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Preface

In this dissertation we shall develop some of the theory of local cohomology modules. This will be done by elementary means via direct limits. It should be understood that we will not touch sheaf theory, Koszul- and Ĉech-complexes which however all belong to the subject in one or the other way.

This dissertation is thought to give beginning postgraduate students an introduction to the subject and so it starts in chapter one with the basic definitions not assuming the reader to know about this. In this first chapter are done the mot obvious results one can get without other tools. One of these tools is described and developed in the second chapter, the concept of *direct limits*. Again the reader is not assumed to know about this and so we give an introduction to direct limits and adjoint pairs. We show how local cohomology is related to direct limits and as a consequence, that local cohomology and direct limits commute.

In the third chapter we show a useful relationship between the local cohomology modules with respect to different ideals, which has its origin in algebraic topology. We will use the *Mayer-Vietoris-sequence* to show that local cohomology is in a certain sense independent of the ring in question, to be more precise we will show that local cohomology "commutes" with ring homomorphisms. These very neat results are then used in chapter four to calculate some local cohomology modules and we will establish in many cases the vanishing of these modules in connection with dimensions and depths of the modules and ideals under consideration. Also in this chapter we will give a short extract to the theory of *secondary representation* without doing proofs to this subject. We show then, that it can be used succesfully in local cohomology to get results which are otherwise rather difficult to achieve.

Since the theory of local cohomology has its origin in algebraic geometry, it is natural that the main interest on the base rings is put onto rings which actually occur in geometry. One large and important class of these is the type of *Gorenstein rings*, which will be discussed in chapter five. We will prove there the "algebraic" part of the main theorem on local Gorenstein rings (that means that we will not develop nor use the concepts "unmixed" or Cohen-Macaulay). In the second part of this chapter we demonstrate a *dualizing* property, these rings have.

This will then be applied in chapter six, where we imitate a proof of *Faltings Annnihilator Theorem.* Also, as preparation to the proof, we will make some comments about the connections of sets *stable under stabilisation* and direct limits over directed index sets as were encountered in chapter two.

Any reader who wants to have a chance to enjoy reading this dissertation should be as familiar as possible with *Steps in commutative Algebra* by R. Y. Sharp and *Introduction to homological Algebra* by D. G. Northcott (see bibliography) (or alternatively with any other book about basic commutative and homological algebra). The results from thes etwo books are widely used throughout and are the basis for our work. Any other concept that is used and does not occur in either of the two mentioned books will be defined and developed from the very beginning – with one exception: since we will deal quite a lot with derived functors, injective modules are frequently under consideration. Many of the results to follow will depend on the structure theory for injective modules due to Eben Matlis. Because of this there is an appendix concerned with injectives only. It did not seem to be convenient to put that material in the text wherever it would be needed, since one easily gets lost in the story if it is changed every line.

CHAPTER 1

Introduction

Notation 1.1. We begin this chapter with some conventions and notations. Throughout this dissertation we always denote by

- A a Noetherian commutative ring with identity,
- $\mathfrak{a}$ an ideal of the ring A,
- M a module over A,
- $\mathbb{N}$ the set 0,1,2, ... of the nonnegative integers,
- $\mathbb{Z}$ the set of integers,
- $\mathcal{C}_A$ the category of A-modules and A-homomorphisms,
- m.c.s. a multiplicatively closed subset which contains 1.

Local cohomology is a branch of mathematics, linking commutative algebra and algebraic geometry. The concept of local cohomology itself originally was defined in 1961 by Grothendieck in the language of algebraic geometry and subsequently developed by him, R. Hartshorne and others. Later it became also part of commutative algebra by the work of Sharp and Macdonald. More recently Faltings has made considerable contributions to the subject. As a consequence, this dissertation will be guided in the parts concerning local cohomology by papers written by Sharp and Macdonald and it will terminate in a rereading of a paper of Faltings in a weakened form (see chapter 6).

After these preliminaries we come to

Definition 1.2. For the A-module M define the subset $\Gamma_{\mathfrak{a}}(M)$ to be the set

$$\{m \in M : \exists n \in \mathbb{N} : \mathfrak{a}^n m = 0\} = \bigcup_{n \in \mathcal{N}} (0 \ :_M \mathfrak{a}^n).$$

This set is called the $\mathfrak{a}$-*torsion* of M.

Remark 1.3. $\Gamma_{\mathfrak{a}}(M)$ has similarities with the usual torsion-part of an Abelian group with one difference: while the torsion of a group is defined to be the set of elements which vanish under multiplication by *some* $0 \neq n \in \mathbb{N}$, i.e. by multiplication by *some* ideal of $\mathbb{Z}$, $\Gamma_{\mathfrak{a}}(M)$ concentrates on $\mathfrak{a}$ and its powers.

We demonstrate this with

EXAMPLE 1.4. Let $A = \mathbb{Z}, M = \mathbb{Z}/12\mathbb{Z}$. Then M is in the group-theoretical sense completely torsion, whereas if $\mathfrak{a} = 2 \cdot \mathbb{Z}, \mathfrak{b} = 3 \cdot \mathbb{Z}$ then

$$\begin{aligned} \Gamma_{\mathfrak{a}}(M) &= \{m \in M : 2^n \cdot m = 0 \text{ for some } n \in \mathbb{N}\} \\ &= \{(0 + 12\mathbb{Z}), \dots, (9 + 12\mathbb{Z})\} \text{ and} \\ \Gamma_{\mathfrak{b}}(M) &= \{m \in M : 3^n \cdot m = 0 \text{ for some } n \in \mathbb{N}\} \\ &= \{(0 + 12\mathbb{Z}), (4 + 12\mathbb{Z}), (8 + 12\mathbb{Z})\}, \end{aligned}$$

and these sets are different.

Let M, N, S be A-modules. A little thought makes clear that $\Gamma_{\mathfrak{a}}(M)$ is not only a subset of M but actually a submodule of M. Also, if $f : M \to N$ is an A-homomorphism, then for $m \in \Gamma_I(M)$ we know that there is an $n \in \mathbb{N}$ such that $\mathfrak{a}^n m = 0$ and hence $\mathfrak{a}^n \cdot f(m) = f(\mathfrak{a}^n m) = 0$ such that $f(m) \in \Gamma_{\mathfrak{a}}(N)$. We have therefore established a map $\Gamma_{\mathfrak{a}}(f) : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N)$. It is further clear that for another A-homomorphism $g : N \to S$ (so $g(\Gamma_{\mathfrak{a}}(N)) \subseteq \Gamma_{\mathfrak{a}}(S)$), $\Gamma_{\mathfrak{a}}(g \circ f) = \Gamma_{\mathfrak{a}}(g) \circ \Gamma_{\mathfrak{a}}(f)$ and $\Gamma_{\mathfrak{a}}(\mathrm{id}_M) = \mathrm{id}_{\Gamma_{\mathfrak{a}}(M)}$ because $\Gamma_{\mathfrak{a}}$ applied to maps is just restriction. So we have proved

LEMMA 1.5. *$\Gamma_{\mathfrak{a}}$ is a (covariant) functor from the category $\mathcal{C}_A$ of A-modules and A-homomorphisms to itself.* □

LEMMA 1.6. *$\Gamma_{\mathfrak{a}}$ is left exact.*

PROOF. Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence. Because $\Gamma_{\mathfrak{a}}(f)$ is restriction, it is clear that $0 \to \Gamma_{\mathfrak{a}}(M') \to \Gamma_{\mathfrak{a}}(M)$ is exact and since $\Gamma_{\mathfrak{a}}$ is functor, $\Gamma_{\mathfrak{a}}(g) \circ \Gamma_{\mathfrak{a}}(f) = 0$. Furthermore $\Gamma_{\mathfrak{a}}(g) : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M")$ has kernel

$$\begin{aligned} \ker(\Gamma_{\mathfrak{a}}(g)) &= \{x \in \Gamma_{\mathfrak{a}}(M) : \Gamma_{\mathfrak{a}}(g)(x) = 0 (\text{i.e. } g(x) = 0)\ \} \\ &= \{x \in \Gamma_{\mathfrak{a}}(M) : x \in \ker(g)\} \\ &= \Gamma_{\mathfrak{a}}(M) \cap \ker(g) \\ &= \Gamma_{\mathfrak{a}}(M) \cap \mathrm{im}(f). \end{aligned}$$

So if $x \in \ker(\Gamma_{\mathfrak{a}}(g))$, then there is $y \in M' : f(y) = x$. Also $x \in \Gamma_{\mathfrak{a}}(M)$ implies the existence of $n \in \mathbb{N} : 0 = \mathfrak{a}^n \cdot x = \mathfrak{a}^n f(y) = f(\mathfrak{a}^n y)$. Therefore $\mathfrak{a}^n y \subseteq \ker(f)$. Since f is a monomorphism, this implies $\mathfrak{a}^n \cdot y = 0$, i.e. $y \in \Gamma_{\mathfrak{a}}(M')$ and so $x \in \mathrm{im}(\Gamma_{\mathfrak{a}} f)$. □

Now that we have this friendly property of the functor $\Gamma_{\mathfrak{a}}$, we may use it after the following

DEFINITION 1.7. Let $A, \mathfrak{a}$ and M be as usual. Then the *i-th local cohomology module* of M with respect to $\mathfrak{a}$, written $H^i_{\mathfrak{a}}(M)$, is defined by

$$H^i_{\mathfrak{a}}(M) = (\mathcal{R}^i \Gamma_{\mathfrak{a}})(M),$$

the i^{th} right derived functor of $\Gamma_{\mathfrak{a}}$ applied to M.

By the previous lemma, we may observe, that $H^0_{\mathfrak{a}}$ and $\Gamma_{\mathfrak{a}}$ are naturally equivalent functors (see e.g. [**9**] , Theorem 6.5).

DEFINITION 1.8. Let M be an A-module. Then the *support* of M with respect to A, written $\mathrm{supp}_A(M)$, is defined to be the set

$$\{p \in \mathrm{Spec}(A) : M_p \neq 0\}$$

where M_p is the localisation of M at p. If there is no misunderstanding about the base ring, we also will use the symbol $\mathrm{supp}(M)$.

Now if $M, A, \mathfrak{a}$ are as usual, then since A is Noetherian for $m \in M$ are equivalent

$$\begin{aligned} m \in \Gamma_{\mathfrak{a}}(M) &\iff \exists n \in \mathbb{N} : \mathfrak{a}^n m = 00 \\ &\Leftrightarrow \mathfrak{a} \subseteq \sqrt{\mathrm{ann}(m)} \\ &\Leftrightarrow \mathrm{Var}(\mathfrak{a}) \supseteq \mathrm{Var}(\mathrm{ann}(A \cdot m)) \\ &\Leftrightarrow \mathrm{Var}(\mathfrak{a}) \supseteq \mathrm{supp}(A \cdot m) \end{aligned}$$

so that

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M : \mathrm{supp}(A \cdot m) \subseteq \mathrm{Var}(\mathfrak{a})\}.$$

This is in fact the historical origin of the local cohomology modules, defined in this way by A. Grothendieck as the local sections of the structure sheaf of A on $\mathrm{Spec}(A)$ tensored by M (see for example in [**3**], Ch. 1).

We will now make the first observation concerning the vanishing of certain local cohomology modules, this being the beginning of a long sequence.

LEMMA 1.9. *Let $A, \mathfrak{a}, M$ be as usual. Then $X := \Gamma_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M)) = 0$.*

PROOF. Let $\xi = x + \Gamma_{\mathfrak{a}}(M) \in X$. Then there is a natural number n : $\mathfrak{a}^n \xi = 0$ by definition of X. Now A is Noetherian, so $\mathfrak{a}^n$ is finitely generated, say by $a_1, \dots, a_t$ and so $\mathfrak{a}^n \xi$ is generated by $a_1 \xi, \dots, a_t \xi$. Since $\mathfrak{a}^n \xi = 0$ in $M/\Gamma_{\mathfrak{a}}(M)$, it follows that $a_1 x, \dots, a_t x$ are all in $\Gamma_{\mathfrak{a}}(M)$. By definition of $\Gamma_{\mathfrak{a}}(M)$ there are $\gamma_1, \dots, \gamma_t$ in $\mathbb{N} : \mathfrak{a}^{\gamma_i}(a_i x) = 0$ for each $i = 1, \dots, t$. Let γ be the biggest of these γ_i, then $\mathfrak{a}^{\gamma}(a_i x) = 0$ for $1 \leq i \leq t$ and so $\mathfrak{a}^{\gamma+n} x = 0$ or $x \in \Gamma_{\mathfrak{a}}(M)$. □

The following lemmata will make easier the actual computation of local cohomology modules and shorten some proofs.

LEMMA 1.10. *Let $A, \mathfrak{a}, M$ be as usual. Then for each $i \geq 0$,*

$$H^i_{\mathfrak{a}}(M) = H^i_{\sqrt{\mathfrak{a}}}(M).$$

PROOF. Since $H^i_{\mathfrak{a}}$ is calculated by taking an injective resolution, applying $\Gamma_{\mathfrak{a}}$ and taking cohomology, it will suffice to show that $\Gamma_{\mathfrak{a}}$ and $\Gamma_{\sqrt{\mathfrak{a}}}$ are equal as functors. This we do.

We want to show, that

$$\{m \in M : \exists n \in \mathbb{N} : \mathfrak{a}^n m = 0\} = \{m \in M : \exists n \in \mathbb{N} : \sqrt{\mathfrak{a}}^n m = 0\}$$

But since $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$ we have $(\sqrt{\mathfrak{a}})^n m = 0 \Rightarrow \mathfrak{a}^n m = 0$ and since A is Noetherian there is $t(\mathfrak{a}) \in \mathbb{N} : (\sqrt{\mathfrak{a}})^{t(\mathfrak{a})} \subseteq \mathfrak{a}$, so that $\mathfrak{a}^n m = 0 \Rightarrow (\sqrt{\mathfrak{a}})^{n \cdot t(\mathfrak{a})} m = 0$ and hence $\Gamma_{\mathfrak{a}}(M) = \Gamma_{\sqrt{\mathfrak{a}}}(M)$. □

COROLLARY 1.11. *Let* $\mathfrak{a}, \mathfrak{b}$ *be two ideals in* A *and* M *a module over* A. *Then*

$$H^i_{\mathfrak{a}}(-) = H^i_{\mathfrak{b}}(-) \text{ as functors } \Leftrightarrow \sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}.$$

PROOF. $\Rightarrow$: Let $\sqrt{\mathfrak{a}} \neq \sqrt{\mathfrak{b}}$. Without loss of generality we can assume the existence of an $x \in \sqrt{\mathfrak{a}} \backslash \sqrt{\mathfrak{b}}$. Let $M := A/\sqrt{\mathfrak{b}}$. Then of course $\Gamma_{\mathfrak{b}}(M) = M$, but $\xi := x + \sqrt{\mathfrak{b}} \notin \Gamma_{\mathfrak{a}}(M)$ since otherwise there would have to be $n \in \mathbb{N}$: $\mathfrak{a}^n \xi = 0_M$, that is, $\mathfrak{a}^n x \subseteq \sqrt{\mathfrak{b}}$ such that $x \in \sqrt{\mathfrak{b}}$, a contradiction.

$\Leftarrow$: Suppose, $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Then there exist numbers α and β such that $(\sqrt{\mathfrak{a}})^\alpha \subseteq \mathfrak{a}$ and $(\sqrt{\mathfrak{b}})^\beta \subseteq \mathfrak{b}$ because A is Noetherian (see [**12**],8.21). So $\mathfrak{a}^\beta \subseteq \mathfrak{b}$ and $\mathfrak{b}^\alpha \subseteq \mathfrak{a}$. This shows that $m \in M$ is annihilated by some power of $\mathfrak{a}$ iff it is annihilated by some power of $\mathfrak{b}$.

Therefore $\Gamma_{\mathfrak{a}}$ and $\Gamma_{\mathfrak{b}}$ are equal as functors and hence so are the derived functors. □

COROLLARY 1.12. *For two ideals* $\mathfrak{a}, \mathfrak{b}$ *we have* $H^i_{\mathfrak{a}\cdot\mathfrak{b}} = H^i_{\mathfrak{a}\cap\mathfrak{b}}$ *as functors.*

PROOF. The result follows directly from [**12**], 2.30. □

In many cases in commutative algebra the "localization of a problem" is a powerful tool, because the local case is easier to handle. However, one needs to show before that the current problem "commutes with localization". We do this now with the formation of the $\mathfrak{a}$-torsion modules.

PROPOSITION 1.13. *Let* $S \subseteq A$ *be a multiplicatively closed subset (in the sequel "m.c.s.") and* $\mathfrak{a}$ *an ideal in* A. *Then the functors from* $\mathcal{C}_A$ *to* $\mathcal{C}_{S^{-1}A}$

$$S^{-1}\Gamma_{\mathfrak{a}}(\cdot), \Gamma_{\mathfrak{a}S^{-1}A}(S^{-1}(\cdot))$$

are naturally equivalent and for the A*-module* M, $S^{-1}\Gamma_{\mathfrak{a}}(M)$ *and* $\Gamma_{\mathfrak{a}S^{-1}A}(M)$ *are equal as subsets of* $S^{-1}(M)$.

PROOF. For each A-module M we define

$$\Pi := S^{-1}\Gamma_{\mathfrak{a}}(M) = \{\frac{m}{s} : s \in S, m \in M, \exists n \in \mathbb{N} : \mathfrak{a}^n m = 0\}$$

and

$$\Omega := \Gamma_{\mathfrak{a}S^{-1}A}(S^{-1}M) = \{\frac{m}{s} : s \in S, m \in M, \exists n \in \mathbb{N} : (\mathfrak{a}S^{-1}A)^n \cdot \frac{m}{s} = 0\}.$$

We will consider both Ω and Π as subsets of $S^{-1}M$. It is obvious that $\Pi \subseteq \Omega$. So let $\frac{m}{s}$ be in Ω. Hence there is an $n \in \mathbb{N} : (\mathfrak{a}S^{-1}A)^n \frac{m}{s} = \mathfrak{a}^n \frac{m}{s} = 0$, considering $S^{-1}M$ as an A-module. Now A is Noetherian and hence $\mathfrak{a}^n$ finitely generated, say by $a_1, \dots, a_t$. Then $a_i \frac{m}{s} = 0$ for each $1 \leq i \leq t$ such that there exist

$$s_i \in S : s_i a_i m = 0 (1 \leq i \leq t)$$

and hence $s_1 s_2 \dots s_t a_i m = 0$ for all $1 \leq i \leq t$. But then

$$\mathfrak{a}^n m s_1 s_2 \dots s_t m = 0 \text{ and } \frac{s_1 s_2 \dots s_t m}{s_1 s_2 \dots s_t s} = \frac{m}{s}$$

so that we have found an element in Ω that equals $\frac{m}{s}$ and has its numerator in $\Gamma_{\mathfrak{a}}(M)$. So $\Omega \subseteq \Pi$, hence $\Omega = \Pi$. But functors which are equal are of course naturally equivalent. $\square$

COROLLARY 1.14. *For any m.c.s. $S \subseteq A$ and every ideal $\mathfrak{a} \subseteq A$ and $i \in \mathbb{N}$, the functors*

$$S^{-1}H^i_{\mathfrak{a}}(\cdot) \text{ and } H^i_{\mathfrak{a}S^{-1}A}(S^{-1}(\cdot))$$

are naturally equivalent functors.

PROOF. Writing "Inj" for taking an injective resolution, it is sufficient to show that

$$S^{-1} \circ H^i \circ \Gamma_{\mathfrak{a}} \circ Inj(M) \text{ and } H^i \circ \Gamma_{\mathfrak{a}\cdot S^{-1}A} \circ S^{-1} \circ Inj(M)$$

are naturally isomorphic modules for the A-module M. Since S^{-1} is an exact functor (see [**12**], 9.9), S^{-1} and taking cohomology commute (see [**9**],6.1). Further we just have seen, that $S^{-1} \circ \Gamma_{\mathfrak{a}}$ and $\Gamma_{\mathfrak{a}S^{-1}A} \circ S^{-1}$ are naturally equivalent. So it remains to show that localization commutes with taking injective resolutions. But [**11**], Thm.3.76 together with the mentioned fact that S^{-1} is exact, shows that applying S^{-1} to an injective resolution gives an injective resolution of the localized object. And this is enough since [**9**], Theorem 6.2 assures that derived functors are independent from the resolution chosen. This proves the corollary. $\square$

DEFINITION 1.15. We call an A-module M $\mathfrak{a}$-*torsion* precisely when $\Gamma_{\mathfrak{a}}(M) = M$, and $\mathfrak{a}$-*torsionfree* whenever $\Gamma_{\mathfrak{a}}(M) = 0$.

The following proposition will use some work of the appendix.

PROPOSITION 1.16. *If $\mathfrak{a}$ is an ideal of A and I an injective module over A, then $\Gamma_{\mathfrak{a}}(I)$ is injective too. Further $\mu(\mathfrak{p}, \Gamma_{\mathfrak{a}}(I))$ equals $\mu(\mathfrak{p}, I)$ if $\mathfrak{a} \subseteq \mathfrak{p}$ and is otherwise zero.*

PROOF. By 7.18 the injective module I is representable as direct sum of indecomposable injective submodules, which by 7.19 are the injective hull of $A/\mathfrak{p}$ for some prime in A or another. By 7.22 each $x \in E(A/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec}(A)$, has its annihilator equal to some $\mathfrak{p}$-primary ideal $\mathfrak{q}$. Since A is Noetherian, this means that $\mathfrak{p}^n \cdot x \subseteq \mathfrak{q} \cdot x = 0$ for some $n \in \mathbb{N}$. So $E(A/\mathfrak{p})$ is $\mathfrak{p}$-torsion.

Let now $\mathfrak{p}$ be containing $\mathfrak{a}$. Then of course any module that is $\mathfrak{p}$-torsion is $\mathfrak{a}$-torsion as well. On the other hand, if there is a direct indecomposable summand $E = E(A/\mathfrak{p})$ of I such that $\mathfrak{p}$ does not contain $\mathfrak{a}$, then $\alpha \in \mathfrak{a} \backslash \mathfrak{p}$ exists and this implies by 7.22 again that no element of E is annihilated by any power of $\mathfrak{a}$. So $\Gamma_{\mathfrak{a}}(I)$ is the direct sum of all indecomposable injective submodules of I whose corresponding prime ideals contain $\mathfrak{a}$, which is injective by 7.2. $\square$

This is a very nice statement, since one can now easily predict what happens to injective resolutions under the application of $\Gamma_{\mathfrak{a}}$.

A natural question is now to ask whether it is possible to give for $\mathfrak{a}$-torsion modules M an injective resolution all modules of which are torsion too. The next proposition settles this question.

PROPOSITION 1.17. *Any $\mathfrak{a}$-torsion module M possesses an injective resolution consisting only of $\mathfrak{a}$-torsion modules.*

PROOF. Let $E(M)$ be the injective hull of M. Then the natural sequence

$$0 \to M \to E(M) \to E(M)/M \to 0$$

gives by the left exactness of $\Gamma_{\mathfrak{a}}$ rise to an exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) = M \to \Gamma_{\mathfrak{a}}(E(M)) \to \Gamma_{\mathfrak{a}}(E(M)/M)$$

and we have just seen that $\Gamma_{\mathfrak{a}}(E(M))$ is injective. So we may embed every module that is $\mathfrak{a}$-torsion into an injective $\mathfrak{a}$-torsion module. But since clearly quotients of $\mathfrak{a}$-torsion modules are $\mathfrak{a}$-torsion too, the cokernel of any such embedding is $\mathfrak{a}$-torsion again so that the proposition is proved. □

The reader should note that we have just shown that the injective hull $E(M)$ of $\mathfrak{a}$-torsion modules M is $\mathfrak{a}$-torsion again because $\Gamma_{\mathfrak{a}}(E(M))$ is injective and $E(M)$ is defined as smallest injective module containing M and $\Gamma_{\mathfrak{a}}(E(M)) \subseteq E(M)$.

COROLLARY 1.18. 1. *For $\mathfrak{a}$-torsion modules M, $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$.*

2. *The natural epimorphism $\pi : N \to N/\Gamma_{\mathfrak{a}}(N)$ induces isomorphisms*

$$H^i_{\mathfrak{a}}(\pi) : H^i_{\mathfrak{a}}(N) \to H^i_{\mathfrak{a}}(N/\Gamma_{\mathfrak{a}}(N))$$

for all $i > 0$.

PROOF. 1. This is clear by the definition of derived functors and 1.17.

2. By applying the connected sequence of functors $H^i_{\mathfrak{a}}(i \in \mathbb{N})$ to

$$0 \to \Gamma_{\mathfrak{a}}(N) \to N \to N/\Gamma_{\mathfrak{a}}(N) \to 0$$

we get exact sequences

$$H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(N)) \to H^i_{\mathfrak{a}}(N) \to H^i_{\mathfrak{a}}(N/\Gamma_{\mathfrak{a}}(N)) \to H^{i+1}_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(N))$$

for all $i > 0$ and this proves (ii) because of (i).

□

CHAPTER 2

Limits

The intention of this chapter is to introduce first a useful notion, with the help of one may solve quite a lot of problems, which otherwise are rather unhandy and non-transparent. It will resemble (and this probably gave the name) usual limits, but now there are not numbers "going to somewhere" but structures. However, before speaking about limits, we must declare what "going towards" means here. The reader interested in a more exhaustive treatment than we intend to do, is referred to [**11**], Ch. 2 and for basic facts and definitions from homological algebra we give [**11**] again and [**2**] as reference.

Definition 2.1. A set I is called *quasi-ordered* precisely if on I works a relation which is reflexive and transitive. We will refer to this couple as $(I, \leq)$ or to I alone if the relation meant is obvious.

The following definition will be explained in the paragraph after it.

Definition 2.2. A *direct* (resp. *inverse*) *system* in a category $\mathcal{C}$ with quasiordered index set $(I, \leq)$ is a covariant (contravariant) functor

$$\mathfrak{F} : I \longrightarrow \mathcal{C}$$

where I is interpreted as category in the sense of [**11**], Ch.1, Ex.7. The collection of all these functors with their natural transformations is a category and will be denoted by $\mathfrak{Dir}_{\mathcal{C}}(I)$. (See [**11**], Ex. 2.40 and the following remarks.)

This definition certainly will seem strange to many readers, and we will explain the meaning a bit: via $\mathfrak{F}$ we are given for each $i \in I$ an object $\mathfrak{F}(i)$ which we will write F_i such that whenever $i, j \in I$ and $i \leq j$ then there is a $\mathcal{C}$-morphism $\varphi^i_j : F_i \to F_j$ (resp. $F_j \to F_i$ in the inverse case) such that

- φ^i_i is the identity on F_i and
- for each triple $i, j, k \in I$ satisfying $i \leq j \leq k$ we have $\varphi^i_k = \varphi^j_k \circ \varphi^i_j$ (resp. $\varphi^i_k = \varphi^i_j \circ \varphi^j_k$).

Usually we will denote $\mathfrak{F}$ by $\{F_i, \varphi^i_j\}$ and call φ^i_j the *inner morphisms of* $\mathfrak{F}$.

Example 2.3. Let M be an A-module. Take $I = \mathbb{N}$ with the natural order and $F_i = M$ for each $i \in I$. Then with $\varphi^i_j = \mathrm{id}_M$ this becomes a direct system in $\mathcal{C}_A$, the category of A-modules.

There is a similar and more useful

EXAMPLE 2.4. If M is a module over A, the family of its finitely generated submodules is quasiordered by inclusion and may therefore serve as index set. Let for each $i \in I$ (which is hence a module) F_i be i itself and define for $i \leq j \in I$ (i.e. $F_i \subseteq F_j$ in M) φ_j^i as the inclusion $\varphi_j^i : F_i \to F_j$. Then this is a direct system in $\mathcal{C}_A$.

PROOF. This is obvious.

We also have an example for an inverse system:

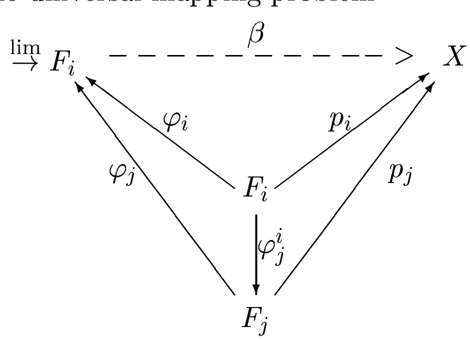
EXAMPLE 2.5. Let $I = \mathbb{N}$ with its natural order and $F_i = \mathfrak{a}^i$ for $\mathfrak{a}$ an ideal in A. Let $\varphi_j^i : \mathfrak{a}^j \to \mathfrak{a}^i$ be the natural inclusion for $i \leq j \in I$. Then this is an inverse system over $\mathbb{N}$ in $\mathcal{C}_A$.

PROOF. That the combination of natural inclusions gives again a natural inclusion is clear and so is item one of the definition of inverse systems. That $\mathfrak{F}$ is contravariant is obvious.

We are now able to define the concept of a direct limit, on which we will dwell during the remainder of the chapter.

DEFINITION 2.6. Let $\mathfrak{F} = \{F_i, \varphi_j^i\}$ be a direct system in a category $\mathcal{C}$ over the quasi-ordered set I. The *direct limit* of this system, written $\varinjlim F_i$, is

1. an object $\varinjlim F_i$ and a family of morphisms $\varphi_i : F_i \to \varinjlim F_i$ in $\mathcal{C}$ which we will usually call the *natural maps* ($i \in I$),
2. satisfying $\varphi_i = \varphi_j \circ \varphi_j^i$ whenever $i, j \in I$ and $i \leq j$ and
3. solving the universal mapping problem



meaning that whenever we are given $X \in Ob(\mathcal{C})$ and morphisms p_i for $i \in I$, such that $p_i = p_j \circ \varphi_j^i$ for all $i \leq j$ in I, then there is a unique morphism $\beta : \varinjlim F_i \to X$ making all triangles commute.

