A; \Bbbk) & \text{if } \beta \in \mathbb{N}A; \\ 0 & \text{if } \beta \notin \mathbb{N}A. \end{cases}$$

Proof. If $\beta \notin \mathcal{E}_A$ then $\rho(\mathcal{M}_A^\beta) = \rho(\mathcal{K}_\bullet(S_A; E-\beta))$. Now use Lemma 2.5. □

The natural morphism $\rho(\mathcal{K}_\bullet(S_A; E-\beta)) \longrightarrow \rho(\mathcal{M}_A^\beta)$ need not be an isomorphism:

Example 2.7. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$ and take $\beta = (1, 2)$, the only parameter with higher Euler–Koszul homology for this A (by [ST98]). The $\Bbbk$-dimension vectors for $\rho(\mathcal{M}_A^\beta)$ and $\rho(\mathcal{K}_\bullet(S_A; E - \beta))$ are $(0, 0, 1, 0, 0)$ and $(0, 0, 0, 0, 0)$ respectively. $\diamond$

In order to better understand the relationship between the restrictions of the A-hypergeometric system and the Euler–Koszul complex respectively, we consider the 3-rd quadrant spectral sequence

$$E^2_{-i,-j} = \rho_j(\mathcal{H}_i(-; E - \beta)) \Longrightarrow (\rho(\mathcal{K}_\bullet(-; E - \beta)))_{i+j}.$$

The k-th differential is $d_k \colon E^k_{-p,-q} \longrightarrow E^k_{-p-k+1,-q+k}$. A toric map $N \longrightarrow N'$ induces a morphism of corresponding spectral sequences.

All our experiments indicate that if $\beta \in \mathbb{N}A$ then $\rho_j(\mathcal{M}_A^\beta) = H^j_{dR}(T_A)$, irrespective of exceptionality. While we cannot show that, we have a one-way estimate:

Lemma 2.8. *If $\beta \in \mathbb{N}A$ then there is a natural inclusion $\bigwedge(\Bbbk A) \hookrightarrow \rho(\mathcal{M}_A^\beta)$.*

Proof. Consider the morphism of spectral sequences attached to the inclusion $S_A \hookrightarrow \tilde{S}_A$ of S_A into its normalization. For any $\beta \in \mathbb{N}A$, the induced map of abutments $\rho(\mathcal{K}_\bullet(S_A; E - \beta)) \longrightarrow \rho(\mathcal{K}_\bullet(\tilde{S}_A; E - \beta))$ is an isomorphism by Lemma 2.5. Since $\tilde{S}_A$ is Cohen–Macaulay, it has no higher Euler–Koszul homology and so the abutment $\rho(\mathcal{K}_\bullet(\tilde{S}_A; E - \beta))$ is stored in the $i = 0$ column of the E^2-term. It follows that the isomorphism on abutments must be coming from the map of the $i = 0$ column, for $k \gg 0$. But $E^k_{0,-j}$ is a submodule of $E^2_{0,-j}$ for $k \geq 2$. In particular, $\rho(\mathcal{K}_\bullet(\tilde{S}_A; E - \beta)) \simeq \bigwedge(\Bbbk A)$ is contained in $\rho(\mathcal{K}_\bullet(S_A; E - \beta)) = \rho(\mathcal{M}_A^\beta)$. $\square$

2.4. **De Rham cohomology.** We consider now the effect of $(D/\partial_A D_A) \otimes^L_{D_A} (-)$ on M_A^β and on the Euler–Koszul complex. This behaves differently since $(D_A/\partial_A D_A) \otimes_{D_A} N$ is not N for most A-graded R_A-modules N.

Definition 2.9. If β is in $\deg_A(\mathrm{R}\Gamma_{\partial_A}(S_A)) = \bigcup_{k<\dim T_A} \deg_A(H^k_{\partial_A}(S_A)) \subseteq \mathcal{E}_A$ it is called *strongly A-exceptional.* $\diamond$

Theorem 2.10. *For any A-graded R_A-module N, $(D/\partial_A D_A) \otimes^L_{D_A} K_\bullet(N; E - \beta)$ vanishes whenever β is not an A-degree of $\mathrm{R}\Gamma_{\partial_A}(N))$. More precisely,*

$$(D/\partial_A D_A) \otimes^L_{D_A} K_\bullet(N; E - \beta) \simeq \left(\bigoplus_i H^i_{\partial_A}(N) \right)_\beta \otimes_\Bbbk \bigwedge(\Bbbk A)[\dim X_A].$$

As in Lemma 2.5, an A-graded map $N \longrightarrow N'$ induces a map of de Rham complexes that is identified with $(\mathrm{R}\Gamma_{\partial_A}(N) \longrightarrow \mathrm{R}\Gamma_{\partial_A}(N'))_\beta \otimes_\Bbbk \bigwedge(\Bbbk A)[\dim X_A]$.

*If β is not strongly exceptional (*e.g, *if S_A is Cohen–Macaulay), then*

$$\mathrm{Tor}^{D_A}_\bullet(D/\partial_A D_A, K_\bullet(S_A; E - \beta)) = H^{\bullet + \dim X_A}_{dR}(T_A; \Bbbk)$$

if β is in $\deg_A(H^{\dim T_A}_{\partial_A}(S_A))$ and zero otherwise.

Proof. As in the proof of Lemma 2.5, we extend the action of the Euler operators to the quotient $(D_A/\partial_A D_A) \otimes_{R_A} N$ for any A-graded N. Hence $(D_A/\partial_A D_A) \otimes^L_{D_A} K_\bullet(N; E - \beta) = K_\bullet((D_A/\partial_A D_A) \otimes^L_{R_A} N; E - \beta)$ for any A-graded R_A-module N.

Recall that $\varepsilon_A = \sum_j \mathbf{a}_j$ and that its components $\varepsilon_{A,i}$ satisfy $E_i + \varepsilon_{A,i} = \sum_j a_{ij} \partial_j x_j$. Take now a free A-graded R_A-resolution $F_\bullet$ for N. Then for any A-graded element $P \otimes f \in (D_A/\partial_A D_A) \otimes_{R_A} F_k$ the cosets of $(E_i - \beta_i) \circ (P \otimes f)$, of

$(E_i - \beta_i + \deg_{A,i}(P \otimes f))P \otimes f$ and of $(-\varepsilon_{A,i} - \beta_i + \deg_{A,i}(P \otimes f))P \otimes f$ coincide. So, as in Lemma 2.5, the Euler–Koszul complex on $(D_A/\partial_A D_A) \otimes F_\bullet$ is in degree α the Koszul complex on $D_A/\partial_A D_A \otimes F_\bullet$ induced by the numbers $\{-\varepsilon_{A_i} - \beta_i + \alpha_i\}$. Hence $K_\bullet((D_A/\partial_A D_A) \otimes^L_{R_A} N; E - \beta)$ can only have cohomology when $(D_A/\partial_A D_A) \otimes^L_{R_A} N$ has a cohomology class in degree $\beta + \varepsilon_A$. By Lemma 2.1 this is equivalent to β being the degree of a nonzero cohomology class in $\check{C}^\bullet_A \otimes_{R_A} N$ which proves the first claim.

If $\mathrm{R}\Gamma_{\partial_A}(N)$ is non-exact in degree β then $(D_A/\partial_A D_A) \otimes^L_{R_A} (K_\bullet(N; E - \beta - \varepsilon_A))$ is $(H^\bullet_{\partial_A}(N))_\beta$ tensored with a Koszul complex (shifted by $\dim X_A$) on $\dim(T_A)$ maps $\Bbbk \longrightarrow \Bbbk$ each of which is the zero map. Hence in this case, the resulting cohomology is $(H^\bullet_{\partial_A}(N))_\beta \otimes_\Bbbk \bigwedge(\Bbbk A)[\dim X_A]$. The indicated naturality condition is clear from the discussion.

