ALGORITHMIC COMPUTATION OF LOCAL COHOMOLOGY MODULES AND THE LOCAL COHOMOLOGICAL DIMENSION OF ALGEBRAIC VARIETIES

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Abstract. In this paper we present algorithms that compute certain local cohomology modules associated to ideals in a ring of polynomials containing the rational numbers. In particular we are able to compute the local cohomological dimension of algebraic varieties in characteristic zero. Our approach is based on the theory of D-modules.

1. Introduction

1.1. Let R be a commutative Noetherian ring, I an ideal in R and M an R-module. The i-th *local cohomology functor* with respect to I is the i-th right derived functor of the functor $H^0_I(-)$ which sends M to the I-torsion $\bigcup_{k=1}^{\infty}(0 :_M I^k)$ of M and is denoted by $H^i_I(-)$. Local cohomology was introduced by A. Grothendieck as an algebraic analog of (classical) relative cohomology. An introduction to local cohomology may be found in [2].

The *cohomological dimension* of I in R, denoted by $\mathrm{cd}(R, I)$, is the smallest integer c such that $H^q_I(M) = 0$ for all $q > c$ and all R-modules M. If R is the coordinate ring of an affine variety X and $I \subseteq R$ is the defining ideal of the Zariski closed subset $V \subseteq X$ then the *local cohomological dimension* of V in X is defined as $\mathrm{cd}(R, I)$. It is not hard to show that if X is smooth, then the integer $\dim(X) - \mathrm{cd}(R, I)$ depends only on V but neither on X nor on the embedding $V \hookrightarrow X$.

1.2. Knowledge of local cohomology modules provides interesting information, illustrated by the following three situations. Let $I \subseteq R$ and $c = \mathrm{cd}(R, I)$. Then I cannot be generated by fewer than c elements. In fact, no ideal J with the same radical as I will be generated by fewer than c elements.

Let H^i_{dR} stand for the i-th de Rham cohomology group. A second application is a family of results commonly known as Barth theorems which are a generalization of the classical Lefschetz theorem that states that if $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a hypersurface then $H^i_{dR}(\mathbb{P}^n_{\mathbb{C}}) \to H^i_{dR}(Y)$ is an isomorphism for $i < \dim(Y) - 1$ and injective for $i = \dim(Y)$. For example, let $Y \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a closed subset and $I \subseteq R = \mathbb{C}[x_0, \dots, x_n]$ the defining ideal of Y. Then $H^i_{dR}(\mathbb{P}^n_{\mathbb{C}}) \to H^i_{dR}(Y)$ is an isomorphism for $i < n - \mathrm{cd}(R, I)$ and injective if $i = n - \mathrm{cd}(R, I)$ ([9], theorem III.7.1).

Another consequence of the work of Ogus and Hartshorne ([22], 2.2, 2.3 and [9], IV.3.1) is the following. If $I \subseteq R = \mathbb{C}[x_0, \dots, x_n]$ is the defining ideal of a complex smooth variety $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ then, for $i < n - \mathrm{codim}(V)$, $\dim_{\mathbb{C}} \mathrm{soc}_R(H^0_{\mathfrak{m}}(H^{n-i}_I(R)))$ equals $\dim_{\mathbb{C}} H^i_x(\tilde{V}, \mathbb{C})$ where $H^i_x(\tilde{V}, \mathbb{C})$ stands for the i-th singular cohomology group of the affine cone $\tilde{V}$ over V with support in the vertex x of $\tilde{V}$ and with coefficients in $\mathbb{C}$ ($\mathrm{soc}_R(M)$ denotes the socle $(0 :_M \mathfrak{m}) \subseteq M$ for any R-module M).

1.3. The cohomological dimension has been studied by many authors, for example R. Hartshorne [8], A. Ogus [22], R. Hartshorne and R. Speiser [10], C. Peskine and L. Szpiro [23], G. Faltings [6], C. Huneke and G. Lyubeznik [11], C. Huneke and R. Y. Sharp [12]. Yet despite this extensive effort, the problem of finding an algorithm for the computation of cohomological dimension remained open in general. For the determination of $\mathrm{cd}(R, I)$ it is in fact enough to find an algorithm to decide whether or not the local cohomology module $H^i_I(R) = 0$ for given i, R, I. This is because $H^q_I(R) = 0$ for all $q > c$ implies $\mathrm{cd}(R, I) \leq c$ (see [8], section 1). In [17], G. Lyubeznik gave an algorithm for deciding whether or not $H^i_I(R) = 0$ for all $I \subseteq R = K[x_1, \dots, x_n]$ where K is a field of positive characteristic. One of the main purposes of this work is to produce such an algorithm in the case where K is a computable field containing the rational numbers and $R = K[x_1, \dots, x_n]$. (By a computable field we mean a subfield K of $\mathbb{C}$ such that K is described by a finite set of data and for which addition, subtraction, multiplication and division as well as the test whether the result of any of these operations is zero in the field can be executed by the Turing machine.)

Since in such a situation the local cohomology modules $H^i_I(R)$ have a natural structure of finitely generated left $D(R, K)$-modules ([15], and in the algebraic context [16]), $D(R, K)$ being the ring of K-linear differential operators of R, explicit computations may be performed. Using this finiteness we employ the theory of Gröbner bases to develop algorithms that give a representation of $H^i_I(R)$ and $H^i_{\mathfrak{m}}(H^j_I(R))$ for all triples $i, j \in \mathbb{N}, I \subseteq R$ in terms of generators and relations over $D(R, K)$ (where $\mathfrak{m} = (x_1, \dots, x_n)$). This also leads to an algorithm for the computation of the invariants $\lambda_{i,j}(R/I) = \dim_K \mathrm{soc}_R(H^i_{\mathfrak{m}}(H^{n-j}_I(R)))$ introduced in [16]. Our algorithms are in part modelled after algorithms due to T. Oaku. In [19], [20], [21] he develops a method that computes explicitly $H^i_I(R)$ if I is a complete intersection and $i = \mathrm{depth}(I)$.

We remark that if R is an arbitrary finitely generated K-algebra and I is an ideal in R then, if R is regular, our algorithms can be used to determine $\mathrm{cd}(R, I)$ for all ideals I of R, but if R is not regular, then the problem of algorithmic determination of $\mathrm{cd}(R, I)$ remains open (see subsection 5.4).

1.4. The outline of the paper is as follows. The next section is devoted to a short survey of results on local cohomology and D-modules as they apply to our work.

In section 3 we review the theory of Gröbner bases for modules over the Weyl algebra. Readers interested in proofs and more details are encouraged to look at the book by D. Eisenbud ([5], chapter 15 for the commutative case) or [7], [4], [24], [13] (for the more general situation).

In section 4 we first present generalizations of some results due to B. Malgrange and M. Kashiwara on D-modules and their localizations. The purpose of that section is to find a representation of $R_f \otimes N$ as a cyclic A_n-module if N is a given holonomic D-module (for a definition and some properties of holonomic modules, see subsection 2.3 below). Most of the algorithmic ideas in this section appear already in T. Oaku's work [19], [20], [21].

In section 5 we describe our main results, namely algorithms that for arbitrary i, j, k, I determine the structure of $H^k_I(R), H^i_{\mathfrak{m}}(H^j_I(R))$ and find $\lambda_{i,j}(R/I)$. Some of these algorithms have been implemented in the programming language C. The final section is devoted to comments on implementations, effectivity and examples.

2. Preliminaries

2.1. **Notation.** Throughout we shall use the following notation: K will denote a field of characteristic zero, $R = K[x_1, \ldots, x_n]$ the ring of polynomials over K in n variables, $A_n = K\langle x_1, \partial_1, \ldots, x_n, \partial_n\rangle$ the Weyl algebra over K in n variables, or, equivalently, the ring of K-linear differential operators on R, $\mathfrak{m}$ will stand for the maximal ideal $(x_1, \ldots, x_n)$ of R, Δ will denote the maximal left ideal $A_n \cdot (\partial_1, \ldots, \partial_n)$ of A_n and I will stand for the ideal $(f_1, \ldots, f_r)$ in R. Every A_n-module becomes an R-module via the embedding $R \hookrightarrow A_n$.