One may therefore imagine a direct limit to be a thing, lying behind the last F_i, being as big as necessary (for the existence of β), as small as possible (for the uniqueness of β) behaving properly under passage (for the commutativity).

We settle now the question, how different direct limits of the same direct system may be.

LEMMA 2.7. *The direct limit of a direct system $\mathfrak{F}$ over the index set I in the category $\mathcal{C}$ is, if it exists, unique up to equivalences in the category $\mathcal{C}$.*

PROOF. Let $\mathfrak{F} = \{F_i, \varphi_j^i\}_{\{i \in I\}}$ be a direct system over I in $\mathcal{C}$ with direct limits L, L' respectively together with natural maps $\varphi_i : F_i \to L$ and $\varphi_i' : F_i \to L'$ such that for all $i \leq j \in I$

$$\varphi_i = \varphi_j \circ \varphi_j^i, \;\; \varphi_i' = \varphi_j' \circ \varphi_j^i.$$

Then the universal property yields two $\mathcal{C}$-morphisms $\beta : L \to L'$ and $\beta' : L' \to L$ such that for all $i \in I$

$$\beta \circ \varphi_i = \varphi_i', \;\; \beta' \circ \varphi_i' = \varphi_i.$$

It follows, that $\beta \circ \beta' \circ \varphi_i' = \varphi_i'$ and $\beta' \circ \beta \circ \varphi_i = \varphi_i$. So if we set in the definition of direct limits $\varinjlim F_i = L, X = L$ and $p_i = \varphi_i$, the maps id_L and $\beta' \circ \beta : L \to L$ make all triangles commute. But by the uniqueness property of the induced map, id_L has to be equal to $\beta' \circ \beta$. Similarily, $\mathrm{id}_{L'}$ has to be equal to $\beta \circ \beta'$ so that β and β' are equivalences in $\mathcal{C}$. ☐

PROPOSITION 2.8. *The direct limit $\varinjlim F_i$ of a direct system $\{F_i, \varphi_j^i\}$ in the category $\mathcal{C}_A$ of A-modules exists and is isomorphic to $\bigoplus F_i/S$ where S is the submodule of $\bigoplus F_i$ generated by all elements of the form $\lambda_j \varphi_j^i(f_i) - \lambda_i(f_i)$, where λ_i means the natural embedding $F_i \to \bigoplus F_i$.*

PROOF. We omit the proof because it is technical and not very interesting. We give as reference [**11**], Theorem 2.16.

We are now going to use example 2.4 and consider its direct limit.

EXAMPLE 2.9. Let M be an A-module and $\{M_i, \varphi_j^i\}$ the direct sytem in $\mathcal{C}_A$ defined in 2.4. Then $\varinjlim M_i \cong M$.

PROOF. Let $\mu_i : M_i \to \bigoplus M_i$ be the natural injections. Let further S be the submodule of $\bigoplus M_i$ generated by all elements of the form $\mu_i(m_i) - \mu_j \circ \varphi_j^i(m_i)$ $(m_i \in M_i)$. Then by 2.8, $\varinjlim M_i \cong \bigoplus M_i/S$. So it is enough to show that $M \cong \bigoplus M_i/S$. This we do.

By definition, elements of $\bigoplus M_i$ have only finitely many nonzero entries, so that we may define

$$\varphi : \bigoplus M_i \to M \text{ by } (\dots, m_i, \dots, m_j, \dots) \to \Sigma_{i \in I} m_i$$

Since all elements of S are mapped to 0 under φ , there is an induced map $\varphi^* : \bigoplus M_i/S \to M$. It is obvious that φ is surjective (and hence φ^*), we investigate the kernel.

Let $m = (\dots, m_i, \dots, m_j, \dots)$ be in $\bigoplus M_i$. Then only finitely many coordinates of M are nonzero, say those with indices $i_r : r = 1, \dots, t$. Then $\Sigma_{1 \leq r \leq t} M_{i_r}$ is a finitely generated submodule of M and hence there is an index $i_0 \in I$ such that

$$M_{i_0} = \Sigma_{1 \leq r \leq t} M_{i_r}.$$

It follows, that $i_r \leq i_0$ for $1 \leq r \leq t$. So $\Sigma_1^t m_{i_r} \in M_{i_0}$. Now suppose $(m+S) \in \bigoplus M_i/S$ is in $\ker(\varphi^*)$, such that

$$0 = \varphi^*(m+S) = (\Sigma_{1\leq r\leq t} m_r) = (\Sigma_{1\leq r\leq t}\varphi_{i_0}^{i_r}(m_{i_j}))$$

Therefore

$$m = \Sigma_{1\leq r\leq t}\mu_{i_r}(m_{i_r}) = \Sigma_{1\leq r\leq t}(\mu_{i_r}(m_{i_r}) - \mu_{i_0}\circ\varphi_{i_0}^{i_r}(m_{i_r}))$$

which is in S. So $\ker\varphi^* \subseteq 0+S$, such that φ^* is injective. Therefore it is an isomorphism. □

REMARK 2.10. Let $\mathcal{C}$ be a category, I a quasi-ordered set and $\{A_i, \varphi_j^i\}, \{B_i, \psi_j^i\}$ and $\{C_i, \theta_j^i\}$ be three direct systems in $\mathcal{C}$ over I. Suppose further that we are given two *morphisms* $\mathfrak{s}, \mathfrak{t}$ *of direct systems*: that is for each $i \in I$ we have a map $s_i : A_i \to B_i$ and $t_i : B_i \to C_i$ such that for all $i, j \in I$ and $i \leq j$

$$\psi_j^i \circ s_i = s_j \circ \varphi_j^i \quad \text{and} \quad \theta_j^i \circ t_i = t_j \circ \psi_j^i$$

Then these maps give rise to well defined A-homomorphisms $A_i \to \varinjlim B_i, B_i \to \varinjlim C_i$ and $A_i \to \varinjlim C_i$ via the combination of $\mathfrak{s}$ and $\mathfrak{t}$ with the natural maps φ^i, ψ^i and θ^i from A_i, B_i and C_i into $\varinjlim A_i, \varinjlim B_i$ and $\varinjlim C_i$ respectively.

By 2.6 there are then induced maps

$$\underset{\to}{\mathfrak{s}} : \varinjlim A_i \to \varinjlim B_i,\ \underset{\to}{\mathfrak{t}} : \varinjlim B_i \to \varinjlim C_i \text{ and } \overset{\mathfrak{t}\circ\mathfrak{s}}{\to}: \varinjlim A_i \to \varinjlim C_i$$

such that for each $i \in I$

$$\underset{\to}{\mathfrak{s}} \circ\varphi_i = \psi_i \circ s_i,\ \underset{\to}{\mathfrak{t}} \circ\psi_i = \theta_i \circ t_i \text{ and } s \underset{\to}{\circ} t \circ\varphi_i = \theta_i \circ t_i \circ s_i.$$

But from this and the uniqueness part of the definition of direct limits follows immediately, that $\underset{\to}{\mathfrak{t}} \circ \underset{\to}{\mathfrak{s}} = \mathfrak{t} \underset{\to}{\circ} \mathfrak{s}$. Obviously whenever $\mathfrak{s}$ is an identity map of direct systems (so $A_i = B_i$ and $s_i = \mathrm{id}_{A_i}$ for all $i \in I$), $\underset{\to}{\mathfrak{s}}$ is identity as well and it follows that $\varinjlim$ is a covariant functor from $\mathfrak{Dir}_{\mathcal{C}}(I)$ to $\mathcal{C}$.

The example 2.9 has been of a rather special nature which makes things very easy. The point is, that the index set was directed. That means the following: since the sum of two finitely generated modules is finitely generated too, for any two finitely generated submodules of M there is another finitely generated submodule of M containing both, a fact we used with delight. This explains the importance of

DEFINITION 2.11. A quasiordered set is called *directed* if for all pairs $i, j \in I$ there is a $k \in I$ with $i \leq k$ and $j \leq k$. A direct system over a directed set in a category $\mathcal{C}$ will be called *directed system in* $\mathcal{C}$.

The importance of directed index sets is layed down in the following lemma, which we will not prove because it does not involve new ideas. The interested reader be referred to [**11**], Theorem 2.17.

LEMMA 2.12. *Let I be a (quasiordered) directed set and $\{A_i, \varphi_j^i\}$ a directed system over I in the category $\mathcal{C}_A$. Denote the injection $A_i \to \bigoplus A_i$ by λ_i and let $\varinjlim A_i$ be represented by $\bigoplus A_i/S$, S being defined as in 2.8. Then (for $i \in I$)*

- *$\varinjlim A_i$ consists entirely of elements of the form $\lambda_i a_i + S$ where $a_i \in A_i$ and*
- *$\lambda_i a_i + S = 0 \Leftrightarrow \exists j \in I : i \leq j, \varphi_j^i a_i = 0.$* □

We have another example for directed systems:

EXAMPLE 2.13. Let M be an A-module. Let $\{M_i\}$ be a family of submodules over the set I satisfying

$$M = \sum_{i \in I} M_i.$$

Let $\{N_\omega\}_{\omega \in \Omega}$ be the family of all possible finite sums of M_i's. Define $\varphi_{\omega'}^{\omega} : N_\omega \to N_{\omega'}$ to be the inclusion map if $\omega \subseteq \omega'$. (Then Ω is the set of finite elements of the powerset of I). Then $M \cong \varinjlim(N_\omega)$.

PROOF. We proceed as in 2.9 and observe first, that the index set is directed since finite unions of finite subsets give finite subsets. It will suffice to show, that M is isomorphic to $\bigoplus(N_\omega)/S$ where S is the submodule of $\bigoplus(N_\omega) =: N$ generated by all elements like $\nu_{\omega_1}(n_{\omega_1}) - \nu_{\omega_2} \circ \varphi_{\omega_2}^{\omega_1}(n_{\omega_1})$ whenever $\omega_1 \leq \omega_2$ (so that $N_{\omega_1} \subseteq N_{\omega_2}$) and $\nu_\omega : N_\omega \to \bigoplus N_\omega$ for all $\omega \in \Omega$ is the natural embedding.

As before we define $\varphi : N \to M$ by adding up all nonzero components. Again this map is surjective, linear and the kernel includes S. On the other hand, if some element $n = (0, \dots, n_{\omega_1}, \dots, n_{\omega_r}, 0, \dots) \in N$ belongs to the kernel of φ, then there is an ω_0 such that

$$N_{\omega_1} + \ldots + N_{\omega_r} = N_{\omega_0}$$

and then

$$0 = \varphi(n) = \Sigma_1^r n_{\omega_i} = \Sigma_1^r \varphi_{\omega_0}^{\omega_i}(n_{\omega_i})$$

so that

$$n + S = 0 + \sum_1^r (\nu_{\omega_i}(n_{\omega_i}) - \nu_{\omega_0} \circ \varphi_{\omega_0}^{\omega_i}(n_{\omega_i})) + S$$

and hence each kernel element of φ^* is in S. Hence $M \cong N/S \cong \varinjlim(N_\omega)$. □

REMARK 2.14. We want to point out to the reader, that one can instead of the system $\{N_\omega\}$ take an arbitrary system of submodules of M subject to the condition, that the index set is directed, the inner morphisms are inclusions and the union of all submodules involved covers M.

One of the features of directed systems shows

PROPOSITION 2.15. *Let I be a directed set. Then $\varinjlim$ (exists for all directed systems $\{A_i, \varphi_j^i\}$ over I in $\mathcal{C}_A$ and) carries exact sequences of objects in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$ into exact sequences in $\mathcal{C}_A$. (For morphisms of direct systems see 2.10).*

PROOF. By [**9**], Theorem 3.3 it is enough to show that $\varinjlim$ takes short exact sequences in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$ to short exact sequences in $\mathcal{C}_A$. So suppose we are given three directed systems over I whose entries are objects and morphisms in $\mathcal{C}_A$, say $\mathfrak{A} = \{A_i, \varphi_j^i\}, \mathfrak{B} = \{B_i, \psi_j^i\}, \mathcal{C} = \{C_i, \theta_j^i\}$. Let further $\mathfrak{s} : \mathfrak{Dir}_{\mathcal{A}} \to \mathfrak{Dir}_{\mathcal{B}}$ and $\mathfrak{t} : \mathfrak{Dir}_{\mathcal{B}} \to \mathfrak{Dir}_{\mathcal{C}}$ be morphisms in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$ such that for each $i \in I$ the sequence

$$0 \to A_i \stackrel{s_i}{\to} B_i \stackrel{t_i}{\to} C_i \to 0$$

is exact. Let further denote λ_i, μ_i and ν_i be the injections $A_i \to \bigoplus A_i, B_i \to \bigoplus B_i$ and $C_i \to \bigoplus C_i$ and imagine $\varinjlim A_i$ as $\bigoplus A_i/S, \varinjlim B_i$ as $\bigoplus B_i/T$ and $\varinjlim C_i$ as $\bigoplus C_i/U$ where S is as in 2.8 and T, U are the analogues to S.

Then as in 2.10 there are maps $\underset{\to}{\mathfrak{s}} : \varinjlim A_i \to \varinjlim B_i$ and $\underset{\to}{\mathfrak{t}} : \varinjlim B_i \to \varinjlim C_i$ such that every possible triangle commutes. Let now $x \in \varinjlim A_i$ be in $\ker(\underset{\to}{\mathfrak{s}})$. By 2.12, x is image under some λ_i followed by the factorisation $\bigoplus A_i \to \bigoplus A_i/S$, say $x = \lambda_i a_i + S$ for some $i \in I$ and $a_i \in A_i$. Then

$$\underset{\to}{\mathfrak{s}}(x) = \mu_i \circ s_i(a_i) + T \text{ by definition of } \underset{\to}{\mathfrak{s}}.$$

Since $\underset{\to}{\mathfrak{s}}(x) = 0$, $\mu_i \circ s_i(a_i) + T = 0$ what by 2.12 again implies $\exists j \in I$: $\psi_j^i \circ s_i(a_i) = 0$ in A_j. Now $\mathfrak{s}$ is morphism, hence $\psi_j^i \circ s_i = s_j \circ \varphi_j^i$ and therefore $s_j \circ \varphi_j^i(a_i) = 0$. By hypothesis s_j is monomorphism and so $\varphi_j^i(a_i) = 0$. But then $x = \lambda_i(a_i) + S = \lambda_j \circ \varphi_j^i(a_i) + S = 0 + S$ whence $\underset{\to}{\mathfrak{s}}$ is shown to be injective.

The remaining parts (exactness in $\varinjlim B_i$ and $\varinjlim C_i$) may be done even without the assumption of I being directed (see [**11**],remarks after Th. 2.18) and we will do only the first assertion, the latter being entirely similar.

Since $\varinjlim$ is a functor, it is immediate that $\ker(\underset{\to}{\mathfrak{t}}) \supseteq \operatorname{im}(\underset{\to}{\mathfrak{s}})$. So let $y = \mu_i(b_i) + T$ be in $\varinjlim B_i$ such that $\underset{\to}{\mathfrak{t}}(y) = 0$. As before $\underset{\to}{\mathfrak{t}}(y) = \theta_i \circ t_i(b_i) + U = 0$ whence by 2.12 $\exists j \in I : \theta_j^i \circ t_i(b_i) = 0$. Again $\mathfrak{t}$ is morphism and it follows $t_j \circ \psi_j^i(b_i) = 0$. So $\psi_j^i(b_i)$ is in $\ker(t_j) = \operatorname{im}(s_j)$. So $\exists a_j : s_j(a_j) = \psi_j^i(b_i)$. But then

$$y = \mu_i(b_i) + T = \mu_j \circ \psi_j^i(b_i) + T = \mu_j \circ s_j(a_j) + T = \underset{\to}{\mathfrak{s}}(\lambda_j(a_j + S))$$

because $\underset{\to}{\mathfrak{s}}$ is limit morphism, and we have shown that $y \in \operatorname{im}(\underset{\to}{\mathfrak{s}})$. As said before, exactness in $\varinjlim C_i$ is left to the reader. □

Before we can apply the theory developed so far to local cohomology, we have to investigate the existence of direct limits in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$. When we do this, we necessarily have to speak about direct systems in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$.

That is, about direct systems of direct systems of modules. To be able to do this, we have to blow up our notation as follows: Given a direct system $\mathfrak{Dir}_{\mathcal{A}} = \{A_i, \varphi_j^i\}$ in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$ we will from now on denote the A-homomorphism $\varphi_j^i : A_i \to A_j$ by $\varphi_{i,j}$ so that we now refer to $\mathfrak{Dir}_{\mathcal{A}}$ as to $\{A_i, \varphi_{i,i'}\}$ (equivalent changes apply to other direct systems in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$ like $\mathfrak{Dir}_{\mathcal{B}}$ or $\mathfrak{Dir}_{\mathcal{C}}$). It will now happen that we have in the same time to deal with two direct systems, say $\mathfrak{Dir}^j_{\mathcal{A}}$ and $\mathfrak{Dir}^{j'}_{\mathcal{A}}$ which are linked by a morphism $\varphi^{j,j'}$. Then there is for each $i \in I$ an A-homomorphism $\varphi_i^{j,j'} : A_{ji} \to A_{ji}$. (Here of course A_{ji} is the module of index i in the direct system $\mathfrak{Dir}^j_{\mathcal{A}}$ etc.)

To say now that $\varphi^{j,j'}$ is a morphism from the direct system $\{\mathfrak{Dir}^j_{\mathcal{A}}, \varphi^j_{i,i'}\}$ to the direct system $\{\mathfrak{Dir}^{j'}_{\mathcal{A}}, \varphi^{j'}_{i,i'}\}$ is therefore to say that for all $i \leq i' \in I$ we have

$$\varphi_i^{j,j'} : A_{ji} \to A_{j'i}, \varphi^j_{i,i'} : A_{ji} \to A_{ji'}, \varphi^{j'}_{i,i'} : A_{j'i} \to A_{j'i'}$$

such that

$$\varphi^{j'}_{i,i'} \circ \varphi_i^{j,j'} = \varphi_{i'}^{j,j'} \circ \varphi^j_{i,i'}$$

by definition of morphisms in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$.

Up to now we have dealt only with two different direct systems, but we take now remedial measures against this. For let J be a second quasi-ordered set. We continue our notation such that for each $j \in J$ $\mathfrak{A}^j$ denotes a direct system of A-modules over one and the same quasi-ordered set I, such that as before A_{ji} is the A-module belonging to the index $i \in I$ in the direct system $\mathfrak{A}^j$ belonging to the index $j \in J$. As in the preceding paragraph, $\varphi^{j,j'}$ will denote the morphism $\mathfrak{A}^j \to \mathfrak{A}^{j'}$ which is a morphism in $\mathfrak{Dir}_{\mathfrak{Dir}_{\mathcal{C}_A}(I)}(J)$. Suppose that for each $i \in I$, $\{A_{ji}, \varphi_i^{j,j'}\}$ is a direct system over J. To say then that $\{\mathfrak{A}^j, \varphi^{j,j'}\}$ is a direct system over J in $\mathfrak{Dir}_{\mathfrak{Dir}_{\mathcal{C}_A}(I)}(J)$ is precisely to say that for all $i \leq i' \in I$ and for all $j \leq j' \in J$

$$\varphi_{i'}^{j,j'} \circ \varphi^j_{i,i'} = \varphi^{j'}_{i,i'} \circ \varphi_i^{j,j'}.$$

It is therefore clear that the family $\{A_{ji}\}$ of A-modules together with the families $\{\varphi^j_{i,i'} : A_{ji} \to A_{ji}\}$ and $\{\varphi_i^{j,j'} : A_{ji} \to A_{ji}\}$ can be interpreted as a direct system of objects of $\mathfrak{Dir}_{\mathcal{C}_A}(I)$ over J if and only if it can be interpreted as a direct system of objects of $\mathfrak{Dir}_{\mathcal{C}_A}(J)$ over I.

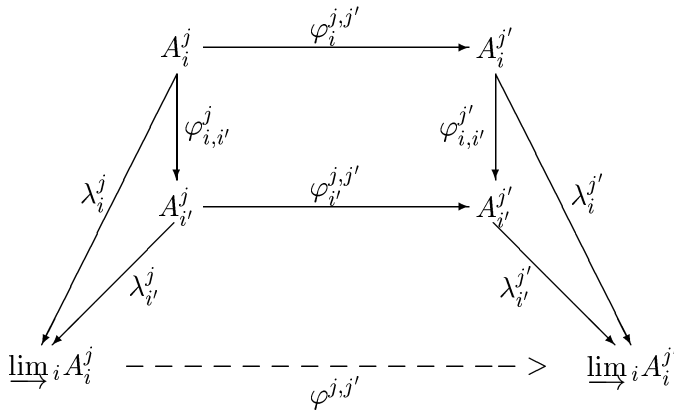
We investigate now the question of direct limits of direct systems of direct systems of A-modules and A-homomorphisms.

In conjunction with the comments above we will denote an object in $\mathfrak{Dir}_{\mathfrak{Dir}_{\mathcal{C}_A}(I)}(J)$ (which is then also an object in $\mathfrak{Dir}_{\mathfrak{Dir}_{\mathcal{C}_A}(J)}(I)$) by $\{A_{ji}, \varphi_i^{j,j'}, \varphi^j_{i,i'}\}$.

Now suppose, that $\mathfrak{A}$ is such an object and assume, that I and J are directed. We will try to find $\varinjlim_i(A_{ji})$, meaning that we try to produce a direct limit over the directed set I with objects in $\mathfrak{Dir}_{\mathcal{C}_A}(J)$.

Fix $j \in J$. Then the A-modules A_{ji}(for varying $i \in I$) make up a direct system over I together with the maps $\varphi^j_{i,i'}$ with varying i, i'. By 2.8

this direct system posesses a direct limit $\varinjlim_i(A_{ji})$ which is representable by $\bigoplus_{i\in I} A_{ji}/S$ where S is the submodule of $\bigoplus_{i\in I} A_{ji}$ generated by all elements of the form $\lambda_i^j(a_i^j) - \lambda_{i'}^j \circ \varphi_{i,i'}^j(a_i^j)$. (Here λ_i^j is the natural injection $A_{ji} \to \bigoplus_{i\in I} A_{ji}$). Then for all $j \le j' \in J$ there is a commutative diagram



which by the fact that $\varinjlim_i A_{ji}$ is a direct limit yields a map $\varphi^{j,j'} : \varinjlim_i(A_{ji}) \to \varinjlim_i(A_{ji})$ (in the picture dashed) such that for all $i \in I$

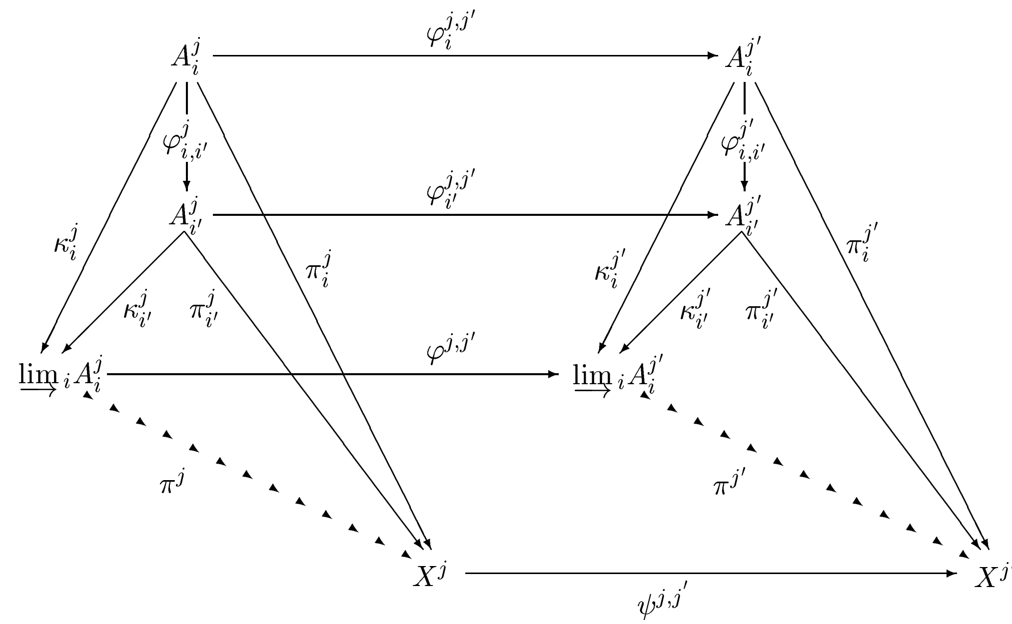
$$\lambda_i^{j'} \circ \varphi_i^{j,j'} = \varphi^{j,j'} \circ \lambda_i^j.$$

From this immediately follows that for all $j \le j' \le j" \in J$, $\varphi^{j,j"} = \varphi^{j',j"} \circ \varphi^{j,j'}$. $\{\varinjlim_i A_{ji}, \varphi^{j,j'}\}$ is then up to isomorphisms in $\mathfrak{Dir}_{\mathcal{C}_A}(J)$ obviously the only possible solution for our direct limit problem.

To show that this construction actually yields a direct limit, we have to show that, given $\{X^j, \psi^{j,j'}\} \in \mathfrak{Dir}_{\mathcal{C}_A}(J)$ together with a family $\{\pi_i^j\}_{i\in I}$ such that

- $\pi_i^j : A_{ji} \to X^j$,
- for all $j \in J$ and all $i, i' \in I$ we have $\pi_i^j = \pi_{i'}^j \circ \varphi_{i,i'}^j$ and
- for all $j \le j' \in J$ and all $i \in I$ is true that $\pi_i^{j'} \circ \varphi_i^{j,j'} = \psi^{j,j'} \circ \pi_i^j$

(which is nothing but the description of a map from a direct system to an object in $\mathfrak{Dir}_{\mathcal{C}_A}(J)$), there exists a unique $\mathfrak{Dir}_{\mathcal{C}_A}(J)$-morphism π from $\{\varinjlim_i A_{ji}, \varphi^{j,j'}\}$ to $\{X^j, \psi^{j,j'}\}$ (such that for each $j \in J$ there is an A-homomorphism π_j satisfying $\psi^{j,j'} \circ \pi_j = \pi_{j'} \circ \varphi^{j,j'}$). The situation is demonstrated in the picture below, in which all straight line polygones commute and we are looking for the dashed maps making everything commute and being unique.



Now the construction of *some* unique maps makes no problem since each of the $\varinjlim_i A_{ji}$ is a direct limit and therefore there exist maps π_j as indicated in the picture making both "sidefaces" of all possible "roofs" commute and we have to care about the bottom in the sequel. At this stage the property of I being directed comes to play an important role. Because of this, for each $j \in J$, $\varinjlim_i A_{ji}$ is not only a direct sum of A-modules factorized by a relation module (see 2.8) but also each element in each of these direct limits is image under some κ_i^j (which is defined by λ_i^j followed by the appropriate factorization: see 2.12).

So fix $j \leq j' \in J$ and take $\alpha \in \varinjlim_i (A_{ji})$. As just outlined, there is an $i \in I$ and a $\beta_i^j \in A_{ji}$ such that $\alpha = \lambda_i^j(\beta_i^j) + S_j$.

By the hypothesis that $\{\pi_i^j\}$ is a map from a direct system to an object (in $\mathfrak{Dir}_{\mathcal{C}_A}(J)$) we have

$$\psi^{j,j'} \circ \pi_i^j(\beta_i^j) = \pi_i^{j'} \circ \varphi_i^{j,j'}(\beta_i^j)$$

and because the side triangles commute,

$$\pi_i^{j'} \circ \varphi_i^{j,j'}(\beta_i^j) = \pi_{j'} \circ \kappa_i^{j'} \circ \varphi_i^{j,j'}(\beta_i^j)$$

and

$$\psi^{j,j'} \circ \pi_j \circ \kappa_i^j(\beta_i^j) = \psi^{j,j'} \circ \pi_i^j(\beta_i^j)$$

while the commutativity of the backside yields

$$\pi_{j'} \circ \kappa_i^{j'} \circ \varphi_i^{j,j'}(\beta_i^j) = \pi_{j'} \circ \varphi^{j,j'} \circ \kappa_i^j(\beta_i^j)$$

so that

$$\begin{aligned}\pi_{j'} \circ \varphi^{j,j'}(\alpha) &= \pi_{j'} \circ \varphi^{j,j'} \circ \kappa_i^j(\beta_i^j)\\ &= \pi_{j'} \circ \kappa_i^{j'} \circ \varphi_i^{j,j'}(\beta_i^j)\\ &= \pi_i^{j'} \circ \varphi_i^{j,j'}(\beta_i^j)\\ &= \psi^{j,j'} \circ \pi_i^j(\beta_i^j)\\ &= \psi^{j,j'} \circ \pi_j \circ \kappa_i^j(\beta_i^j)\\ &= \psi^{j,j'} \circ \pi_j(\alpha)\end{aligned}$$

whence the bottom commutes for all $j, j' \in J$ and $i, i' \in I$. As said before, the $\varinjlim_i A_{ji}$ are unique up to isomorphism and the π_j are unique for chosen A_{ji} and hence

$$\{\varinjlim_i A_{ji}, \varphi^{j,j'}\}_{i \in I, j \in J}$$

is a direct limit in $\mathfrak{Dir}_{\mathcal{C}_A}(J)$ for the direct system $\{\mathfrak{A}^j, \varphi_i^{j,j'}, \varphi_{i,i'}^j\}$. We have proved

Proposition 2.16. *Let I, J be two quasiordered sets and I be directed and let $\{\mathfrak{A}^j, \varphi^{j,j'}\}$ be a direct system over J in $\mathfrak{Dir}_{\mathcal{C}_A}(I)$ as explained above. Then $\varinjlim_j \mathfrak{A}^j$ exists.* □

Definition 2.17. Let I, J be directed sets and suppose we are given for each $i \in I, j \in J$ an A-module A_{ji} and for all $i \leq i' \in I$ and $j \leq j' \in J$ A-homomorphisms $\varphi_{i,i'}^j : A_{ji} \to A_{ji'}$ and $\varphi_i^{j,j'} : A_{ji} \to A_{j'i}$ satisfying $\varphi_{i,i'}^{j'} \circ \varphi_i^{j,j'} = \varphi_{i'}^{j,j'} \circ \varphi_{i,i'}^j$. Suppose further, that for each $i \in I \{A_{ji}, \varphi_i^{j,j'}\}$ is a directed system over J in $\mathcal{C}_A$ and that for each $j \in J, \{A_{ji}, \varphi_{i,i'}^j\}$ is a direct system over I in $\mathcal{C}_A$. Then this collection of data $\mathfrak{A} = \{A_{ji}, \varphi_{i,i'}^j, \varphi_i^{j,j'}\}$ may be interpreted as an object of both $\mathfrak{Dir}_{\mathfrak{Dir}_{\mathcal{C}_A}(J)}(I)$ and $\mathfrak{Dir}_{\mathfrak{Dir}_{\mathcal{C}_A}(I)}(J)$. Then we will say that $\mathfrak{A}$ is a *directed bisystem* over I and J in $\mathcal{C}_A$.