If β is not strongly exceptional then $(\mathrm{R}\Gamma_{\partial_A}(S_A))_\beta \simeq (H^{\dim T_A}_{\partial_A}(S_A))_\beta$. The latter is a subquotient of $\Bbbk[\mathbb{Z}A]$ and hence its A-graded Hilbert function takes values in $\{0, 1\}$. The final claim follows. ☐

Example 2.11. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$. Then $(H^2_{\partial_A}(S_A))_\beta$ is nonzero exactly if β is an interior lattice point of $-\mathbb{R}_+ A$, while $H^1_{\partial_A}(S_A)$ is a 1-dimensional vector space concentrated in degree $(1, 2)$. It follows that $\mathrm{Tor}^{D_A}_\bullet(D/\partial_A D_A, K_\bullet(S_A; E - \beta))$ is $H^{\bullet+4}(T_A; \Bbbk)$ when β supports $H^2_{\partial_A}(S_A)$; it is the shifted $H^{\bullet+4}(T_A; \Bbbk)[1] = H^{\bullet+5}(T_A; \Bbbk)$ when $\beta = (1, 2)$; it is zero in all other cases.

In particular, no simple general formula (not appealing to local cohomology modules) for $\mathrm{Tor}^{D_A}_\bullet(D/\partial_A D_A, K_\bullet(S_A; E - \beta))$ comes to mind. ⋄

Corollary 2.12. $\mathrm{Tor}^{D_A}_\bullet(D/\partial_A D_A, M^\beta_A)$ *and* $\mathrm{Ext}^\bullet_{D_A}(O_A, M^\beta_A)$ *are nonzero only if*

$$\beta \in \mathcal{E}_A \cup (\deg_A(H^{\dim(S_A)}_{\partial_A}(S_A))).$$

Proof. Note first that resolving O_A over D_A and dualizing the resolution gives a resolution of (a cohomologically shifted) $D_A/\partial_A D_A$, so that the Ext- and Tor-claims are equivalent.

By [MMW05], the Euler–Koszul complex resolves M^β_A whenever $\beta \notin \mathcal{E}_A$. So, for such β not in $\deg_A(H^{\dim(S_A)}_{\partial_A}(S_A))$, the indicated Ext- and Tor-groups vanish by Theorem 2.10. ☐

Definition 2.13. Let N_A be the *interior ideal* of S_A, generated by the monomials whose degrees are in the topological interior of $\mathbb{R}_+ A$. ⋄

Corollary 2.14. *If S_A is normal, then*

$$\begin{aligned} \mathrm{Tor}^{D_A}_\bullet(D/\partial_A D_A, M^\beta_A) &= \mathrm{Tor}^{D_A}_\bullet(D/\partial_A D_A, K_\bullet(S_A; E - \beta)) \\ &= \begin{cases} H^{\bullet+\dim X_A}_{dR}(T_A; \Bbbk) & \text{if } -\beta \in \deg_A(N_A); \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Proof. The exceptional locus is here empty. The interior ideal is the canonical module ω_{S_A} in the A-graded category by [BH93, Cor. 6.3.6] while also in the A-graded category $\omega_{S_A} = \mathrm{Ext}^{\dim X_A - \dim T_A}_{R_A}(S_A, \omega_{R_A})$, [BH93, Prop. 3.6.12]. Then graded local duality [BH93, Thm. 3.6.19] yields that $\deg_A(H^{\dim T_A}_{\partial_A}(S_A)) = -\deg_A(N_A)$. Now use Theorem 2.10. ☐

For our applications, it is interesting to know that $\mathbb{N}A$ does not meet the strongly A-exceptional locus where for $\tau \subseteq A$ a face we write ∂_τ for $\{\partial_j\}_{j\in\tau}$:

Lemma 2.15. *For any face τ of A, no element of $\mathbb{N}A$ is a degree of $\mathrm{R}\Gamma_{\partial_\tau}(S_A)$.*

Proof. For $j \in \tau$, $\bigotimes_\tau(S_A \to S_A[\partial_j^{-1}]) \simeq (S_A \to S_A[\partial_j^{-1}]) \otimes \bigotimes_{\tau\ni j'\neq j}(S_A \to S_A[\partial_{j'}^{-1}])$. The corresponding double complex spectral sequence starts on the E^1-page with modules of the form $S_A[(\partial_j \cdot \prod_{j'\in\tau'} \partial_{j'})^{-1}]/S_A[(\prod_{j'\in\tau'} \partial_{j'})^{-1}]$ for all possible $\tau' \subseteq \tau \smallsetminus \{j\}$. The $\mathbb{k}$-dimension of A-graded localizations of S_A in each A-degree is zero or one, and S_A is a domain,. So, $S_A[(\partial_j \cdot \prod_{j'\in\tau'} \partial_{j'})^{-1}]/S_A[(\prod_{j'\in\tau'} \partial_{j'})^{-1}]$ is of dimension zero in each degree $\beta \in \mathbb{N}A$. Hence the same holds for the abutment. ☐

In contrast, elements of $\mathbb{N}A$, including the origin 0, can indeed be *quasi*-degrees of lower local cohomology (and hence exceptional parameters):

Example 2.16. Let $A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. The exceptional locus is the line $\mathbb{k}\cdot\mathbf{a}_1$. ◇

The following corollary will be used in Section 3

Corollary 2.17. *Suppose S_A is Cohen–Macaulay, and put $M = H_0(S_A\partial_A; E-\beta)$. Then for $\beta = 0$,*

$$\mathrm{Tor}_i^{D_A}(D_A/\partial_A D_A, M) = \begin{cases} \mathbb{k}^{\dim T_A} & \text{if} \quad i = \dim X_A; \\ \mathbb{k} & \text{if} \quad i = \dim X_A - 1; \\ 0 & \text{if} \quad i < \dim X_A - 1; \end{cases}$$

while all Tor-groups vanish if $0 \neq \beta \in \mathbb{N}A$.

Proof. Consider the toric sequence $0 \longrightarrow S_A\partial_A \longrightarrow S_A \longrightarrow \mathbb{k} \longrightarrow 0$. Cohen–Macaulayness ensures, by [MMW05], that the Euler–Koszul functor produces an exact sequence

(2.4.1)
$$0 \longrightarrow H_1(\mathbb{k}; E-\beta) \longrightarrow H_0(S_A\partial_A; E-\beta) \longrightarrow M_A^\beta \longrightarrow H_0(\mathbb{k}; E-\beta) \longrightarrow 0.$$

For $\beta \neq 0$, the outer modules are zero. For $\beta = 0$, the right module is O_A and the left is $O_A^{\dim T_A}$. The claim then follows from Theorem 2.10 and Lemma 2.15: apply $\mathrm{Tor}_\bullet^{D_A}(D_A/\partial_A D_A, -)$ to $0 \longrightarrow M/O_A^{\dim T_A} \longrightarrow M_A^\beta \longrightarrow O_A \longrightarrow 0$ and $0 \longrightarrow O_A^{\dim T_A} \longrightarrow M \longrightarrow M/O_A^{\dim T_A} \longrightarrow 0$. ☐

2.5. **Ext and the polynomial solution functor.** Dualizing a D_A-resolution of O_A gives a resolution of (a cohomologically shifted) $D_A/\partial_A D_A$. Hence, up to shift by ε_A in the A-grading, $\mathrm{Ext}^\bullet_{D_A}(O_A, M_A^\beta) = \mathrm{Tor}^{D_A}_{\dim X_A - \bullet}(D_A/\partial_A D_A, M_A^\beta)$. In particular, the vanishing results in the previous section apply to $\mathrm{Ext}^\bullet_{D_A}(O_A, M_A^\beta)$.

In this subsection we consider the behavior of the solution functor $\mathrm{Hom}_{D_A}(-, O_A)$ with values in the ring of polynomials on the class of A-hypergeometric modules M_A^β. It is immediately clear that $\mathrm{Hom}_{D_A}(M_A^\beta, O_A)$ can only be nonzero if $\beta \in \mathbb{N}A$, and it is an old result that $\beta \in \mathbb{N}A$ implies that $\mathrm{Hom}_{D_A}(M_A^\beta, O_A)$ is 1-dimensional, see [SST00, Prop. 3.4.11]. We investigate here the derived polynomial solution functor and prove

Theorem 2.18. *If $\beta \notin \mathcal{E}_A$ (for example, if S_A is Cohen–Macaulay) then*

$$\operatorname{Ext}^i_{D_A}(M_A^\beta, O_A) = \begin{cases} H^i_{dR}(T_A; \Bbbk) & \text{if} \quad \beta \in \mathbb{N}A; \\ 0 & \text{else.} \end{cases}$$

(All experiments indicate this to be true even if $\beta \in \mathcal{E}_A$.)