All tensor products in this work will be over R and all A_n-modules (resp. ideals) will be left modules (resp. left ideals).

2.2. **Local cohomology.** It turns out that $H^k_I(M)$ may be computed as follows. Let $C^\bullet(f_i)$ be the complex $0 \to R \xrightarrow{1 \to \frac{1}{1}} R_{f_i} \to 0$. Then

$H^k_I(M)$ is the k-th cohomology group of the *Čech complex* defined by $C^\bullet(M; f_1, \dots, f_r) = \bigotimes_1^r C^\bullet(f_i) \otimes M$. Unfortunately, explicit calculations are complicated by the fact that $H^k_I(M)$ is rarely finitely generated as R-module. This difficulty disappears for $H^k_I(R)$ if we enlarge the ring to A_n, in essence because R_f is finitely generated over A_n for all $f \in R$.

2.3. ***D*-modules.** A good introduction to D-modules is the book by Björk, [1].

Let $f \in R$. Then the R-module R_f has a structure as left A_n-module: $x_i(\frac{g}{f^k}) = \frac{x_i g}{f^k}, \partial_i(\frac{g}{f^k}) = \frac{\partial_i(g)f - k\partial_i(f)g}{f^{k+1}}$. This may be thought of as a special case of localizing an A_n-module: if M is an A_n-module and $f \in R$ then $R_f \otimes_R M$ becomes an A_n-module via $\partial_i(\frac{g}{f^k} \otimes m) = \partial_i(\frac{g}{f^k}) \otimes m + \frac{g}{f^k} \otimes \partial_i m$. Localization of A_n-modules lies at the heart of our algorithms.

Of particular interest are the *holonomic* modules which are those finitely generated A_n-modules N for which $\mathrm{Ext}^j_{A_n}(N, A_n)$ vanishes unless $j = n$. Our standard example of a holonomic module is $R = A_n/\Delta$. Holonomic modules are always cyclic and of finite length over A_n. Besides that, if $N = A_n/L$ is holonomic, $f \in R$, s is an indeterminate and $\overline{1}$ is the coset of $1 \in A_n$ in N, then there is a nonzero polynomial $b(s)$ in $K[s]$ and an operator $P(s) \in A_n[s]$ such that $P(s)(f \cdot f^s \otimes \overline{1}) = b(s) \cdot f^s \otimes \overline{1}$. The unique monic polynomial that divides all other polynomials satisfying an identity of this type is called the *Bernstein polynomial* of L and f and denoted by $b^L_f(s)$. Any operator $P(s)$ that satisfies $P(s) f^{s+1} \otimes \overline{1} = b^L_f(s) f^s \otimes \overline{1}$ we shall call a *Bernstein operator* and refer to the roots of $b^L_f(s)$ as *Bernstein roots* of f on A_n/L.

Any localization of a holonomic module $N = A_n \cdot g$ at a single element (and hence at any finite number of elements) of R are holonomic ([1], 1.5.9) and in particular cyclic over A_n, generated by $f^{-a}g$ for sufficiently large $a \in \mathbb{N}$. So the complex $C^\bullet(N; f_1, \dots, f_r)$ consists of holonomic A_n-modules whenever N is holonomic. This facilitates the use of Gröbner bases as computational tool for maps between holonomic modules and their localizations. As a special case we note that localizations of R are holonomic, and hence finite, over A_n.

2.4. **The Čech complex.** Local cohomology modules are D-modules and in fact holonomic: we know already that the modules in the Čech complex are holonomic, it suffices to show that the maps are A_n-linear. All maps in the Čech complex are direct sums of localization maps. Suppose R_f is generated by f^s and R_{fg} by $(fg)^t$. We may replace s, t by their minimum u and then we see that the inclusion $R_f \to R_{fg}$ is nothing but the map $A_n/\operatorname{ann}(f^u) \to A_n/\operatorname{ann}((fg)^u)$ sending $P + \operatorname{ann}(f^u)$

to $P \cdot g^{-u} + \operatorname{ann}((fg)^u)$. So $C^i(N; f_1, \dots, f_r) \to C^{i+1}(N; f_1, \dots, f_r)$ is an A_n-linear map between holonomic modules for every holonomic N. One can prove that kernels and cokernels of A_n-linear maps between holonomic modules are holonomic. Holonomicity of $H^k_I(R)$ follows.

Similarly, $H^i_{\mathfrak{m}}(H^j_I(R))$ is holonomic for $i, j \in \mathbb{N}$ (since $H^j_I(R)$ is holonomic).

3. Gröbner bases of modules over the Weyl algebra

In this section we review some of the concepts and results related to the Buchberger algorithm in modules over Weyl algebras. It turns out that with a little care many of the important constructions from the theory of commutative Gröbner bases carry over to our case. For an introduction into non-commutative monomial orders and related topics, [13] is a good source.

Let us agree that every time we write an element in A_n, we write it as a sum of terms $c_{\alpha\beta} x^\alpha \partial^\beta$ in multi-index notation. That is, $\alpha, \beta \in \mathbb{N}^n$, $c_{\alpha\beta} \in K$, $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}, \partial^\beta = \partial_1^{\beta_1} \cdot \ldots \cdot \partial_n^{\beta_n}$ and in every monomial we write first the powers of x and then the powers of the differentials. Further, if $m = c_{\alpha\beta} x^\alpha \partial^\beta, c_{\alpha\beta} \in K$, we will say that m has degree $\deg m = |\alpha + \beta|$ and an operator $P \in A_n$ has degree equal to the largest degree of any monomial occuring in P.

Recall that a *monomial order* $<$ in A_n is a total order on the monomials of A_n, subject to $m < m' \Rightarrow mm'' < m'm''$ for all nonzero monomials m, m', m''. Since the product of two monomials in our notation is not likely to be a monomial (as $\partial_i x_i = x_i \partial_i + 1$) it is not obvious that such orderings exist at all. However, the commutator of any two monomials m_1, m_2 will be a polynomial of degree at most $\deg(m_1) + \deg(m_2) - 2$. That means that the degree of an operator and its component of maximal degree is independent of the way it is represented. Thus we may for example introduce a monomial order on A_n by taking any monomial order on $\tilde{A}_n = K[x_1, \dots, x_n, \partial_1, \dots, \partial_n]$ (the polynomial ring in $2n$ variables) that refines the partial order given by total degree, and defining $m_1 > m_2$ in A_n if and only if $m_1 > m_2$ in $\tilde{A}_n$.

Let $<$ be a monomial order on A_n. Let $G = \bigoplus_1^d A_n \cdot \gamma_i$ be the free A_n-module on the symbols $\gamma_1, \dots, \gamma_d$. We define a monomial order on G by $m_i\gamma_i > m_j\gamma_j$ if either $m_i > m_j$ in A_n, or $m_i = m_j$ and $i > j$. The largest monomial $m\gamma$ in an element $g \in G$ will be denoted by $\operatorname{in}(g)$, called the *initial term of* g. Of fundamental importance is

Algorithm 3.1 (Remainder)**.** Let h and $\underline{g} = \{g_i\}_1^s$ be elements of G. Set $h_0 = h, \sigma_0 = 0, j = 0$ and let $\varepsilon_i = \varepsilon(g_i)$ be symbols. Then

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Repeat
        If in(g_i)|in(h_j) set
                        {h_{j+1} := h_j - (in(h_j)/in(g_i)) g_i,
                         σ_{j+1} := σ_j + (in(h_j)/in(g_i)) ε_i,
                         j := j + 1}
Until   No in(g_i) divides in(h_j).
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The result is h_a, called a *remainder* $\Re(h, \underline{g})$ *of* h *under division by* $\underline{g}$, and an expression $\sigma_a = \sum_{i=1}^s a_i\varepsilon_i$ with $a_i \in A_n$ and $\text{in}(a_ig_i) < \text{in}(g)$ for all i. By Dickson's lemma ([13], 1.1) the algorithm terminates. It is worth mentioning that $\Re(h, \underline{g})$ is not uniquely determined, it depends on which g_i we pick amongst those whose initial term divides the initial term of h_j.