Corollary 2.18. *Let I and J be two directed sets. Let $\mathfrak{A}$ be a directed bisystem over I and J in $\mathcal{C}_A$, then both $\varinjlim_i A_{ji}$ and $\varinjlim j A_{ji}$ exist.*

Proof. The interpretation has been outlined in the text before 2.16, the existence of $\varinjlim_i A_{ji}$ is proposition 2.18 and the existence of $\varinjlim j A_{ji}$ follows from this by exchange of the sets I and J. □

As in 2.10 one may verify that $\varinjlim_i$ and $\varinjlim_j$ are covariant functors.

Being able now from a directed bisystem $\{A_{ji}, \varphi_{i,i'}^j, \varphi_i^{j,j'}\}$ over the directed sets I and J respectively to produce two direct systems $\{\varinjlim_i A_{ji}\}$ and $\{\varinjlim_j A_{ji}\}$ to which one can apply the functors $\varinjlim_j$ and $\varinjlim_i$ respectively, naturally the question arises

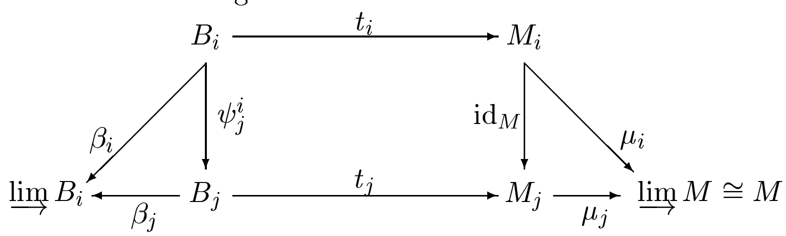
$$\text{Is } \varinjlim_i(\varinjlim_j(A_{ji})) = \varinjlim_j(\varinjlim_i(A_{ji})) \text{ ?}$$

We will deal now with this question.

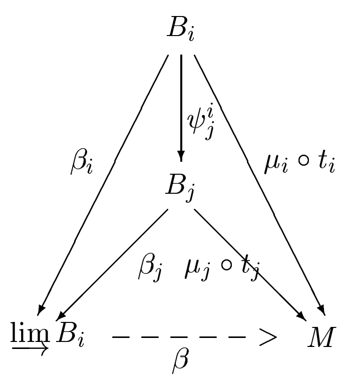
For this we introduce a functor defined on arbitrary categories - a functor opposite to the $\varinjlim$-functor and very related to 2.3. So let $\mathcal{C}$ be a category and $c \in Ob(\mathcal{C})$. If then I is a quasiordered set one can produce a direct system over I in $\mathfrak{Dir}_{\mathcal{C}}(I)$ as follows: For all $i \in I$ define c_i to be equal to c and for all $i \leq j \in I$ let φ^i_j be the identity map on c. Then $\{c_i, \varphi^i_j\}$ is clearly a direct system in $\mathcal{C}$ over I. Again it is easy to see that this is a covariant functor and we will denote it by $|(.)|(I)$ such that in our case $|c|(I) = \{c_i, \varphi^i_j\}$. We now specify $\mathcal{C}$ to be the category of A-modules and A-homomorphisms $\mathcal{C}_A$.

Let $\{B_i, \psi^i_j\}$ be a direct system over the directed set I. Let further M be an A-module. Then we can form $\mathrm{Hom}_A(\varinjlim B_i, M)$ as well as $\mathrm{Hom}_{\mathfrak{Dir}_{\mathcal{C}_A}(I)}(\{B_i, \varphi^i_j\}, |M|(I))$. There are extraordinary relationships between these two Hom's:

Let $\mathfrak{t}$ be in $\mathrm{Hom}_{\mathfrak{Dir}_{\mathcal{C}_A}(I)}(\{B_i, \psi^i_j\}, |M|(I))$ such that for each $i \leq j \in I$ we have a commutative diagram



which may be interpreted as



which gives a unique map β making the diagram commute.

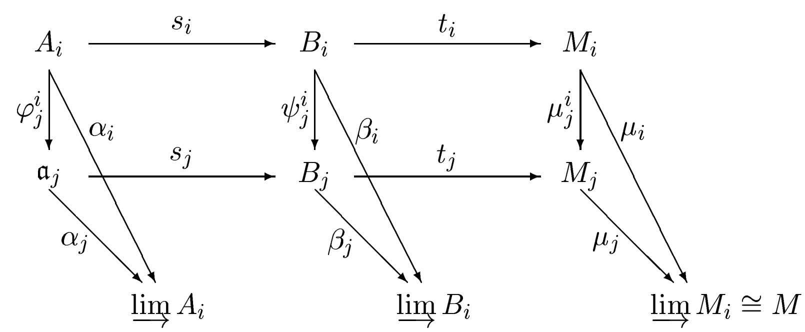
So we have established a map $\tau : \mathrm{Hom}_{\mathfrak{Dir}_{\mathcal{C}_A}(I)}(\{B_i, \psi^i_j\}, |M|(I)) \rightarrow \mathrm{Hom}_A(\varinjlim B_i, M)$. This map is surjective, since a given homomorphism $\beta : \varinjlim B_i \rightarrow M$ can be used to produce $\mathfrak{t} : \{B_i, \psi^i_j\} \rightarrow |M|(I)$ by defining $t_i : B_i \rightarrow M_i = M$ as to be $\beta \circ \beta_i$ what of course is a morphism of directed systems with $\underset{\rightarrow}{\mathfrak{t}} = \beta$.

LEMMA 2.19. *τ is injective.*

PROOF. We have just seen, that from any homomorphism $\beta : \varinjlim B_i \rightarrow M$ one can get a corresponding morphism of direct systems $\mathfrak{t} : \{B_i, \psi^i_j\} \rightarrow$

$|M|(I)$ by defining $t_i = \beta \circ \beta_i$. So we have to show that any morphism of direct systems $t : \{B_i, \psi^i_j\} \to |M|(I)$ can be produced in this way. Now in the first of the two pictures above by properties of direct limits (with an imagined $\beta : \varinjlim B_i \to \varinjlim(M_i) \cong M$), $\mu_i \circ t_i = \beta \circ \beta_i$. Also the reader may easily verify that each μ_i is exactly the inverse to the isomorphism φ^* established in 2.9. So whenever $t_i \neq \varphi^* \circ \beta \circ \beta_i$, then $\mu_i \circ t_i \neq \mu_i \circ \varphi * \circ \beta \circ \beta_i = \beta \circ \beta_i$ which is a contradiction. So always to a homomorphism of the limits belongs a unique morphism of the direct systems and τ is injective. □

Now we investigate the behaviour of τ under variation of $\{B_i, \psi^i_j\}$ and M respectively. So suppose first we have a second direct system in $\mathcal{C}_A$, $\{A_i, \varphi^i_j\}$ together with a morphism of direct systems $\mathfrak{s} : \{A_i, \varphi^i_j\} \to \{B_i, \psi^i_j\}$. Then we might calculate $\tau(\mathfrak{t} \circ \mathfrak{s})$ and $\tau(\mathfrak{t}) \circ \tau(\mathfrak{s})$. (The situation is pictured below.)



Since $\varinjlim$ is a functor, the two homomorphisms coincide.

Secondly we might vary the module M (and therefore $|M|(I)$) by means of a homomorphism $f : M \to M'$, and look what happens now but the corresponding picture is exactly the one above only replacing (A, B, M, t, s) by $(B, M, M', s, \tau^{-1}(f))$ wherever they occur. Again by the fact that $\varinjlim$ is functor, $\tau(\tau^{-1}(f) \circ s)$ is welldefined. We conclude, that τ is bijective and natural in both variables. We have now set the stage for

THEOREM 2.20. *Let $\{A_{ji}, \varphi^j_{i,i'}, \varphi^{j,j'}_i\}_{i\in I, j\in J}$ be a directed bisystem over the directed sets I and J. Then $\varinjlim_i(\varinjlim_j(A_{ji})) \cong \varinjlim_j(\varinjlim_i(A_{ji}))$.*

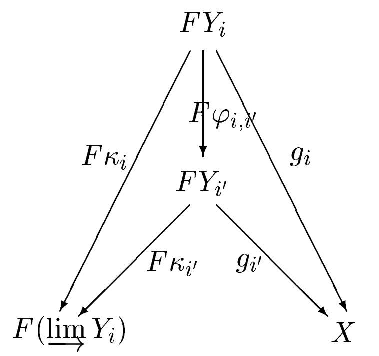
PROOF. In the preceding paragraphs we have explained, that $|(.)|(I)$ is for every directed system I a functor working on any category $\mathcal{C}$, so for example we can apply it to $\mathcal{C}_A$. Further we have shown that $\varinjlim$ is a functor working at least on the categories $\mathfrak{Dir}_{\mathcal{C}_A}(I), \mathfrak{Dir}_{\mathcal{C}_A}(J)$ and $\mathfrak{Dir}_{\mathfrak{Dir}_{\mathcal{C}_A}(J)}(I) =: \mathcal{D}$. We will for the remainder of this proof denote $\varinjlim_j$ by F and $|.|(J)$ by G. Then $F : \mathfrak{Dir}_{\mathcal{C}_A}(J) \to \mathcal{C}_A$ and $G : \mathcal{C}_A \to \mathfrak{Dir}_{\mathcal{C}_A}(J)$.

Further we have a natural bijection (with $\mathcal{C} := \mathfrak{Dir}_{\mathcal{C}_A}(J)$)

$$\tau : \mathrm{Hom}_A(F(Y_i), X)) \to \mathrm{Hom}_{\mathcal{C}}(Y_i, G(X))$$

for all $X \in Ob(\mathcal{C}_A)$ and $Y_i \in Ob(\mathcal{C})$.

Our aim is now to show for an element Y in $Ob(\mathcal{D})$, that $F(\varinjlim_i(Y))$ and $\varinjlim_i(F(Y))$ are isomorphic. So let Y be in $Ob(\mathcal{D})$, so that it is a direct system over I with entries Y_i and inner morphisms $\varphi_{i,i'}$ in $\mathfrak{Dir}_{\mathcal{C}_A}(J)$. Let further X be an object of $\mathcal{C}_A$ and let be given a family of $\mathcal{C}_A$-morphisms $\{g_i : i \in I\}$ with $g_i : FY_i \to X$ such that

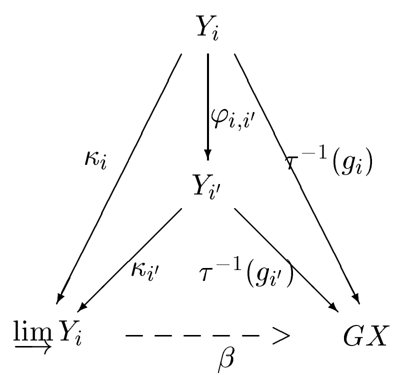


where the κ_i are the natural maps $Y_i \to \varinjlim_i(Y_i)$, commutes for all $i, i' \in I$. If we can show for each such X and each such $\{g_i : i \in I\}$ that there exists a unique morphism $F(\varinjlim_i Y_i) \to X$ then by the uniqueness of direct limits we have shown that

$$F(\varinjlim_i Y_i) \cong \varinjlim_i(F(Y_i))$$

because both constructions solve the same universal mapping problem.

Consider now the commutative diagram for all $i \leq j \in I$:



in which the left part trivially commutes and the the right part commutes by the naturality of τ.

By definition of direct limits there is $\beta \in \mathrm{Hom}_{\mathcal{C}}(\varinjlim_i Y_i, G(X))$ making all these diagrams commute. Now define in the original picture $\gamma : F(\varinjlim_i Y_i) \to X$ as to be $\tau(\beta)$.

Since τ is natural, so is τ^{-1} and therefore γ makes all diagrams in question commute. Also, γ is unique since the existence of of two such morphisms by the bijectivity of τ would contradict the uniqueness of β.

We have shown that $F(\varinjlim_i Y_i)$ and $\varinjlim_i(F(Y_i))$ are solutions for the same universal mapping problem and hence are isomorphic. $\square$

REMARK 2.21. As the reader probably has noticed, the proof of 2.20 is based in a singular manner on τ. Without it, there were no such proof possible. The existence of such a connecting τ for the functors F and G is usually expressed by saying, that *F and G are a dual or adjoint pair.* The above established fact that the left brother of an adjoint pair always commutes with direct limits is not the only property, these couples enjoy. For example, F is always right exact and G always left exact. The interested reader is referred to the subsections on Limits and Watts theorems in [**11**].

In the hope that not all readers are run away frustrated by the amount of homological algebra we now will show the way in which local cohomology is related to direct limits. To this end we return to the notation of direct systems, we introduced first.

DEFINITION 2.22. Let M be an A-module, I be a directed set and $\mathcal{A} = \{\mathfrak{a}_i, \varphi^i_j\}$ an inverse directed system (see 2.2) of ideals of A over I (so that $i \geq j \in I \Rightarrow \mathfrak{a}_i \subseteq \mathfrak{a}_j$), then $\Gamma'_{\mathcal{A}}(M)$ is defined to be $\bigcup(0 :_M \mathfrak{A}\mathfrak{a}_i)$.

REMARK 2.23. It follows from 2.14 that $\Gamma'_{\mathcal{A}}(M)$ is isomorphic to $\varinjlim(0 :_M \mathfrak{a}_i)$.

LEMMA 2.24. *Let I be a directed set and $\mathcal{A} = \{\mathfrak{a}_i, \varphi^i_j\}$ be an inverse directed system of ideals over I. Then there is a natural isomorphism from $\varinjlim \mathrm{Hom}_A(A/\mathfrak{a}_i, M)$ to $\Gamma'_{\mathcal{A}}(M)$.*

PROOF. For $i \in I$ let φ_i denote the A-homomorphism $\mathrm{Hom}_A(A/\mathfrak{a}_i, M) \to M$ that is defined by $f \to f(1 + \mathfrak{a}_i)$. Then from $a \cdot f(1 + \mathfrak{a}_i) = f(a + \mathfrak{a}_i) = 0$ for $a \in \mathfrak{a}_i$ it follows that φ_i maps actually into $(0 :_M \mathfrak{a}_i)$. Further, if $m \in (0 :_M \mathfrak{a}_i)$, one can define a map $f_m : A/\mathfrak{a}_i \to M$ by $f_m(1 + \mathfrak{a}_i) = m$. It follows that φ_i is an epimorphism.

If $\varphi_i(f)$ happens to be zero, then necessarily $f(1 + \mathfrak{a}_i) = 0$ and so $f(A/\mathfrak{a}_i) = 0$ and hence $f = 0$. We conclude that φ_i is an isomorphism from $\mathrm{Hom}_A(A/\mathfrak{a}_i, M)$ to $(0 :_M \mathfrak{a}_i)$.

All these isomorphisms are natural in M since given $\mu : M \to M'$, the diagram

$$
\begin{array}{ccccccc}
\mathrm{Hom}(A/\mathfrak{a}_i, M) & \longrightarrow & \mathrm{Hom}(A/\mathfrak{a}_i, M') & & f & \longrightarrow & \mu \circ f \\
\downarrow & & \downarrow & \text{commutes because of} & \downarrow & & \downarrow \\
(0 :_M \mathfrak{a}_i) & \longrightarrow & (0 :_{M'} \mathfrak{a}_i) & & f(1+\mathfrak{a}_i) & \longrightarrow & \mu(f(1+\mathfrak{a}_i)) = (\mu \circ f)(1+\mathfrak{a}_i)
\end{array}
$$

Let for $i \geq j, \psi^i_j : A/\mathfrak{a}_i \to A/\mathfrak{a}_j$ denote the natural projection. Then $\mathfrak{A}' := \{A/\mathfrak{a}_i, \psi^i_j\}$ is an inverse system over I, hence a contravariant functor from I to $\mathcal{C}_A$, $\mathrm{Hom}_A(\cdot, M) \circ \mathfrak{A}'$ is a covariant functor from I to $\mathcal{C}_A$. So by application of the functor $\varinjlim$ we get a natural isomorphism φ between

$$\varinjlim \mathrm{Hom}_A(A/\mathfrak{a}_i, M) \text{ and } \varinjlim(0 :_M \mathfrak{a}_i).$$

$\square$

To have any use of this lemma we should show that $\Gamma_{\mathfrak{a}}$ and $\Gamma'_{\mathcal{A}}$ are by some means related things. This we do.

PROPOSITION 2.25. *Let* $\mathfrak{a}$ *be an ideal of* A *and consider*

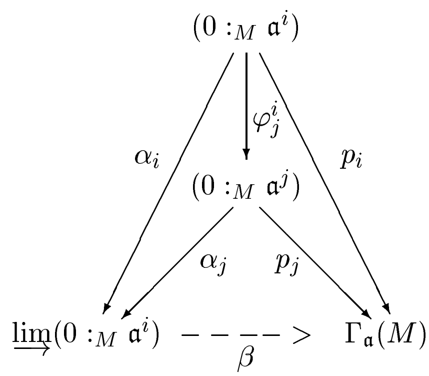
$$\mathcal{A} = \{\mathfrak{a}_i, \varphi^i_j\} = \{\mathfrak{a}^i, \hookrightarrow\}_{i \in \mathbb{N}}$$

as inverse directed system over $I = \mathbb{N}$.

Then $\Gamma_{\mathfrak{a}}(\cdot)$ *and* $\Gamma'_{\mathcal{A}}(\cdot)$ *are naturally equivalent.*

PROOF. We observe first that the indexing set is directed.

Let M be an A-module. For each $i \in I$ we have $(0 :_M \mathfrak{a}^i) \subseteq \Gamma_{\mathfrak{a}}(M)$ such that we can think for $j \geq i$ of a commutative diagram



where all p_i and φ^i_j are inclusion maps and the α_i are the natural ones. By 2.6 there is a unique β making all these pictures commute. Now by 2.12 every element in $\varinjlim(0 :_M \mathfrak{a}^i)$ is image of some element in one module in the "middle axis". So especially are all elements in the kernel of β. But then $\ker \beta$ must be equal to zero since the corresponding diagram commutes and each p_i is injective.

On the other hand, whenever $m \in \Gamma_{\mathfrak{a}}(M)$, $\exists i \in I : \mathfrak{a}^i \cdot m = 0$. So $m \in (0 :_M \mathfrak{a}^i)$ and therefore M is image under some p_i and hence under β too. So β is an isomorphism.

If $f : M \to M'$ is a homomorphism, then there is a map $\overline{f}$ of direct systems

$$\overline{f} : \{0 :_M \mathfrak{a}^i, \varphi^i_j\} \to \{(0 :_{M'} \mathfrak{a}^i), \varphi^{j'}_i\}$$

induced by f and defined by restriction. ($\varphi^{j'}_i : (0 :_{M'} \mathfrak{a}^j) \to (0 :_{M'} \mathfrak{a}^i)$ for $j \leq i \in I$). Then $\alpha \in \varinjlim(0 :_M \mathfrak{a}^i)$ is image of (say) $m \in (0 :_M \mathfrak{a}^i)$ under α_i

and so

$$\begin{aligned}\Gamma_{\mathfrak{a}}(f)\circ\beta\circ\alpha_i(m) &= \Gamma_{\mathfrak{a}}(f)\circ p_i(m)\\ &= p'_i\circ\overline{f}_i(m)\\ &= \beta'\circ\alpha'_i\circ\overline{f}_i(m)\\ &= \beta'\circ\varinjlim(\overline{f})\circ\alpha_i(m)\end{aligned}$$

because of (in this order): the definition of direct limits, $\Gamma_{\mathfrak{a}}(f)$ is restriction, the definition of direct limits. And of course $\varinjlim(\overline{f}) = f$. (We have expanded the notation by dashes to the corresponding things of M'.) It follows, that the isomorphism is natural. □

We therefore can unambiguously take Γ instead of Γ' whenever both functors are defined and will in the sequel denote both by Γ.

COROLLARY 2.26. *Γ is always left exact.*

PROOF. This follows immediately from 2.25, 2.15 and the fact that Hom is left exact. □

We are now going to produce a statement similar to 2.25 involving the $H^i_{\mathfrak{a}}$.

Suppose, T is an A-linear functor $\mathcal{C}_A \times \mathcal{C}_A \to \mathcal{C}_A$ which is contravariant in the first variable and covariant in the second. Let further I be a directed set. Also let $\{\mathfrak{a}_i\}$ be an inverse system of ideals over I such that for all $j \le i \in I$ the natural morphism nat^i_j is the projection $A/\mathfrak{a}_i \to A/\mathfrak{a}_j$. Then $\{A/\mathfrak{a}_i, nat^i_j\}$ are an inverse system of A-modules over I . We then may apply the functor $T(\cdot, M)$ with M an A-module to this inverse system to get a direct system $\{T(A/\mathfrak{a}_i, M), T(nat^i_j, M)\}$ over the directed set I. So we then can apply $\varinjlim$. If T happens to be left exact, it is a consequence of 2.15 that this procedure if applied to an exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

yields an exact sequence

$$0 \to \varinjlim T(A/\mathfrak{a}_i, M') \to \varinjlim T(A/\mathfrak{a}_i, M) \to \varinjlim T(A/\mathfrak{a}_i, M").$$

Of course the standard example for T is Hom, but the Ext^i are available too. Since $\varinjlim$ is a functor and Hom and Ext^0 are naturally equivalent, $\varinjlim(\mathrm{Hom}(A/\mathfrak{a}_i, \cdot))$ and $\varinjlim(\mathrm{Ext}^0(A/\mathfrak{a}_i, \cdot))$ are as well.

Now if

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is an exact sequence of A-modules, there are for all $i \in I$ connecting homomorphisms $\mathrm{Ext}^n(A/\mathfrak{a}_i, M'') \to \mathrm{Ext}^{n+1}(A/\mathfrak{a}_i, M')$, which are natural in both variables (see [**9**], Theorem 6.3) and make the long exact Ext-sequence with $\mathfrak{a}_i$ in the first variable exact. Then application of $\varinjlim$ yields connecting homomorphisms

$$\varinjlim \mathrm{Ext}^n(A/\mathfrak{a}_i, M'') \to \varinjlim \mathrm{Ext}^{n+1}(A/\mathfrak{a}_i, M')$$

such that for all $n \in \mathbb{N}$

$$\varinjlim \operatorname{Ext}^n(A/\mathfrak{a}_i, M) \to \varinjlim \operatorname{Ext}^n(A/\mathfrak{a}_i, M'') \to$$
$$\to \varinjlim \operatorname{Ext}^n(A/\mathfrak{a}_i, M') \to \varinjlim \operatorname{Ext}^{n+1}(A/\mathfrak{a}_i, M)$$

is exact.

So $\{\varinjlim \operatorname{Ext}^n(A/\mathfrak{a}_i, \cdot)\}$ is a connected sequence of covariant functors.

THEOREM 2.27. *Let $\mathfrak{A} = \{\mathfrak{a}_i, \varphi^i_j\}$ be an inverse system of ideals over the directed set I in A. For all $0 \le n \in \mathbb{N}$,*

$$H^n_{\mathcal{A}}(\cdot) = \mathcal{R}^n\Gamma_{\mathcal{A}}(\cdot) \text{ and } \varinjlim \operatorname{Ext}^n(A/\mathfrak{a}_i, \cdot)$$

are naturally equivalent functors.

PROOF. By 2.25 and 2.24, this is the case for $n = 0$. Also, $H^i_{\mathcal{A}}(I) = 0$ whenever I is an injective module and i is a positive integer. Similarily $\operatorname{Ext}^n(A/\mathfrak{a}_i, I) = 0$ for injective I and $n > 0$. So $\varinjlim \operatorname{Ext}^n(A/\mathfrak{a}_i, I) = 0$ for such I, n. Further $\{\varinjlim \operatorname{Ext}^n(A/\mathfrak{a}_i, \cdot)\}_{n\in\mathbb{N}}$ is a connected right sequence of functors since $\varinjlim$ is exact and trivially so are the $\mathcal{R}^n\Gamma_{\mathfrak{a}}(\cdot)$. By [**9**], Corollary 6.10 we are done. □

The work previously done in this chapter brings now

COROLLARY 2.28. *The local cohomology functors commute with direct limits on directed sets.*

PROOF. This follows directly from 2.20 and 2.27. □

CHAPTER 3

Sequences

As the title of the chapter promises, the main effort in this part will be spent on sequences and some consequences of these investigations.

The sequences arise in the most cases by the application of the connected sequence of functors $\{H^i_{\mathfrak{a}}\}_{i\epsilon\mathbf{N}}$ to a module M with two different ideals $\mathfrak{a}$ and $\mathfrak{b}$ say. This may remind some reader of relative homology where the homology groups of three different spaces, which are however related to each other by set theoretic conditions, are put into a long exact sequence. We will here produce an analogue.

While in algebraic topology for two topological spaces X_1, X_2 the *Mayer-Vietoris-sequence* provides an exact sequence

$$\ldots \to H_n(X_1 \cap X_2) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X_1 \cup X_2) \to H_{n-1}(X_1 \cap X_2) \to \ldots$$

for each $n \in \mathbb{N}$, we will produce for two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A an exact sequence

$$\ldots \to H^n_{\mathfrak{a}+\mathfrak{b}}(M) \to H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) \to H^n_{\mathfrak{a}\cap\mathfrak{b}}(M) \to H^{n+1}_{\mathfrak{a}+\mathfrak{b}}(M) \to \ldots$$

for every $n \in \mathbb{N}$. If one now looks back to chapter 1, 1.5 and uses the definition of $\Gamma_{\mathfrak{a}}$ described there, takes further $X_1 = \mathrm{Var}(\mathfrak{a})$ and $X_2 = \mathrm{Var}(\mathfrak{b})$ (then $\mathrm{Var}(\mathfrak{a} + \mathfrak{b}) = X_1 \cap X_2$ and $\mathrm{Var}(\mathfrak{a} \cap \mathfrak{b}) = X_1 \cup X_2$) the similarities between both sequences become extraordinary. However the reader should be aware, that the indices go up instead of down.

In the previous chapter we saw that one can define local cohomology for inverse systems of ideals over directed sets. Our first lemma here deals with the comparability of those functors.

LEMMA 3.1. *Let $(I, \leq)$ and $(J, \leq)$ be two directed sets. Let further $\mathfrak{A} = \{\mathfrak{a}_i\}$ and $\mathfrak{B} = \{\mathfrak{b}_i\}$ be two inverse systems of ideals over I and J respectively.*

Suppose that for all $i \in I \exists j \in J : \mathfrak{a}_i \subseteq \mathfrak{b}_j$ and for all $j \in J$ $\exists i \in I : \mathfrak{b}_j \subseteq \mathfrak{a}_i$. Then

(i) $\Gamma_{\mathcal{A}} = \Gamma_{\mathcal{B}}$ as functors,

(ii) the negative strongly connected sequences

$$\{\varinjlim \mathrm{Ext}^n_A(A/\mathfrak{a}_i, -)\}_{n\epsilon\mathbb{N}}$$

and

$$\{\varinjlim \mathrm{Ext}^n_A(A/\mathfrak{b}_j, -)\}_{n\epsilon\mathbb{N}}$$

are isomorphic.

PROOF. (i) is clear by the definition of Γ and 2.23. (ii) follows from (i). □

EXAMPLE 3.2. Let the index set be $\mathbb{N}$ with the natural order, $\mathfrak{B} = \{\mathfrak{a}^i + \mathfrak{b}^i\}$ an inverse system of ideals over $\mathbb{N}$ with inclusions as inner morphisms.

Then because of $\mathfrak{a}^i + \mathfrak{b}^i \subseteq (\mathfrak{a}+\mathfrak{b})^i$ and $(\mathfrak{a}+\mathfrak{b})^{2i-1} \subseteq \mathfrak{a}^i + \mathfrak{b}^i$, it follows from 3.1 that the functors

$$H^n_{\mathfrak{a}+\mathfrak{b}}(-) \text{ and } H^n_{\mathcal{B}}(-)$$

are naturally equivalent.

This example will play an important part in the construction of the Mayer-Vietoris sequence.

LEMMA 3.3. *4.3*
Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in A. Then $\{\mathfrak{a}^i \cap \mathfrak{b}^i\}_{\{i \epsilon N\}}$ is an inverse system over $\mathbb{N}$ and the connected sequences

$$\{\varinjlim \operatorname{Ext}^n(A/(\mathfrak{a}^i \cap \mathfrak{b}^i), -)\}_{n\epsilon\mathcal{N}}$$

and

$$\{H^n_{\mathfrak{a}\cap\mathfrak{b}}(-)\}_{n\epsilon\mathcal{N}}$$

are naturally isomorphic.