Proof. Write $\tau(-)$ for the transposition $x^{\mathbf{u}}\partial^{\mathbf{v}} \mapsto \partial^{\mathbf{v}}(-x)^{\mathbf{u}}$ on D_A. Let $F_\bullet$ be an A-graded R_A-free resolution of S_A and denote $\omega_{R_A} = D_A/\partial_A D_A[\dim X_A]$. Then we have the following equalities, where $(-)^\vee$ is the vector space dual:

$$\begin{aligned}
\Big(\mathrm{R}\operatorname{Hom}_{D_A}(M_A^\beta, O_A)\Big)^\vee &\overset{(a)}{\simeq} \mathrm{R}\operatorname{Hom}_{D_A}(O_A, \mathbb{D}M_A^\beta) \\
&\overset{(b)}{\simeq} \omega_{R_A} \otimes^L_{D_A} \mathbb{D}M_A^\beta \\
&\overset{(c)}{=} \omega_{R_A} \otimes^L_{D_A} \mathbb{D}K_\bullet(S_A; E-\beta) \\
&\overset{(d)}{=} \omega_{R_A} \otimes^L_{D_A} \mathbb{D}K_\bullet(F_\bullet; E-\beta) \\
&\overset{(e)}{=} \omega_{R_A} \otimes^L_{D_A} K_\bullet(\operatorname{Hom}_{R_A}(F_\bullet, R_A); E+\beta+\varepsilon_A) \\
&\simeq \omega_{R_A} \otimes^L_{D_A} K_\bullet(\mathrm{R}\operatorname{Hom}_{R_A}(S_A, R_A); E+\beta+\varepsilon_A) \\
&\overset{(f)}{\simeq} K_\bullet(\omega_{R_A} \otimes^L_{R_A} \mathrm{R}\operatorname{Hom}_{R_A}(S_A, R_A); E+\beta+\varepsilon_A) \\
&\overset{(g)}{\simeq} (\mathrm{R}\Gamma_{\partial_A}\mathrm{R}\operatorname{Hom}_{R_A}(S_A, R_A))_{-\beta-\varepsilon_A} \otimes \bigwedge(\Bbbk A) \\
&\overset{(h)}{\simeq} \Big((\mathrm{R}\operatorname{Hom}_{R_A}(\mathrm{R}\operatorname{Hom}_{R_A}(S_A, R_A), R_A))_\beta \otimes \bigwedge(\Bbbk A)\Big)^\vee \\
&= \Big((S_A)_\beta \otimes \bigwedge(\Bbbk A)\Big)^\vee.
\end{aligned}$$

The following notes justify the above transformations:

- (a) Duality gives $\mathrm{R}\operatorname{Hom}_{D_A}(M, M') \simeq (\mathrm{R}\operatorname{Hom}_{D_A}(\mathbb{D}M', \mathbb{D}M))^\vee$, [HTT08, §2.6].
- (b) Resolve O_A and dualize the resolution, incurring a cohomological shift.
- (c) By [MMW05], the hypergeometric system is resolved by the Euler–Koszul complex as long as β is not exceptional.
- (d) The Euler–Koszul functor can be applied to any A-graded resolution.
- (e) $K_\bullet(F_\bullet; E-\beta)$ is a free complex. Applying $\operatorname{Hom}_{D_A}(-, D_A)$ and the transposition τ turns $D_A \otimes_{R_A} F_\bullet$ into $D_A \otimes_{R_A} \operatorname{Hom}_{R_A}(F_\bullet, R_A)$ and the Euler–Koszul complex on β into that on $-\beta-\varepsilon_A$ since $x_j\partial_j$ turns into $-\partial_j x_j$.
- (f) As in the proof of Theorem 2.10.
- (g) Theorem 2.10 works for A-graded complexes just as well.
- (h) Apply local A-graded duality (responsible for the dual).

∎

3. Three four-term sequences

Notation. From now on, A is a $(d+1)\times(n+1)$ matrix and $\mathbb{N}A$ is assumed to be saturated, in addition to the conventions in Notation 1.2 and Definition 1.3.

All products of $\Bbbk$-schemes are by default over $\Bbbk$. ⋄

Consider the exact toric sequence $0 \longrightarrow S_A\partial_A \longrightarrow S_A \longrightarrow \Bbbk \longrightarrow 0$. Normality ensures, by [MMW05, Prop. 5.3, Thm. 6.6], that the Euler–Koszul functor produces the exact sequence (2.4.1), and, for $i \geq 1$, isomorphisms

$$H_i(\Bbbk; E-\beta) \simeq \begin{cases} O_A^{\binom{d+1}{i}} & \text{for } \beta = 0; \\ 0 & \text{else.} \end{cases} \tag{3.0.1}$$

In this section we will show that the sequence (2.4.1) has a geometric interpretation when A is homogeneous. Our approach is inspired by [Sti98], where Stienstra defined on the torus T_A a family F of Laurent polynomials using the matrix A. He showed that one term in the long exact cohomology sequence of the pair (T_A, fiber of F) could be naturally identified with a fiber in the A-hypergeometric system M_A^0 when F is smooth. We will extend this identification to the non-smooth fibers of F.

We will proceed as follows. First we identify the second term of (2.4.1) as a concatenation of (proper) direct image functors applied to the structure sheaf $\mathcal{O}_{T_A}$. The third term already has such an interpretation by Corollary 2.3 above. The remaining terms are identified as the cohomology of the cone of a natural adjunction morphism between the second and third term.

As a second step we show in Lemma 3.7 that the sequence (2.4.1) is part of a long exact sequence coming from a triangle of elementary $\mathcal{D}$-modules on the line $\hat{\mathbb{A}}^1$. We also show in Proposition 3.8 that the Fourier–Laplace transform of (2.4.1) is induced by the FL-transformed triangle of elementary $\mathcal{D}$-modules from Lemma 3.7. This enables us to give a geometric interpretation of the exact sequence in Theorem 3.10 in terms of Gauß–Manin systems of the pair (T_A, fiber of F) as alluded to above.

As a preparatory result we begin with an identification of two functors on certain sheaves.

3.1. **Quasi-equivariant bundles.** Denote $\mathbb{G}_m$ the scheme of units of $\Bbbk$. A $\mathbb{G}_m$-*action on the variety* Y is a multiplicative morphism $\mu\colon \mathbb{G}_m \times Y \longrightarrow Y$ where $1 \in \mathbb{G}_m$ acts as identity. That is, μ is a morphism, $\mu(g, \mu(g', y)) = \mu(gg', y)$ and $\mu(1, y) = y$.

Let X be an affine smooth variety and $\pi : E = X \times \mathbb{A}^n \to X$ be a trivial vector bundle on X. Write $E^* = E \setminus (X \times \{0\})$ and let E_x be the fiber over $x \in X$. The zero section is identified with X as closed subvariety via the embedding

$$i\colon X \hookrightarrow E.$$

Definition 3.1. A $\mathbb{G}_m$-action $\mu\colon \mathbb{G}_m \times E \longrightarrow E$ on E is *fibered* if
(1) μ preserves fibers, $\mu\colon \mathbb{G}_m \times E_x \longrightarrow E_x$;
(2) μ is the restriction of a morphism $\mu\colon \mathbb{A}^1 \times E \longrightarrow E$ under $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$;
(3) $0 \in \mathbb{A}^1$ multiplies into the zero section, $\mu\colon 0 \times E_x \longrightarrow i(X)$;
(4) $\mathbb{A}^1$ fixes the zero section, $\mu\colon \mathbb{A}^1 \times i(X) \longrightarrow i(X)$.

◇

Definition 3.2. Let $\mu : \mathbb{G}_m \times E \to E$ be a fibered $\mathbb{G}_m$-action on E. Write $p\colon \mathbb{G}_m \times E \longrightarrow E$ for the projection and denote by μ' and p' the restrictions of μ and p to $\mathbb{G}_m \times E^*$.