Note that if h_a is zero, σ_a tells us how to write h in terms of $\underline{g}$. Such a σ_a is called a *standard expression for* h *with respect to* $\{g_1, \ldots, g_s\}$.

Definition 3.2. If $\text{in}(g_i)$ and $\text{in}(g_j)$ involve the same basis element of G, then we set $s_{ij} = m_{ji}g_i - m_{ij}g_j$ and $\sigma_{ij} = m_{ji}\varepsilon_i - m_{ij}\varepsilon_j$ where $m_{ij} = \frac{\text{lcm}(\text{in}(g_j),\text{in}(g_i))}{\text{in}(g_j)}$. Otherwise, σ_{ij} and s_{ij} are defined to be zero. s_{ij} is called the *S-polynomial* to g_i and g_j.

Suppose $\Re(s_{ij}, \underline{g})$ is zero for all i, j. Then we call $\underline{g}$ a *Gröbner basis* for the module $A_n \cdot (g_1, \ldots, g_s)$.

The following proposition ([13], Lemma 3.8) indicates the usefulness of Gröbner bases.

Proposition 3.3. *Let* $\underline{g}$ *be a finite set of elements of* G*. Then* $\underline{g}$ *is a Gröbner basis if and only if* $h \in A_n\underline{g}$ *implies* $\exists i : \text{in}(g_i)|\,\text{in}(h)$. □

Computation of Gröbner bases over the Weyl algebra works just as over polynomial rings:

Algorithm 3.4 (Buchberger)**.** Input: $\underline{g} = \{g_1, \ldots, g_s\} \subseteq G$.
Output: a Gröbner basis for $A_n \cdot (g_1, \ldots, g_s)$.
Begin.

```
Repeat
        If h := ℜ(s_ij, g) ≠ 0
                        add h to g
Until   ℜ(s_ij, g) = 0 for all {i, j}.
Return g.
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End.

3.1. Now we shall describe the construction of kernels of A_n-linear maps using Gröbner bases. We first consider the case of a map between free A_n-modules.

Let $E = \bigoplus_1^s A_n\varepsilon_i, G = \bigoplus_1^r A_n\gamma_j$ and $\phi : E \to G$ be a A_n-linear map. Assume $\phi(\varepsilon_i) = g_i$. Suppose that in order to make $\underline{g}$ a Gröbner basis we have to add $g'_1, \dots, g'_{s'}$ to $\underline{g}$ which satisfy $g'_i = \sum_{k=1}^s a_{ik}g_k$. We get an induced map

$$\begin{array}{ccc} \bigoplus_1^{s+s'} A_n\varepsilon_i & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\tilde{\phi}} & \\ \bigoplus_1^s A_n\varepsilon_i & \xrightarrow[\phi]{} & \bigoplus_1^r A_n\gamma_j \end{array}$$

where π is the identity on ε_i for $i \leq s$ and sends ε_{i+s} into $\sum_{k=1}^s a_{ik}\varepsilon_k$. Of course, $\tilde{\phi} = \phi\pi$.

The kernel of ϕ is just the image of the kernel of $\tilde{\phi}$ under π. So in order to find kernels of maps between free modules one may assume that the generators of the source are mapped to a Gröbner basis and replace ϕ by $\tilde{\phi}$. Recall from definition 3.2 that $\sigma_{ij} = m_{ji}\varepsilon_i - m_{ij}\varepsilon_j$ or zero, depending on the leading terms of g_i and g_j.

Proposition 3.5. *Assume that $\{g_1, \dots, g_s\}$ is a Gröbner basis. Let $s_{ij} = \sum d_{ijk}g_k$ be standard expressions for the S-polynomials. Then $\{\sigma_{ij} - \sum_k d_{ijk}\varepsilon_k\}_{1 \leq i < j \leq s}$ generate the kernel of $\phi : \bigoplus_1^s A_n\varepsilon_i \to \bigoplus_1^r A_n\gamma_j$, sending ε_i to g_i.*

The proof proceeds exactly as in the commutative case, see for example [5], section 15.10.8.

3.2. We explain now how to find a set of generators for the kernel of an arbitrary A_n-linear map. Let E, G be as in subsection 3.1 and suppose $P \subseteq E, A_n(q_1, \dots, q_b) = Q \subseteq G$ and $\phi : \bigoplus_1^s A_n\varepsilon_i/P \to \bigoplus_1^r A_n\gamma_j/Q$. It will be sufficient to consider the case $P = 0$ since we may lift ϕ to the free module E surjecting onto E/P.

Let as before $\phi(\varepsilon_i) = g_i$. A kernel element in E is a sum $\sum_i a_i\varepsilon_i, a_i \in A_n$, which if ε_i is replaced by g_i can be written in terms of the generators q_j of Q. Let $\underline{\beta} = \{\beta_1, \dots, \beta_c\}$ be such that $\underline{g} \cup \underline{q} \cup \underline{\beta}$ is a Gröbner basis for $A_n(\underline{g}, \underline{q})$, obtained from $\underline{g} \cup \underline{q}$ by application of algorithm 3.4. Then from algorithm 3.1 we have expressions

$$\beta_i = \sum_j c_{ij}g_j + \sum_k c'_{ik}q_k, \tag{3.1}$$

with $c_{ij}, c'_{ik} \in A_n$. Furthermore, by proposition 3.5, algorithm 3.4 returns a generating set $\underline{\sigma}$ for the syzygies on $\underline{g} \cup \underline{q} \cup \underline{\beta}$. Write

$$\sigma_i = \sum_j a_{ij}\varepsilon_{g_j} + \sum_k a'_{ik}\varepsilon_{q_k} + \sum_l a''_{il}\varepsilon_{\beta_l} \tag{3.2}$$

and eliminate the last sum using the relations (3.1) to obtain syzygies

$$\tilde{\sigma}_i = \sum_j a_{ij}\varepsilon_{g_j} + \sum_k a'_{ik}\varepsilon_{q_k} + \sum_l a''_{il}\left(\sum_v c_{lv}\varepsilon_{g_v} + \sum_w c'_{lw}\varepsilon_{q_w}\right). \tag{3.3}$$

These will then form a generating set for the syzygies on $\underline{g} \cup \underline{q}$. Cutting off the q-part of these syzygies we get a set of forms

$$\left\{\sum_j a_{ij}\varepsilon_{g_j} + \sum_l a''_{il}\left(\sum_v c_{lv}\varepsilon_{g_v}\right)\right\}$$

which generate the kernel of the map $E \to G/Q$.

3.3. The comments in this subsection will find their application in algorithm 5.2 which computes the structure of $H^i_{\mathfrak{m}}(H^j_I(R))$ as A_n-module. Let

$$\begin{array}{ccccc}
M'_3 & \xrightarrow{\alpha} & M_3 & \xrightarrow{\alpha'} & M''_3 \\
\uparrow{\scriptstyle \phi'} & & \uparrow{\scriptstyle \psi'} & & \uparrow{\scriptstyle \rho'} \\
M'_2 & \xrightarrow{\beta} & M_2 & \xrightarrow{\beta'} & M''_2 \\
\uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \rho} \\
M'_1 & \xrightarrow{\gamma} & M_1 & \xrightarrow{\gamma'} & M''_1
\end{array}$$

be a commutative diagram of A_n-modules. Note that the row cohomology of the column cohomology at M_2 is given by

$$\left[\ker(\psi') \cap {\beta'}^{-1}(\operatorname{im}\rho) + \operatorname{im}(\psi)\right] \ / \ \left[\beta(\ker(\phi')) + \operatorname{im}(\psi)\right].$$

In order to compute this we need to be able to find:

- preimages of submodules,
- kernels of maps,
- intersections of submodules.

It is apparent that the second and third calculation is a special case of the first: kernels are preimages of zero, intersections are images of preimages (if ${A_n}^r \stackrel{\phi}{\to} {A_n}^s/M \stackrel{\psi}{\leftarrow} {A_n}^t$ is given, then $\operatorname{im}(\phi) \cap \operatorname{im}(\psi) = \psi(\psi^{-1}(\operatorname{im}(\phi)))$).

So suppose in the situation $\phi : {A_n}^r/M \to {A_n}^s/N$, $\psi : {A_n}^t/P \to {A_n}^s/N$ we want to find $\psi^{-1}(\operatorname{im}(\phi))$. We may reduce to the case where

M and P are zero and then lift ϕ, ψ to maps into ${A_n}^s$. The elements x in $\psi^{-1}(\operatorname{im}\phi) \subseteq {A_n}^t$ are exactly the elements in $\ker({A_n}^t \xrightarrow{\psi} {A_n}^s/N \to {A_n}^s/(N + \operatorname{im}\phi))$ and this kernel can be found according to the comments in 3.2.

4. D-modules after Kashiwara, Malgrange and Oaku

The purpose of this section is as follows. Given $f \in R$ and an ideal $L \subseteq A_n$ such that A_n/L is holonomic and L is f-saturated (i.e. $f \cdot P \in L$ only if $P \in L$), we want to compute the structure of the module $R_f \otimes A_n/L$. It turns out that it is useful to know the ideal $J^L(f^s)$ which consists of the operators $P(s) \in A_n[s]$ that annihilate $f^s \otimes \overline{1} \in M := R_f[s]f^s \otimes A_n/L$ where the bar denotes cosets in A_n/L. In order to find $J^L(f^s)$, we will consider the module M over the ring $A_{n+1} = A_n\langle t, \partial_t\rangle$. It will turn out in 4.1 that one can easily compute the ideal $J^L_{n+1}(f^s) \subseteq A_{n+1}$ consisting of all operators that kill $f^s \otimes \overline{1}$. In proposition 4.3 we will then explain how to compute $J^L(f^s)$ from $J^L_{n+1}(f^s)$.

Another important result in this section (proposition 4.2) shows how to compute the structure of $R_f \otimes A_n/L$ as A_n-module once $J^L(f^s)$ is known.

4.1. Consider $A_{n+1} = A_n\langle t, \partial_t\rangle$, the Weyl algebra in $x_1, \dots, x_n$ and the new variable t. B. Malgrange ([18]) has defined an action of t and ∂_t on $M = R_f[s] \cdot f^s \otimes_R A_n/L$ by $t(g(x,s) \cdot f^s \otimes \overline{P}) = g(x, s+1)f \cdot f^s \otimes \overline{P}$ and $\partial_t(g(x,s) \cdot f^s \otimes \overline{P}) = \frac{-s}{f} g(x, s-1) \cdot f^s \otimes \overline{P}$ for $\overline{P} \in A_n/L$. A_n acts on M as expected, the variables by multiplication on the left, the ∂_i by the product rule. One checks that this actually defines a structure of M as a left A_{n+1}-module and that $-\partial_t t$ acts as multiplication by s.