PROOF. It is clear that $(\mathfrak{a}\cap\mathfrak{b})^i \subseteq \mathfrak{a}^i \cap \mathfrak{b}^i$. By 3.1 and 2.25 it suffices to show that for each $i \in \mathbb{N}$ there exists a $q(i) \in \mathbb{N}$ such that $\mathfrak{a}^{q(i)} \cap \mathfrak{b}^{q(i)} \subseteq (\mathfrak{a}\cap\mathfrak{b})^i$. To do this we use the Artin-Rees lemma. (See for example in [**10**], Chapter 4, par. 7). This Artin-Rees lemma states that for all $i \in \mathbb{N}$ there is a $c \in \mathbb{N}$ such that

$$\mathfrak{a}^{i'+c} \cap \mathfrak{b}^i = \mathfrak{a}^{i'}(\mathfrak{a}^c \cap \mathfrak{b}^i)$$

for all $0 \leq i'$. But then, for $i = i'$:

$$\mathfrak{a}^{i+c} \cap \mathfrak{b}^{i+c} \subseteq \mathfrak{a}^{i+c} \cap \mathfrak{b}^i = \mathfrak{a}^i(\mathfrak{a}^c \cap \mathfrak{b}^i) \subseteq \mathfrak{a}^i \cdot \mathfrak{b}^i \subseteq (\mathfrak{a}\cap\mathfrak{b})^i$$

and the proof is completed. □

The following construction will so to say be the basis on which we will build our sequence.

LEMMA 3.4. *Let $N_1, N_2 \subseteq M$ be three A-modules. Writing in the displayed sequence below*

$$\alpha(m+N_1\cap N_2) = (m+N_1, m+N_2), \beta(x+N_1, y+N_2) = x-y+(N_1+N_2),$$

the sequence

$$0 \to M/(N_1\cap N_2) \xrightarrow{\alpha} M/N_1 \oplus M/N_2 \xrightarrow{\beta} M/(N_1+N_2) \to 0$$

is exact.

PROOF. It is obvious that the sequence is a zero complex, that α is injective and that β is surjective. So let $(x+N_1, y+N_2)$ be in $\ker(\beta)$. Then $x-y = n+n'$ for some $n \in N_1$ and $n' \in N_2$.

So $(x+N_1, y+N_2) = (x-n+N_1, y-n'+N_2) = \alpha(x-n+N_1\cap N_2) \in \operatorname{im}(\alpha)$. □

As the reader might suspect we we will apply to the sequence

$$0 \to A/(\mathfrak{a}^i \cap \mathfrak{b}^i) \to A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i \to A/(\mathfrak{a}^i + \mathfrak{b}^i) \to 0$$

(which is exact by 3.4) the functors $\mathrm{Ext}^n(-, M)$ for $n \in \mathbb{N}$, take the direct limit and use then 3.2 and 3.3 to get informations about the local cohomology modules. In this procedure obviously the term $\varinjlim \mathrm{Ext}^n_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i, M)$ occurs. This rather looks like $H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M)$ but we can use here neither 3.2 nor 3.3.

LEMMA 3.5.

$$\varinjlim \mathrm{Ext}^n_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i, -) \ \textit{and}\ H^n_{\mathfrak{a}}(-) \oplus H^n_{\mathfrak{b}}(-)$$

are naturally equivalent for all $n \in \mathbb{N}$.

PROOF. Let L, N, M be modules and $0 \to L \overset{\sigma}{\to} M \overset{\pi}{\to} N \to 0$ be a split exact A-sequence. This is to say that $\exists \sigma'$ and π' such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\sigma} & M & \xrightarrow{\pi} & N & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & L & \xleftarrow{\pi'} & M & \xleftarrow{\sigma'} & N & \longleftarrow & 0 \end{array}$$

is a commutative diagram with exact rows and $\sigma' \circ \pi + \sigma \circ \pi' = \mathrm{id}_M$. Then the application of $\mathrm{Ext}^n_A(-, X)$ gives by the A-linearity and half-exactness of Ext the (commutative) diagram with exact rows

$$\begin{array}{ccccc} \mathrm{Ext}^n_A(N, X) & \xrightarrow{\widetilde{\pi}} & \mathrm{Ext}^n_A(M, X) & \xrightarrow{\widetilde{\sigma}} & \mathrm{Ext}^n_A(L, X) \\ \| & & \| & & \| \\ \mathrm{Ext}^n_A(N, X) & \xleftarrow{\widetilde{\sigma'}} & \mathrm{Ext}^n_A(M, X) & \xleftarrow{\widetilde{\pi'}} & \mathrm{Ext}^n_A(L, X) \end{array}$$

again with exact rows and $\tilde{\sigma}' \circ \tilde{\pi} + \tilde{\sigma} \circ \tilde{\pi}' = \mathrm{Hom}_A(\mathrm{id}_M, X)$ showing that both $\tilde{\sigma} = \mathrm{Ext}^n_A(\sigma, X)$ and $\tilde{\sigma}' = \mathrm{Ext}^n_A(\sigma', X)$ are surjective and $\tilde{\pi} = \mathrm{Hom}_A(\pi, X)$ and $\tilde{\pi}' = \mathrm{Hom}_A(\pi', X)$ are injective. So the application of $\mathrm{Ext}^n_A(-, X)$ gives again a split exact sequence.

Now let for all $j \leq i \in \mathbb{N}$ and all ideals $\mathfrak{a}, \mathfrak{b}$ in A

$$0 \to A/\mathfrak{a}^i \to A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i \to A/\mathfrak{b}^i \to 0$$

denote the natural, split sequence and $h^i_j : A/\mathfrak{a}^i \to A/\mathfrak{a}^j$ and $k^i_j : A/\mathfrak{b}^i \to A/\mathfrak{b}^j$ the natural projections. Then we have for all such $i, j, \mathfrak{a}, \mathfrak{b}$ a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A/\mathfrak{a}^j & \longrightarrow & A/\mathfrak{a}^j \oplus A/\mathfrak{b}^j & \longrightarrow & A/\mathfrak{b}^j & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h^j_i} & & \downarrow{\scriptstyle h^j_i \oplus k^j_i} & & \downarrow{\scriptstyle k^j_i} & & \\
0 & \longrightarrow & A/\mathfrak{a}^i & \longrightarrow & A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i & \longrightarrow & A/\mathfrak{b}^i & \longrightarrow & 0
\end{array}$$

with split exact rows and application of $\mathrm{Ext}^n_A(-, M)$ yields

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}^n(A/\mathfrak{b}^j, M) & \longrightarrow & \mathrm{Ext}^n(A/\mathfrak{b}^j, M) \oplus (A/\mathfrak{a}^j, M) & \longrightarrow & \mathrm{Ext}^n(A/\mathfrak{a}^j, M) & \to & 0 \\
 & & \uparrow{\scriptstyle \mathrm{Ext}^n(k^j_i, M)} & & \uparrow{\scriptstyle \mathrm{Ext}^n(k^j_i \oplus h^j_i, M)} & & \uparrow{\scriptstyle \mathrm{Ext}^n(h^j_i, M)} & & \\
0 & \to & \mathrm{Ext}^n(A/\mathfrak{b}^i, M) & \longrightarrow & \mathrm{Ext}^n(A/\mathfrak{b}^i, M) \oplus (A/\mathfrak{a}^i, M) & \longrightarrow & \mathrm{Ext}^n(A/\mathfrak{a}^i, M) & \to & 0
\end{array}$$

with again split exact rows.

Now we take direct limits in each row and observe that by 2.15 this gives a split exact sequence

$$0 \to \varinjlim \mathrm{Ext}^n_A(A/\mathfrak{a}^i, M) \to \varinjlim(\mathrm{Ext}^n_A(A/\mathfrak{a}^i \oplus (A/\mathfrak{b}^i, M)) \to \varinjlim \mathrm{Ext}^n_A(A/\mathfrak{b}^i, M) \to 0$$

by [**9**] Th. 1.4 and 2.24 proves that for all A-modules M

$$H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) \text{ and } \varinjlim \mathrm{Ext}^n_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i, M)$$

are isomorphic. Now Ext and $\varinjlim$ are functors so that from a map $f : M \to M'$ we get translations of the above diagrams into similar ones with M' instead of M which then yield a translation of the final sequence into one with M replaced by M' showing that the stated isomorphism is natural. □

We now have collected enough information to prove the first and basic

THEOREM 3.6. *For all A-modules M and all pairs of ideals in A there is a long exact sequence*

$$\begin{array}{rl}
0 & \to H^0_{\mathfrak{a}+\mathfrak{b}}(M) \to H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) \to H^0_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \\
 & \to H^1_{\mathfrak{a}+\mathfrak{b}}(M) \to H^1_{\mathfrak{a}}(M) \oplus H^1_{\mathfrak{b}}(M) \to H^1_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \\
 & \qquad\qquad \ldots \to \ldots \\
 & \to H^n_{\mathfrak{a}+\mathfrak{b}}(M) \to H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) \to H^n_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \\
 & \qquad\qquad \ldots \to \ldots
\end{array}$$

such that for $f : M \to M'$ *a homomorphism, the diagram*

$$
\begin{array}{ccccccc}
H^n_{\mathfrak{a}+\mathfrak{b}}(M) & \longrightarrow & H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) & \longrightarrow & H^n_{\mathfrak{a}\cap\mathfrak{b}}(M) & \longrightarrow & H^{n+1}_{\mathfrak{a}+\mathfrak{b}}(M) \\
\downarrow {\scriptstyle H^n_{\mathfrak{a}+\mathfrak{b}}(f)} & & \downarrow {\scriptstyle H^i_{\mathfrak{a}}(f) \oplus H^n_{\mathfrak{b}}(f)} & & \downarrow {\scriptstyle H^n_{\mathfrak{a}\cap\mathfrak{b}}(f)} & & \downarrow {\scriptstyle H^{n+1}_{\mathfrak{a}+\mathfrak{b}}(f)} \\
H^n_{\mathfrak{a}+\mathfrak{b}}(M') & \longrightarrow & H^n_{\mathfrak{a}}(M') \oplus H^n_{\mathfrak{b}}(M') & \longrightarrow & H^n_{\mathfrak{a}\cap\mathfrak{b}}(M') & \longrightarrow & H^{n+1}_{\mathfrak{a}+\mathfrak{b}}(M')
\end{array}
$$

commutes for all $n \in \mathbb{N}$.

Proof. Let $i \le j \in \mathbb{N}$ and let $h^j_i : A/\mathfrak{a}^j \to A/\mathfrak{a}^i$ and $k^j_i : A/\mathfrak{b}^j \to A/\mathfrak{b}^i$ denote the natural homomorphisms. Applying 3.4 to the submodules $\mathfrak{a}^l, \mathfrak{b}^l$ of A for $l = i, j$ one checks easily that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A/(\mathfrak{a}^i \cap \mathfrak{b}^i) & \longrightarrow & A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i & \longrightarrow & A/(\mathfrak{a}^i + \mathfrak{b}^i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle h^i_j \bigoplus k^i_j} & & \downarrow & & \\
0 & \longrightarrow & A/(\mathfrak{a}^j \cap \mathfrak{b}^j) & \longrightarrow & A/\mathfrak{a}^j \oplus A/\mathfrak{b}^j & \longrightarrow & A/(\mathfrak{a}^j + \mathfrak{b}^j) & \longrightarrow & 0
\end{array}
$$

commutes and has exact rows. So we may apply the functor $\mathrm{Hom}_A(-, M)$ to it to get a long exact sequence

$$
\begin{array}{rl}
0 \to & \mathrm{Hom}_A(A/(\mathfrak{a}^i + \mathfrak{b}^i), M) \to \mathrm{Hom}_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i, M) \to \mathrm{Hom}_A(A/(\mathfrak{a}^i \cap \mathfrak{b}^i), M) \to \\
& \mathrm{Ext}^1_A(A/(\mathfrak{a}^i + \mathfrak{b}^i), M) \to \mathrm{Ext}^1_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i, M) \to \mathrm{Ext}^1_A(A/(\mathfrak{a}^i \cap \mathfrak{b}^i), M) \to \\
& \ldots \to \\
& \mathrm{Ext}^n_A(A/(\mathfrak{a}^i + \mathfrak{b}^i), M) \to \mathrm{Ext}^n_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i, M) \to \mathrm{Ext}^n_A(A/(\mathfrak{a}^i \cap \mathfrak{b}^i), M) \to \\
& \ldots \to
\end{array}
$$

together with a translation of this sequence into the same sequence with j instead of i. This sequence is natural in M since Hom is a functor. So passing to direct limits this naturality is conserved and the result is the long exact sequence

$$\begin{aligned}0 \to \quad & \varinjlim(\operatorname{Hom}_A(A/(\mathfrak{a}^i + \mathfrak{b}^i), M)) \to \varinjlim(\operatorname{Hom}_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i), M)) \to \\ & \to \varinjlim(\operatorname{Hom}_A(A/(\mathfrak{a}^i \cap \mathfrak{b}^i), M) \to \\ & \varinjlim(\operatorname{Ext}^1_A(A/(\mathfrak{a}^i + \mathfrak{b}^i), M)) \to \varinjlim(\operatorname{Ext}^1_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i), M)) \to \\ & \to \varinjlim(\operatorname{Ext}^1_A(A/(\mathfrak{a}^i \cap \mathfrak{b}^i), M)) \to \\ & \to \dots \to \\ & \varinjlim(\operatorname{Ext}^n_A(A/(\mathfrak{a}^i + \mathfrak{b}^i), M)) \to \varinjlim(\operatorname{Ext}^n_A(A/\mathfrak{a}^i \oplus A/\mathfrak{b}^i), M)) \to \\ & \to \varinjlim(\operatorname{Ext}^n_A(A/(\mathfrak{a}^i \cap \mathfrak{b}^i), M)) \to \\ & \to \dots\end{aligned}$$

Using 3.5, 3.2 and 3.3 the proof is complete. □

Theorem 3.6 is the most important tool we will encounter in this chapter. We will in the remainder of the chapter deal with the question how far the calculation of local cohomology modules is dependent of the base ring under special homomorphisms. To this we first introduce the notion of acyclic modules and show a property acyclic modules have.

Definition 3.7. An A-module M is called T*-acyclic* for an additive functor $T : \mathcal{C}_A \to \mathcal{C}_A$ whenever all right derived functors of order greater than zero of T applied to M are 0.

The actual use of acyclic modules is illustrated by the following

Lemma 3.8. *Let T be a left exact additive covariant functor from the category of A-modules to itself. Denote by T^i its i-th right derived functor. Suppose,*

$$0 \to M \to M^0 \to M^1 \to M^2 \to \dots$$

is an exact sequence and $T^i(M^j) = 0$ for all $1 \leq i$ and $j \in \mathbb{N}$. Then $T^i(M) \cong \ker(T(M^i \to M^{i+1}))/\operatorname{im}(T(M^{i-1} \to M^i))$. Thus the right derived functors of T may be calculated by means of T-acyclic resolutions.

Proof. Let

$$0 \to M \overset{\alpha}{\to} A^0 \overset{d^0}{\to} A^1 \overset{d^1}{\to} \dots$$

be a T-acyclic resolution (that is, the displayed sequence is exact and all A^i are T-acyclic) for the A-module M. By the left exactness of T this gives that

$$0 \to T(M) \overset{T(\alpha)}{\to} T(A^0) \overset{T(d^0)}{\to} T(A^1)$$

is exact and so $T(M) \cong \ker(T(d^0))$ so that we have proved the claim for $i = 0$.

Let now $i > 0$ and suppose that for all $j \leq i - 1$ we have proved that $T^j(M) \cong H^j(T(\mathbf{A}^\cdot))$ where $A^\cdot$ denotes the given acyclic resolution of M and H^j the j^{th} cohomology module. Since the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha} & A^0 & \xrightarrow{d^0} & A^1 & \xrightarrow{d^1} & A^2 & \xrightarrow{d^2} & \cdots \\ & & & & \pi\searrow & & \nearrow\iota & & & & \\ & & & & & \ker(d^1) & & & & & \\ & & & & \nearrow & & \searrow & & & & \\ & & & & 0 & & 0 & & & & \end{array}$$

commutes, with π and ι the obvious natural maps,

$$0 \to \ker(d^1) \xrightarrow{\iota} A^1 \xrightarrow{d^1} A^2 \dots$$

is an acyclic resolution for $\ker(d^1)$.

By the inductive hypothesis we have then

$$T^{i-1}(\ker(d^1)) \cong \begin{cases} H^i(T(A^{\cdot})) & \text{for i¿1} \\ \ker T(d^1) & \text{for i=1} \end{cases}$$

Further from the exactness of

$$0 \to M \to A^0 \xrightarrow{\pi} \ker(d^1) \to 0$$

follows, for $i > 1$,

$$T^i(M) \cong T^{i-1}(\ker(d^1)) \cong H^i(T(\mathbf{A}^{\cdot}))$$

and for $i = 1$ that

$$0 \to T(M) \xrightarrow{T\alpha} T(A^0) \xrightarrow{T\pi} T(\ker(d^1)) \to T^1(M) \to 0$$

is exact. So $T^1(M) \cong T(\ker(d^1))/\operatorname{im}(T(\pi))$. But $T(\ker(d^1)) = \ker(T(d^1))$ by the left exactness of T in the sequence

$$0 \to \ker(d^0) \xrightarrow{\iota} A^1 \xrightarrow{d^1} A^2$$

and $\operatorname{im}(T(\pi)) = \operatorname{im}(T(\iota \circ \pi)) = \operatorname{im}(T(d^0))$. $\square$

The following two results pave the way for a statement that is needed for the theorem after it, but from its nature belongs into the next chapter. It will therefore be stated explicitely in there.

LEMMA 3.9. *Let $a \in A$. Then there is a natural isomorphism*

$$\varinjlim \operatorname{Hom}_A(A \cdot a^n, M) \to M_a$$

where M_a means the localization of M with respect to the m.c.s. $\{a^n\}_{n\epsilon\mathbb{N}}$.

PROOF. For all $n \in \mathbb{N}$ there is an A-homomorphism $\tau_n : \operatorname{Hom}_A(A \cdot a^n, M) \to M_a$ given by $\tau_n(f)_n = f_n(a^n)/a^n$. If one takes $\iota_n^{n'} : A \cdot a^n \to A \cdot a^n$ to be the inclusion map whenever $n \leq n' \in \mathbb{N}$, then clearly $\tau_{n'} \circ \operatorname{Hom}_A(\iota_n^{n'}, M) = \tau_n$. Taking $\varinjlim$ induces therefore a homomorphism

$$\tau : \varinjlim \operatorname{Hom}_A(A \cdot a^n, M) \to M_a.$$

We try to show that this homomorphism is injective and surjective.

So let $f \in \ker(\tau)$. Since the direct system $\{\mathrm{Hom}_A(A{\cdot}a^n, M), \mathrm{Hom}_A(\iota_n^{n'}, M)\}$ has directed index set, it follows from 2.12 that there is an $n \in \mathbb{N}$ and a $f_n \in \mathrm{Hom}_A(A \cdot \mathfrak{a}^n, M)$ such that f is image under an inner morphism of f_n. By property 3 of direct limits, then $\tau_n(f_n) = f_n(a^n)/a^n = 0$. By the definition of M_a there is an $s \in \mathbb{N} : a^s f_n(a^n) = 0$ whence $\mathrm{Hom}_A(\iota_n^{n+s}, M)(f_n) = f_n \circ \iota_n^{n+s} = 0$. So the image of f_n in $\varinjlim \mathrm{Hom}_A(A \cdot a^n, M)$ is zero and τ is injective.

Let now $\mu \in M_a$. So $\mu = \frac{m}{a^n}$ for some $n \in \mathbb{N}, m \in M$. As A is Noetherian, the sequence

$$(0 :_A a) \subseteq (0 :_A a^2) \subseteq \ldots \subseteq (0 :_A a^i) \ldots$$

is eventually stationary so that there exists a $c \in \mathbb{N} : (0 :_A a^c) = (0 :_A a^{c+i})$ for all $i \in \mathbb{N}$. Hence $r \cdot a^{c+n} = r' \cdot a^{c+n}$ for $r, r' \in A \Rightarrow r - r' \in (0 :_A a^{c+n}) = (0 :_A a^c)$ and so $r \cdot a^c m = r' \cdot a^c m$. So we can define

$$h : A \cdot a^{c+n} \to M_a \text{ by } h(ra^{c+n}) = ra^c m.$$

Then $\tau_{c+n}(h) = h(a^{c+n})/a^{c+n} = a^c m/a^{c+n} = m/a^n = \mu$. So τ is surjective.

Given an A-homomorphism $\varphi : M \to M'$ from M into another A-module M', then each $f \in \varinjlim \mathrm{Hom}_A(A \cdot a^n, M)$ is image of some $f_n \in \mathrm{Hom}_A(A \cdot a^n, M)$ for some n. So for the naturality of τ we have by property 3 of direct limits and the fact that $\varinjlim$ is functor to show that $\varphi_a(\tau_n(f_n)) = \tau'_n(\varphi \circ f_n)$, where $\varphi_a(x/y) = \varphi(x)/y$ for $x/y \in M_a$ and τ' denotes the isomorphism $\varinjlim \mathrm{Hom}_A(A \cdot a^n, M') \to M'_a$. But this is obvious. $\square$

LEMMA 3.10. *For the ideal $\mathfrak{a}$ of A, denote the inclusion map $\mathfrak{a}^{i'} \hookrightarrow \mathfrak{a}^i$ for $i \leq i'$ by $\iota_i^{i'}$. The family of A-modules and A-homomorphisms*

$$\{\mathrm{Hom}_A(\mathfrak{a}^i, M), \mathrm{Hom}_A(\iota_i^{i'}, M)\}_{i,i' \epsilon \mathcal{N}}$$

is a direct system over $\mathbb{N}$, so that we can form its direct limit. Then

$$H^n_{\mathfrak{a}}(-) and (\mathcal{R}^{n-1}(\varinjlim \mathrm{Hom}_A))(A \cdot \mathfrak{a}^i, -)$$

are naturally equivalent functors for $n > 1$.

PROOF. For all $i \in \mathbb{N}$ we have an exact sequence

$$0 \to \mathfrak{a}^i \to A \to A/\mathfrak{a}^i \to 0.$$

Further, for $1 \leq j \in \mathbb{N}$, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a}^j & \longrightarrow & A & \longrightarrow & A/\mathfrak{a}^j & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longleftarrow & \mathfrak{a}^i & \longleftarrow & A & \longleftarrow & A/\mathfrak{a}^j & \longleftarrow & 0
\end{array}$$

Applying the functor $\mathrm{Hom}(-, M)$ and its right derived functors yields long exact sequences

$$\begin{aligned} 0 \quad & \to \mathrm{Hom}_A(A/\mathfrak{a}^i, M) \to \mathrm{Hom}_A(A, M) \cong M \to \mathrm{Hom}_A(A \cdot \mathfrak{a}^i, M) \to \\ & \to \mathrm{Ext}_A{}^1(A/\mathfrak{a}^i, M) \to \mathrm{Ext}_A{}^1(A, M) = 0 \to \mathrm{Ext}_A{}^1(A \cdot \mathfrak{a}^i, M) \to \\ & \to \dots \\ & \to \mathrm{Ext}_A{}^n(A/\mathfrak{a}^i, M) \to \mathrm{Ext}_A{}^n(A, M) = 0 \to \mathrm{Ext}_A{}^n(A \cdot \mathfrak{a}^i, M) \to \\ & \to \dots \end{aligned}$$

for all $i \in \mathbb{N}$. By [**9**], theorem 6.7, the above picture assures, that there is for all $i \leq j \in \mathbb{N}$ a chain map from the displayed long sequence to one where i is replaced by j and by this theorem again, the long sequence is natural in M. Now we take direct limits to get for all $n \geq 1$

$$\varinjlim \mathrm{Ext}^n(\mathfrak{a}^i, M) \cong \varinjlim \mathrm{Ext}^{n+1}(A/\mathfrak{a}^i, M)$$

by the exactness of $\varinjlim$, and the isomorphisms are clearly natural. It follows from 2.27 that $H_{\mathfrak{a}}^{n+1}(M) \cong \varinjlim \mathrm{Ext}^{n+1}(A/A \cdot \mathfrak{a}^i, M)$ and [**9**], theorem 6.1 together with 2.15 shows that $\varinjlim \mathrm{Ext}^n(\mathfrak{a}^i, M)$ is isomorphic to $\mathcal{R}^n(\varinjlim \mathrm{Hom}_A(\mathfrak{a}^i, M))$. □

THEOREM 3.11. *Let M be an A-module. Suppose, Π is a set of ideals in A with the properties that*

- *Π is closed under the formation of finite sums and products,*
- *$0 \in \Pi$,*
- *each ideal in Π is the sum of finitely many principal ideals which belong to Π.*

Then from $H_J^1(M) = 0$ for all $J \in \Pi$ follows that M is Γ_J-acyclic.

PROOF. Since Γ_{0A} is the identity functor on $\mathcal{C}_A$, resolutions are unchanged under application of it and so each M is Γ_0-acyclic.

Now let x be in A and consider $\mathfrak{a} := x \cdot A$, an ideal. Suppose, $H_{Ax}^1(M) = 0$. We show that then M is Γ_{Ax}-acyclic.

From 3.10 we get that for $n \geq 2$, $H_{A \cdot x}^n(M) \cong \mathcal{R}^{n-1}(\varinjlim \mathrm{Hom}_A(A \cdot x^i, M))$ and this is by 3.9 isomorphic to $\mathcal{R}^{n-1}(M_x)$, meaning that the $(n-1)$-st right derived functor of $(-)_x$ is applied to M. But localization is an exact functor and hence its derived functors of order unequal to zero vanish. So $H_{A \cdot x}^n(M) \cong 0$ for all $n > 1$ and by hypothesis $H_{A \cdot x}^1(M) = 0$, so M is $A \cdot x$-acyclic. This proves the theorem for principal ideals in Π.

Now suppose inductively, that $t > 1$ and we have proved that whenever $J' \in \Pi$ can be expressed as sum of at most $t-1$ principal ideals which all belong to Π , then M is acyclic for $\Gamma_{J'}$. We saw this to be correct for $t = 2$. Now let $J = Ax_1 + Ax_2 + \ldots + Ax_t$ where all the principal ideals on the right side belong to Π.

Set $K = Ax_1 + \ldots + Ax_{t-1}$ and $K' = Ax_t$. By hypothesis K, K' and $K \cdot K'$ belong to Π. Since $K, K', K \cdot K'$ are all expressible as sum of less than t principal ideals of Π, it follows from the inductive hypothesis, that

M is acyclic with respect to $\Gamma_K, \Gamma_{K'}$ and $\Gamma_{K\cdot K'}$. Using the Mayer-Vietoris sequence we see that for each $i > 1$ the sequence

$$H^{i-1}_{K\cdot K'}(M) \to H^i_{K+K'}(M) \to H^i_K(M) \oplus H^i_{K'}(M)$$

is exact, the first and last term being 0 by inductive hypothesis and $K+K' = J$. Therefore M is Γ_J-acyclic. □

We will now assume until further notice that there is a second commutative ring B together with a homomorphism $f : A \to B$, such that B considered as an A-module by means of f is flat. We will then investigate how far the local cohomology modules with respect to an ideal $I \subseteq A$ are related to the local cohomology modules with respect to the ideal $f(I) \cdot B$. We will need some preparatory statements.

LEMMA 3.12. *Let A' be another commutative ring and let $T, U : \mathcal{C}_A \to \mathcal{C}_{A'}$ be two additive contravariant left exact functors. Suppose $\mu : T \to U$ is a morphism of functors such that $\mu_A : T(A) \to U(A)$ is an isomorphism. Then $\mu_M : T(M) \to U(M)$ is an isomorphism for all finitely generated A-modules M.*

PROOF. Since T and U are additive, it follows that for a split exact sequence

$$0 \to F' \to F \to A \to 0$$

of free A-modules F, F' the induced diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(A) & \longrightarrow & T(F) & \longrightarrow & T(F') & \longrightarrow & 0 \\
 & & \cong\downarrow\mu_A & & \downarrow\mu_F & & \downarrow\mu_{F'} & & \\
0 & \longrightarrow & U(A) & \longrightarrow & U(F) & \longrightarrow & U(F') & \longrightarrow & 0
\end{array}$$

has split exact rows (see e.g. 3.5). Since free modules of rank one are isomorphic to A, it follows by the five lemma (see [**11**], Lemma 3.32) through induction on rank that μ_F is an isomorphism for all free modules of finite rank.

If now M is an arbitrary finitely generated A-module, M has a free resolution where all modules involved are finitely generated (see e.g. [**9**], Theorem 5.10). So taking the last three terms of this resolution, and applying T and U to it, we get the (commutative) diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T(M) & \longrightarrow & T(F_0) & \longrightarrow & T(F_1) \\
 & & \downarrow\mu_M & & \cong\downarrow\mu_{F_0} & & \cong\downarrow\mu_{F_1} \\
0 & \longrightarrow & U(M) & \longrightarrow & U(F_0) & \longrightarrow & U(F_1)
\end{array}$$

with exact rows (since T, U are left exact). Now the two right vertical maps are isomorphisms and it follows from the five lemma again that μ_M is an isomorphism. □

The reader should note that these isomorphisms are natural since μ is transformation.

LEMMA 3.13. *Let the ring homomorphism $f : A \to B$ be flat, meaning that B is A-flat by means of f. Let E be an A-module. Then by* [**11**], *Theorem 1.8, $E \otimes_A B$ can be made into a B-module via $s \cdot (\Sigma_{1 \le i \le t}(m_i \otimes b_i)) = \Sigma_{1 \le i \le t}(m_i \otimes s \cdot b_i)$. Moreover if $g : E \to F$ is an A-homomorphism, then both $E \otimes_A B$ and $F \otimes_A B$ are B-modules and $g \otimes_A \mathrm{id}_B : E \otimes_A B \to F \otimes_A B$ is a B-homomorphism.*

It is clear that $g = \mathrm{id}_E$ induces $g \otimes \mathrm{id}_B = \mathrm{id}_{E \otimes B}$ and for a combined map we have $(g \circ g') \otimes \mathrm{id}_B = (g \otimes \mathrm{id}_B) \circ (g' \otimes \mathrm{id}_B)$. So $(-) \otimes_A B$ may be viewed as (additive) covariant functor from $\mathcal{C}_A$ to $\mathcal{C}_B$, the category of B-modules and B-homomorphisms.