A holonomic $\mathcal{D}_E$-module $\mathcal{M}$ is called $\mathbb{G}_m$-*quasi-equivariant* if $(\mu')^+\mathcal{M}_{|E^*} \simeq (p')^+\mathcal{M}_{|E^*}$. ◇

We consider the derived category of bounded complexes of $\mathcal{D}_E$-modules with holonomic and quasi-equivariant cohomology.

Lemma 3.3. *Let $\pi : E \to X$ be fibered and denote $i : X \to E$ the inclusion of the zero section. For every $\mathbb{G}_m$-quasi-equivariant $\mathcal{D}_E$-module $\mathcal{M}$,*

$$\pi_+\mathcal{M} \simeq i^\dagger\mathcal{M} \qquad \textit{and} \qquad \pi_\dagger\mathcal{M} \simeq i^+\mathcal{M}$$

Proof. By duality, it suffices to prove the first claim. Denote by $j : E^* \to E$ the open embedding of the complement of the zero section and let π be the projection to the base X. We have the exact triangles

$$j_\dagger j^{-1}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow i_+ i^\dagger\mathcal{M} \xrightarrow{+1} \tag{3.1.1}$$

$$\pi_+ j_\dagger j^{-1}\mathcal{M} \longrightarrow \pi_+\mathcal{M} \longrightarrow i^\dagger\mathcal{M} \xrightarrow{+1} \tag{3.1.2}$$

and the Cartesian diagram

$$\begin{array}{ccc} \mathbb{G}_m \times E^* & \xrightarrow{j'} & \mathbb{A}^1_m \times E \\ \Big\downarrow{\scriptstyle \mu'} & & \Big\downarrow{\scriptstyle \mu} \\ E^* & \xrightarrow{j} & E \end{array}$$

where μ' is the restriction of μ to $\mathbb{G}_m \times E^*$ and j' is the canonical inclusion. The morphism $s : E \to \mathbb{A}^1 \times E$ with $s(x) = (1, x)$ is a section of μ. Thus, the composition (induced by the natural transformation $\mathrm{id}_E \longrightarrow \mu_+\mu^\dagger$)

$$\pi_+ j_\dagger j^{-1}\mathcal{M} \to \pi_+\mu_+\mu^\dagger j_\dagger j^{-1}\mathcal{M} \to \pi_+\mu_+ s_+ s^\dagger \mu^\dagger j_\dagger j^{-1}\mathcal{M} = \pi_+ j_\dagger j^{-1}\mathcal{M}$$

is an isomorphism by (3.1.2); hence it is enough to prove $\pi_+\mu_+\mu^\dagger j_\dagger j^{-1}\mathcal{M} = 0$. By base change,

$$\pi_+\mu_+\mu^\dagger j_\dagger j^{-1}\mathcal{M} \simeq \pi_+\mu_+ j'_\dagger (\mu')^\dagger j^{-1}\mathcal{M}.$$

Since $\mathcal{M}$ is $\mathbb{G}_m$-quasi-equivariant, we have

$$(\mu')^\dagger j^{-1}\mathcal{M} \simeq p^\dagger\, j^{-1}\mathcal{M} \simeq \mathcal{O}_{\mathbb{G}_m} \boxtimes j^{-1}\mathcal{M}\,.$$

Therefore (letting $a\colon \mathbb{A}^1 \to \{pt\}$ be the map to the point) we get

$$\begin{aligned} \pi_+\mu_+ j'_\dagger (\mu')^\dagger j^{-1}\mathcal{M} &\simeq \pi_+\mu_+ j'_\dagger(\mathcal{O}_{\mathbb{G}_m} \boxtimes j^{-1}\mathcal{M}) \\ &\simeq \pi_+\mu_+(j_{1\dagger}\mathcal{O}_{\mathbb{G}_m} \boxtimes j_\dagger j^{-1}\mathcal{M}) \\ &\simeq \pi_+\, p_+(j_{1\dagger}\mathcal{O}_{\mathbb{G}_m} \boxtimes j_\dagger j^{-1}\mathcal{M}) \\ &\simeq a_+ j_{1\dagger}\mathcal{O}_{\mathbb{G}_m} \boxtimes \pi_+ j_\dagger j^{-1}\mathcal{M} \end{aligned}$$

where $j_1 : \mathbb{G}_m \to \mathbb{A}^1$ is the canonical inclusion. Since $a_+ j_{1\dagger}\mathcal{O}_{\mathbb{G}_m} = 0$ in $\mathrm{D}^b(\mathcal{D}_{\{pt\}})$ we have $\pi_+\mu_+\mu^\dagger j_\dagger j^{-1}\mathcal{M} = 0$. □

Recall that A is a $(d+1) \times (n+1)$ matrix. Let $T_A = \mathrm{Spec}\,(\Bbbk[t_0^\pm, \ldots, t_d^\pm])$ and consider the ring homomorphism

$$\begin{aligned} \Bbbk[y_0, \ldots, y_n] &\longrightarrow \Bbbk[t_0^\pm, \ldots, t_d^\pm] \\ y_i &\mapsto t^{\mathbf{a}_i} \end{aligned}$$

which gives rise to a morphism

$$h_A : T_A \to \hat{\mathbb{A}}^{n+1},$$

where $\hat{\mathbb{A}}^{n+1} = \mathrm{Spec}\,(\Bbbk[y_0, \ldots, y_n])$. We factorize this embedding as

$$T_A \xrightarrow{h_1} \hat{\mathbb{A}}^{n+1} \setminus \{0\} \xrightarrow{h_2} \hat{\mathbb{A}}^{n+1}\,.$$

We are now ready to show a useful property of A-hypergeometric systems and their Fourier–Laplace transforms.

Lemma 3.4. *The $\mathcal{D}_{\hat{\mathbb{A}}^{n+1}}$-module $h_{A+}\mathcal{O}_{T_A}$ is $\mathbb{G}_m$-quasi-equivariant.*

Proof. We view $\hat{A}^{n+1}$ as trivial bundle over itself. Since A is pointed, there is $\mathbf{u} \in \mathbb{Z}^{d+1}$ with $\mathbf{v} = \mathbf{u}^T \cdot A$ componentwise positive. Let $\mu' \colon \mathbb{G}_m \times \hat{\mathbb{A}}^{n+1} \longrightarrow \hat{\mathbb{A}}^{n+1}$ be the monomial action induced by $\mathbf{v}$ and let $\tilde{\mu} \colon \mathbb{G}_m \times T_A \longrightarrow T_A$ be the action induced by $\mathbf{u}$. (Compare the discussion on the Euler space in [RSW17].) Consider the Cartesian diagram

$$\begin{array}{ccccc} \mathbb{G}_m \times T_A & \xrightarrow{h_1 \times id} & \mathbb{G}_m \times \hat{\mathbb{A}}^{n+1} \setminus \{0\} & & \\ p \downarrow\downarrow \tilde{\mu} & & p \downarrow\downarrow \mu' & & \\ T_A & \xrightarrow{h_1} & \hat{\mathbb{A}}^{n+1} \setminus \{0\} & \xrightarrow{h_2} & \hat{\mathbb{A}}^{n+1} \end{array}$$

and note that the positivity of $\mathbf{v}$ allows to extend μ' to $\mathbb{A}^1 \times \hat{\mathbb{A}}^{n+1}$. Then

$$p^+ \, h_{1+}\mathcal{O}_{T_A} \simeq (h_1 \times id)_+ \, p^+ \, \mathcal{O}_{T_A} \simeq (h_1 \times id)_+ \tilde{\mu}^+ \mathcal{O}_{T_A} \simeq (\mu')^+ h_{1+}\mathcal{O}_{T_A}$$

and so $p^+ \, h_{1+}\mathcal{O}_{T_A} \simeq (\mu')^+ h_{1+}\mathcal{O}_{T_A}$. □

3.2. **The four-term sequence in terms of direct images.** Recall Notation 1.6 regarding Fourier–Laplace transforms and consider the inverse Fourier–Laplace transformation of the sequence (2.4.1):