We denote by $J^L_{n+1}(f^s)$ the ideal in A_{n+1} that annihilates the element $f^s \otimes \overline{1}$ in M. Then we have an induced morphism of A_{n+1}-modules $A_{n+1}/J^L_{n+1}(f^s) \to M$ sending $P + J^L_{n+1}(f^s)$ to $P(f^s \otimes \overline{1})$.

The following lemma generalizes lemma 4.1 in [18] (as well as part of the proof given there) where the special case $L = (\partial_1, \dots, \partial_n)$, $A_n/L = R$ is considered. Note that $J^L_{n+1}(f^s)$ makes perfect sense even if L is not holonomic.

Lemma 4.1. *Suppose that $L = A_n \cdot (P_1, \dots, P_r)$ is f-saturated (i.e., if $f \cdot P \in L$, then $P \in L$). With the above definitions, $J^L_{n+1}(f^s)$ is the ideal generated by $f - t$ together with the images of the P_j under the automorphism ϕ of A_{n+1} induced by $x \to x, t \to t - f$.*

Proof. The automorphism sends ∂_i to $\partial_i + f_i\partial_t$ and ∂_t to ∂_t. So if we write $P_j = P_j(\partial_1, \dots, \partial_n)$, then $\phi P_j = P_j(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t)$.

One checks that $(\partial_i + f_i\partial_t)(f^s \otimes \overline{Q}) = f^s \otimes \overline{\partial_i Q}$ for all $Q \in A_{n+1}$ so that $\phi(P_j(\partial_1, \dots, \partial_n))(f^s \otimes \overline{1}) = f^s \otimes \overline{P_j(\partial_1, \dots, \partial_n)} = 0$. By definition, $f \cdot (f^s \otimes \overline{1}) = t \cdot (f^s \otimes \overline{1})$. So $t - f \in J^L_{n+1}(f^s)$ and $\phi(P_j) \in J^L_{n+1}(f^s)$ for $i = 1, \dots, r$.

Conversely let $P(f^s \otimes \overline{1}) = 0$. We may assume, that P does not contain any t since we can eliminate t using $f - t$. Now rewrite P in terms of ∂_t and the $\partial_i + f_i\partial_t$. Say, $P = \sum c_{\alpha\beta}\partial_t^\alpha x^\beta Q_{\alpha\beta}(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t)$, where the $Q_{\alpha\beta}$ are polynomials in n variables and $c_{\alpha\beta} \in K$. Application to $f^s \otimes \overline{1}$ results in $\sum \partial_t^\alpha(f^s \otimes c_{\alpha\beta}x^\beta \overline{Q_{\alpha\beta}(\partial_1, \dots, \partial_n)})$.

Let $\overline{\alpha}$ be the largest $\alpha \in \mathbb{N}$ for which there is a nonzero $c_{\alpha\beta}$ occuring in $P = \sum c_{\alpha\beta}\partial_t^\alpha x^\beta Q_{\alpha\beta}(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t)$. We show that the sum of terms that contain $\partial_t^{\overline{\alpha}}$ is in $A_{n+1} \cdot \phi(L)$ as follows. In order for $P(f^s \otimes \overline{1})$ to vanish, the sum of terms with the highest s-power, namely $s^{\overline{\alpha}}$, must vanish, and so $\sum_\beta c_{\overline{\alpha}\beta}(-1/f)^{\overline{\alpha}} f^s \otimes x^\beta Q_{\overline{\alpha}\beta}(\partial_1, \dots, \partial_n) \in R_f f^s \otimes L$ as R_f is R-flat. It follows, that $\sum_\beta c_{\overline{\alpha}\beta}x^\beta Q_{\overline{\alpha}\beta}(\partial_1, \dots, \partial_n) \in L$ (L is f-saturated!) and hence $\sum_\beta \partial_t^{\overline{\alpha}} c_{\overline{\alpha}\beta}x^\beta Q_{\overline{\alpha}\beta}(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t) \in A_{n+1} \cdot \phi(L)$.

So by the first part, $P - \sum_\beta c_{\overline{\alpha}\beta}\partial_t^{\overline{\alpha}} x^\beta Q_{\overline{\alpha}\beta}(\partial_1 + f_1\partial_t, \dots, \partial_n + f_n\partial_t)$ kills $f^s \otimes \overline{1}$, but is of smaller degree in ∂_t than P was.

The claim follows. □

4.2. Let $J^L(f^s)$ stand for the ideal in $A_n[s] \cong A_n[-\partial_t t]$ that kills $f^s \otimes \overline{1} \in R_f[s]f^s \otimes_R A_n/L$. Note that $J^L(f^s) = J^L_{n+1}(f^s) \cap A_n[-\partial_t t]$. Again, we may talk about $J^L(f^s)$ independently of the holonomicity of L.

Recall that the Bernstein polynomial $b^L_f(s)$ is defined to be the monic generator of the ideal of polynomials $b(s) \in K[s]$ for which there exists an operator $P(s) \in A_n[s]$ such that $P(s)(f^{s+1} \otimes \overline{1}) = b(s)f^s \otimes \overline{1}$, and that $b^L_f(s)$ will exist for example if L is holonomic. The following proposition already appears as Proposition 7.3 in [20].

Proposition 4.2. *If L is holonomic and $a \in \mathbb{Z}$ is such that no integer root of $b^L_f(s)$ is smaller than a, then we have isomorphisms*

$$R_f \otimes A_n/L \cong A_n[s]/J^L(f^s)|_{s=a} \cong A_n \cdot f^a \otimes \overline{1}. \quad \square \tag{4.1}$$

We remark that if any $a \in \mathbb{Z}$ satisfies the conditions of the proposition, then so does every integer smaller than a.

4.3. The purpose of this subsection is to review some algorithms due to T. Oaku. In [21] Theorem 19, Oaku showed how to construct a generating set for $J^L(f^s)$ in the case where $L = (\partial_1, \dots, \partial_n)$. According to 4.2, $J^L(f^s)$ is the intersection of $J^L_{n+1}(f^s)$ with $A_n[-\partial_t t]$. We shall explain how one may calculate $J \cap A_n[-\partial_t t]$ whenever $J \subseteq A_{n+1}$ is any given ideal and as a corollary develop an algorithm that for f-saturated A_n/L computes $J^L(f^s)$.

On $A_{n+1}[y_1, y_2]$ define weights $w(t) = w(y_1) = 1, w(\partial_t) = w(y_2) = -1, w(x_i) = w(\partial_i) = 0$. If $P = \sum_i P_i \in A_{n+1}[y_1, y_2]$ and all P_i are monomials, then we will write $(P)^h$ for the operator $\sum_i P_i \cdot y_1^{d_i}$ where $d_i = \max_j(w(P_j)) - w(P_i)$ and call it the y_1-*homogenization* of P.

Note that the Buchberger algorithm preserves homogeneity in the following sense: if a set of generators for an ideal is given and these generators are homogeneous with respect to the weights above, then any new generator for the ideal constructed with the classical Buchberger algorithm will also be homogeneous. (This is a consequence of the facts that the y_i commute with all other variables and that $\partial_t t = t\partial_t + 1$ is homogeneous of weight zero.)

Proposition 4.3. *Let $J = A_{n+1} \cdot (Q_1, \ldots, Q_r)$. Let I be the left ideal in $A_{n+1}[y_1]$ generated by the y_1-homogenizations $(Q_i)^h$ of the Q_i, relative to the weight w above, and set $\tilde{I} = A_{n+1}[y_1, y_2] \cdot (I, 1 - y_1 y_2)$. Let G be a Gröbner basis for $\tilde{I}$ under a monomial order that eliminates y_1, y_2. For each $P \in G$ set $P' = t^{-w(P)}P$ if $w(P) < 0$ and $P' = \partial_t^{w(P)}P$ if $w(P) \geq 0$ and let $G' = \{P' : P \in G\}$. Then $G_0 = G' \cap A_n[-\partial_t t]$ generates $J \cap A_n[-\partial_t t]$.*

Proof. This is in essence Theorem 18 of [21]. □

So we have

Algorithm 4.4. Input: $f \in R, L \subseteq A_n$ such that L is f-saturated.
Output: Generators for $J^L(f^s)$.
Begin

1. For each generator Q_i of $A_{n+1} \cdot (L, t)$ compute the image $\phi(Q_i)$ under $x_i \to x_i, t \to t - f, \partial_i \to \partial_i + f_i\partial_t, \partial_t \to \partial_t$.
2. Homogenize all $\phi(Q_i)$ with respect to the new variable y_1 relative to the weight w introduced before proposition 4.3.
3. Compute a Gröbner basis for the ideal generated by $(\phi(Q_1))^h, \ldots, (\phi(Q_r))^h$, $1 - y_1 y_2$ in $A_{n+1}[y_1, y_2]$ using an order that eliminates y_1, y_2.