We may therefore consider $T = \mathrm{Hom}_A(-, E) \otimes_A B$ and $U = \mathrm{Hom}_B((-) \otimes_A B, E \otimes_A B)$ to be additive contravariant functors from $\mathcal{C}_A$ to $\mathcal{C}_B$. Then there is a morphism of functors $\mu : T \to U$ such that

1. *for each A-module M, each $g \in \mathrm{Hom}_A(M, E)$ and each $b \in B$ we have*
$$\mu_M(g \otimes b) = b \cdot (g \otimes \mathrm{id}_B),$$
2. *the composition*
$$E \otimes_A B \xrightarrow{\gamma} \mathrm{Hom}_A(A, E) \otimes_A B \xrightarrow{\mu_A} \mathrm{Hom}_B(A \otimes_A B, E \otimes_A B) \xrightarrow{\delta} E \otimes_A B$$
where γ and δ are the natural isomorphisms (see [**9**], *Th.2.6) is the identity map,*
3. *μ_M is an isomorphism for all finitely generated A-modules M.*

PROOF. As one easily verifies, the μ_M as stated constitute a homomorphism from $\mathrm{Hom}_A(M, E) \otimes_A B$ to $\mathrm{Hom}_B((M \otimes_A B), (E \otimes_A B))$.

1. Using the displayed formula, we have to show that it gives a morphism of functors. That is, that it is natural in M. So suppose, we have a map $f : M' \to M$. Then we have to show that for each element $g \in \mathrm{Hom}_A(M, E) \otimes_A B$ and all $b \in B$
$$\mu_{M'} \circ (\mathrm{Hom}_A(f, E) \otimes_A B)(g \otimes b) = \mathrm{Hom}_B(f \otimes_A B, E \otimes_A B) \circ \mu_M(g \otimes b);$$
but this is obvious.
2. Under the given composition we follow the element $e \otimes b$ with $e \in E$ and $b \in B$:
$$e \otimes b \to f_e \otimes b \to b(f_e \otimes \mathrm{id}_B) \to b(e \otimes 1) = e \otimes b$$
(where $f_e : A \to E$ by $a \to a \cdot e$) and we are done.
3. Since tensoring with B over A and $\mathrm{Hom}_A(-, E)$ are additive exact functors and $\mathrm{Hom}_B(-, E \otimes_A B)$ is left exact we can invoke lemma 3.13 since (ii) proves that μ_A is an isomorphism. □

Proposition 3.14. *Assume that the ring homomorphism $f : A \to B$ is flat. Let E be an injective A-module and be I an ideal of A. Then the natural B-homomorphism*

$$\beta : E \otimes_A B \to \mathrm{Hom}_B(I \cdot B, E \otimes_A B)$$

with $\beta(g)(b) = b \cdot g$ for all $g \in E \otimes_A B$ and all $b \in I \cdot B$ is surjective.

Proof. Since f is flat, B is A-flat and so the natural B-homomorphism

$$\gamma : I \otimes_A B \to I \cdot B$$

for which $\gamma(\Sigma_{i=1}^t a_i \otimes b_i) = \Sigma_{i=1}^t f(a_i) b_i$ for all $t \in \mathbb{N}, a_1, \dots, a_t \in I$ and $b_1, \dots, b_t \in B$ is an isomorphism: from the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I \otimes_A B & \longrightarrow & A \otimes_A B \\ & & \downarrow \gamma & & \cong \downarrow \varepsilon \\ 0 & \longrightarrow & I \cdot B & \longrightarrow & B \end{array}$$

where ε is the natural isomorphism and the top row is induced by the exact sequence $0 \to I \to A$, follows that γ is injective.

That γ is surjective follows from the fact that $\sum_1^t f(a_i) \cdot b_i$ is the general form of an element of $I \cdot B$ with $t \in \mathbb{N}, a_i \in I$ and $b_i \in B$. Now I is finitely generated and so the B-homomorphism from 3.13

$$\mu_I : \mathrm{Hom}_A(I, E) \otimes_A B \to \mathrm{Hom}_B(I \otimes_A B, E \otimes_A B)$$

is an isomorphism. Since E is A-injective, it follows from the exact sequence $0 \to I \to A$ and the natural isomorphism $\alpha : E \cong \mathrm{Hom}_A(A, E)$, that $\sigma : E \to \mathrm{Hom}_A(I, E)$ by $\sigma(x)(a) = a \cdot x$ for all $x \in E$ and all $a \in I$ is surjective, so that application of the rightexact functor $(-) \otimes_A B$ shows that

$$\sigma \otimes \mathrm{id}_B : E \otimes_A B \to \mathrm{Hom}_A(I, E) \otimes_A B$$

is a (B-)epimorphism. So

$$\mathrm{Hom}_B(\gamma^{-1}, E \otimes_A B) \circ \mu_I \circ (\sigma \otimes \mathrm{id}_B) : E \otimes_A B \to \mathrm{Hom}_B(I \cdot B, E \otimes_A B)$$

is surjective. But this maps $e \otimes b \to \alpha(f) \otimes b \to b \cdot (\alpha(f) \otimes \mathrm{id}_B) \to b \cdot (\alpha(f) \otimes \mathrm{id}_B) \circ \gamma^{-1}$ which is exactly β. So we are done. □

The next theorem will pave the way for what is called the 'flat base change'.

Theorem 3.15. *Assume that the ring homomorphism $f : A \to B$ is flat and let E be an injective A-module. Then $E \otimes_A B$ is $\Gamma_{I \cdot B}$-acyclic for all ideals I of A.*

PROOF. For each $n \in \mathbb{N}$ the natural B-homomorphism $E \otimes_A B \to \mathrm{Hom}_B(I^n B, E \otimes_A B)$ of 3.14 is surjective. Because of $I^n B = (I \cdot B)^n$ and since $\varinjlim$ is exact, the induced homomorphism

$$\varinjlim(E \otimes_A B) \to \varinjlim(\mathrm{Hom}_B(I^n \cdot B, E \otimes_A B))$$

is surjective. So from the natural short exact sequences $0 \to I^n \cdot B \to B \to B/I^n \cdot B \to 0$ together with the translations into a similar one with n' for n, follows by application of $\varinjlim \mathrm{Hom}_B(-, E \otimes_A B)$ that

$$0 \to \varinjlim \mathrm{Hom}_B(B/I^n \cdot B, E \otimes_A B) \to \varinjlim \mathrm{Hom}_B(B, E \otimes_A B)$$
$$\to \varinjlim \mathrm{Hom}_B(I^n \cdot B, E \otimes_A B)$$
$$\to \varinjlim \mathrm{Ext}^1_B(B/I^n \cdot B, M) \to \varinjlim \mathrm{Ext}^1_B(B, M) = 0$$

is exact. Since the third map is surjective, the fourth term is zero and this is just saying that $H^1_{IB}(E \otimes_A B) = 0$. Since this is true for all ideals I of A, we conclude by 2.6 that $E \otimes_A B$ is Γ_{IB}-acyclic for all ideals I of A. $\square$

We come now to one of the more important statements of this chapter for the work in the following chapters. It is concerned with the change of the base ring under flat homomorphisms. We remind the reader that he has seen before one result to this topic, namely 1.14.

LEMMA 3.16. *Let the ring homomorphism $f : A \to B$ be flat. Then there is a natural equivalence of functors*

$$\varphi : \Gamma_{\mathfrak{a}}(-) \otimes_A B \to \Gamma_{\mathfrak{a}B}((-) \otimes_A B)$$

from $\mathcal{C}_A$ to $\mathcal{C}_B$ which maps $m \otimes b$ into $m \otimes b$ for $m \in \Gamma_{\mathfrak{a}}(M)$ and $b \in B$.

PROOF. Let M be an A-module. Since f is flat, $\Gamma_{\mathfrak{a}}(M) \hookrightarrow M$ induces a B-monomorphism $\Gamma_{\mathfrak{a}}(M) \otimes_A B \to M \otimes_A B$ and its image is $\mathfrak{a}$-torsion since $\otimes_A$ is A-linear.

So we can for all A-modules M give a φ as stated such that φ_M is injective. It is further clear that φ is natural in M because $\Gamma_{\mathfrak{a}}$ is 'taking subsets'. So we have to show that φ is always surjective.

For let $n \in \mathbb{N}$. $A/\mathfrak{a}^n$ is a finitely generated A-module and therefore 3.13 yields a B-isomorphism

$$\mu_{A/\mathfrak{a}^n} : \mathrm{Hom}_A(A/\mathfrak{a}^n, M) \otimes_A B \to \mathrm{Hom}_B(A/\mathfrak{a}^n \otimes_A B, M \otimes_A B)$$

such that $\mu_{A/\mathfrak{a}^n}(f \otimes b) = b(f \otimes \mathrm{id}_B)$ for all $f \in \mathrm{Hom}_A(A/\mathfrak{a}^n, M)$ and all $b \in B$. Then we may produce a composite

$$(0 :_M \mathfrak{a}^n) \otimes_A B \to \mathrm{Hom}_A(A/\mathfrak{a}^n, M) \otimes_A B \xrightarrow{\mu_{A/\mathfrak{a}^n}} \mathrm{Hom}_B((A/\mathfrak{a}^n) \otimes_A B, M \otimes_A B)$$
$$\to \mathrm{Hom}_B(B/\mathfrak{a}^n B, M \otimes_A B) \to (0 :_{M \otimes B} (\mathfrak{a}B)^n)$$

in which all undescribed maps are the natural isomorphisms. Under this composition, $m \otimes b$ is mapped

$$m \otimes b \to f_m \otimes b \to b(f_m \otimes \mathrm{id}_B) \to b \cdot (\Sigma a_i b_i + (\mathfrak{a}^n)) \to \Sigma f_m(\mathfrak{a}_i) \otimes \mathfrak{b}_i \to b(m \otimes 1) = m \otimes b.$$

So the composition is exactly φ_M restricted to $(0 :_M \mathfrak{a}^n) \otimes_A B$. Since for each $m \in \Gamma_{\mathfrak{a}}(M)$ there is an n such that $\mathfrak{a}^n \cdot m = 0$, each $m \otimes_A b$ for $b \in B$ is image under φ_M. So φ must be always surjective. $\square$

THEOREM 3.17 (Flat Base Change). *Assume that the ring homomorphism $f : A \to B$ is flat. Then the connected right sequences of covariant functors*

$$(H^n_{\mathfrak{a}}(-) \otimes_A B)_{n\epsilon\mathcal{N}} \text{ and } (H^n_{\mathfrak{a}\cdot B}((-) \otimes_A B)_{n\epsilon\mathcal{N}}$$

from $\mathcal{C}_A$ to $\mathcal{C}_B$ are isomorphic.

PROOF. Since $(-) \otimes_A B$ is exact, it is clear that both the sets of functors are connected sequences, of course right, covariant and from $\mathcal{C}_A$ to $\mathcal{C}_B$. Also for $n > 0$ trivially $H^n_{\mathfrak{a}}(E) = 0$ whenever E is injective A-module and by 3.15 we see that $H^n_{\mathfrak{a}\cdot B}(E \otimes_A B)$ is zero for such an E if $n > 0$. Also by 3.16 we have the statement proved for $n = 0$. The result follows then from [**9**], theorem 6.10 and its corollary. $\square$

This theorem will be a basic tool in chapters 4 and 6.

We will finish this chapter with the preparation and proof of a statement that is similar to 3.17 and will find application in the next chapter. What follows is taken from [**24**], paragraph 4. We will make clear the notation used in the remainder of the chapter.

A, B will denote commutative Noetherian rings, such that $f : A \to B$ is a ring homomorphism. If $\mathfrak{a}$ is an ideal of A then $\mathfrak{a}^e$ will denote the ideal $f(\mathfrak{a}) \cdot B$ of B, whereas if $\mathfrak{b}$ is an ideal of B, $\mathfrak{b}^c$ will denote the ideal $f^{-1}(\mathfrak{b})$ of A. Every B-module M can be considered as A-module by means of f. So we can calculate for an ideal $\mathfrak{a}$ of A and a B-module M the local cohomology by considering M as an A-module via f and then calculating the local cohomology of this A-module. On the other hand, we could calculate the local cohomology of the B-module M with respect to the B-ideal $\mathfrak{a}^e$ and then consider these B-modules as A-modules via f. It would be nice if these two ways would lead to the same result, wouldn't it ?

LEMMA 3.18. *Suppose, that E is an indecomposable injective B-module. Then E is $\Gamma_{\mathfrak{a}}$-acyclic for all ideals $\mathfrak{a}$ of A when regarded as A-module via f.*

PROOF. By 7.17 $E = E(B/\mathfrak{p})$ for some prime $\mathfrak{p}$ of B. We have as usual two cases:

a) If $\mathfrak{a} \not\subseteq \mathfrak{p}^c$, then there is an a in $\mathfrak{a}\backslash\mathfrak{p}^c$ and by 7.22 multiplication by a provides an A-automorphism on E, which by the A-linearity of $\Gamma_{\mathfrak{a}}$ implies that multiplication by a induces an A-automorphism of $H^i_{\mathfrak{a}}(E)$ for all $i \in \mathbb{N}$. But each $z \in H^i_{\mathfrak{a}}(E)$ is represented by $\zeta + \operatorname{im}\Gamma_{\mathfrak{a}}(E^{i-1} \to E^i)$ where E^i and E^{i-1} are the i^{th} and $(i-1)^{st}$ term of an injective resolution of M respectively and $\zeta \in \Gamma_{\mathfrak{a}}(E^i)$. So there exists $n \in \mathbb{N}$ for which $\mathfrak{a}^n\zeta = 0$ such that $\mathfrak{a}^n z = 0$. So $z = 0$. So $H^i_{\mathfrak{a}}(E) = 0$ in this case for all $i \in \mathbb{N}$.

b) If $\mathfrak{a} \subseteq \mathfrak{p}^c$, then E is $\mathfrak{a}$-torsion by 1.16 and by 1.18 E is $\Gamma_{\mathfrak{a}}$-acyclic. $\square$

COROLLARY 3.19. *Suppose now, that E is an arbitrary injective B-module. Then if E is considered to be an A-module via f, E is $\Gamma_{\mathfrak{a}}$-acyclic for all ideals $\mathfrak{a}$ of A.*

PROOF. By 7.21 E is direct sum of injective indecomposable B-modules which by 3.18 are all $\Gamma_{\mathfrak{a}}$-acyclic for all ideals $\mathfrak{a}$ in A. By the fact that all $H^i_{\mathfrak{a}}$ are A-linear functors they commute with finite direct sums. So whenever E is finite direct sum of indecomposable injective modules, it is $\Gamma_{\mathfrak{a}}$-acyclic for all $\mathfrak{a} \subseteq A$. If E is infinite direct sum of injective indecomposable submodules, we invoke lemma 2.13 together with 3.18 and 2.28 to see that for all $\mathfrak{a}$ and $i \in \mathbb{N}^+, H^i_{\mathfrak{a}}(E)$ is direct limit of zeros, hence zero itself. □

THEOREM 3.20. *Let $\mathfrak{a}$ be an ideal of A. The connected right sequences of covariant functors from $\mathcal{C}_B$ to $\mathcal{C}_A$, $\{H^i_{\mathfrak{a}}(-)\}_{i\epsilon\mathbb{N}}$ and $\{H^i_{\mathfrak{a}^e}(-)\}_{i\epsilon\mathbb{N}}$ are isomorphic, where the latter is to be interpreted as 'first calculate local cohomology with respect to $\mathfrak{a}^e$ and then restrict the result to A' whereas the first is to be interpreted as 'first restrict to A and then form $H^i_{\mathfrak{a}}$ of the result'.*

PROOF. We first show, that the claim is true for $i = 0$. Let M be a B-module. Then $\Gamma_{\mathfrak{a}}(M) = \{m \in M : \exists n \in \mathbb{N} : \mathfrak{a}^n \cdot m = 0\} = \{m \in M : \exists n \in \mathbb{N} : (\mathfrak{a}^e)^n m = 0\} = \Gamma_{\mathfrak{a}^e}(M)$ restricted to A since multiplication of m by elements of $\mathfrak{a}$ is defined via $f(\mathfrak{a}) \cdot B$. Further, if $g : M \to N$ is a B-homomorphism, then $\Gamma_{\mathfrak{a}^e}(g)$ is the restriction of g to $\Gamma_{\mathfrak{a}}(M)$ since $\Gamma_{\mathfrak{a}^e}(g)$ is restriction of g and restriction to A is to take the same map. Also $\Gamma_{\mathfrak{a}}(g)$ is the restriction of a B-map to an A-map followed by the restriction to $\Gamma_{\mathfrak{a}}(M)$. So both maps coincide. Hence $\Gamma_{\mathfrak{a}}(.)$ and $\Gamma_{\mathfrak{a}^e}(.)$ are naturally equivalent functors.

Now for a B-injective module E it is trivial that $H^i_{\mathfrak{a}^e}(E)$ is zero for all $i > 0$ and from 3.19 we know this for $H^i_{\mathfrak{a}}(E)$ for $i > 0$. By [**9**], Theorem 6.10 and its corollary we are done. □

CHAPTER 4

Zeros

In this chapter we will have a large account concerning the calculation of local cohomology modules. This will involve to a large extent the work done in the previous chapter. Also we will make good use of the results of chapter 2, where we established close relationships between local cohomology modules and the functors Hom and Ext. Since these functors are the concern of many branches of algebra, there are some auxiliary results, we will invoke. To this we first give some reminders concerning regular sequences and related things.

DEFINITION 4.1. Let M be an A-module. An element $a \in A$ is said to be *M-regular* if $a \cdot x \neq 0$ for all $0 \neq x \in M$. A sequence $a_1, \dots, a_n$ is said to be an *M-sequence*, or an *M-regular sequence*, if the following two conditions hold:

1. a_1 is M-regular, a_2 is M/a_1M-regular,..., a_n is $M/(a_1, \dots, a_{n-1})M$-regular;
2. $M \neq (a_1, \dots, a_n) \cdot M$.

If $(\mathfrak{a}) = (a_1, a_2, \dots, a_n)$ is an M-sequence, we will denote by $(\mathfrak{a})$ both the sequence and the ideal generated by the members of the sequence.

We cite now some theorems about M-sequences without proof. The references given are always with respect to [**7**] so that the interested reader is referred to this book.

THEOREM 4.2. *(see 16.1) If $a_1, \dots, a_n$ is an M-sequence, then so is $a_1^{\gamma_1}, a_2^{\gamma_2}, \dots, a_n^{\gamma_n}$ for any set of positive integers $\{\gamma_i\}_{1 \leq i \leq n}$.* □

THEOREM 4.3. *(see 16.6) Let $\mathfrak{a}$ be an ideal of A, A a Noetherian ring, M a finitely generated A-module and $\mathfrak{a} \cdot M \neq M$. Then for a given integer n the following are equivalent:*

1. $\mathrm{Ext}_A^i(N, M) = 0$ *for all $i \leq n - 1$ and for any finitely generated A-module N such that* $\mathrm{supp}(N) \subseteq Var(\mathfrak{a})$,
2. $\mathrm{Ext}_A^i(A/\mathfrak{a}, M) = 0$ *for all $i \leq n - 1$*,
3. $\mathrm{Ext}_A^i(N, M) = 0$ *for all $i < n$ and some finitely generated A-module N with* $\mathrm{Var}(\mathfrak{a}) = \mathrm{supp}(N)$,
4. *There exists an M-sequence of length n contained in $\mathfrak{a}$.* □

It is a straightforward consequence of the latter theorem, that for a finite A-module M (meaning finitely generated) the length of all maximal

M-sequences contained in $\mathfrak{a}$ (maximal means, that there is no other M-sequence contained in $\mathfrak{a}$ which is a proper superset of it) is a well determined integer, whenever $\mathfrak{a} \cdot M \neq M$.(See [**7**], Theorem 16.7)

DEFINITION 4.4. Let M be a finite module over A, $\mathfrak{a}$ an ideal in A such that $\mathfrak{a} \cdot M \neq M$. Then the well defined integer that represents the length of a maximal M-sequence contained in $\mathfrak{a}$ is called the *depth of M with respect to* $\mathfrak{a}$ or the $\mathfrak{a}$-*depth of* M, written $\operatorname{depth}(\mathfrak{a}, M)$. If it happens, that $\mathfrak{a} \cdot M = M$, we will write $\operatorname{depth}(\mathfrak{a}, M) = \infty$. In the light of theorems 4.2 and 4.3 we have

$$\operatorname{depth}(\mathfrak{a}, M) = \inf\{i \in \mathbb{N} : \operatorname{Ext}^i_A(A/\mathfrak{a}, M) \neq 0\}.$$

Before we now start to use these things we will restate a result we proved earlier to finish an inductive argument but had the opinion, it would be placed better in this chapter by nature.

THEOREM 4.5. *Let $A, M, \mathfrak{a}$ be as usual. Then $\mathfrak{a}$ is finitely generated. If n denotes the number of elements of $\mathfrak{a}$ which is at least needed to generate it, then for all $i \in \mathbb{N}$,*

$$H^{n+i+1}_{\mathfrak{a}}(M) = 0.$$

PROOF. By the first two paragraphs of the proof of 3.11, the theorem is proved for $n = 0$ and 1.

Suppose inductively, that $r > 1$ and the theorem is proved for all smaller values of n than r. Let $\mathfrak{a}$ be generated by r elements, say $\mathfrak{a} = (g_1, \dots, g_r)$. Set $\mathfrak{b} = (g_1, \dots, g_{r-1})$. Then $\mathfrak{b}$ and $g_r \cdot A$ are generated by less than r elements and we can apply the inductive hypothesis to them. Further we know from 1.12 that $H^i_{\mathfrak{b}\cdot(g_r \cdot A)}(-)$ and $H^i_{\mathfrak{b}\cap g_r \cdot A}(-)$ are naturally equivalent functors for all $i \in \mathbb{N}$. The Mayer-Vietoris-sequence 3.6 yields then the existence of an exact sequence

$$H^{r+i}_{\mathfrak{b}\cdot(g_r\cdot A)}(M) \to H^{r+i+1}_{\mathfrak{b}+g_r\cdot A}(M) \to H^{r+i+1}_{\mathfrak{b}}(M) \oplus H^{r+i+1}_{g_r\cdot A}(M)$$

for all $i \in \mathbb{N}$. By inductive hypothesis, the left hand term is zero for all $i \in \mathbb{N}$ since $\mathfrak{b}$ is generated by $r-1$ elements and the right hand term is zero by the inductive hypothesis again and the fact that $r + i + 1 > 1$. It follows, that the middle term is zero for all $i \in \mathbb{N}$, that is, $H^{r+i+1}_{\mathfrak{a}}(M) = 0$. The theorem follows by induction. □

This theorem providing an upper bound for local cohomology modules being nonzero is in contrast to the following, which is taken from [**22**], Theorem 2.1. In this following statement we adopt the convention, that for a local ring A with maximal ideal $\mathfrak{m}$ we denote $\operatorname{depth}(\mathfrak{m}, M)$ just by $\operatorname{depth}(M)$.

PROPOSITION 4.6. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Suppose M is a nonzero finitely generated A-module. Then* $\operatorname{depth}(M)$ *is the least integer i for which $H^i_{\mathfrak{m}}(M) \neq 0$.*

PROOF. By definition, $H^i_{\mathfrak{m}}(M)$ is isomorphic to the ith cohomology module of the complex obtained by application of $\Gamma_{\mathfrak{m}}$ to a minimal injective resolution of M. Let r denote depth(M). Then by 7.23 and 4.3, r is the smallest integer i for which $\mu^i(\mathfrak{m}, M) > 0$. Then by 1.16 all $H^i_{\mathfrak{m}}(M) = 0$ for $i \leq r-1$. Also

$$H^r_{\mathfrak{m}}(M) \cong \ker(\Gamma_{\mathfrak{m}}(E^r(M) \to E^{r+1}(M))/\operatorname{im}(\Gamma_{\mathfrak{m}}(E^{r-1}(M) \to E^r(M)))$$

$= \{\ker(E^r(M) \to E^{r+1}(M))\} \cap \Gamma_{\mathfrak{m}}(E^r(M))$ since $\Gamma_{\mathfrak{m}}(E^{r-1}) = 0$. Now $K := \ker(E^r(M) \to E^{r+1}(M))$ cannot be zero since E^r would otherwise have to be zero because the latter is an essential extension of the former by definition. But from $\mu^r(\mathfrak{m}, M) \neq 0$ follows E^r nonzero. Further $\Gamma_{\mathfrak{m}}(E^r)$ is nonzero because $\mu^r(\mathfrak{m}, M) \neq 0$. Hence K and $\Gamma_{\mathfrak{m}}(E^r)$ meet nontrivially. So $H^r_{\mathfrak{m}}(M) \neq 0$. □

We give now a nonlocal version to this proposition, using a different approach. This proof is taken from [**1**], Ch. 7.

THEOREM 4.7. *Let A be a Noetherian commutative ring. Let M be a nonzero finitely generated A-module. Suppose $\mathfrak{a}$ is an ideal of A such that $\mathfrak{a} \cdot M \neq M$. Then* depth$(\mathfrak{a}, M)$ *is finite and equal to the least integer i for which $H^i_{\mathfrak{a}}(M) \neq 0$.*

PROOF. Let $(\mathfrak{x}) = (x_1, \dots, x_r)$ be a maximal M-sequence contained in $\mathfrak{a}$ such that $r = \operatorname{depth}(\mathfrak{a}, M)$. Then, by 4.2, $(x_1^n, \dots, x_r^n)$ is an M-sequence for all $n \in \mathbb{N}^+$ and of course contained in $\mathfrak{a}^n$. So depth$(\mathfrak{a}^n, M) \geq r$. On the other hand, depth$(\mathfrak{a}^n, M)$ cannot exceed r, since $\mathfrak{a}^n$ is subset of $\mathfrak{a}$. So for all $n \in \mathbb{N}^+$, depth$(\mathfrak{a}^n, M) = r$.

So we have, by 4.3,

$$\operatorname{Ext}^i_A(A/\mathfrak{a}^n, M) = 0 \text{ for all } i < r \text{ and } n \in \mathbb{N}.$$

By 2.27 this implies that

$$H^i_{\mathfrak{a}}(M) \cong \varinjlim(\operatorname{Ext}^i_A(A/\mathfrak{a}^n, M)) = \varinjlim(0) \text{ for all } i < r.$$

So it remains to show that $H^r_{\mathfrak{a}}(M) \neq 0$. We will do this by induction on r.

If $r = \operatorname{depth}(\mathfrak{a}, M) = 0$, then by definition of depth there is no nonzerodivisor on M in $\mathfrak{a}$ and so $\mathfrak{a}$ has to be contained in the union of the associated primes of M. Since ass(M) is finite, by [**12**], Theorem 3.61 follows that $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{ass}(M)$. So there exists a submodule of M isomorphic to $A/\mathfrak{p}$ by [**12**], 9.33. Since $A/\mathfrak{p}$ is annihilated by $\mathfrak{a}$, it follows that $\Gamma_{\mathfrak{a}}(M) \cong H^0_{\mathfrak{a}}(M) \neq 0$.

Now suppose that depth$(\mathfrak{a}, M) = r \geq 1$ and we have proved the theorem for all values of depth$(\mathfrak{a}, M)$ smaller than r. Let x_1 be a nonzerodivisor on M in $\mathfrak{a}$ which exists by the fact that depth$(\mathfrak{a}, M) \geq 1$. Suppose $(M/x_1M) = \mathfrak{a}(M/x_1M)$, then $\mathfrak{a}M + x_1M = M$ such that $\mathfrak{a}M = M$, a contradiction. So depth$(\mathfrak{a}, M/x_1M) = r-1$ by the remark following 4.3. By the inductive hypothesis,

$$H^{r-1}_{\mathfrak{a}}(M/x_1M) \neq 0.$$

Since x is M-regular, there is an exact sequence $0 \to M \overset{x_1}{\to} M \to M/x_1{\cdot}M \to 0$ giving rise to a long exact H-sequence, a part of which is

$$H^{r-1}_{\mathfrak{a}}(M) \to \ H^{r-1}_{\mathfrak{a}}(M/x_1M) \to \ H^{r}_{\mathfrak{a}}(M).$$

The first part of the proof gives that the left term is zero and the inductive hypothesis assures that the middle term is *non*zero such that by the exactness $H^r_{\mathfrak{a}}(M)$ cannot vanish. This completes the inductive step and the proof is complete. □

While 4.5 gave a criterion for the vanishing of certain local cohomology modules using an invariant of the corresponding ideal alone, and 4.6 one using both the ideal and the module, we will now have a statement involving only information from the module in question.(See [**24**], Theorem 6.1)

PROPOSITION 4.8. *Let A be a Noetherian ring, M a finitely generated A-module of Krulldimension 0 and $\mathfrak{a}$ an ideal of A. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i \geq 1$. (Remember, that the Krulldimension of an A-module M is defined to be the length n of the longest sequence of primes of A, $\mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n \in \operatorname{Spec} A$ such that each prime of the chain belongs to* supp(M). *See e.g.* [**7**], *Chapter 2, paragraph 5.)*

PROOF. The assumptions imply by [**12**], 9.20 that a prime ideal $\mathfrak{p}$ of A belongs to the support of M if and only if it contains the annihilator of M. Since the Krulldimension of M is 0, this implies that the dimension of M is 0 so that ann(M) has only maximal ideals of A in its variety. By [**12**], 8.38 then $A/\operatorname{ann}(M)$ is Artinian so that by [**12**], 7.20 M is Artinian and by [**12**], 7.36 there exists a finite set of submodules of M

$$0 \subsetneq M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_r = M$$

such that for all $1 \leq i \leq r$ we have $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i of A$. By the following argument each of these $\mathfrak{p}_i$ is in supp(M):

The above sequence shows that

$$0 \neq \ (A/\mathfrak{p}_{i+1})_{\mathfrak{p}_{i+1}} \cong \ (M_{i+1}/M_i)_{\mathfrak{p}_{i+1}} \cong \ (M_{i+1})_{\mathfrak{p}_{i+1}}/(M_i)_{\mathfrak{p}_{i+1}}$$

by [**12**], 9.12. Hence $(M_{i+1})_{\mathfrak{p}_{i+1}}$ is nonzero and so $\mathfrak{p}_{i+1}$ is in supp(M). It follows, that all $\mathfrak{p}_i$ are maximal. We show now that if $M' \cong A/\mathfrak{p}$ for $\mathfrak{p}$ maximal in Spec(A), then $H^i_{\mathfrak{a}}(M') = 0$ for $i \geq 1$.

