ON A CONJECTURE OF LYNCH

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The following conjecture has recently attracted attention, e.g., [BE1, BE2, DSZ, HJ]:

Conjecture 1. [Ly, Conjecture 1.2] Let R be a local ring, and I an ideal of R. If the cohomological dimension of I is a positive integer c, then

$$\dim R/\operatorname{ann}_R H^c_I(R) = \dim R/H^0_I(R).$$

The conjecture is know to be false: the first counterexamples were constructed in [Ba]; these are nonequidimensional, with $\dim R \geqslant 5$. We present here a modification—with a short, elementary proof—that serves as a counterexample with $\dim R = 3$. This is a counterexample, as well, to [Ly, Proposition 4.3] and to [Ly, Theorem 4.4]; the error there is in the chain of inequalities in the proof of Proposition 4.3, [Ly, page 550], in the reduction from a complete local ring to a complete local unmixed ring: the cohomological dimension may change under the reduction step.

Example 2. Let k be a field, and set $R := k[x,y,z_1,z_2]/(xyz_1,\ xyz_2)$. Consider the local cohomology module $H^2_{(x,y)}(R)$. Using a Čech complex on x and y, one sees that

$$H^2_{(x,y)}(R) = R_{xy}/\operatorname{image}(R_x + R_y).$$

The images of z_1 and z_2 are zero in R_{xy}, so the local cohomology module above agrees with $S_{xy}/\operatorname{image}(S_x+S_y)$, where $S := R/(z_1,z_2)$ is isomorphic to the polynomial ring $k[x,y]$. Hence $H^2_{(x,y)}(R)$ is a nonzero R-module, with annihilator (z_1,z_2). On the other hand, since the ideal (x,y) contains nonzerodivisors, $H^0_{(x,y)}(R) = 0$. Hence one has

$$\dim R/\operatorname{ann}_R H^2_{(x,y)}(R) = 2 \qquad \text{whereas} \qquad \dim R/H^0_{(x,y)}(R) = 3.$$

References

[Ba] K. Bahmanpour, *A note on Lynch's conjecture*, Comm. Algebra **45** (2017), 2738–2745.

[BE1] A. F. Boix, M. Eghbali, *Annihilators of local cohomology modules and simplicity of rings of differential operators*, Beitr. Algebra Geom. **59** (2018), 665–684.

[BE2] A. F. Boix, M. Eghbali, *Correction to: Annihilators of local cohomology modules and simplicity of rings of differential operators*, Beitr. Algebra Geom. **59** (2018), 685–688.

[DSZ] R. Datta, N. Switala, W. Zhang, *Annihilators of D-modules in mixed characteristic*, `arXiv:1907.09948`.

[HJ] M. Hochster, J. Jeffries, *Faithfulness of top local cohomology modules in domains*, `arXiv:1909.08770`.

[Ly] L. R. Lynch, *Annihilators of top local cohomology modules*, Comm. Algebra **40** (2012), 542–551.

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