(3.2.1)

$$0 \longrightarrow H_1(\mathbb{k}; \hat{E} + \beta) \longrightarrow H_0(\hat{S}_A \cdot y_A; \hat{E} + \beta) \longrightarrow \hat{M}_A^\beta \longrightarrow H_0(\mathbb{k}; \hat{E} + \beta) \longrightarrow 0.$$

Definition 3.5. Let $\mathcal{B}_0$ be the unique simple $\mathcal{D}_{\hat{\mathbb{A}}^{n+1}}$-module supported in $0 \in \hat{\mathbb{A}}^{n+1}$. ◇

Proposition 3.6. *For $\beta = 0$ there is an isomorphism of exact 4-term sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_1(\mathbb{k}; \hat{E}) & \longrightarrow & \mathcal{H}_0(\hat{S}_A \cdot y_A; \hat{E}) & \longrightarrow & \hat{\mathcal{M}}_A^0 & \longrightarrow & \mathcal{H}_0(\mathbb{k}; \hat{E}) & \longrightarrow & 0 \\ & & \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq & & \\ 0 & \longrightarrow & {\mathcal{B}_0}^{d+1} & \longrightarrow & \mathcal{H}^0(h_{2\dagger} h_{1+}\mathcal{O}_{T_A}) & \longrightarrow & \mathcal{H}^0(h_+\mathcal{O}_{T_A}) & \longrightarrow & \mathcal{B}_0 & \longrightarrow & 0 \end{array}$$

Proof. By Corollary 2.3, $h_{A+}\mathcal{O}_{T_A} \simeq \hat{\mathcal{M}}_A^0$. By (3.0.1), $\mathcal{H}_i(\mathbb{k}; \hat{E}) \simeq \mathcal{B}_0^{\binom{d+1}{i}}$. Restricting to $\hat{\mathbb{A}}^{n+1} \setminus \{0\}$ we see that

$$\mathcal{H}_0(\hat{S}_A \cdot y_A; \hat{E})_{|\hat{\mathbb{A}}^{n+1}\setminus\{0\}} \simeq (\hat{\mathcal{M}}_A^0)_{|\hat{\mathbb{A}}^{n+1}\setminus\{0\}} \simeq h_{1+}\mathcal{O}_{T_A} \qquad \text{in} \quad \mathrm{Mod}_h(\mathcal{D}_{\hat{\mathbb{A}}^{n+1}}).$$

Since $\mathcal{H}_{>0}(S_A \cdot y_A; \hat{E})_{|\hat{\mathbb{A}}^{n+1}\setminus\{0\}} = 0$, we have

$$\mathcal{H}_0(S_A \cdot y_A; \hat{E})_{|\hat{\mathbb{A}}^{n+1}\setminus\{0\}} \simeq \mathcal{K}_\bullet(S_A \cdot y_A; \hat{E})_{|\hat{\mathbb{A}}^{n+1}\setminus\{0\}} \qquad \text{in} \quad \mathrm{D}_h^b(\mathcal{D}_{\hat{\mathbb{A}}^{n+1}}).$$

By adjunction this gives a morphism

$$h_{2\dagger} h_{1+}\mathcal{O}_{T_A} \xrightarrow{\simeq} h_{2\dagger} h_2^{-1} \mathcal{K}_\bullet(\hat{S}_A \cdot y_A; \hat{E}) \longrightarrow \mathcal{K}_\bullet(\hat{S}_A \cdot y_A; \hat{E})$$

and so induces a morphism $\mathcal{H}^0(h_{2\dagger} h_{1+}\mathcal{O}_{T_A}) \longrightarrow \mathcal{H}_0(\hat{S}_A \cdot y_A; \hat{E})$ such that the center and right squares in our diagram commute. We need to prove that the morphism

$$h_{2\dagger} h_2^{-1} \mathcal{K}_\bullet(\hat{S}_A \cdot y_A; \hat{E}) \longrightarrow \mathcal{K}_\bullet(\hat{S}_A \cdot y_A; \hat{E}) \tag{3.2.2}$$

is an isomorphism. (While we know that $\mathcal{H}_1(\Bbbk;\hat{E})$ and ${\mathcal{B}_0}^{d+1}$ are isomorphic, it is not yet clear that $\mathcal{H}^0(h_{2\dagger}h_{1+}\mathcal{O}_{T_A}) \to \mathcal{H}_0(\hat{S}_A \cdot y_A;\hat{E})$ induces such isomorphism.)

In order to prove that the morphism (3.2.2) is an isomorphism we have to show that the third term in the adjunction triangle

$$(3.2.3)\qquad h_{2\dagger}h_2^{-1}\mathcal{K}_\bullet(\hat{S}_A\cdot y_A;\hat{E}) \longrightarrow \mathcal{K}_\bullet(\hat{S}_A\cdot y_A;\hat{E}) \longrightarrow i_+i^\dagger\mathcal{K}_\bullet(\hat{S}_A\cdot y_A;\hat{E}) \xrightarrow{+1}$$

vanishes. By Kashiwara equivalence it is enough to show that $i^\dagger\mathcal{K}_\bullet(\hat{S}_A\cdot y_A;\hat{E})$ is isomorphic to zero. Since $\mathcal{K}_\bullet(\widehat{S}_A\cdot y_A;\hat{E})_{|\hat{\mathbb{A}}^{n+1}\setminus\{0\}} \simeq (h_{A+}\mathcal{O}_T)_{|\hat{\mathbb{A}}^{n+1}\setminus\{0\}}$, the complex $\mathcal{K}_\bullet(\widehat{S}_A\cdot y_A;\hat{E})$ is $\mathbb{G}_m$-quasi-equivariant. By Lemma 3.3, $i^\dagger\mathcal{K}_\bullet(\widehat{S}_A\cdot y_A;\hat{E}) \simeq a_+\mathcal{K}_\bullet(\widehat{S}_A\cdot y_A;\hat{E})$ where a is the map to a point. Now $a_+\mathcal{K}_\bullet(\widehat{S}_A\cdot y_A;\hat{E})$ is dual to $\rho\mathcal{K}_\bullet(S_A\partial_A;E)$ which allows us to use Lemma 2.5 to conclude. □

3.3. **The four-term sequence with Gauß–Manin systems.**

Notation. From now on, in addition to the assumptions in Notation 1.2 and Definition 1.3 as well as normality, we assume that the matrix A is homogeneous, *i.e.* that $(1,\dots,1)$ is in the row span of A.

Furthermore, for the remainder of the paper, $\beta = 0$. ◇

One may put A into the following shape by elementary row operations

$$(3.3.1)\qquad A = (\mathbf{a}_1,\dots,\mathbf{a}_n) = \begin{pmatrix} 1 & 1 & \dots & 1\\ 0 & a_{11} & \dots & a_{1n}\\ \vdots & \vdots & & \vdots\\ 0 & a_{d1} & \dots & a_{dn}\end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1\\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$$

where $B = (\mathbf{b}_1,\dots,\mathbf{b}_n)$ is the $d\times n$-matrix with entries $(a_{ij})_{1\le i\le d, 1\le j\le n}$.

Using this homogeneity assumption, we will here give a geometric interpretation to our 4-term sequence (2.4.1). For this we will need a variant of a comparison theorem of d'Agnolo and Eastwood [DE03], between the Radon and Fourier–Laplace transform, and several other preparatory statements.