For, we consider a minimal injective resolution of M'

$$0 \to A/\mathfrak{p} \to E^0(A/\mathfrak{p}) \to E^1(A/\mathfrak{p}) \to \ldots \to E^n(A/\mathfrak{p}) \to \ldots .$$

$A/\mathfrak{p}$ is $\mathfrak{p}$-torsion, so by 1.17 and 7.22, if $\mathfrak{p}'$ is a prime of A different from $\mathfrak{p}$, then $\mu^i(\mathfrak{p}', A/\mathfrak{p}) = 0$ for $\mathfrak{p}' \subsetneq \mathfrak{p}$ and all i, such that by the maximality of $\mathfrak{p}$

$$E^i(A/\mathfrak{p}) \cong \bigoplus_{1}^{\mu^i(\mathfrak{p}, A/\mathfrak{p})} E(A/\mathfrak{p})$$

It may now happen

a) $\mathfrak{a} \not\subseteq \mathfrak{p}$.

Then there exists $x \in \mathfrak{a} \backslash \mathfrak{p}$, and by 7.22 we are done.

b) $\mathfrak{a} \subseteq \mathfrak{p}$.

By 1.16 $E(A/\mathfrak{p})$ is then $\mathfrak{a}$-torsion such that by 1.18 $A/\mathfrak{p}$ is $\Gamma_{\mathfrak{a}}$-acyclic.

So for $\mathfrak{p}$ maximal in $\mathrm{Spec}(A)$, $H^i_{\mathfrak{a}}(A/\mathfrak{p}) = 0$ for $i \geq 1$. So in our chain

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M,$$

$H^i_{\mathfrak{a}}(M_0) = H^i_{\mathfrak{a}}(M_1) = 0$ for $i \geq 1$. Assume inductively that we have shown for $0 \leq j \leq t \in \mathbb{N}$ that $H^i_{\mathfrak{a}}(M_j) = 0$ for $i > 0$. This is certainly true for $t = 1$. Then from

$$0 \to M_t \to \ M_{t+1} \to \ M_{t+1}/M_t \to \ 0,$$

the natural exact sequence with $M_{t+1}/M_t \cong A/\mathfrak{p}_{t+1}$, follows by application of the functors $\{H^i_{\mathfrak{a}}\}_{i\epsilon\mathcal{N}}$ that for all $1 \leq i$

$$0 = H^i_{\mathfrak{a}}(M_t) \to H^i_{\mathfrak{a}}(M_{t+1}) \to H^i_{\mathfrak{a}}(A/\mathfrak{p}) = 0$$

is exact, proving that M_{t+1} is $\Gamma_{\mathfrak{a}}$-acyclic as well. By induction we have proved that for $t = n$, $H^i_{\mathfrak{a}}(M_n) = H^i_{\mathfrak{a}}(M) = 0$ for all $1 \leq i$. □

We now extend this result to many more modules. (Taken again from [**24**], Theorem 6.1)

THEOREM 4.9. *Let A be a Noetherian commutative local ring. Let M be an arbitrary A-module of Krulldimension $s < \infty$. Then $H^i_{\mathfrak{a}}(M) = 0$ for all ideals $\mathfrak{a}$ and all integers $i \geq s + 1$.*

PROOF. The case $s = 0$ has been proved for finitely generated modules in 4.8. We consider the direct system of 2.4 with respect to M and observe, that any submodule of M has Krulldimension not exceeding the Krulldimension of M itself. By 2.28, taking local cohomology modules commutes with other direct limits, so that whenever i is a positive integer and the Krulldimension of M is zero, $H^i_{\mathfrak{a}}(M)$ is isomorphic to the direct limit of zeros, hence zero itself. So suppose inductively that $1 \leq k \in \mathbb{N}$ and that whenever N is an A-module with Krulldimension $s \leq k - 1$, then $H^i_{\mathfrak{a}}(N) = 0$ for all $i \geq s + 1$. Suppose, M is a finite A-module with Krulldimension k. By [**12**], 9.40 there is a finite chain of submodules of M

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

such that for all $1 \leq i \leq r$, $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$ in A. Then all these $\mathfrak{p}_i$ belong to $\mathrm{supp}(M)$ by the argument in 4.8. So $\dim(A/\mathfrak{p}) \leq k$ for all $\mathfrak{p}_i$ in this chain.

By an inductive argument similar to that at the end of 4.8 it is then enough to show that for all prime ideals $\mathfrak{p}$ of A with $\dim(A/\mathfrak{p}) = k$ we have $H^i_{\mathfrak{a}}(A/\mathfrak{p}) = 0$ for all $i \geq k + 1$. This we do.

Let $\kappa = (A/\mathfrak{p})_{\mathfrak{p}}$ be regarded as A-module in the natural way. Let further $\alpha : A/\mathfrak{p} \to \kappa$ be the natural homomorphism, which is injective by the fact that $A/\mathfrak{p}$ is integral domain. Let $H = \kappa/(A/\mathfrak{p})$ be the cokernel of α. Then we have an exact sequence

$$0 \to A/\mathfrak{p} \to \kappa \to H \to 0.(*)$$

We will show now that $H^i_{\mathfrak{a}}(\kappa) = 0$ for all $1 \leq i$.

a) $\mathfrak{a} \not\subseteq \mathfrak{p}$

then there exists $x \in \mathfrak{a}\backslash\mathfrak{p}$ and multiplication by it provides an isomorphism on κ. This is due to the fact that $\mathfrak{p}$ is prime: x provides a monomorphism on $A/\mathfrak{p}$ trivially and $x + \mathfrak{p}$ being not the zero element in $A/\mathfrak{p}$ becomes unit in $(A/\mathfrak{p})_{\mathfrak{p}}$. By the fact that $\Gamma_{\mathfrak{a}}$ is A-linear, multiplication by x induces an automorphism on $H^i_{\mathfrak{a}}(\kappa)$. But each $z \in H^i_{\mathfrak{a}}(\kappa)$ for $1 \leq i$ is representable as $\zeta + \mathrm{im}(\Gamma_{\mathfrak{a}}(E^{i-1}(\kappa) \to E^i(\kappa)))$ where the E^i constitute a minimal injective resolution for M and ζ is an element of $\Gamma_{\mathfrak{a}}(E^i(\kappa))$ and hence annihilated by some power of $\mathfrak{a}$ and so by some power of x as well. This only being possible if $z = 0$ shows that $H^i_{\mathfrak{a}}(\kappa) = 0$ if $1 \leq i$.

b) $\mathfrak{a} \subseteq \mathfrak{p}$

then κ is $\mathfrak{a}$-torsion and we saw in 1.18 that then $H^i_{\mathfrak{a}}(\kappa) = 0$ for $1 \leq i$.

So always $H^i_{\mathfrak{a}}(\kappa) = 0$ for $1 \leq i$. Now consider again the exact sequence (*). Since localisation is an exact functor and $\alpha_{\mathfrak{p}} : (A/\mathfrak{p})_{\mathfrak{p}} \to \kappa_{\mathfrak{p}}$ is an isomorphism, $H_{\mathfrak{p}} = 0$. So $\mathrm{supp}_A(H) \subseteq \mathrm{supp}_A(A/\mathfrak{p})\backslash\{\mathfrak{p}\}$. Since of course $\mathfrak{p}$ is the unique minimal ideal in $\mathrm{supp}_A(A/\mathfrak{p})$, the Krulldimension of H is $\leq k-1$. By the inductive hypothesis $H^i_{\mathfrak{a}}(H) = 0$ whenever $i \geq k$. Now (*) induces a long exact sequence, and a part of it is

$$\ldots \to H^k_{\mathfrak{a}}(H) \to H^{k+1}_{\mathfrak{a}}(A/\mathfrak{p}) \to H^{k+1}_{\mathfrak{a}}(\kappa) \to \ldots$$

Now for all $1 \leq i$, $H^i_{\mathfrak{a}}(\kappa) = 0$ and for all $i \geq k$, $H^i_{\mathfrak{a}}(H) = 0$ such that by the exactness of the displayed sequence $H^i_{\mathfrak{a}}(A/\mathfrak{p}) = 0$ for all $i \geq k+1$.

As indicated above it is now clear that this implies the correctness of the statement of the theorem for all finitely generated modules of Krulldimension k.

Now we can consider the direct system of 2.4 for an arbitrary A-module M and observe that any submodule of M has Krulldimension at most equal to that of M itself. It follows that whenever i exceeds the Krulldimension k of M, then $H^i_{\mathfrak{a}}(M') = 0$ for all finitely generated submodules of M. So the direct limit of these $H^i_{\mathfrak{a}}(M')$ is zero and equals on the other hand $H^i_{\mathfrak{a}}(\varinjlim(M'))$ since we know that taking local cohomology modules commutes with other direct limits over directed sets. (See 2.28.) But this is exactly $H^i_{\mathfrak{a}}(M)$ by 2.9. We have therefore for all A-modules M of Krulldimension k proved that $i \geq k+1$ implies that $H^i_{\mathfrak{a}}(M) = 0$ for all ideals $\mathfrak{a}$ of A. The theorem follows by induction. □

We want to point out here that there are other proofs of this theorem, one for example is to be found in [**3**], 1.12. However the reader should not expect to understand the proof there without having either experiences in sheaf theory or a lot of patience.

If we now put together what we have learned in the previous three statements, then if A is a Noetherian local ring, $\mathfrak{a}$ the maximal ideal of A and M an arbitrary module then whenever $i \in \mathbb{N}$ is such that $H^i_{\mathfrak{m}}(M) \neq 0$, we must have $i \leq$ Krulldimension(M); if M is finitely generated, $H^i_{\mathfrak{m}}(M) \neq 0$ implies $i \geq \operatorname{depth}(\mathfrak{m}, M)$ for such i and $H^{\operatorname{depth}(\mathfrak{m},M)}_{\mathfrak{m}}(M) \neq 0$. The reader may wonder what about the upper bound for such i - can it be improved? We answer this question now with a theorem to be found in [**1**], Chapter 6.

At this point we want to tell the reader, that there are yet more methods to investigate local cohomology than we have encountered up to now. For example, local cohomology modules occur as (co-)homology modules of suitable Koszul- or Čech-complexes. We will not further explain this and the interested reader is best referred to [**1**].

Another way of investigation arises by what is called *secondary representation of a module* This is a tool of commutative algebra introduced and developed by I. G. Macdonald, which is dual to the usual concept of primary dcomposition. We will give some results of this theory in the next definition but we do not intend to prove these statements.

Definition 4.10. We say that the A-module M is *secondary* if and only if multiplication of M by $a \in A$ provides either a surjective or a nilpotent endomorphism on M for each $a \in A$. If M happens to be secondary, then $\sqrt{\operatorname{ann}_A(M)}$ is a prime ideal and M is said to be $\mathfrak{p}$-secondary.

Further, if there is a finite family of $\mathfrak{p}$-secondary modules $\{M_i\}_{1 \leq i \leq n}$ for $n \in \mathbb{N}$, then the sum of these A-modules is again $\mathfrak{p}$-secondary. (Remember: the finite intersection of $\mathfrak{q}$-primary submodules was again $\mathfrak{q}$-primary.)

If for an A-module M there are submodules $M_i, 1 \leq i \leq n$ for some $n \in \mathbb{N}$ such that

$$M = M_1 + \ldots + M_n$$

where M_i is $\mathfrak{p}_i$-secondary for $1 \leq i \leq n$, then the above line will be called a *secondary decomposition* for M. Such a decomposition is said to be *minimal* whenever all prime ideals involved are different and no module of the sum is superfluous. The reader may note that by the fact that finite sums of secondary modules are secondary, one can produce a minimal decomposition from a given arbitrary decomposition.

As for primary decomposition, the prime ideals which occur in a minimal secondary decomposition for the A-module M satisfy the uniqueness property one might hope: in any two minimal secondary decompositions for the A-module M, the sets of prime ideals involved are identical. This set (with respect to the module M) is called the *attached prime ideals of M* and is written $\operatorname{att}_A(M)$.

To have any use of these things one has to show, that there are modules which have secondary decomposition. In fact there are and it is no surprise that one can show, that all Artinian modules do. (The proof is similar to the primary case and uses sum-irreducible modules.) However, the class of modules that actually have secondary decomposition is larger than the class of Artinian modules: for example in [**25**] is shown that injective modules over Noetherian rings have secondary decompositions and of course there are injective non-Artinian modules over Noetherian rings. The interested reader is also referred to [**22**].

We give now a theorem taken from [**22**] (Theorem 2.2), which has local cohomology as topic and involves in statement as well as in proof the concept just described.

THEOREM 4.11. *Suppose, $(A, \mathfrak{m})$ is a local commutative Noetherian ring. Let M be a nonzero finitely generated A-module of Krulldimension $s = \dim(M)$.*
Then

$$H^s_{\mathfrak{m}}(M) \neq 0 \text{ and } \operatorname{Att}_A(H^s_{\mathfrak{m}}(M)) = \{\mathfrak{p} \in \operatorname{Ass}(M) : \dim(A/\mathfrak{p}) = s\}.$$

□

The reader will admit that especially the second part of this statement is rather striking. The following is an application of this theorem.

COROLLARY 4.12. *Let A be an arbitrary Noetherian commutative ring and M a nonzero finitely generated A-module. Let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a} + \operatorname{ann}(M)$ is a proper ideal of A. Let $\mathfrak{p}$ be a minimal prime of $\mathfrak{a} + \operatorname{ann}(M)$ and let $r = \dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$. Then $H^r_{\mathfrak{a}}(M) \neq 0$.*

PROOF. Let $\mathfrak{b} = \operatorname{ann}(M)$. Then M can be viewed as an $A/\mathfrak{b}$-module in the natural way. By 3.20 applied to the natural projection $A \to A/\mathfrak{b}$ we can calculate the local cohomology over $A/\mathfrak{b}$ so that

$$H^r_{\mathfrak{a}}(M) \cong H^r_{(\mathfrak{a}+\mathfrak{b})/\mathfrak{b}}(M)$$

and again we use 3.20 to get

$$H^r_{(\mathfrak{a}+\mathfrak{b})/\mathfrak{b}}(M) \cong H^r_{\mathfrak{a}+\mathfrak{b}}(M),$$

M on the right side again being considered as A-module.

From the fact that $\mathfrak{a} + \mathfrak{b}$ has $\mathfrak{p}$ as minimal prime follows that $(\mathfrak{a} + \mathfrak{b})_{\mathfrak{p}}$ has $\mathfrak{p}A_{\mathfrak{p}}$ as minimal prime. But the latter is maximal, so the former is $\mathfrak{p}A_{\mathfrak{p}}$-primary. So

$$H^r_{(\mathfrak{a}+\mathfrak{b})A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = H^r_{\mathfrak{p}A_{\mathfrak{p}}}(M))$$

by 1.11 and by 4.11 the right hand term is nonzero. If we put together the information we have collected, we obtain

$$\begin{aligned}(H^r_{\mathfrak{a}}(M))_{\mathfrak{p}} &\cong (H^r_{\mathfrak{a}+\mathfrak{b}}(M))_{\mathfrak{p}} \text{by the above argument}\\ &\cong H^r_{(\mathfrak{a}+\mathfrak{b})A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \text{by the flat base change}\\ &\cong H^r_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \text{by} 1.11\\ &\neq 0\end{aligned}$$

whence $H^r_{\mathfrak{a}}(M) \neq 0$. □

We conclude that for a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and M a nonzero finitely generated A-module the possible integers i for which $H^i_{\mathfrak{m}}(M) \neq 0$, have to be between $\operatorname{depth}(M)$ and $\operatorname{Krulldim.}(M) = \dim(M)$ and the values of i on either side correspond to nonzero local cohomology modules. Since every nonzero finitely generated module has a dimension, this means that for such a module M over such a ring there always exists an i such that the i^{th} right derived functor of $\Gamma_{\mathfrak{m}}(-)$ evaluated at M is nonzero. So the question arises, whether there are other general constraints we have not yet treated which force $H^i_{\mathfrak{m}}(M)$ to be zero. That means, we are interested in the structure of the subset $\Theta(\mathfrak{m}, M) \subseteq \mathbb{N}$ which is defined to consist exactly of these $i \in \mathbb{N}$ for which $H^i_{\mathfrak{m}}(M) \neq 0$. We will show that if A is allowed to vary over all Noetherian local rings, then for each finite $\Theta \subseteq \mathbb{N}$ there is a Noetherian ring A and a finitely generated module M over A such that $\Theta = \Theta(\mathfrak{m}, M)$. The idea and proof are due to I. G. Macdonald and to be found in [**21**].

THEOREM 4.13. *Let Θ be a finite subset of $\mathbb{N}$. Then there is a Noetherian local A-algebra $(B, \mathfrak{n})$ over a Noetherian local ring $(A, \mathfrak{m})$ such that for $i \in \mathbb{N}$ we have*

$$H^i_{\mathfrak{n}}(B) \neq 0 \Leftrightarrow i \in \Theta$$

PROOF. Let n be the largest integer in Θ. Let κ be any field. Then let A be the power series ring over κ in the n indeterminates $x_1, \dots, x_n$. Then A is a regular Noetherian local ring with dimension n and maximal ideal $\mathfrak{m} = (x_1, \dots, x_n) \cdot A$. (See [**12**], 8.13, 3.19 and 15.29.) Further it is clear that $x_1, \dots, x_n$ is an A-sequence in $\mathfrak{m}$. So $\operatorname{depth}(A) \geq n$. Hence $\Theta(\mathfrak{m}, A) = \{n\}$. For each $i \in \mathbb{N}$ let $\mathfrak{p}_i$ be the ideal generated by $x_{i+1}, \dots, x_n$. Then by [**12**], 15.38 each $\mathfrak{p}_i$ is prime. Let further M be the A-module defined by

$$M = \bigoplus_{i \epsilon \Theta \backslash \{n\}} A/p_i.$$

We are interested in $\Theta(\mathfrak{m}, M)$. By 4.6 and the fact that $x_1, \dots, x_i$ is an $A/\mathfrak{p}_i$-sequence contained in $\mathfrak{m}$, all $j \in \Theta(\mathfrak{m}, A/\mathfrak{p}_i)$ have to be at least i. On the other hand, the annihilator of $A/\mathfrak{p}_i$ is equal to $\mathfrak{p}_i$ and so $\dim(A/\mathfrak{p}_i) = i$. Since A/p_i is finitely generated, $\dim(A/\mathfrak{p}_i) = \operatorname{Krulldim}(A/\mathfrak{p}_i)$ so that by 4.8 and the previous comment $\Theta(\mathfrak{m}, A/\mathfrak{p}_i) = \{i\}$.

Now we know that Γ is additive for every A-ideal $\mathfrak{a}$ and therefore the same is true for the $H^i_{\mathfrak{m}}$. From this it follows, (see e.g. the proof of 3.5) that $H^i_{\mathfrak{a}}(N \oplus N') = H^i_{\mathfrak{a}}(N) \oplus H^i_{\mathfrak{a}}(N')$ and therefore $\Theta(\mathfrak{m}, N \oplus N') = \Theta(\mathfrak{m}, N) \cup \Theta(\mathfrak{m}, N')$ for all A-modules N, N'. Hence

$$\Theta(\mathfrak{m}, M) = \bigcup_{i \epsilon \mathcal{N} \backslash \{n\}} \Theta(\mathfrak{m}, A/p_i) = \Theta \backslash \{n\}.$$

We will now explain, how one can on the direct sum of a ring and a module over this ring define a new ring structure. For let R be a Noetherian local ring and N an R-module. Let $S = R \oplus N$. Define a multiplication on S by

$$(r \oplus x) \cdot (s \oplus y) = rs \oplus ry + sx.$$

This is clearly commutative with identity (1,0),

$$(r \oplus x) \cdot [(s \oplus y) \cdot (t \oplus z)] = (r \oplus x)(st \oplus sz + ty) = rst \oplus rsz + rty + stx$$

which is obviously symmetric in r, s, t and x, y, z so that $\cdot$ is associative. Finally,

$$(r \oplus x) \cdot [(s + t \oplus y + z)] = rs + rt \oplus ry + rz + sx + tx = (r \oplus x)(s \oplus y) + (r \oplus x)(t \oplus z)$$

and hence S is commutative ring.

Further, $0 \oplus N$ is an ideal of S since $R \cdot N = N$ and $N^2 = 0$. By the latter token, $0 \oplus N \subseteq \sqrt{(0)}$. Since $R \cong S/N$, it follows, that S, R, N are all S-modules and especially N and R are Noetherian because each S-submodule of either of them is R-submodule as well and is zero as S-module iff it is zero as R-module. By [**12**], 7.17 it follows that S is Noetherian S-module, hence Noetherian ring.

If $\mathfrak{a}$ denotes the maximal ideal of R, we consider the submodule $\mathfrak{a} \oplus N$ of S. One easily detects this to be an ideal. Further, if $x \oplus n$ is in $S \backslash \mathfrak{a} \oplus N$, then x has to be a unit in R and so has an inverse. Then $(x \oplus n) \cdot (x^{-1} \oplus (-x^{-1} \cdot n)) = 1_R \oplus 0$, the identity of S. Hence each element outside $\mathfrak{a} \oplus N$ is unit in S such that with the observation that $\mathfrak{a} \oplus N$ is ideal, we see that S is a Noetherian local ring with maximal ideal $\mathfrak{a} \oplus N$.

We apply this to the local Noetherian ring A and the finitely generated A-module M to get the Noetherian local ring $A \oplus M =: B$. Since $B/M \cong A$, A can be considered as B-algebra. So every A-module X can be considered as B-module via $\pi : B \to A$ and of course the other way round.

So for our investigations about the *vanishing* of local cohomology modules it does not matter, whether we consider for a B-module X, $H^i_{\mathfrak{m} \oplus M}(X)$ or the restriction of this to an A-module by means of the natural embedding $A \to B$. We consider the exact sequence of B-modules

$$0 \to M \to B \to A \to 0$$

and the resulting long exact cohomology sequence

$$\ldots \to H^i_{\mathfrak{m} \oplus M}(M) \to H^i_{\mathfrak{m} \oplus M}(B) \to H^i_{\mathfrak{m} \oplus M}(A) \to H^{i+1}_{\mathfrak{m} \oplus M}(M) \to \ldots$$

for each $i \in \mathbb{N}$. If we now restrict this sequence to A-modules, theorem 3.20 allows us to calculate instead the local cohomology of M, B, A, considered as A-modules by means of the inclusion map $A \to B$. It follows then from the part of the long exact H^i-sequence

$$H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(B) \to H^i_{\mathfrak{m}}(A)$$

(now as A-modules) and the considerations at the beginning of the proof, that whenever i is in $\Theta(\mathfrak{m}, M)$, then since $i < n$ that $H^i_{\mathfrak{m}}(M) \neq 0$ and $H^i_{\mathfrak{m}}(A) = 0$ so that $H^i_{\mathfrak{m}}(B) \neq 0$ or if $i = n$, that $H^i_{\mathfrak{m}}(A) \neq 0$ and $H^i_{\mathfrak{m}}(M) = 0$ and so $H^i_{\mathfrak{m}}(B) \neq 0$ again. On the other hand, if i is neither equal to n nor in $\Theta(\mathfrak{m}, M)$, then $H^i_{\mathfrak{m}+M}(B) = H^i_{\mathfrak{m}}(B) = 0$ since both neighbouring terms of $H^i_{\mathfrak{m}}(B)$ are zero. Therefore $\Theta(\mathfrak{m} \oplus M, B) = \Theta \backslash \{n\} \cup n = \Theta$. □

REMARK 4.14. We have even shown that for each finite $\Theta \subseteq \mathbb{N}$, there is a Noetherian ring which, considered as module over itself gives the desired Θ.

Having come so far, we want to make a connection between the things already done and what is to follow in the next two chapters.

We showed in 4.12 that whenever M is a finitely generated A-module and nonzero, and $\mathfrak{a}$ an ideal of A with $\mathfrak{a} + \operatorname{ann}(M) \neq A$, then for each minimal prime $\mathfrak{q}$ of $\mathfrak{a} + \operatorname{ann}(M)$ we have

$$H_{\mathfrak{a}}^{\dim_{A_\mathfrak{q}} M_\mathfrak{q}}(M) \neq 0.$$

Now suppose, $\mathfrak{p}$ is a minimal prime of $\operatorname{ann}(M)$ contained in $\mathfrak{q}$ and $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = \dim_{A_\mathfrak{q}}(M_\mathfrak{q}) = r$. Then $\mathfrak{p}$ consists only of zerodivisors on M by [**12**], 9.36. It follows, $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$. So if we invoke the notation of 4.13 to denote by $\Theta(\mathfrak{a}, M)$ the set of integers i, for which $H^i_{\mathfrak{a}}(M) \neq 0$, and let $\theta_{\mathfrak{a}}(M)$ be the smallest element of $\Theta(\mathfrak{a}, M)$, then we can write

$$i < \theta_{\mathfrak{a}}(M) \Rightarrow i < \operatorname{depth}(\mathfrak{p} \cdot A_\mathfrak{p}, M_\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p}).$$

since $\operatorname{depth}(\mathfrak{p} \cdot A_\mathfrak{p}, A_\mathfrak{p}) = 0, \operatorname{ht}(\mathfrak{q}/\mathfrak{p}) = r$ and $H^r_{\mathfrak{a}}(M) = H_{\mathfrak{a}}^{\dim_{A_\mathfrak{q}} M_\mathfrak{q}}(M) \neq 0$. Now suppose, there is a nonzerodivisor x on M. Suppose further, that there is a minimal prime $\mathfrak{q}'$ of $\mathfrak{a} + \operatorname{ann}(M/x \cdot M)$ such that $\dim_{A_{\mathfrak{q}'}}(M/x \cdot M)_{\mathfrak{q}'} = r'$ is greater than zero. By 4.12 again, $H^{r'}_{\mathfrak{a}}(M/x \cdot M) \neq 0$ and from the exact sequence

$$0 \to M \xrightarrow{x} M \to M/x \cdot M \to 0$$

which induces

$$H^{r'}_{\mathfrak{a}}(M) \to H^{r'}_{\mathfrak{a}}(M/x \cdot M) \to H^{r'+1}_{\mathfrak{a}}(M)$$

follows that at least one of the outer terms has to be nonzero. So $\theta_{\mathfrak{a}}(M)$ cannot exceed $r' + 1$.

Suppose now, that $\mathfrak{p}'$ is a minimal prime of $\operatorname{ann}(M/x \cdot M)$ contained in $\mathfrak{q}'$ with $\operatorname{ht}(\mathfrak{q}'/\mathfrak{p}') = \dim_{A'_\mathfrak{q}}(M_{\mathfrak{q}'}) = r'$. Then $\mathfrak{p}'$ containes x because x annihilates $M/x \cdot M$, and x is M-regular. It follows that $\operatorname{depth}_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}) \geq 1$. Again we put our knowledge in a new way:

$$i < \theta_{\mathfrak{a}}(M) \Rightarrow i < r' + 1 \Rightarrow i < \operatorname{ht}(\mathfrak{q}'/\mathfrak{p}') + \operatorname{depth}_{A_{\mathfrak{p}'}}(M_{\mathfrak{p}'}).$$

because $\operatorname{depth}(\mathfrak{p}' A'_\mathfrak{p}, M_{\mathfrak{p}'}) \geq 1$, $\operatorname{ht}(\mathfrak{q}'/\mathfrak{p}') = r'$ and $H^{r'+1}_{\mathfrak{a}}(M)$ or $H^{r'}_{\mathfrak{a}}(M)$ are nonzero. One becomes interested in these sums of depths and heights and especially the question arises, whether we can say something about the other implication: is there an integer i, smaller than all possible sums of $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}/\mathfrak{p})$ for $\mathfrak{q}$ in the variety of $\mathfrak{a}$ and $\mathfrak{p} \subseteq \mathfrak{q}$ for which $H^i_{\mathfrak{a}}(M) = 0$?

We will try to find an answer to this problem for quite a large class of rings, according to geometrical problems. This will involve the concept of Gorenstein rings, which will be developed in the next chapter.

CHAPTER 5

Gorenstein rings and Duality

At the end of the last chapter we made a promise to investigate a question concerning the relationships between the set of integers $\Theta(\mathfrak{a}, M)$ which indicate the nonvanishing of local cohomology modules of the A-module M with respect to the ideal $\mathfrak{a}$ of A on one side and A sum of certain depths and dimensions on the other. The answer we will provide, which will be given in the next chapter, involves to a great deal Gorenstein rings and a special property these rings enjoy. So this chapter is essentially devoted to the introduction of Gorenstein rings and the proof, that two certain modules have the same annihilator.

The work in the first half of the chapter is based on the structure of injective modules and we have to make some additional comments to this topic.

It is shown in 7.21 that every injective module over a Noetherian commutative ring A can be decomposed into the direct sum of indecomposable injective modules which all are the injective hull of $A/\mathfrak{p}$ for some prime $\mathfrak{p} \in \mathrm{Spec}(A)$ or another. 7.20 states that this decomposition is unique in the sense that given two compositions of the same injective module, then for all $\mathfrak{p} \in \mathrm{Spec}(A)$ the number of summands which equal $E(A/\mathfrak{p})$ is the same for both decompositions. So each injective module E is up to isomorphism uniquely described by a set of cardinals $\{\mu(\mathfrak{p}, E)\}_{\mathfrak{p} \epsilon spec(A)}$, which stand for the number of terms $E(A/\mathfrak{p})$ in each decomposition of E into a direct sum of indecomposable injective modules. So given an A-module M, we saw in 7.23 how M defines a sequence of sets of cardinals $\{\mu^i(\mathfrak{p}, M)\}_{\mathfrak{p} \epsilon spec(A), i \epsilon \mathcal{N}}$ such that $\{\oplus_{\mathfrak{p} \epsilon spec(A)} E(A/\mathfrak{p})^{M_i pmo}\}_{i \epsilon \mathcal{N}}$ are the modules which occur in every minimal injective resolution of M (of course up to isomorphisms).