4. Select the operators $\{P_j\}_1^b$ in this basis which do not contain y_1, y_2.
5. For each P_j, $1 \leq j \leq b$, if $w(P_j) > 0$ replace P_j by $P'_j = \partial_t^{w(P_j)} P_j$. Otherwise replace P_j by $P'_j = t^{-w(P_j)} P_j$.
6. Return the new operators $\{P'_j\}_1^b$.

End.

This algorithm was already stated in Proposition 7.1 of [20].

In order to guarantee existence of the Bernstein polynomial $b_f^L(s)$ we assume for our next result that L is holonomic. Let $|s|$ denote the complex absolute value of s.

Corollary 4.5. *Suppose L is a holonomic ideal in A_n. If $J^L(f^s)$ is known or it is known that L is f-saturated, then the Bernstein polynomial $b_f^L(s)$ of $R_f \otimes_R A_n/L$ can be found from $(b_f^L(s)) = A_n[s] \cdot (J^L(f^s), f) \cap K[s]$.*

Moreover, suppose $b_f^L(s) = s^d + b_{d-1}s^{d-1} + \ldots + b_0$ and define $B = \max_i\{|b_i|^{1/(d-i)}\}$. In order to find the smallest integer root of $b_f^L(s)$, one only needs to check the integers between $-2B$ and $2B$.

If in particular $L = (\partial_1, \ldots, \partial_n)$, it suffices to check the integers between $-b_{d-1}$ and -1.

Proof. If L is f-saturated, propositions 4.1 and 4.3 enable us to find $J^L(f^s)$. The first part follows then easily from the definition of $b_f^L(s)$: as $(b_f^L(s) - P_f^L \cdot f)(f^s \otimes \overline{1}) = 0$ it is clear that $b_f^L(s)$ is in $K[s]$ and in $A_n[s](J^L(f^s), f)$. Using an elimination order on $A_n[s]$, $b_f^L(s)$ will be (up to a scalar factor) the unique element in the reduced Gröbner basis for $A_n[s] \cdot (J^L(f^s), f)$ that contains no x_i nor ∂_i.

Now suppose $|s_0| = 2B\rho$ where B is as defined above and $\rho > 1$. Assume also that s_0 is a root of $b_f^L(s)$. We find

$$\begin{aligned}(2B\rho)^d = |s_0|^d &= |-\sum_0^{d-1} b_i {s_0}^i| \leq \sum_0^{d-1} B^{d-i}|s|^i \\ &= B^d \sum_0^{d-1}(2\rho)^i \leq B^d((2\rho)^d - 1),\end{aligned}$$

using $\rho \geq 1$. By contradiction, s_0 is not a root.

The final claim is a consequence of Kashiwara's work [14] where it is proved that if $L = (\partial_1, \ldots, \partial_n)$ then all roots of $b_f^L(s)$ are negative and hence $-b_{n-1}$ is a lower bound for each single root. □

For purposes of reference we also list algorithms that compute the Bernstein polynomial to a holonomic module and the localization of a holonomic module.

Algorithm 4.6. Input: $f \in R, L \subseteq A_n$ such that A_n/L is holonomic and f-torsionfree.

Output: The Bernstein polynomial $b_f^L(s)$.

Begin

1. Determine $J^L(f^s)$ following algorithm 4.4.

2. Find a reduced Gröbner basis for the ideal $J^L(f^s)+A_n[s]\cdot f$ using an elimination order for x and ∂.
3. Pick the unique element $b(s)\in K[s]$ contained in that basis and return it.

End.

Algorithm 4.7. Input: $f\in R, L\subseteq A_n$ such that A_n/L is holonomic and f-torsionfree.

Output: Generators for an ideal J such that $R_f\otimes A_n/L\cong A_n/J$.

Begin

1. Determine $J^L(f^s)$ following algorithm 4.4.
2. Find the Bernstein polynomial $b^L_f(s)$ using algorithm 4.6.
3. Find the smallest integer root a of $b^L_f(s)$.
4. Replace s by a in all generators for $J^L(f^s)$ and return these generators.

End.

Algorithms 4.6 and 4.7 appear already in [20] as Theorem 6.14 and Proposition 7.3. Correctness of step 4 in algorithm 4.7 follows from proposition 4.2.

5. Local cohomology as A_n-module

In this section we will combine the results from the previous sections to obtain algorithms that compute for given $i,j,k\in\mathbb{N}, I\subseteq R$ the local cohomology modules $H^k_I(R), H^i_{\mathfrak{m}}(H^j_I(R))$ and the invariants $\lambda_{i,j}(R/I)$ associated to I.

5.1. **Computation of $H^k_I(R)$.** Here we will describe an algorithm that takes in a finite set of polynomials $\underline{f}=\{f_1,\dots,f_r\}\subset R$ and returns a presentation of $H^k_I(R)$ where $I=(\underline{f_1},\dots,f_r)$. In particular, if $H^k_I(R)$ is zero, then the algorithm will return the zero presentation.

Consider the Čech complex associated to $f_1,\dots,f_r$ in R,

$$0\to R\to\bigoplus_1^r R_{f_i}\to\bigoplus_{1\le i<j\le r}R_{f_if_j}\to\cdots\to R_{f_1\cdot\ldots\cdot f_r}\to 0. \tag{5.1}$$

Its k-th cohomology group is $H^k_I(R)$. The map

$$C^k=\bigoplus_{1\le i_1<\cdots<i_k\le r}R_{f_{i_1}\cdot\ldots\cdot f_{i_k}}\to\bigoplus_{1\le j_1<\cdots<j_{k+1}\le r}R_{f_{j_1}\cdot\ldots\cdot f_{j_{k+1}}}=C^{k+1} \tag{5.2}$$

is the sum of maps

$$R_{f_{i_1}\cdot\ldots\cdot f_{i_k}}\to R_{f_{j_1}\cdot\ldots\cdot f_{j_{k+1}}} \tag{5.3}$$

which are either zero (if $\{i_1, \ldots, i_k\} \not\subseteq \{j_1, \ldots, j_{k+1}\}$) or send $\frac{1}{1}$ to $\frac{1}{1}$, up to sign. Recall that $A_n/\Delta \cong R$ and identify $R_{f_{i_1}\cdot\ldots\cdot f_{i_k}}$ with $A_n/J^\Delta((f_{i_1}\cdot\ldots\cdot f_{i_k})^s)|_{s=a}$ and $R_{f_{j_1}\cdot\ldots\cdot f_{j_{k+1}}}$ with $A_n/J^\Delta((f_{j_1}\cdot\ldots\cdot f_{j_{k+1}})^s)|_{s=b}$ where a, b are sufficiently small integers. By proposition 4.2 we may assume that $a = b \leq 0$. Then the map (5.2) is in the nonzero case multiplication from the right by $(f_l)^{-a}$ where $l = \{j_1, \ldots, j_{k+1}\} \backslash \{i_1, \ldots, i_k\}$, again up to sign.

It follows that the matrix representing the map $C^k \to C^{k+1}$ in terms of A_n-modules is very easy to write down once the annihilator ideals and Bernstein polynomials for all k- and $(k+1)$-fold products of the f_i are known: the entries are 0 or $\pm f_l^{-a}$ where f_l is the new factor.

Let Θ^r_k be the set of k-element subsets of $1, \ldots, r$ and for $\theta \in \Theta^r_k$ write F_θ for the product $\prod_{i\in\theta} f_i$. We have demonstrated the correctness and finiteness of the following algorithm.

Algorithm 5.1. Input: $f_1, \ldots, f_r \in R; k \in \mathbb{N}$.

Output: $H^k_I(R)$ in terms of generators and relations as finitely generated A_n-module.

Begin

1. Compute the annihilator ideal $J^\Delta((F_\theta)^s)$ and the Bernstein polynomial $b^\Delta_{F_\theta}(s)$ for all $(k-1)$-, k- and $(k+1)$-fold products F_θ of $f_1, \ldots, f_r$ as in 4.4 and 4.6 (so θ runs through $\Theta^r_{k-1} \cup \Theta^r_k \cup \Theta^r_{k+1}$).
2. Compute the smallest integer root a_θ for each $b^\Delta_{F_\theta}(s)$, let a be the minimum of all a_θ and replace s by a in all the annihilator ideals.
3. Compute the two matrices M_{k-1}, M_k representing the A_n-linear maps $C^{k-1} \to C^k$ and $C^k \to C^{k+1}$ as explained in subsection 5.1.
4. Compute a Gröbner basis G for the kernel of the map
$$\bigoplus_{\theta\in\Theta^r_k} A_n \to \bigoplus_{\theta\in\Theta^r_k} A_n/J^\Delta((F_\theta)^s)|_{s=a} \stackrel{M_k}{\longrightarrow} \bigoplus_{\theta\in\Theta^r_{k+1}} A_n/J^\Delta((F_\theta)^s)|_{s=a}$$
as in 3.2.
5. Compute a Gröbner basis G_0 for a lift to $\bigoplus_{\theta\in\Theta^r_k} A_n$ of the module
$$\mathrm{im}(M_{k-1}) \subseteq \bigoplus_{\theta\in\Theta^r_k} A_n/J^\Delta((F_\theta)^s)|_{s=a}.$$
6. Compute the remainders of all elements of G with respect to G_0.
7. Return these remainders and G_0.

End.

The nonzero elements of G generate the quotient $G/G_0 \cong H^k_I(R)$ so that $H^k_I(R) = 0$ if and only if all returned remainders are zero.