Set $T_B := \operatorname{Spec}(\Bbbk[t_1^\pm,\dots,t_d^\pm])$ and $\hat{\mathbb{A}}^1 := \operatorname{Spec}(\Bbbk[t_0])$. We will identify T_A with $T_B\times(\hat{\mathbb{A}}^1\setminus\{0\})$. From the ring homomorphism

$$\begin{aligned}\Bbbk[y_0,\dots,y_n] &\longrightarrow \Bbbk[t_0,t_1^\pm,\dots,t_d^\pm]\\ (y_0,\dots,y_n) &\mapsto (t_0, t_0 t^{\mathbf{b}_1},\dots,t_0 t^{\mathbf{b}_n})\end{aligned}$$

we get a map

$$(3.3.2)\qquad k\colon T_B\times\hat{\mathbb{A}}^1 \longrightarrow \hat{\mathbb{A}}^{n+1}$$

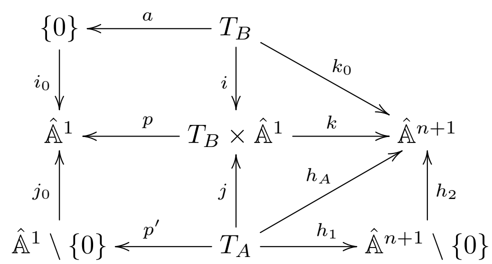
whose restriction to T_A is just our old morphism h_A. Let $\tilde{k}$ be the closed embedding

$$\tilde{k} := (id_{T_B}\times k) : T_B\times\hat{\mathbb{A}}^1 \to T_B\times\hat{\mathbb{A}}^{n+1},$$

let j, i be the embedding and inclusion

$$j\colon T_A = T_B\times(\hat{\mathbb{A}}^1\setminus\{0\}) \to T_B\times\hat{\mathbb{A}}^1, \qquad i\colon T_B\times\{0\}\to T_B\times\hat{\mathbb{A}}^1.$$

Then there is a commutative diagram



where $p\colon T_B \times \hat{\mathbb{A}}^1 \to \hat{\mathbb{A}}^1$ is the projection and k_0 sends T_B to the origin. Define the following $\mathcal{D}$-modules on $\hat{\mathbb{A}}^1$:

$$\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1 := \mathcal{D}_{\hat{\mathbb{A}}^1}/(\partial_t), \qquad \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \mathfrak{H} := \mathcal{D}_{\hat{\mathbb{A}}^1}/(t\partial_t),$$
$$\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1/t := \mathcal{D}_{\hat{\mathbb{A}}^1}/(\partial_t t), \qquad \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta := \mathcal{D}_{\hat{\mathbb{A}}^1}/(t).$$

(The module $\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \mathfrak{H}$ encodes the Heaviside distribution).

Lemma 3.7. *We have the following isomorphisms:*

$$k_+\mathcal{O}_{T_B\times\hat{\mathbb{A}}^1} \simeq k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1), \qquad h_{A+}\mathcal{O}_{T_A} \simeq k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1/t),$$
$$k_{0+}\mathcal{O}_{T_B} \simeq k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta), \qquad h_{2\dagger}h_{1+}\mathcal{O}_{T_A} \simeq k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \mathfrak{H}).$$

The adjunction morphism $h_{2\dagger}h_{1+}\mathcal{O}_{T_A} \to h_{A+}\mathcal{O}_{T_A}$ is induced by the adjunction morphism $\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \mathfrak{H} \to \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1/t$.

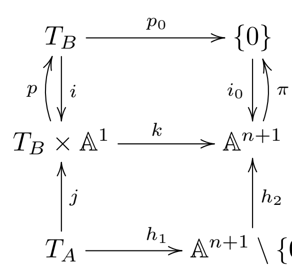
Proof. The first three isomorphisms follow from

$$\begin{aligned} k_+\mathcal{O}_{T_B\times\hat{\mathbb{A}}^1} &\simeq k_+p^+\mathcal{O}_{\hat{\mathbb{A}}^1} = k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1), \\ h_{A+}\mathcal{O}_{T_A} &\simeq k_+j_+\mathcal{O}_{T_A} \simeq k_+j_+p'^+\mathcal{O}_{\hat{\mathbb{A}}^1\setminus\{0\}} \simeq k_+p^+j_{0+}\mathcal{O}_{\hat{\mathbb{A}}^1\setminus\{0\}} \simeq k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1/t), \\ k_{0+}\mathcal{O}_{T_B} &\simeq k_+i_+\mathcal{O}_{T_B} \simeq k_+i_+a^+\mathcal{O}_{\{0\}} \simeq k_+p^+i_0\mathcal{O}_{\{0\}} \simeq k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta). \end{aligned}$$

For the last one we have

$$k_+p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \mathfrak{H}) \simeq k_+p^+j_{0\dagger}\mathcal{O}_{\hat{\mathbb{A}}^1\setminus\{0\}} \simeq k_+j_\dagger p'^+\mathcal{O}_{\hat{\mathbb{A}}^1\setminus\{0\}} \simeq k_+j_\dagger\mathcal{O}_{T_A}.$$

So it remains to prove that $k_+j_\dagger\mathcal{O}_{T_A} \simeq h_{2\dagger}h_{1+}\mathcal{O}_{T_A}$. For this consider the diagram with Cartesian squares



Base change in the lower square gives $h_{1+}\mathcal{O}_{T_A} \xrightarrow{\simeq} h_{1+}j^{-1}j_\dagger\mathcal{O}_{T_A} \xrightarrow{\simeq} h_2^{-1}k_+j_\dagger\mathcal{O}_{T_A}$. Adjunction yields a morphism $h_{2\dagger}h_{1+}\mathcal{O}_{T_A} \to k_+j_\dagger\mathcal{O}_{T_A}$. In order to prove that this is an isomorphism, it is hence enough to show that $h_{2\dagger}h_{1+}\mathcal{O}_{T_A} \simeq h_{2\dagger}h_2^{-1}k_+j_\dagger\mathcal{O}_{T_A} \to k_+j_\dagger\mathcal{O}_{T_A}$ is an isomorphism. Using the triangle

$$h_{2\dagger}h_2^{-1}k_+j_\dagger\mathcal{O}_{T_A} \longrightarrow k_+j_\dagger\mathcal{O}_{T_A} \longrightarrow i_{0+}i_0^\dagger k_+j_\dagger\mathcal{O}_{T_A} \xrightarrow{+1},$$

it remains to show that $i_{0+}i_0^\dagger k_+ j_\dagger \mathcal{O}_{T_A}$ is zero. For this we observe that

$$h_2^+ k_+ j_\dagger \mathcal{O}_{T_A} \simeq h_{1+} j^+ j_\dagger \mathcal{O}_{T_B} \simeq h_{1+}\mathcal{O}_{T_B}$$

is the restriction of a quasi-equivariant module. This shows, via Lemma 3.4, that $k_+ j_\dagger \mathcal{O}_{T_A}$ is $\mathbb{G}_m$-quasi-equivariant. We therefore have

$$i_{0+}i_0^\dagger k_+ j_\dagger \mathcal{O}_{T_A} \simeq i_{0+}\pi_+ k_+ j_\dagger \mathcal{O}_{T_A} \simeq i_{0+}p_{0+}p_+ j_\dagger \mathcal{O}_{T_A} \simeq i_{0+}p_{0+}i^\dagger j_\dagger \mathcal{O}_{T_A}\,,$$

using Lemma 3.3 to substitute p_+ by $i^!$. Since $i^\dagger j_\dagger \mathcal{O}_{T_A}$ is zero, the claim follows. □

Consider the diagram

$$\mathbb{A}^1 \xleftarrow{F} T_B \times \mathbb{A}^{n+1} \xrightarrow{q} \mathbb{A}^{n+1}$$

where q is the projection and

$$F(t_1,\dots,t_n,\lambda_0,\dots,\lambda_n) = \lambda_0 + \sum_{i=1}^n \lambda_i t^{\mathbf{b}_i}.$$

Denote $\Gamma = \mathrm{Var}(F)$ and write

$$i_\Gamma\colon \Gamma \subset T_B \times \mathbb{A}^{n+1}, \quad j_U\colon U \to T_B \times \mathbb{A}^{n+1}$$

for the inclusion of Γ and its complement U. The Gauß–Manin system $q_{U+}\mathcal{O}_U$ is of interest since it carries a mixed Hodge structure by Saito's work in [Sai90]. Our article gives evidence to our belief that many D-modules arising from Euler–Koszul complexes also carry such structure, and that they relate to interesting geometric information.