Definition 5.1. For all prime ideals $\mathfrak{p}$ in A, $\kappa(\mathfrak{p})$ is defined as $(A/\mathfrak{p})_{\mathfrak{p}}$.

Lemma 5.2. *Let M be A finite A-module. We claim that if there are two primes $\mathfrak{p}, \mathfrak{p}'$ in A with* $\mathrm{ht}(\mathfrak{p}'/\mathfrak{p}) = 1$ *(that means that $\mathfrak{p}' \supseteq \mathfrak{p}$ and this inclusion allows no prime between them), then $\mu^r(\mathfrak{p}, M) \neq 0$ implies $\mu^{r+1}(\mathfrak{p}', M) \neq 0$. (See* [**16**]*,3.1.)*

Proof. Let

$$0 \to M \to E^0(M) \to E^1(M) \to \ldots \to E^i(M) \to \ldots$$

be a minimal injective resolution for M. Then $AE^i \cong \bigoplus_{q \epsilon spec(A)} (E(A/\mathfrak{q}))^{M_i qmo}$.

If we apply to this exact sequence the functor $(\cdot)_{\mathfrak{p}}$, then we get an acyclic complex which is by [**11**], Theorem 3.76 an injective resolution of $M_{\mathfrak{p}}$ and by 7.11 even a minimal injective resolution. Let $x \in \mathfrak{p}' \backslash \mathfrak{p}$, $B = A_{\mathfrak{p}'}/\mathfrak{p}_{\mathfrak{p}'}$ and $C = B/xB = (A/(\mathfrak{p}, x))_{\mathfrak{p}'}$. Then $\mathfrak{p} \cdot B$ is properly contained in the radical of $(\mathfrak{p}, x) \cdot B$ so that this radical equals $\mathfrak{p}' \cdot B$. So $(\mathfrak{p}'/(\mathfrak{p}, x))_{\mathfrak{p}'}$ is nilpotent what means that C is Artinian.

Further, x is not a zerodivisor on B. Then

$$0 \to B \xrightarrow{x} B \to C \to 0$$

is exact and induces the exact sequence

$$Ext^r_A(B, M) \xrightarrow{x} Ext^r_A(B, M) \to Ext^{r+1}_A(C, M)$$

We use now the fact established in 7.23 that the hypothesis implies that

$$Ext^r_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) = (\mathrm{Ext}^r_A(A/\mathfrak{p}, M))_{\mathfrak{p}} \neq 0$$

where $\kappa(\mathfrak{q})$ denotes for all primes $\mathfrak{q}$ in A the quotient $A_{\mathfrak{q}}/(\mathfrak{q} \cdot A_{\mathfrak{q}})$.

It follows, that $(\mathrm{Ext}^r_A(B, M))_{\mathfrak{p}} \neq 0$ since $\kappa(\mathfrak{p}) \cong B_{\mathfrak{p}}$. Hence $(\mathrm{Ext}^r_A(B, M))_{\mathfrak{p}'} \neq 0$ since $\mathfrak{p}' \supseteq \mathfrak{p}$. If we now localize the above Ext-sequence at $\mathfrak{p}'$ and observe that x is in the Jacobson radical of A, $(Ext^{r+1}_A(C, M))_{\mathfrak{p}'}$ cannot be zero by Nakayamas lemma.

Now $\kappa(\mathfrak{p}') \cong C/(\mathfrak{p}' \cdot C)$. Further C is an Artinian ring, so of finite length. If the length of $C \lambda(C) = 1$, then $C \cong \kappa(\mathfrak{p}')$ and we are done. We show that $Ext^{r+1}_{A_{\mathfrak{p}'}}(\kappa(\mathfrak{p}'), M_{\mathfrak{p}'})$ is nonzero. Let k be a minimal submodule of C (by the definition of multiplication of A on C it does not matter whether we say A-submodule or C-submodule). Since C is local, it follows that $k \subseteq \mathfrak{p}' \cdot C$. We have then a short exact sequence

$$0 \to k \to C \to C/k \to 0$$

which gives rise to the exact triple

$$Ext^{r+1}_{A_{\mathfrak{p}'}}(C/k, M_{\mathfrak{p}'}) \to Ext^{r+1}_{A_{\mathfrak{p}'}}(C, M_{\mathfrak{p}'}) \to Ext^{r+1}_{A\mathfrak{p}'}(k, M_{\mathfrak{p}'})$$

Since the the middle term is nonzero, either of the two outer terms has to be nonzero as well. If it is the right one, we are done: k has by definition length 1, is therefore isomorphic to $A_{\mathfrak{p}'}/\mathfrak{q}$ for some maximal ideal $\mathfrak{q}$ of $A_{\mathfrak{p}'}$. Since this is unique and equals $\mathfrak{p}' \cdot A_{\mathfrak{p}'}$, $k \cong \kappa(\mathfrak{p}')$. If it is the left one, then we may reformulate the problem with C/k instead of C which is still local Artinian ring, but $\lambda(C/k) = \lambda(C) - 1$. Proceding in this way, either at some stage a right hand term in the various Ext-sequences is nonzero or left one is never zero. In the first case we know that then $\mathrm{Ext}^{r+1}_{A_{\mathfrak{p}'}}(\kappa(\mathfrak{p}'), M_{\mathfrak{p}'}) \neq 0$ whereas in the latter the procedure terminates with the result that C/k has length 1 and is hence again isomorphic to $\kappa(\mathfrak{p}')$. We conclude that in every case $\mathrm{Ext}^{r+1}_{A_{\mathfrak{p}'}}(\kappa(\mathfrak{p}'), M_{\mathfrak{p}'})$ cannot be zero. Again applying lemma 7.23 we see that $\mu^{r+1}(\mathfrak{p}', M) \neq 0$. $\square$

The attentive reader will have noticed, that we proved in chapter 4 the theorems 4.6 and 4.7, which look quite similar, in two different looking ways. The reference to 7.23 throws light on that affair: the difference has not been that big.

We come now to the first major ingredience of a theorem that is to follow in 5.8.

DEFINITION 5.3. Given an A-module M we will say, that the *injective dimension of* M, written $\mathrm{injdim}(M)$ (or $\mathrm{injdim}_A(M)$ if the base ring is in question), equals r if and only if there is an injective resolution

$$0 \to M \to E^0 \to E^1 \to \ldots \to E^r \to 0$$

and there is no exact such sequence with fewer than r terms. If there is no such 'terminating' resolution of M, we will say, that M has *infinite injective dimension* and write $\mathrm{injdim}(M) = \infty$.

PROPOSITION 5.4. *(See* [**16**]*3.3) Let* $(A, \mathfrak{p})$ *be local,* $M \neq 0$ *a finite A-module with finite injective dimension* r*. Then* $r = depth(\mathfrak{p}, A)$*.*

PROOF. Let $\mathfrak{q} \in \mathrm{Spec}(A)$ be such that $\mu^r(\mathfrak{p}, A) \neq 0$. Such $\mathfrak{q}$ exists since otherwise by 7.21 $E^r(M) = 0$, a contradiction to $\mathrm{injdim}(M) = r$. Suppose $\mathfrak{q}$ were not maximal. Then there is a $\mathfrak{q}'$ in $\mathrm{Spec}(A)$ such that $\mathrm{ht}(\mathfrak{q}'/\mathfrak{q}) = 1$. By 5.2 then $\mu^{r+1}(\mathfrak{q}', M) \neq 0$ such that $E^{r+1}(M) \neq 0$ contradicting $\mathrm{injdim}(M) = r$ again. So all such $\mathfrak{q}$ have to be maximal, that is $\mathfrak{q} = \mathfrak{p}$. It follows from 7.23, that $\mathrm{Ext}^r_A(A/\mathfrak{p}, M) \neq 0$.

Now the injective dimension of M is r such that $ext^{r+1}_A(\cdot, M)$ is the zero functor. Via the long exact Ext-sequence with M as second argument it turns out that $\mathrm{Ext}^r_A(\cdot, M)$ is rightexact. So

$$0 \to A/\mathfrak{p} \to N \to (N/(A/\mathfrak{p})) \to 0$$

exact for some A-module N implies

$$\mathrm{Ext}^r_A(N, M) \to \mathrm{Ext}^r_A(A/\mathfrak{p}, M) \to 0$$

to be exact, proving that then $\mathrm{Ext}^r_A(N, M) \neq 0$ for such N.

Let now $(\mathfrak{x})$ be a maximal A-sequence in A such that since A local, $(\mathfrak{x}) \subseteq \mathfrak{p}$. We show how one can embed $A/\mathfrak{p}$ in $A/(\mathfrak{x})$. (Remember, we have agreed, that $(\mathfrak{x})$ denotes the A-sequence and the ideal generated by the elements of this sequence as well.) Let $\eta \in ((\mathfrak{x}) : \mathfrak{p}) \backslash (\mathfrak{x}) where (\mathfrak{x}) : \mathfrak{p}$ denote the ideal quotient.

(η exists because $(\mathfrak{x}) : \mathfrak{p} = (\mathfrak{x})$ implies that $\mathfrak{p}$ is not in $\mathrm{ass}((\mathfrak{x}))$ and then each element in $\mathfrak{p} \backslash A \cdot (\mathfrak{x})$ would be a nonzerodivisor in $A/(\mathfrak{x})$ by [**12**], 9.33 and 9.36 and hence eligible to extend $(\mathfrak{x})$ to an A-sequence of length exceeding $\mathrm{depth}(\mathfrak{p}, A)$. Finally $\mathfrak{p} = (\mathfrak{x})$ implies $((\mathfrak{x}) : \mathfrak{p}) = A \neq \mathfrak{p}$.)

Define now $\varphi : A/\mathfrak{p} \to A/(\mathfrak{x})$ by $\varphi(a + \mathfrak{p}) = \eta \cdot a + (\mathfrak{x})$. We have to check several things:

If $\pi \in \mathfrak{p}$, then $\varphi(\pi + \mathfrak{p}) = \pi \cdot \eta + (\mathfrak{x}) = 0 + (\mathfrak{x})$. Further, if for a, a' we have $a + \mathfrak{p} = a' + \mathfrak{p}$, then $a - a' \in \mathfrak{p}$ and hence $\varphi(a + \mathfrak{p}) - \varphi(a' + \mathfrak{p}) =$

$a \cdot \eta + a' \cdot \eta + \mathfrak{p} = (a - a') \cdot \eta + \mathfrak{p} = 0 + \mathfrak{p}$ by the definition of η. $\pi + \mathfrak{p}$ so that φ is a map.

If $a, b \in A$, then $\varphi(a + b + \mathfrak{p}) = \eta \cdot (a + b) + (\mathfrak{x}) = \eta \cdot a + \eta \cdot b + (\mathfrak{x}) = \varphi(a + \mathfrak{p}) + \varphi(b + \mathfrak{p})$ so that φ is additive. Similarily one shows φ to be A-linear.

If $\varphi(a + \mathfrak{p}) = 0$, then $\eta \cdot a + (\mathfrak{x}) = 0$, that is, $\eta \cdot a \in (\mathfrak{x})$. If here $a \in \mathfrak{p}$, then $a + \mathfrak{p} = 0$. Otherwise, A is unit and η must be in $(\mathfrak{x})$, a contradiction. Hence $a \in \mathfrak{p}$ and therefore φ is injective.

We have shown that $A/(\mathfrak{x})$ admits a momomorphism $A/\mathfrak{p} \hookrightarrow A/(\mathfrak{x})$ so that

$$\mathrm{Ext}^r_A(A/(\mathfrak{x}), M) \neq 0.$$

This proves, that the homological dimension of $A/(\mathfrak{x})$ is at least r. By [**9**],Th.9.20, this homological dimension is equal to the length of $(\mathfrak{x})$. Hence

$$\mathrm{depth}(A) \geq r$$

So we have to show the other inequality, which will be drawn from a more general result taken from [**15**], 4.10:

Sublemma 5.5. *(Auslander/Goldman): Let $(A, \mathfrak{m})$ be a local Noetherian commutative ring and N a finite A-module of finite homological dimension n (see* [**9**]*, paragraph 7.5). If M is a nonzero finitely generated A-module , then $\mathrm{Ext}^n_A(N, M) \neq 0$. Proof of sublemma: Let*

$$0 \to P_n \stackrel{d_n}{\to} P_{n-1} \stackrel{d_{n-1}}{\longrightarrow} \ldots \to P_0 \stackrel{d_0}{\to} N \to 0$$

be a projective resolution constructed as follows: N is finitely generated by hypothesis, say by at least n_0 elements. Let then $P_0 = A^{n_0}$ and d_0 the natural projection $(x_1, \ldots, x_{n_0}) \to \Sigma_1^{n_0} x_i \cdot \nu_i$ where the ν_i constitute a generating set of minimal cardinality n_0 for N. Of course P_0 is Noetherian and hence so is $\ker(d_0)$. So we can apply this procedure to $\ker(d_0)$ instead of N and get Noetherian P_1 and $\ker(d_1)$. We proceed in this way until we have an exact sequence

$$P_{n-1} \stackrel{d_{n-1}}{\longrightarrow} \ldots P_0 \stackrel{d_0}{\to} N \to 0$$

where all involved modules P_i are finitely generated free. Then by [**9**]*, Theorem 7.11 and the fact that the homological dimension of N is n, $\ker(d_{n-1})$ is projective. Since A is local this gives by* [**9**]*, Theorem 9.12 that $\ker(d_{n-1}) =: P_n$ is even free and, since Noetherian, a finite direct sum of A's, just as all other modules in the sequence with possible exception of N.*

Let now $i \in \mathbb{N}$ be fixed between 0 and n and let $x \in P_i$. Then $x = (x_1, \ldots, x_{n_i})$ with $x_j \in A$ for $1 \leq j \leq n_i$. Suppose $4d_i(x) = 0$. Let $\{\pi_j\}_{1 \leq j \leq n_i}$ be the generating set for $\ker(d_{i-1})$ that was used to construct our resolution for N. Then $0 = d_i(x) = \Sigma_{l=1}^{n_i} x_l \cdot \pi_l$. If now one of the x_l happens to be outside $\mathfrak{m}$, then it is unit, hence has an inverse. In this case (if this x_l has $l = k$)

$$\pi_k = (\Sigma_{l=1, l \neq k}^{n_i} x_l \pi_l) \cdot x_k^{-1}$$

showing that π_k is superfluous in the generating set for $\ker(d_{i-1})$. *This contradiction shows, that for all $i \in \mathbb{N}$ we have*

$$ker(d_i) \subseteq \mathfrak{m} \cdot P_i.$$

After these preliminaries we observe that

$$\mathrm{Hom}_A(P_{n-1}, M) \overset{Hom_A(d_n,M)}{\longrightarrow} \mathrm{Hom}_A(P_n, M) \overset{nat}{\to} \mathrm{Ext}^n_A(N, M) \to 0$$

is exact by definition of Ext and since the homological dimension of M is n. We want to show, that $\mathrm{Ext}^n_A(N, M) \neq 0$, *so suppose this were not the case. Then each* $f \in \mathrm{Hom}_A(P_n, M)$ *is restriction of some* $g \in \mathrm{Hom}_A(P_{n-1}, M)$. *Since*

$$d_n(P_n) \subseteq \mathfrak{m} \cdot P_{n-1},$$

this means that $f(P_n) \subseteq \mathfrak{m}\cdot M$ *for all* $f \in \mathrm{Hom}_A(P_n, M)$. *Since the homological dimension of N is n, P_n cannot be zero. Since it is free, each $\mu \in M$ is image under some* $f \in \mathrm{Hom}_A(P_n, M)$. *Since* $f(P_n) \subseteq \mathfrak{m} \cdot M$, *this gives that* $\mathfrak{m} \cdot M \supseteq M$, *a contradiction to Nakayamas lemma. Hence there must exist* $f \in \mathrm{Hom}_A(P_n, M)$ *which are not restrictions of a map* $g \in \mathrm{Hom}_A(P_{n-1}, M)$ *whence* $\mathrm{Hom}_A(d_n, M)$ *is not surjective and hence* $\mathrm{Ext}^n_A(N, M) \neq 0$. *End of proof of Sublemma.*

We apply this now to $N = A/(\mathfrak{x})$, such that the homological dimension of N equals the depth of A by [**9**], Theorem 9.20. For M we take the M of the sublemma. Then, since both M and $A/(\mathfrak{x})$ are finitely generated, the sublemma gives

$$Ext_A^{depth(\mathfrak{p},A)}(A/(\mathfrak{x}), M) \neq 0$$

showing that the injective dimension of M has to be at least equal to $\mathrm{depth}(\mathfrak{p}, A)$. This together with the inequality immediately preceeding the sublemma, proves

$$r = depth(\mathfrak{p}, M).$$

□

This result (especially in the form of the following proposition) will certainly give rise to some surprise, showing that there is nothing like an analogue to homological codimension: 'cohomological codimension ' is either 0 or not finite.

Corollary 5.6. *If the injective dimension of the local ring $(A, \mathfrak{m})$, considered as module over itself, is finite, then*

$$\mathrm{injdim}_A(A) = \dim(A) = \mathrm{depth}(\mathfrak{m}, A).$$

Proof. It follows from 5.4 that $\mathrm{injdim}(A) = \mathrm{depth}(A)$. Of course $\mathrm{depth}(A) \leq \dim(A)$.

On the other hand, let $\mathfrak{p}$ be a minimal prime of A such that $\mathrm{ht}(\mathfrak{m}/\mathfrak{p}) = \dim(A)$. Then $\mathfrak{p} \cdot A_\mathfrak{p} \in \mathrm{Ass}(A_\mathfrak{p})$ since no other prime is there. So $A_\mathfrak{p}$ contains a submodule isomorphic to $\kappa(\mathfrak{p})$. Therefore

$$\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), A_\mathfrak{p}) \neq 0$$

and by 7.23 $\mu^0(\mathfrak{p}\cdot A_{\mathfrak{p}}, A_{\mathfrak{p}}) \neq 0$. By 7.11 this implies that $\mu^0(\mathfrak{p}, A) \neq 0$. Applying 5.2, $\mathrm{ht}(\mathfrak{m}/\mathfrak{p})$ times, we get

$$\mu^{ht(\mathfrak{m}/\mathfrak{p})}(\mathfrak{m}, A) \neq 0,$$

which by 7.23 gives that $\mathrm{Ext}_A^{\mathrm{ht}(\mathfrak{m}/\mathfrak{p})}(A/\mathfrak{m}, A) = \mathrm{Ext}_A^{\dim(A)}(A/\mathfrak{m}, A) \neq 0$. Hence $\dim(A) \leq \mathrm{injdim}(A)$. So

$$\dim(A) \leq \mathrm{injdim}(A) = \mathrm{depth}(A) \leq \dim(A).$$

This completes the proof. □

The following is the last preparatory result for the introduction of Gorenstein rings.

LEMMA 5.7. *(See* [**16**], *2.6) Let A be a commutative Noetherian ring. Let M be an A-module and x a nonzero divisor on both M and A. Then*

$$\mu^i(\mathfrak{p}/x\cdot A, M/x\cdot M) = M_i plpm$$

for all primes $\mathfrak{p}$ of A which contain x and all $i \geq 0$.

PROOF. Let $\mathfrak{p}$ be in $\mathrm{Spec}(A)$ and $x \in \mathfrak{p}$. By 7.11, 7.23 and the fact that localisation is exact (so that $(\mathrm{Ext}^i_A(M,N))_{\mathfrak{p}} \cong Ext^i_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$) it is clear that we may assume that $\mathfrak{p}$ is maximal. This we do.

By 7.23 we have then

$$M_i pm = \dim_{\kappa(\mathfrak{p})} \mathrm{Ext}^i_A(A/\mathfrak{p}, M).$$

Now $A/\mathfrak{p}$ can be considered as $A/x\cdot A$-module, and each homomorphism starting in $A/\mathfrak{p}$ is annihilated by x. So $\mathrm{Ext}^i_A(A/\mathfrak{p}, M)$ is $A/x\cdot A$ module. Then by [**9**], Theorem 9.6

$$\mathrm{Ext}^{i+1}_A(A/\mathfrak{p}, M) \cong \mathrm{Ext}^i_{A/x\cdot A}(A/\mathfrak{p}, M/x\cdot M)$$

as $A/x\cdot A$-modules and hence as A-modules as well. Now $A/\mathfrak{p} \cong ((A/x\cdot A)/(\mathfrak{p}/x\cdot A))$ and hence

$$\begin{aligned}
\mu^i(\mathfrak{p}/x\cdot A, M/x\cdot A) &= \dim_{\kappa(\mathfrak{p}/x\cdot A)} /ext^i_{A/x\cdot A}(((A/x\cdot A)/(\mathfrak{p}/x\cdot A)), M/x\cdot M) \\
&= \dim_{\kappa(\mathfrak{p})} \mathrm{Ext}^i_{A/x\cdot A}(((A/x\cdot A)/(\mathfrak{p}/x\cdot A)), M/x\cdot M) \\
&= \dim_{\kappa(\mathfrak{p})} \mathrm{Ext}^{i+1}_A(A/\mathfrak{p}, M) \\
&= \mu^{i+1}(\mathfrak{p}, M)
\end{aligned}$$

by 7.23 again. □

We have spoken frequently about the following

THEOREM 5.8. *(See* [**16**],*3.6) Let A be Noetherian commutative ring. Then the following conditions on a prime ideal $\mathfrak{p}$ in A are equivalent:*

1. $\mathrm{injdim}_{A_{\mathfrak{p}}} A_{\mathfrak{p}} < \infty$;
2. $\mathrm{injdim}_{A_{\mathfrak{p}}} A_{\mathfrak{p}} = \mathrm{ht}(\mathfrak{p}) (= \dim A_{\mathfrak{p}})$;
3. $M_i pa = 0$ *for all* $i > \mathrm{ht}(\mathfrak{p})$;

4. $M_ipa = 0$ *for some* $i > \mathrm{ht}(\mathfrak{p})$*;*
5. $M_ipa = 0$ *for* $i < \mathrm{ht}(\mathfrak{p})$ *and* $\mu^{ht(\mathfrak{p})}(\mathfrak{p}, A) = 1$*;*
6. $M_iqa = \delta_{i,ht(\mathfrak{q})}$ *for all prime ideals* $\mathfrak{q} \subseteq \mathfrak{p}$*.(Kronecker delta)*

PROOF. 1⇒ 2 is 5.6,
2⇒ 3 is trivial,
3⇒ 4 as well.

4⇒ 1 : Let $\mathrm{injdim}_{A_\mathfrak{p}} A_\mathfrak{p}$ be infinite. By 7.11, $M_ipa = \mu^i(\mathfrak{p}\cdot A_\mathfrak{p}, A_\mathfrak{p})$. If $\mathrm{ht}(\mathfrak{p}) = 0$, then $\mathfrak{p}\cdot A_\mathfrak{p}$ is the only prime in $A_\mathfrak{p}$ so that $\mu^i(\mathfrak{p}\cdot A_\mathfrak{p}, A_\mathfrak{p}) = 0$ implies $E^i_{A_\mathfrak{p}}(A_\mathfrak{p}) = 0$. If this happens for $i > \mathrm{ht}(\mathfrak{p})$, $\mathrm{injdim}_{A_\mathfrak{p}} A_\mathfrak{p} = \infty$ is not possible. So in the case $\mathrm{ht}(\mathfrak{p}) = 0$ we are done.

Assume inductively, that $\mathrm{ht}(\mathfrak{p}) = h > 0$ and the statement has been proved for all values of $\mathrm{ht}(\mathfrak{p})$ smaller than h. This is the case for $h = 1$.

Let $\mathfrak{q}$ be a prime ideal contained in but different from $\mathfrak{p}$, the existence of which is guaranteed by $h > 0$. Defining $s := \mathrm{ht}(\mathfrak{p}/\mathfrak{q})$, we observe that $s > 0$ and

$$s + \mathrm{ht}(\mathfrak{q}) \leq h$$

In the case that $\mathrm{injdim}_{A_\mathfrak{q}} A_\mathfrak{q} = \infty$, we know $\mu^i(\mathfrak{q}\cdot A_\mathfrak{q}, A_\mathfrak{q}) \neq 0$ for all $i > \mathrm{ht}(\mathfrak{q})$ by the inductive hypothesis. So by 5.2, $\mu^{i+s}(\mathfrak{p}\cdot A_\mathfrak{p}, A_\mathfrak{p}) \neq 0$ for $i > \mathrm{ht}(\mathfrak{q})$ such that by the above displayed inequality $M_ipa \neq 0$ for $i > h$.

So whenever there exists an ideal $\mathfrak{q} \subset \mathfrak{p}$ in A with $\mathrm{injdim}_{A_\mathfrak{q}} A_\mathfrak{q} = \infty$, then we have proved that $M_ipa \neq 0$ for all $i > \mathrm{ht}(\mathfrak{p})$.

There remains the case, where no such ideal exists. Then for $\mathfrak{q} \subset \mathfrak{p}$, we have

$$\mu(\mathfrak{q}\cdot A_\mathfrak{p}, E^i_{A_\mathfrak{p}}(A_\mathfrak{p})) = \mu(\mathfrak{q}\cdot A_\mathfrak{q}, (E^i_{A_\mathfrak{p}}(A_\mathfrak{p}))_{\mathfrak{q}\cdot A_\mathfrak{p}}) = M_iqa$$

by 7.11 where

$$0 \to A_\mathfrak{p} \to E^0_{A_\mathfrak{p}}(A_\mathfrak{p}) \to \ldots \to E^i_{A_\mathfrak{p}}(A_\mathfrak{p}) \to \ldots$$

is a minimal injective resolution of the $A_\mathfrak{p}$-module $A_\mathfrak{p}$.

$\mathrm{injdim}_{A_\mathfrak{q}} A_\mathfrak{q}$ is finite, so by 5.6 these μ^i's have to be zero whenever i exceeds the height of $\mathfrak{q}$. So for $i \geq ht(\mathfrak{p})$, $E^i_{A_\mathfrak{p}}(A_\mathfrak{p})$ is direct sum of copies of $E(\kappa(\mathfrak{p}))$. If the cardinality of the set of these copies happens to be zero, the injective resolution displayed terminates and hence $\mathrm{injdim}_{A_\mathfrak{p}} A_\mathfrak{p}$ is finite, again contradicting the hypothesis. So in each case we have the required conclusion or the needed contradiction.

1⇒ 5: By 5.6, the hypothesis implies that $\operatorname{depth} A_\mathfrak{p} = \dim A_\mathfrak{p} = \mathrm{ht}(\mathfrak{p})$. So

$$\mathrm{Ext}^i_{A_\mathfrak{p}}(\kappa(\mathfrak{p})), A_\mathfrak{p}) = 0 \text{ for } 0 \leq i < \mathrm{ht}(\mathfrak{p})$$

by 4.3 and this gives by 7.23, that $\mu^i(\mathfrak{p}\cdot A_\mathfrak{p}, A_\mathfrak{p}) = 0$ for all such i.

Let $(\mathfrak{x})$ be a maximal $A_\mathfrak{p}$-sequence (of length $\mathrm{ht}(\mathfrak{p})$). Then by 5.7,

$$\mu^i(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}), A_\mathfrak{p}/(\mathfrak{x})) = \mu^{i+\mathrm{ht}(\mathfrak{p})}(\mathfrak{p}\cdot A_\mathfrak{p}, A_\mathfrak{p}) \text{ for all } i \in \mathbb{N}.$$

It follows that $\mathrm{injdim}_{A/(\mathfrak{x})} A/(\mathfrak{x}) < \infty$. Since 1 implies 3, $E^i_{A_\mathfrak{p}/(\mathfrak{x})}(A_\mathfrak{p}/(\mathfrak{x}))$ does not contain any submodule isomorphic to $E(\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x})))$ for $i > 0$

so that we can by 5.2 conclude that $\operatorname{injdim}_{A_\mathfrak{p}/(\mathfrak{x})} A_\mathfrak{p}$ is zero. It follows, that $E^0_{A_\mathfrak{p}/(\mathfrak{x})}(A_\mathfrak{p}/(\mathfrak{x}))$ is homomorphic image of $A_\mathfrak{p}/(\mathfrak{x})$ under the natural inclusion. Therefore these two modules are isomorphic and $A_\mathfrak{p}/(\mathfrak{x})$ is self injective. So $A_\mathfrak{p}/(\mathfrak{x})$ has to be isomorphic to

$$E_{A_\mathfrak{p}/(\mathfrak{x})}((A_\mathfrak{p}/(\mathfrak{x}))/(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}))) = E_{A_\mathfrak{p}/(\mathfrak{x})}(\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x})))$$

since $A_\mathfrak{p}/(\mathfrak{x})$ is local and hence indecomposable. We conclude that

$$\operatorname{Hom}_{A_\mathfrak{p}/(\mathfrak{x})}(\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x})), A_\mathfrak{p}/(\mathfrak{x})) \cong \kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}))$$

since these are vectorspaces and $E(\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x})))$ is essential over $\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}))$. By 5.2 and 5.7 again we conclude

$$\mu^{\operatorname{ht}(\mathfrak{p})}(\mathfrak{p}, A) = \mu^{\operatorname{ht}(\mathfrak{p}\cdot A_\mathfrak{p})}(\mathfrak{p}\cdot A_\mathfrak{p}, A_\mathfrak{p}) = \mu^0(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}), A_\mathfrak{p}/(\mathfrak{x})) = 1.$$

5$\Rightarrow$ 3: From the hypothesis, 7.23 and 4.3 follows, that $\operatorname{depth} A_\mathfrak{p} = \operatorname{ht}(\mathfrak{p}) = \dim A_\mathfrak{p}$. As above the consequence is that for a maximal $A_\mathfrak{p}$-sequence $(\mathfrak{x})$, $(\mathfrak{x})$ is $\mathfrak{p}\cdot A_\mathfrak{p}$-primary. This time from the hypothesis (and 5.2), $\mu^0(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}), A_\mathfrak{p}/(\mathfrak{x})) = 1$. Since there is no other prime in $A_\mathfrak{p}/(\mathfrak{x})$, $E(A_\mathfrak{p}/(\mathfrak{x})) = E(\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x})))$. So we may embed $A_\mathfrak{p}/(\mathfrak{x})$ and $\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}))$ in one and the same injective indecomposable $A_\mathfrak{p}/(\mathfrak{x})$-module $E(A_\mathfrak{p}/(\mathfrak{x}))$. On this point we will shorten the notation: we know from $A_\mathfrak{p}/(\mathfrak{x})$, that it is local Artinian and the injective hull of its residue field contains a carbon copy of it. We denote in the following $A_\mathfrak{p}/(\mathfrak{x})$ by R, $\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x})$ by $\mathfrak{m}$, $\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x}))$ by κ and $E(\kappa(\mathfrak{p}\cdot A_\mathfrak{p}/(\mathfrak{x})))$ by E. We will in the sequel by R denote the ring R as well as its isomorphic image in E and the same applies to κ.