5.2. **Computation of $H^i_{\mathfrak{m}}(H^j_I(R))$.** As a second application of Gröbner basis computations over the Weyl algebra we show now how to compute $H^i_{\mathfrak{m}}(H^j_I(R))$. Note that we cannot apply lemma 4.1 to $A_n/L = H^j_I(R)$ since $H^j_I(R)$ may well contain some torsion.

As in the previous sections, $C^j(R; f_1, \ldots, f_r)$ denotes the j-th module in the Čech complex to R and $\{f_1, \ldots, f_r\}$. Let $C^{\bullet\bullet}$ be the double complex with $C^{i,j} = C^i(R; x_1, \ldots, x_n) \otimes_R C^j(R; f_1, \ldots, f_r)$, the vertical maps $\phi^{\bullet\bullet}$ induced by the identity on the first factor and the usual Čech maps on the second, whereas the horizontal maps $\xi^{\bullet\bullet}$ are induced by the Čech maps on the first factor and the identity on the second. Since $C^i(R; x_1, \ldots, x_n)$ is R-flat, the column cohomology of $C^{\bullet\bullet}$ at (i, j) is $C^i(R; x_1, \ldots, x_n) \otimes_R H^j_I(R)$ and the induced horizontal maps in the j-th row are simply the maps in the Čech complex $C^\bullet(H^j_I(R); x_1, \ldots, x_n)$. It follows that the row cohomology of the column cohomology at (i_0, j_0) is $H^{i_0}_{\mathfrak{m}}(H^{j_0}_I(R))$.

Now $C^{i,j}$ is a direct sum of modules R_g where $g = x_{\alpha_1} \cdot \ldots \cdot x_{\alpha_i} \cdot f_{\beta_1} \cdot \ldots \cdot f_{\beta_j}$. So the whole double complex can be rewritten in terms of A_n-modules and A_n-linear maps using 4.7:

$$\begin{array}{ccccc}
C^{i-1,j+1} & \xrightarrow{\xi^{i-1,j+1}} & C^{i,j+1} & \xrightarrow{\xi^{i,j+1}} & C^{i+1,j+1} \\
\uparrow{\scriptstyle \phi^{i-1,j}} & & \uparrow{\scriptstyle \phi^{i,j}} & & \uparrow{\scriptstyle \phi^{i+1,j}} \\
C^{i-1,j} & \xrightarrow{\xi^{i-1,j}} & C^{i,j} & \xrightarrow{\xi^{i,j}} & C^{i+1,j} \\
\uparrow{\scriptstyle \phi^{i-1,j-1}} & & \uparrow{\scriptstyle \phi^{i,j-1}} & & \uparrow{\scriptstyle \phi^{i+1,j-1}} \\
C^{i-1,j-1} & \xrightarrow{\xi^{i-1,j-1}} & C^{i,j-1} & \xrightarrow{\xi^{i,j-1}} & C^{i+1,j-1}
\end{array}$$

Using the comments in subsection 3.3, we may now compute the modules $H^i_{\mathfrak{m}}(H^j_I(R))$. More concisely we have, denoting by $X_{\theta'}$ in analogy to F_θ the product $\prod_{\alpha \in \theta'} x_\alpha$, the following

Algorithm 5.2. Input: $f_1, \ldots, f_r \in R; i_0, j_0 \in \mathbb{N}$.

Output: $H^{i_0}_{\mathfrak{m}}(H^{j_0}_I(R))$ in terms of generators and relations as finitely generated A_n-module.

Begin.

1. For $i = i_0 - 1, i_0, i_0 + 1$ and $j = j_0 - 1, j_0, j_0 + 1$ compute the annihilators $J^\Delta((F_\theta \cdot X_{\theta'})^s)$ and Bernstein polynomials $b^\Delta_{F_\theta \cdot X_{\theta'}}(s)$ of $F_\theta \cdot X_{\theta'}$ where $\theta \in \Theta^r_j, \theta' \in \Theta^n_i$.
2. Let a be the minimum integer root of the product of all these Bernstein polynomials and replace s by a in all the annihilators computed in the previous step.

3. Compute the matrices to the A_n-linear maps $\phi^{i,j} : C^{i,j} \to C^{i,j+1}$ and $\xi^{i,j} : C^{i,j} \to C^{i+1,j}$, again for $i = i_0 - 1, i_0, i_0 + 1$ and $j = j_0 - 1, j_0, j_0 + 1$.
4. Compute Gröbner bases for the modules

$$G = \ker(\phi^{i_0,j_0}) \cap \left[(\xi^{i_0,j_0})^{-1}(\mathrm{im}(\phi^{i_0+1,j_0-1}))\right] + \mathrm{im}(\phi^{i_0,j_0-1})$$

and $G_0 = \xi^{i_0-1,j_0}(\ker(\phi^{i_0-1,j_0})) + \mathrm{im}(\phi^{i_0,j_0-1})$.
5. Compute the remainders of all elements of G with respect to G_0 and return these remainders together with G_0.

End.

The elements of G will be generators for $H^{i_0}_{\mathfrak{m}}(H^{j_0}_I(R))$ and the elements of G_0 generate the relations that are not syzygies.

5.3. **Computation of** $\lambda_{i,n-j}(R/I)$**.** In [16] it has been shown that $H^i_{\mathfrak{m}}(H^j_I(R))$ is an injective $\mathfrak{m}$-torsion R-module of finite socle dimension $\lambda_{i,n-j}$ (which depends only on i, j and R/I) and so isomorphic to $(E_R(K))^{\lambda_{i,n-j}}$ where $E_R(K)$ is the injective hull of K over R. We are now in a position that allows computation of these invariants of R/I.

Algorithm 5.3. Input: $f_1, \dots, f_r \in R; i, j \in \mathbb{N}$.
Output: $\lambda_{i,n-j}(R/(f_1, \dots, f_r))$.
Begin.

1. Using Algorithm 5.2 find $g_1, \dots, g_l \in {A_n}^d$ and $h_1, \dots, h_e \in {A_n}^d$ such that $H^i_{\mathfrak{m}}(H^j_I(R))$ is isomorphic to $A_n(g_1, \dots, g_l)$ modulo $H = A_n(h_1, \dots, h_e)$.
2. Assume that g_1 is not in H. If such a g_1 cannot be chosen, quit.
3. Find a monomial $m \in R$ such that $m \cdot g_1 \notin H$ but $x_i m g_1 \in H$ for all x_i.
4. Replace H by $A_n m g_1 + H$ and reenter at step 2.
5. $\lambda_{i,n-j}(R/I)$ equals the number of times step 3 was executed.

End.

The justification of the correctness of the algorithm is as follows. We know that $(A_n \cdot g_1 + H)/H$ is $\mathfrak{m}$-torsion (as $H^i_{\mathfrak{m}}(H^j_I(R))$ is) and so it is possible (with trial and error) to find the monomial m in step 3. Then the element $mg_1 + H/H$ has annihilator equal to $\mathfrak{m}$ and therefore generates an A_n-module isomorphic to $A_n/A_n \cdot \mathfrak{m} \cong E_R(K)$. The injection $(A_n \cdot mg_1 + H)/H \hookrightarrow (A_n \cdot (g_1, \dots, g_l) + H)/H$ splits as map of R-modules because of injectivity and so the cokernel $A_n(g_1, \dots, g_l)$ modulo $A_n(mg_1, h_1, \dots, h_e)$ is isomorphic to $(E_R(K))^{\lambda_{i,n-j}-1}$.

Reduction of the g_i with respect to a Gröbner basis of the new relation module and repetition will lead to the determination of $\lambda_{i,n-j}$.

5.4. **Local cohomology in ambient spaces different from $\mathbb{A}^n_K$.** If A equals $K[x_1,\ldots,x_n]$, $I \subseteq A$, $X = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(A/I)$, knowledge of $H^i_I(A)$ for all $i \in \mathbb{N}$ answers of course the question about the local cohomological dimension of V in X. It is worth mentioning, that if $W \subseteq X$ is a smooth variety containing V then our algorithm 5.1 for the computation of $H^i_I(A)$ also leads to a determination of the local cohomological dimension of V in W. Namely, if J stands for the defining ideal of W in X so that $R = A/J$ is the affine coordinate ring of W and if we set $c = \operatorname{ht}(J)$, then it can be shown that $H^{i-c}_I(R) = \operatorname{Hom}_A(R, H^i_I(A))$ for all $i \in \mathbb{N}$. As $H^i_I(A)$ is I-torsion (and hence J-torsion), $\operatorname{Hom}_A(R, H^i_I(A))$ is zero if and only if $H^i_I(A) = 0$. It follows that the local cohomological dimension of V in W equals $\operatorname{cd}(A, I) - c$ and in fact $\{q \in \mathbb{N} : H^q_I(A) \neq 0\} = \{q \in \mathbb{N} : H^{q-c}_I(R) \neq 0\}$.

If however W is not smooth, no algorithms for the computation of either $H^i_I(R)$ or $\operatorname{cd}(R, I)$ are known, irrespective of the characteristic of the base field.

6. Implementation and examples

Some of the algorithms described above have been implemented as C-scripts and tested on some examples.