Proposition 3.8. *With $u = 1, \delta, 1/t, \mathfrak{H}$ and $\hat u = \delta, 1, \mathfrak{H}, 1/t$, and with k as in (3.3.2) we have the following isomorphisms*

$$\mathrm{FL}(k_+ p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat u)) \simeq q_+ F^+(\mathcal{D}_{\mathbb{A}^1} \bullet u).$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
\hat{\mathbb{A}}^1 & \xleftarrow{p} & T_B \times \hat{\mathbb{A}}^1 & \xrightarrow{k} & \hat{\mathbb{A}}^{n+1} \\
\uparrow p_1 & & \uparrow p_{12} & & \uparrow q_1 \\
\hat{\mathbb{A}}^1 \times \mathbb{A}^1 & \xleftarrow{id_{\hat{\mathbb{A}}^1}\times F} & T_B \times \hat{\mathbb{A}}^1 \times \mathbb{A}^{n+1} & \xrightarrow{k\times id_{\mathbb{A}^{n+1}}} & \hat{\mathbb{A}}^{n+1} \times \mathbb{A}^{n+1} \\
\downarrow p_2 & & \downarrow p_{13} & & \downarrow q_2 \\
\mathbb{A}^1 & \xleftarrow{F} & T_B \times \mathbb{A}^{n+1} & \xrightarrow{q} & \mathbb{A}^{n+1}
\end{array}$$

where p_{ij} are the projections to the factors i and j. Recall the Fourier–Laplace sheaf $\mathcal{L}$ on $\mathbb{A}^{n+1}$ from Definition 1.5 and denote $\mathcal{L}_1$ the Fourier–Laplace sheaf on

$\mathbb{A}^1 \times \mathbb{A}^1$. Then

$$\begin{aligned}
\mathrm{FL}(k_+ p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \cdot \hat{u})) &= q_{2+}\big((q_1^+ k_+ p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat{u})) \otimes \mathcal{L}\big)\\
&\simeq q_{2+}((k \times id)_+ \, p_{12}^+ \, p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat{u}) \otimes \mathcal{L})\\
&\simeq q_{2+}(k \times id)_+ (p_{12}^+ p^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat{u}) \otimes (k \times id)^+ \mathcal{L})\\
&\simeq q_+ p_{13+}((id \times F)^+ p_1^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat{u}) \otimes (k \times id)^+ \mathcal{L})\\
&\simeq q_+ p_{13+}((id \times F)^+ p_1^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat{u}) \otimes (id \times F)^+ \mathcal{L}_1)\\
&\simeq q_+ p_{13+}(id \times F)^+ (p_1^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat{u}) \otimes \mathcal{L}_1)\\
&\simeq q_+ F^+ p_{2+} p_1^+(\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \hat{u}) \otimes \mathcal{L}_1)\\
&\simeq q_+ F^+(\mathcal{D}_{\mathbb{A}^1} \bullet u).
\end{aligned}$$

□

Now consider the diagram

$$\begin{array}{ccccc}
\{0\} & \xleftarrow{\ a\ } & \Gamma & \overset{q_\Gamma}{\searrow} & \\
\downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_\Gamma} & & \\
\mathbb{A}^1 & \xleftarrow{\ F\ } & T_B \times \mathbb{A}^{n+1} & \xrightarrow{\ q\ } & \mathbb{A}^{n+1}\\
\uparrow{\scriptstyle j_0} & & \uparrow{\scriptstyle j_U} & & \\
\mathbb{A}^1 \setminus \{0\} & \xleftarrow{\ F|_U\ } & U & \underset{q_U}{\nearrow} &
\end{array}$$

where q is the projection. We have, writing F for F_U,

$$\begin{aligned}
q_+ F^+(\mathcal{D}_{\mathbb{A}^1} \bullet 1/t) &\simeq q_+ F^+ j_{0+} \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}} \simeq q_+ j_{U+} F^+ \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}} \simeq q_{U+} \mathcal{O}_U,\\
q_+ F^+(\mathcal{D}_{\mathbb{A}^1} \bullet \mathfrak{H}) &\simeq q_+ F^+ j_{0\dagger} \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}} \simeq q_+ j_{U\dagger} F^+ \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}} \simeq q_+ j_{U\dagger} \mathcal{O}_U,\\
q_+ F^+(\mathcal{D}_{\mathbb{A}^1} \bullet \delta) &\simeq q_+ F^+ i_{0+} \mathcal{O}_{\{0\}} \simeq q_+ i_{\Gamma+} a^+ \mathcal{O}_{\{0\}} \simeq q_+ i_{\Gamma+} \mathcal{O}_\Gamma \simeq q_{\Gamma+} \mathcal{O}_\Gamma,\\
q_+ F^+(\mathcal{D}_{\mathbb{A}^1} \bullet 1) &\simeq q_+ \mathcal{O}_{T_B \times \mathbb{A}^{n+1}},
\end{aligned}$$

where the second isomorphism in the second line follows from the smoothness of F.

Notation 3.9. If W is a $\Bbbk$-space (for example, $H^i_{dR}(T_B; \Bbbk)$) then $\underline{W}$ denotes the trivial vector bundle $W \otimes_\Bbbk \mathcal{O}_{\mathbb{A}^{n+1}}$. ⋄

Consider the following exact sequence of $\mathcal{D}_{\hat{\mathbb{A}}^1}$-modules

$$0 \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \mathfrak{H} \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1/t \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta \longrightarrow 0 \tag{3.3.3}$$

Theorem 3.10. *The exact sequence (3.3.3) induces an isomorphism of exact sequences*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{H}_1(\Bbbk; E) & \longrightarrow & \mathcal{H}_0(S_A \partial_A; E) & \longrightarrow & \mathcal{M}^0_A & \longrightarrow & \mathcal{H}_0(\Bbbk; E) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} & & \\
0 & \longrightarrow & \begin{array}{c}\underline{H^d_{dR}(T_B; \Bbbk)}\\ \oplus \\ \underline{H^{d-1}_{dR}(T_B; \Bbbk)}\end{array} & \longrightarrow & \mathcal{H}^0(q_{U+} \mathcal{O}_U) & \longrightarrow & \mathcal{H}^0(q_+ j_{U\dagger} \mathcal{O}_U) & \longrightarrow & \underline{H^d_{dR}(T_B; \Bbbk)} & \longrightarrow & 0
\end{array}$$

Proof. The sequence (3.3.3) is part of the long exact sequence coming from the triangle

$$j_{0\dagger}j_0^{-1}\mathcal{O}_{\hat{\mathbb{A}}^1} \longrightarrow j_{0+}j_0^{-1}\mathcal{O}_{\hat{\mathbb{A}}^1} \longrightarrow i_{0+}i_0^{\dagger}j_{0+}j_0^{-1}\mathcal{O}_{\hat{\mathbb{A}}^1} \xrightarrow{+1}$$

which is isomorphic to

$$\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \mathfrak{H} \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1/t \longrightarrow (\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta) \oplus (\mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta[1]) \xrightarrow{+1}$$

Applying the concatenated functor $\mathrm{FL} \circ k_+p^+$ to the triangle above and using Lemma 3.7, Proposition 3.6, and the fact that $\mathcal{H}^i(k_{0+}\mathcal{O}_{T_B}) = \mathcal{B}_0{}^{\binom{d}{i}}$ we obtain the upper sequence in the theorem. (Recall that k_0 sends T_B to the origin in $\hat{\mathbb{A}}^{n+1}$). Applying $q_+F^+ \circ \mathrm{FL}$ instead gives the lower sequence. □

If one applies $q_+F^+ \circ \mathrm{FL}$ to the exact sequence

$$0 \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1 \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet 1/t \longrightarrow \mathcal{D}_{\hat{\mathbb{A}}^1} \bullet \delta \longrightarrow 0 \tag{3.3.4}$$

one obtains as a part of the resulting long exact sequence the piece

(3.3.5)

$$0 \longrightarrow \underline{H_{dR}^{d-1}(T_B;\Bbbk)} \longrightarrow \mathcal{H}^0(q_{\Gamma+}\mathcal{O}_\Gamma) \longrightarrow \mathcal{H}^0(q_+j_{U\dagger}\mathcal{O}_U) \longrightarrow \underline{H_{dR}^{d}(T_B;\Bbbk)} \longrightarrow 0,$$

We now determine how this sequence relates to the two sequences in Theorem 3.10.