6.1. The algorithm 4.4 with $L = \Delta$ has been implemented by Oaku using the package Kan (see [25]) which is a postscript language for computations in the Weyl algebra and in polynomial rings. An implementation for general L is written by the current author and part of a program that deals exclusively with computations around local cohomology ([26]). [26] is theoretically able to compute $H^i_I(R)$ for arbitrary $i, R = \mathbb{Q}[x_1,\ldots,x_n], I \subseteq R$ in the above described terms of generators and relations, using algorithm 5.1. It is expected that in the future a Kan-based implementation will work for $R = K[x_1,\ldots,x_n]$ where K is any computable field of characteristic zero and also algorithms for computation of $H^i_{\mathfrak{m}}(H^j_I(R))$ and $\lambda_{i,j}(R)$ will be implemented, but see the comments in 6.2 below.

Example 6.1. Let I be the ideal in $R = K[x_1,\ldots,x_6]$ that is generated by the 2×2 minors f, g, h of the matrix $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}$. Then $H^i_I(R) = 0$ for $i < 2$ and $H^2_I(R) \neq 0$ because I is a height 2 prime and $H^i_I(R) = 0$ for $i > 3$ because I is 3-generated, so the only remaining case is $H^3_I(R)$. This module in fact does not vanish, but until the discovery of our algorithm, its non-vanishing was a rather non-trivial fact. Our algorithm provides the first known proof of this fact by direct calculation.

Previously, Hochster pointed out that $H^3_I(R)$ is nonzero, using the fact that the map $K[f,g,h] \to R$ splits (compare [11], Remark 3.13) and Bruns and Schwänzl ([3], the corollary to Lemma 2) provided a topological proof of the nonvanishing of $H^3_I(R)$ via étale cohomology. Both of these proofs are quite ingenious and work only in very special situations.

Using the program [26], one finds that $H^3_I(R)$ is isomorphic to a cyclic A_6-module generated by $1 \in A_6$ subject to relations $x_1 = \ldots = x_6 = 0$. This is a straightforward computational proof of the non-vanishing of $H^3_I(R)$. Of course this proof gives more than simply the non-vanishing. Since $A_6/A_6(x_1,\ldots,x_6)$ is isomorphic to $E_R(R/(x_1,\ldots,x_6))$, the injective hull of $R/(x_1,\ldots,x_6) = K$ in the category of R-modules, our proof implies that $H^3_I(R) \cong E_R(K)$.

6.2. Computation of Gröbner bases in many variables is in general a time- and space consuming enterprise. Already in (commutative) polynomial rings the worst case performance for the number of elements in reduced Gröbner bases is doubly exponential in the number of variables and the degrees of the generators. In the (relatively small) example above R is of dimension 6, so that the intermediate ring $A_{n+1}[y_1,y_2]$ contains 16 variables. In view of these facts the following idea has proved useful.

The general context in which lemma 4.1 and proposition 4.2 were stated allows successive localization of R_{fg} in the following way. First one computes R_f according to algorithm 4.7 as quotient of A_n by a certain holonomic ideal $L = J^\Delta(f^s)|_{s=a}, a \ll 0$. Then R_{fg} may be computed using 4.7 again since $R_{fg} \cong R_g \otimes A_n/L$. (Note that all localizations of R are automatically f-torsion free for $f \in R$ as R is a domain.) This process may be iterated for products with any finite number of factors. Note also that the exponents for the various factors might be different. This requires some care as the following situations illustrate. Assume first that -1 is the smallest integer root of the Bernstein polynomials of f and g (both in R) with respect to the holonomic module R. Assume further that $R_{fg} \cong A_n \cdot f^{-2}g^{-1} \supsetneq A_n \cdot (fg)^{-1}$. Then $R_f \to R_{fg}$ can be written as $A_n/\operatorname{ann}(f^{-1}) \to A_n/\operatorname{ann}(f^{-2}\cdot g^{-1})$ sending $P \in A_n$ to $P \cdot f \cdot g$.

Suppose on the other hand that we are interested in $H^2_I(R)$ where $I = (f,g,h)$ and we know that $R_f = A_n \cdot f^{-2} \supsetneq A_n \cdot f^{-1}, R_g = A_n \cdot g^{-2}$ and $R_{fg} = A_n \cdot f^{-1}g^{-2}$. (In fact, the degree 2 part of the Čech complex of example 6.1 consists of such localizations.) It is tempting to write the embedding $R_f \to R_{fg}$ with the use of a Bernstein operator (if $P_f(s)f^{s+1} = b^\Delta_f(s)f^s$ then take $s = -2$) but as f^{-1} is not a generator for R_f, $b_f^\Delta(-2)$ will be zero. In other words, we must write R_{fg} as $A_n/\operatorname{ann}((fg)^{-2})$ and then send $P \in \operatorname{ann}(f^{-2})$ to $P \cdot g^2$.

The two examples indicate how to write the Čech complex in terms of generators and relations over A_n while making sure that the maps $C^k \to C^{k+1}$ can be made explicit using the f_i - the exponents used in C^k have to be at least as big as those in C^{k-1} (for the same f_i).

Remark 6.2. One might hope that for all holonomic fg-torsionfree modules $M = A_n/L$ we have (with $R_g \otimes M \cong A_n/L'$):

$$\min\{s \in \mathbb{Z} : b_f^L(s) = 0\} \leq \min\{s \in \mathbb{Z} : b_f^{L'}(s) = 0\}.$$

This would guarantee, that successive localization at the factors of a product does not lead to matrices in the Čech complex with entries of higher degree than localization at the product at once.

However, as was pointed out by one of the referees, the following example shows that this hope is unfounded. Let $R = \mathbb{C}[x_1, \ldots, x_5]$, $f = x_1^2+x_2^2+x_3^2+x_4^2+x_5^2$. One may check that then $b_f^\Delta(s) = (s+1)(s+5/2)$. Hence $R_f = A_5 \cdot f^{-1}$, let $L = \ker(A_5 \to A_5 \cdot f^{-1})$. Set $g = x_1$. Then $b_g^\Delta(s) = s+1$, let $L' = \ker(A_5 \to A_5 \cdot g^{-1})$.

Computations with [26] or [25] show that $b_f^{L'}(s) = (s+1)(s+2)(s+5/2)$ and $b_g^L(s) = (s+1)(s+3)$. This shows that R_{fg} is generated by $f^{-2}g^{-1}$ or $f^{-1}g^{-3}$ but not by $f^{-1}g^{-2}$ and in particular not by $f^{-1}g^{-1}$. Notice that this example not only disproves the above inequality but also shows the inequality to be wrong if $\mathbb{Z}$ is replaced by $\mathbb{R}$ (as $-3 < \min(-5/2, -1)$).

Nonetheless, localizing R_{fg} as $(R_f)_g$ is advantageous, heuristically. For one, it allows the exponents of the various factors to be distinct which is useful for the subsequent cohomology computation: it helps to keep the degrees of the maps small. (So for example $R_{x\cdot(x^2+y^2)}$ can be written as $A_n \cdot x^{-1}(x^2+y^2)^{-2}$ instead of $A_n \cdot (x^{-2} \cdot (x^2+y^2)^{-2})$. On the other hand, since the computation of Gröbner bases is doubly exponential it seems to be advantageous to break a big problem (localization at a product) into many "easy" problems (successive localization).

An interesting case of this behaviour is our example 6.1. If we compute R_{fgh} as $((R_f)_g)_h$, the calculation uses approximately 2.5 kB and lasts 10 seconds on a Sun workstation using [26]. If one tries to localize R at the product of the three generators at once, [26] needs about 30 minutes.

So, one should proceed as follows for the computation of the Čech complex $C^\bullet(R; f_1, \ldots, f_r)$. First compute $J^\Delta((f_i)^s)$ for all i, find all minimal integer Bernstein roots β_i of f_i on R and substitute them into the appropriate annihilator ideals. Iteratively use algorithm 4.7 in order to compute $R_{f_{i_1}\cdot\ldots\cdot f_{i_k}\cdot f_{i_{k+1}}}$ from $R_{f_{i_1}\cdot\ldots\cdot f_{i_k}}$ If all Bernstein roots for $f_{i_{k+1}}$ on $R_{f_1\cdot\ldots\cdot f_k}$ are greater or equal to $\beta_{i_{k+1}}$ we are fine. As soon as one of them is smaller than the corresponding β, we need to replace β_i by that root and start from the beginning.

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