Proposition 3.11. *The exact sequence* (3.3.5) *is the quotient of the exact sequence*

$$0 \longrightarrow \underline{H_{dR}^{d}(T_B;\Bbbk)} \oplus \underline{H_{dR}^{d-1}(T_B;\Bbbk)} \longrightarrow \mathcal{H}^0(q_{U+}\mathcal{O}_U) \longrightarrow \mathcal{H}^0(q_+j_{U\dagger}\mathcal{O}_U) \longrightarrow \underline{H_{dR}^{d}(T_B;\Bbbk)} \longrightarrow 0$$

by the exact sequence

$$0 \longrightarrow \underline{H_{dR}^{d}(T_B;\Bbbk)} \longrightarrow \underline{H_{dR}^{d}(T_B;\Bbbk)} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0.$$

Proof. Consider the Fourier–Laplace transforms of the sequences (3.3.3) and (3.3.5). We get a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet 1 & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet 1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet 1 & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet 1/t & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet \mathfrak{H} & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet 1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet \delta & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet \mathfrak{H} & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet 1 & \longrightarrow & 0
\end{array}$$

and morphisms of triangles

$$\begin{array}{ccccccc}
\mathcal{D}_{\mathbb{A}^1} \bullet 1 & \longrightarrow & 0 & \longrightarrow & (\mathcal{D}_{\mathbb{A}^1} \bullet 1)[1] & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{D}_{\mathbb{A}^1} \bullet 1/t & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet \mathfrak{H} & \longrightarrow & (\mathcal{D}_{\mathbb{A}^1} \bullet 1) \oplus (\mathcal{D}_{\mathbb{A}^1} \bullet 1)[1] & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{D}_{\mathbb{A}^1} \bullet \delta & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet \mathfrak{H} & \longrightarrow & \mathcal{D}_{\mathbb{A}^1} \bullet 1 & \xrightarrow{+1} &
\end{array}$$

From this, we get an exact sequence of exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \underline{H^d_{dR}(T_B;\Bbbk)} & \longrightarrow & \underline{H^d_{dR}(T_B;\Bbbk)} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \underline{H^d_{dR}(T_B;\Bbbk)} \oplus \underline{H^{d-1}_{dR}(T_B;\Bbbk)} & \longrightarrow & \mathcal{H}^0(q_{U+}\mathcal{O}_U) & \longrightarrow & \mathcal{H}^0(q_+ j_{U\dagger}\mathcal{O}_U) & \longrightarrow & \underline{H^d_{dR}(T_B;\Bbbk)} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \underline{H^{d-1}_{dR}(T_B;\Bbbk)} & \longrightarrow & \mathcal{H}^0(q_{\Gamma+}\mathcal{O}_\Gamma) & \longrightarrow & \mathcal{H}^0(q_+ j_{U\dagger}\mathcal{O}_U) & \longrightarrow & \underline{H^d_{dR}(T_B;\Bbbk)} & \longrightarrow & 0
\end{array}$$

The lower middle maps are surjective since F^+ is exact and q_+ is right exact. □

We now introduce a family of Laurent polynomials defined on $T_B \times \mathbb{A}^n$ using the columns of the matrix B. For this, recall Definition 3.3.1 and consider the ring homomorphism

$$\begin{aligned} \Bbbk[\lambda_0,\dots,\lambda_n] &\longrightarrow \Bbbk[t_1^\pm,\dots,t_d^\pm]\otimes_\Bbbk \Bbbk[\lambda_1,\dots,\lambda_n] \\ \lambda_i &\mapsto \begin{cases} -\sum_{i=1}^n t^{\mathbf{b}_i}\otimes\lambda_i & \text{for } i=0; \\ \lambda_i & \text{for } i=1,\dots,n, \end{cases} \end{aligned} \tag{3.3.6}$$

which induces a family of Laurent polynomials

$$\varphi_B\colon T_B\times\mathbb{A}^n \longrightarrow \mathbb{A}^{n+1} = \mathbb{A}^1\times\mathbb{A}^n\,. \tag{3.3.7}$$

and an isomorphism

$$i_\varphi : T_B\times\mathbb{A}^n \longrightarrow \Gamma \subseteq T_B\times\mathbb{A}\times\mathbb{A}^n$$

onto the graph Γ. Hence $\varphi_B = q_\Gamma\circ i_\varphi$ and therefore $\mathcal{H}^0(\varphi_{B+}\mathcal{O}_{T_B\times\mathbb{A}^n}) \simeq \mathcal{H}^0(q_{\Gamma+}\mathcal{O}_\Gamma)$.

This recovers a special case of a theorem of [Rei14], *i.e.* there is an exact sequence

(3.3.8)

$$0\longrightarrow \underline{H^{d-1}_{dR}(T_B;\Bbbk)} \longrightarrow \mathcal{H}^0(\varphi_{B+}\mathcal{O}_{T_B\times\mathbb{A}^n}) \longrightarrow \mathcal{M}^0_A \longrightarrow \underline{H^d_{dR}(T_B;\Bbbk)} \longrightarrow 0$$

which is isomorphic to the sequence (3.3.5).

3.4. **Vanishing Gauß–Manin system and the extension class.** In this section we show that the A-hypergeometric system is an extension of a trivial vector bundle of rank one by the quotient of a Gauß–Manin system modulo its flat sections. We show that this extension does not split.

As before, $\beta = 0$ (and A is saturated, homogeneous, and pointed).

Definition 3.12. The *vanishing Gauß–Manin system* $\mathcal{V}$ with respect to the map φ_B is the cokernel of the map $\underline{H^{d-1}_{dR}(T_B\Bbbk)} \longrightarrow \mathcal{H}^0(\varphi_{B+}\mathcal{O}_{T_B\times\mathbb{A}^n})$. In other words,

$$0\longrightarrow\mathcal{V}\longrightarrow\mathcal{M}^0_A\longrightarrow\underline{H^d_{dR}(T_B;\Bbbk)}\longrightarrow 0 \tag{3.4.1}$$

is exact. We write $V_A = \Gamma(\mathbb{A}^{n+1},\mathcal{V}_A)$ and note the short exact sequence

$$0\longrightarrow V_A\longrightarrow M^0_A\longrightarrow O_A\longrightarrow 0.$$

◇

The terminology is borrowed from the vanishing cohomology of a hyperplane section $j : X \hookrightarrow Y$ of an n-dimensional projective variety Y which is a direct summand $H^{n-1}(X) = H^{n-1}(X)_{van} \oplus j^*H^{n-1}(Y)$.

The sheaf $\mathcal{V}$ appears perhaps for the first time in Stienstra's article [Sti98, Formula (61)], essentially as a restriction of (3.4.1) to the smooth locus (where all sheaves in (3.3.5) become vector bundles). However, our situation is more general even in Stienstra's set-up since in [Sti98] the matrix B is assumed to be homogeneous while it is arbitrary for us.

A natural question is: what is the extension class of $\mathcal{M}_A^0$ inside the sequence (3.4.1)? Our next result answers this question, confirming a prediction of Duco van Straten.

Theorem 3.13. *Write $\mathcal{O}$ for $H^d_{dR}(T_B; \Bbbk) \otimes \mathcal{O}_{\mathbb{A}^{n+1}}$. There are natural (in $\Bbbk$) isomorphisms*

$$\operatorname{Ext}^i_{\mathcal{D}}(\mathcal{O}, \mathcal{V}) \simeq \begin{cases} \Bbbk & \text{for } i = 1 \\ 0 & \text{else.} \end{cases}$$

The class of the sequence (3.4.1) is nonzero and induced by the identity on $\mathcal{O}$ under the connecting morphism.

Proof. Since $\mathbb{A}^{n+1}$ is affine it suffices to compute on the level of global sections. By Corollary 2.12, $\operatorname{Ext}^\bullet_{D_A}(O_A, M_A^\beta)$ vanishes for $\beta \in \mathbb{N}A$. Hence, $\operatorname{Ext}^i_{D_A}(O_A, V_A) = \operatorname{Ext}^{i-1}_{D_A}(O_A, O_A)$ and so has exactly the prescribed $\Bbbk$-space structure. In particular, (3.4.1) does not split.

The class of (3.4.1) inside $\operatorname{Ext}^1_D(O_A, V) \simeq \operatorname{Ext}^0_D(O_A, O_A)$ is the image of the identity on $\mathcal{O}_A$ under the connecting morphism induced by (3.4.1), compare [Wei94, Sec. 3.4]. Since the connecting morphism is an isomorphism, this element is nontrivial. ☐

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