We want then to show, that the injective dimension of R is 0. This can be done by proving, that the natural embedding $R \hookrightarrow E$ is surjective. This we do.

Since κ is simple, it is generated by one element, say by g. Let $k \in \kappa$ be nonzero. Since E is essential over R, $R \cap R\cdot k \neq 0$. So $0 \neq r = k\cdot r'$ for $r, r' \in R$. Since κ is annihilated by $\mathfrak{m}$, r' has to be in $R\backslash\mathfrak{m}$ such that r' is unit and $k \in R$. So $\kappa \subseteq R$.

Let $f \in \operatorname{Hom}_R(\kappa, E)$. Then $\mathfrak{m}\cdot f(k) = F(\mathfrak{m}\cdot k) = f(0) = 0$ for all $k \in \kappa$. By 7.23, $f(g)$ has then to be in κ and hence $f(\kappa) \subseteq \kappa \subseteq R$. So $E \supseteq A \supseteq \kappa$ implies together with $f(\kappa) \subseteq \kappa$ for all $f \in \operatorname{Hom}_A(\kappa, E)$, that $\operatorname{Hom}_R(\kappa, E) = \operatorname{Hom}_R(\kappa, R) = \operatorname{Hom}_R(\kappa, \kappa) \cong \kappa$.

It follows, $\operatorname{Hom}_R(hom_R(\kappa, E), E) \cong \kappa$. Now R is Artinian and so there is a finite chain of submodules of R

$$0 = R_0 \subseteq R_1 \subseteq \ldots \subseteq R_{n-1} \subseteq R_n = R$$

for some $n \in \mathbb{N}$ and $R_i/R_{i-1} \cong \kappa$ for $1 \leq i \leq n$. Denoting $\operatorname{Hom}_R(\operatorname{Hom}_R(X, E), E)$ by X" for each R-module X, we can for all R-modules define a homomorphism $\varphi_M : M \to M$" by the rule $\varphi_M(m)(f) = f(m)$ for $m \in M$, $f \in \operatorname{Hom}_R(M, E)$. It is obvious from the definition that this is a natural map in the sense that given two R-modules M, N and an R-map $f : M \to N$, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_N} \\ M" & \xrightarrow{\mathrm{Hom}_R(\mathrm{Hom}_R(f,E),E)} & N" \end{array}$$

commutes.

Since E is injective, $\mathrm{Hom}_R(\cdot, E)$ is exact contravariant and hence $(\cdot)"$ an exact covariant functor. So φ is a natural transformation from the (exact) identity functor to $(\cdot)"$.

If $\varphi_\kappa(k) = 0$ for $k \in \kappa$ then $f(k) = 0$ for all $f \in \mathrm{Hom}_A(\kappa, E)$. For f the inclusion map this proves that φ_κ is injective. Since κ is simple and as we just have established its image is nonzero, the image of κ is a vectorspace of dimension one over κ. Since the same is true for $\kappa"$ and the latter contains the former, φ_κ is surjective. Since in the above displayed sequence $\kappa \cong R_1$, we can make the inductive hypothesis, that φ_M is an isomorphism for all R_i for $0 \leq i \leq t$. This we know is true for $t = 1$. We have an exact sequence

$$0 \to R_{t+1} \to R_t \to R_{t+1}/R_t \cong \kappa \to 0$$

which gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_t & \longrightarrow & R_{t+1} & \longrightarrow & R_{t+11}/R_t \cong \kappa & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{R_t}} & & \downarrow{\scriptstyle \varphi_{R_{t+1}}} & & \downarrow{\scriptstyle \varphi_{R_{t+1}/R_t}} & & \\ 0 & \longrightarrow & R_t" & \longrightarrow & R_{t+1}" & \longrightarrow & (R_{t+1}/R_t)" \cong \kappa" & \longrightarrow & 0 \end{array}$$

Now by inductive hypothesis, the right and left vertical maps are isomorphisms, so that by the five lemma $\varphi_{R_{t+1}}$ has to be an isomorphism as well. It follows by induction, that φ_R is an isomorphism. Further $R" = \mathrm{Hom}_R(\mathrm{Hom}_R(R, E), E)$ is isomorphic to $\mathrm{Hom}_R(E, E)$ in the natural manner.

Now the exact sequence

$$0 \to R \to E \to E/R \to 0$$

gives rise to

$$0 \to \mathrm{Hom}_R(E/R, E) \to \mathrm{Hom}_R(E, E) \to \mathrm{Hom}_R(R, E) \to 0$$

Since we have proved that the middle term is isomorphic to R and the right hand term is clearly isomorphic to E, we have established a surjective map from R to E. Since R is of finite length n, its image E cannot have length greater than n. It follows from $R \subseteq E$, that $R = E$. Hence R is of injective dimension 0.

So in our original problem, $A_{\mathfrak{p}}/(\mathfrak{x})$ is selfinjective. So $M_i pa = 0$ for $i > \mathrm{ht}(\mathfrak{p})$ by 5.2 and 5.7.

1⇒ 6: 1⇒3 and 5 , so $M_i pa = \delta_{i,\mathrm{ht}(\mathfrak{p})}$. By [**11**], 3.76 property 1 is inherited by subideals of $\mathfrak{p}$, so we are done.

6⇒1 is obvious. □.

DEFINITION 5.9. Let A be as usual and $\mathfrak{p}$ an ideal in A that satisfies the equivalent conditions of 5.8. Then $A_{\mathfrak{p}}$ is said to be a *local Gorenstein ring.* In the case that for all maximal ideals $\mathfrak{m}$ of $A_{\mathfrak{m}}$ is local Gorenstein, we will say that A *is Gorenstein.*

At this point it is convenient, to make some remarks:

- If a ring A is Gorenstein, then for each maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is local Gorenstein by definition and hence of finite injective dimension. By 5.8 this is inherited by all further localisations, such that A Gorenstein implies that all localisations $A_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Spec}(A)$ are local Gorenstein.
- If A is Gorenstein and finite dimensional, then there is a minimal injective resolution
$$0 \to A \to E^0 \to E^1 \to \ldots \to E^{dim(A)} \to 0.$$
(Since $\mu^{i+\dim(A)}(\mathfrak{p}, A) = 0$ for all $i \in \mathbb{N}^+$ by 7.11 and 5.8.) Localizing at a prime ideal $\mathfrak{p}$, that makes a contribution to E^n, one sees that $\mathrm{injdim}_{A_{\mathfrak{p}}} A_{\mathfrak{p}} \geq n$ by 7.11 and by the theorem $n = \mathrm{ht}(\mathfrak{p}) = \mathrm{injdim}(A)$. So $\dim_A A \geq n$. On the other hand, a maximal ideal $\mathfrak{m}$ of height $\dim_A A$ has to make a contribution to $E^{dim(A)}$ since otherwise the localisation of the above sequence at $\mathfrak{m}$ would produce the injective resolution of a local Gorenstein ring with injective dimension smaller than the height of its maximal ideal. So $\mathrm{injdim}(A) \geq \dim(A)$. Hence $\dim_A A = \mathrm{injdim}_A A$.
- The concept of local Gorenstein rings is closely related to local Cohen-Macaulay rings, which are defined by the equation $\mathrm{depth}\, A = \dim A$. So we see that local Gorenstein ⇒ local Cohen-Macaulay. The difference is in that for all $\mathfrak{p}$ in the spectrum of a Gorenstein ring $\mu^{\mathrm{ht}(\mathfrak{p})}(\mathfrak{p} \cdot A_{\mathfrak{p}}, A_{\mathfrak{p}}) = 1$, what is not necessarily true for Cohen-Macaulay rings: let κ be a field, $R = \kappa[x, y]/(x, y)^2$ for two indeterminants x, y. Then R is local Noetherian with nilpotent maximal ideal $\mathfrak{m}$. Hence $\mathrm{depth}(\mathfrak{m}, R) = \dim(r) = 0$ so that R is Cohen-Macaulay. Now the quotient field of R is isomorphic to κ. Further is R considered as κ-module isomorphic to $\kappa \oplus \kappa \cdot x \oplus \kappa \cdot y$. Also, $x \cdot R$ and $y \cdot R$ are annihilated by $\mathfrak{m}$ and are not R-multiples of each other. So $\dim_\kappa(\mathrm{Hom}_R(\kappa, R) \geq 2$. Hence either R is not the injective hull of κ or $\mu^0(\mathfrak{m}, \kappa) \neq 1$. Since $\dim(R) = 0$, in both cases follows, that R is not Gorenstein.

  Coming along this way one can describe Gorenstein rings also as Cohen-Macaulay rings, in which some (or each) system of parameters (see [**12**],15.19) in $A_{\mathfrak{p}}$ generates an irreducible ideal in $A_{\mathfrak{p}}$ or, equivalently, by the criterion that each ideal of height r which can be generated by r elements is unmixed (that is all associated primes have the same height, r) and the primary components are all irreducible. So there are various ways of defining Gorenstein rings, which may be found and studied for example in [**16**], or in texts on geometry as [**3**] or [**4**]. We will neither develop nor use other descriptions of Gorenstein rings in this work.

- Regular rings are Gorenstein: it follows from [**7**], Theorem 19.3 that it suffices to prove, that local regular rings are local Gorenstein rings. This we do. In a local regular ring $(A, \mathfrak{m})$ of dimension n exists by definition a set Π of n elements, which generates $\mathfrak{m}$. Since by [**12**], 15.38 all ideals generated by subsets of Π are prime, it follows that Π is an A-regular sequence such that regular rings are Cohen-Macaulay. Now $\mathrm{Ext}^n_{A/\Pi}(A/\mathfrak{m}, A/\Pi) \cong \mathrm{Ext}^0_A(A/\mathfrak{m}, A)$ by [**9**], Theorem 9.6. The former module is of dimension 1 over $A/\mathfrak{m}$ so that regular rings satisfy condition 5 of the theorem and are Gorenstein.

  As final remark we want tell the reader that not each Gorenstein ring is regular: Let κ be any field and $\kappa[x]$ the ring of polynomials over κ in one indeterminate x. Then $R = \kappa[x]/(x)^2$ is Noetherian local ring with nilpotent maximal ideal $\mathfrak{m}$, hence Artinian. As above follows, that R is Cohen-Macaulay. Now any R-homomorphism that starts in κ is annihilated by $\mathfrak{m}$. So the range of $f \in Hom_r(\kappa, R)$ has to be inside $0 :_R \mathfrak{m}$. This is exactly $R{\cdot}x \cong \kappa$ as R-modules. Therefore,

  $$\dim_\kappa(\mathrm{Hom}_R(\kappa, R)) = 1.$$

  Consider the injective hull $E(R)$. Since $x \cdot R$ is the only prime ideal in R, $E(R)$ is direct sum of copies of $E(\kappa)$. The above equality shows then that $E(R) = E(\kappa)$. For local Artinian modules of this kind we have shown in the proof of 5.8, 5⇒3, that they are Gorenstein. On the other hand, R is not integral domain and hence not regular.

As the reader probably has noticed, we have used in the proof of 5.8 part 5⇒3 a very peculiar property of $E(A_\mathfrak{p}/\mathfrak{p} \cdot A_\mathfrak{p})$ - it produced isomorphisms

$$M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, E), E)$$

for modules of finite length M over local Gorenstein rings A. The remainder of this chapter will be devoted to some features of these isomorphisms. To this end we will follow an idea of Yoneda, which is to be found in [**4**], Chapter 4, concerning the interpretation of Ext-functors.

By definition, for the calculation of $\mathrm{Ext}^i_A(M, N)$ for two A-modules (we do not assume A to be Gorenstein here) M, N one has to take a projective resolution $\mathbf{P}^{\cdot}$ for M and an injective resolution $\mathbf{I}^{\cdot}$ for N, then to build the complex $\mathrm{Hom}_A(\mathbf{P}^{\cdot}, \mathbf{I}^{\cdot})$ and to take cohomology. This procedure may be replaced by using M instead of $\mathbf{P}^{\cdot}$ and calculating $H^n(\mathrm{Hom}_A(M, \mathbf{I}^{\cdot}))$, as is outlined in [**9**], Theorem 7.9. In this way each $\overline{f} \in \mathrm{Ext}^n_A(M, N)$ can be interpreted as $f + im\{(I^{n-1} \to I^n) \circ \mathrm{Hom}_A(M, I^{n-1})\}$ where $f \in \mathrm{Hom}_A(M, I^n)$

and $(I^n \to I^{n+1}) \circ f$ is zero. This identification will prove to be very useful in the sequel.

At this point we take for each A-module M an arbitrary injective resolution and fix it. If in the sequel we say 'take the injective resolution of M' this always will refer to the now selected one.

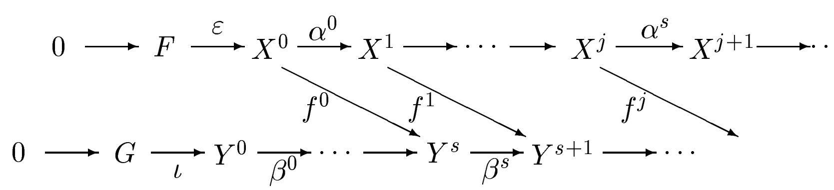
Let now again F, G be two A-modules. Let further be the injective resolutions just selected be

$$0 \to G \overset{\iota}{\to} Y^0 \overset{\beta^0}{\to} Y^1 \overset{\beta_1}{\to} \ldots \to Y^n \overset{\beta_n}{\to} \ldots$$

and

$$0 \to F \overset{\varepsilon}{\to} X^0 \overset{\alpha_0}{\to} X^1 \overset{\alpha_1}{\to} \ldots \to X^n \overset{\alpha_n}{\to} \ldots .$$

Suppose we are given a set of A-homomorphisms $\{f^i\}_{i \epsilon \mathcal{N}}$ such that $f^i : X^i \to Y^{i+s}$ for all $i \in \mathbb{N}$ and a fixed $s \in \mathbb{N}$ such that the diagram



commutes . Under these circumstances we will write, that $f \in \mathrm{Hom}^s_A(\mathbf{X}^\cdot, \mathbf{Y}^\cdot)$ with an obvious interpretation for varying $s, \mathbf{X}^\cdot, \mathbf{Y}^\cdot$. It is then clear that f^0 combined with ε gives a homomorphism from F to Y^s. We will write in the sequel for this, that $f^0 \circ \varepsilon \in \mathrm{Hom}^s_A(F, \mathbf{Y}^\cdot)$, again with an obvious interpretation for other $s, F, \mathbf{Y}^\cdot$.

We observe, that if $f \in \mathrm{Hom}^s_A(\mathbf{X}^\cdot, \mathbf{Y}^\cdot)$, then $f^0 \circ \varepsilon \in \mathrm{Hom}^s_A(F, \mathbf{Y}^\cdot)$ and $\beta^s \circ f^0 \circ \varepsilon = f^1 \circ \alpha^0 \circ \varepsilon = 0$ by the commutativity of the diagram. So $f^0 \circ \varepsilon \in ker(\mathrm{Hom}_A(F, \beta^0)$ and so defines in this way as cocycle an element in $H^s(\mathrm{Hom}_A(F, \mathbf{Y}^\cdot)) \cong \mathrm{Ext}^s_A(F, G)$. So there is a map

$$\varphi^s : \mathrm{Hom}^s_A(\mathbf{X}^\cdot, \mathbf{Y}^\cdot) \to \mathrm{Ext}^s_A(F, G)$$

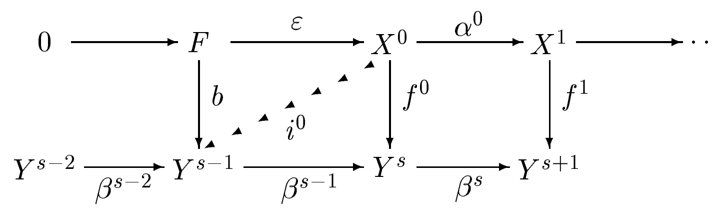
for all $s \in \mathbb{N}$ working by $f \to f^0 \circ \varepsilon$.

Now two different maps $f, g \in \mathrm{Hom}^s_A(\mathbf{X}^\cdot, \mathbf{Y}^\cdot)$ which are homotopic, give the same representative in $\mathrm{Ext}^s_A(F, G)$ under this procedure: if $f^0 = g^0 + i^1 \circ \alpha^0 + (0 \to Y^0) \circ i^0$ for $i^1 : X^1 \to Y^0$ and $i^0 : X^0 \to 0$, then visibly $f^0 \circ \varepsilon = g^0 \circ \varepsilon$ by the exactness of $\mathbf{X}^\cdot$. We therefore put together all maps in $\mathrm{Hom}^s_A(\mathbf{X}^\cdot, \mathbf{Y}^\cdot)$ which are homotopic. We call the result $\mathfrak{H}om^s_A(F, \mathbf{Y}^\cdot)$. Since the corresponding representative in $\mathrm{Ext}^s_A(F, G)$ of an element $f \in \mathrm{Hom}^s_A(F, \mathbf{Y}^\cdot)$ is independent on the chosen member of the homotopy class, we have established a map

$$\varphi'^s : \mathfrak{H}om^s_A(\mathbf{X}^\cdot, \mathbf{Y}^\cdot) \to \mathrm{Ext}^s_A(F, G).$$

Let now again $f \in \mathrm{Hom}^s_A(\mathbf{X}^\cdot, \mathbf{Y}^\cdot)$ and suppose further that φ^s applied to f gives the zero map, that is $f^0 \circ \varepsilon \in \mathrm{im}(\mathrm{Hom}_A(F, \beta^{s-1}))$. Then there is a

$b \in \mathrm{Hom}_A(F, Y^{s-1})$ with $f^0 \circ \varepsilon = \beta^{s-1} \circ b$. So we have

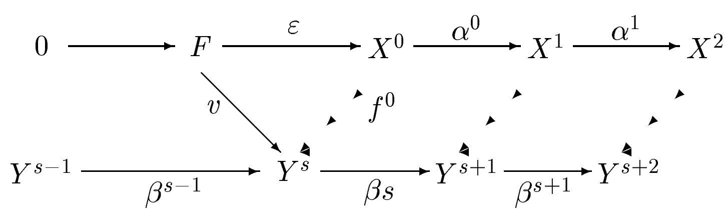


and Y^{s-1} is injective. So there exists $i^0 : X^0 \to Y^{s-1}$ such that $i^0 \circ \varepsilon = b$. Then $\omega := f^0 - \beta^{s-1} \circ i^0$ maps $X^0 into Y^s$, $\omega \circ \varepsilon$ is zero, $F \to X^0 \to X^1$ is exact and Y^s is injective. So there exists $i^1 : X^1 \to Y^s$ such that $i^1 \circ \alpha^0 = f^0 - \beta^{s-1} \circ i^0$. Proceeding in this way, we establish the existence of maps $i^j : X^j \to Y^{j+s-1}$ for $j \in \mathbb{N}$ with

$$f^j = \beta^{j-s+1} \circ i^j + i^{j+1} \circ \alpha^i.$$

Therefore f is nullhomotopic. (In the case $s = 0$, $f^0 \circ \varepsilon$ is in $\mathrm{Hom}_A(F, 0)$ by the left exactness of the Hom-functor, so $f^0 \circ \varepsilon = 0$ such that we can build up the family $\{i^j\}$ beginning with $i^1 : X^1 \to Y^0$ since $i^0 : X^0 \to 0$ is unique.)

Now let $\overline{v}$ be an element of $\mathrm{Ext}^s_A(F, G)$ in the sense we have adopted. Then we can write $\overline{v} = v + im(\mathrm{Hom}_A(F, \beta^{s-1}))$, $v \in \mathrm{Hom}_A(F, Y^s)$, $\beta^s \circ v = 0$. We have the diagram



showing that v factors through X^0 by the injectivity of Y^s. So f^0 exists. Since $\beta^s \circ v = 0$, $\beta^s \circ f^0 \circ \varepsilon = 0$.

Hence

- $\beta^s \circ f^0 : X^0 \to Y^{s+1}$,
- $F \to X^0 \to Y^{s+1}$ is zero,
- $F \to X^0 \to X^1$ is exact,
- Y^{s+1} is injective.

It follows the existence of $f^1 \in \mathrm{Hom}^s_A(X^1, Y^{s+1})$ such that $f^1 \circ \alpha^0 = \beta^s \circ f^0$.

Working our way along the resolution of F we get A-homomorphisms $\{f^i\}_{i \epsilon \mathcal{N}}$ which by construction constitute an $f \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$. It is obvious that φ^s applied to f gives $\overline{v}$. Then φ^s (homotopy class of f) equals $\overline{v}$ as well. So φ'^s is surjective.

We show now how to turn $\mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ into a module. Suppose, we have $f, g \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$. We define the sum $(f+g)$ by $(f+g)^i(x) = f^i + g^i$ for $x \in X^i$. Since all f^i, g^i are A-homomorphisms, this gives again a chain map. We also can define, for $f \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ the negative $(-f)$ where $(-f)^i(x) = -f^i(x)$ for $x \in X^i$. Since again all f^i are A-homomorphisms, this gives a chain map and of course $f + (-f)$ is the zero transformation.

Lastly for $a \in A$ and $f \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ we define $(a \cdot f)$ by $(a \cdot f)^i(x) = af^i(x) = f^i(a{\cdot}x)$ which is possible because f^i is A-linear and A commutative. So $\mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ is an A-module.

Now that we know that domain and range of φ^s are A-modules there is the natural question whether φ^s is possibly an A-homomorphism. Since φ^s applied to $f \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ is taking the cohomology class of f^0, it is clear that by definition of addition in $\mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$, φ^s is additve. Also for $a \in A$,

$$\varphi^s(a \cdot f) = \varphi^s(\{x^i \to a \cdot f^i(x^i)\}_{i\epsilon\mathcal{N}, x^i \epsilon X^i}) = \overline{af^0} = a\overline{f^0} = a\varphi^s(f)$$

such that φ^s is A-linear. Now our first investigations concerning φ^s have brought to light that $f \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ maps to zero under φ^s if and only if f is nullhomotopic. By the now established fact that φ^s is an A-homomorphism, it follows that the set of nullhomotopic translations of degree s is actually the kernel of φ^s, $\mathfrak{H}om^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ is as the qoutient an A-module and $\varphi'^s : \mathfrak{H}om^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot}) \to \mathrm{Ext}^s_A(F, G)$ is an isomorphism.

Suppose now, we have a third A-module H together with the corresponding injective resolution $0 \to H \xrightarrow{\zeta} \mathbf{Z}^{\cdot}$ and an A-homomorphism $\eta : H \to F$. Then we can build over η a chain map $\eta^{\cdot} : \mathbf{Z}^{\cdot} \to \mathbf{X}^{\cdot}$ in the sense of [**9**], Chapter 5. By [**9**], Theorem 5.13 all possible $\eta^{\cdot}$ belong to the same homotopy class.

We may now take one of these and fix it. Then for any $f \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ we can combine f^i and η^i to a homomorphism $f^i \circ \eta^i : Z^i \to Y^{i+s}$ for all $i \in \mathbb{N}$. It is then clear that this collection of A-maps commutes with the coboundaries of $\mathbf{Y}^{\cdot}$ and $\mathbf{Z}^{\cdot}$. So $\{f^i \circ \eta^i\}_{i\epsilon\mathcal{N}}$ represents an element of $\mathrm{Hom}^s_A(\mathbf{Z}^{\cdot}, \mathbf{Y}^{\cdot})$. This means, that starting with an $f \in \mathrm{Hom}^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot})$ we can first apply φ^s and then combine the result with η or we consider φ^s applied to the combination of $\eta^{\cdot}$ and f, in both cases finishing in $\mathrm{Ext}^s_A(H, G)$. So the question is about the commutativity of the diagram

$$\begin{array}{ccc}
Hom^s_A(\mathbf{Z}^{\cdot}, \mathbf{Y}^{\cdot}) & \longrightarrow & H^s(Hom(H, \mathbf{Y}^{\cdot})) \\
\circ\eta^{\cdot} \uparrow & & \uparrow \circ\eta \\
Hom^s_A(\mathbf{X}^{\cdot}, \mathbf{Y}^{\cdot}) & \longrightarrow & H^s(Hom(F, \mathbf{Y}^{\cdot}))
\end{array}$$

If we follow one element each time, we get

$$\begin{array}{ccc} g & \longrightarrow & \overline{g^0 \circ \zeta} \\ f \circ \eta^{\cdot} & & \overline{h \circ \eta} \\ \Big\uparrow & & \Big\uparrow \\ & & \overline{h} \\ f & \longrightarrow & \overline{f^0 \circ \varepsilon} \end{array}$$

and the combined maps work as follows

$$\begin{array}{ccc} f \circ \eta^{\cdot} & \longrightarrow & \overline{(f \circ \eta^{\cdot})^0 \circ \zeta} = \overline{f^0 \circ \eta^0 \circ \zeta} = \overline{f^0 \circ \varepsilon \circ \eta} \\ & & \overline{f^0 \circ \varepsilon \circ \eta} \\ \Big\uparrow & & \Big\uparrow \\ \overline{f} & \longrightarrow & \overline{f^0 \circ \varepsilon} \end{array}$$

by the construction of $\eta^{\cdot}$. It follows, that φ^s is natural in the first variable of the Hom^s_A-functor.

Since we do not want to investigate what happens to φ under change of the injective resolutions, we stick to the agreement, that for one module we have selected just one injective resolution.

Now let T be a covariant A-linear functor which is left exact and denote by T^i its right derived functors. Let M, N be two A-modules and $0 \to M \overset{\varepsilon}{\to} \mathbf{M}^{\cdot}$ and $0 \to N \overset{\iota}{\to} \mathbf{N}^{\cdot}$ be the two corresponding injective resolutions. Then as explained there is an A-isomorphism

$$\varphi'^{s}_{M,N} : \mathfrak{H}om^s_A(\mathbf{M}^{\cdot}, \mathbf{N}^{\cdot}) \to \mathrm{Ext}^s_A(M, N)(s \in \mathbb{N}).$$

So to each $\overline{f} \in \mathrm{Ext}^s_A(M, N)$ corresponds a unique class of homotopic translations $\overline{f}^{\cdot}$ of degree s from $\mathbf{M}^{\cdot}$ to $\mathbf{N}^{\cdot}$. The functor T, if applied to a translation $g^{\cdot} : M^{\cdot} \to N^{\cdot}$ of degree s yields a translation $T(\mathbf{g}^{\cdot}) : T(\mathbf{M}^{\cdot}) \to T(\mathbf{N}^{\cdot})$ of the same degree and homotopic translations are carried into homotopic translations. It follows that each such $T(g^{\cdot})$ induces a family of A-homomorphisms on the cohomology modules in a unique fashion, where homotopic translations induce the same family of morphisms by [**9**], Theorem 4.7.

Since the cohomology modules of $T(\mathbf{M}^{\cdot})$ are just the derived functors of T applied to M and the same is true for N instead of M , we conclude that each $\overline{f}$ in $\mathrm{Ext}^s_A(M, N)$ gives rise to a unique family of A-homomorphisms from $T^i(M)$ to $T^{s+i}(N)$. This is to say, that for $\overline{f} \in \mathrm{Ext}^s_A(M, N)$ and $\overline{m} \in T^i(M)$ there is a unique element $\overline{n} \in T^{i+s}(N)$ such that the latter is the unique result of the action on the cohomology induced by $\overline{f}$ via φ^s applied to $\overline{m}$. We conclude, that there is a map

$$\times : T^i(M) \times \mathrm{Ext}^s_A(M, N) \to T^{i+s}(M)$$

for all A-modules M, N and all $i, s \in \mathbb{N}$. Is is clear from the definition, that this map is A-linear in the first component. What about the second?

Let $\overline{f}, \overline{g} \in \operatorname{Ext}^s_A(M, N)$. By the definition of $\times$, $\overline{m} \times \overline{g} + \overline{m} \times \overline{f}$ is the result of the cohomology map induced by $\overline{f}$ applied to $\overline{m}$ added to the result of the cohomology map induced by $\overline{g}$ applied to $\overline{m}$. But this is just the result of $(\overline{f} + \overline{g}) \times \overline{m}$, so $\times$ is additive in the second argument. Similarily, if $f \in \operatorname{Hom}_A(\mathbf{M}^{\cdot}, \mathbf{N}^{\cdot})$ corresponds to $\overline{f} \in \operatorname{Ext}^s_A(M, N)$, then $\overline{m} \times \overline{(a \cdot f)}$ is just A times $\overline{m} \times \overline{f}$ so that $\times$ is A-linear in both arguments. Since this is the case, each $\overline{m} \in T^i(M)$ gives rise to a unique A-homomorphism $\operatorname{Ext}^s_A(M, N) \to T^{i+s}(N)$. By the linearity in the first argument, we have established an A-homomorphism

$$\Omega^T_{M,N} : T^i(M) \to \operatorname{Hom}_A(\operatorname{Ext}^s_A(M, N), T^{i+s}(N))$$

for which

$$\overline{m} \to (\overline{f} \to \overline{m} \times \overline{f}).$$

We consider now the case of varying M. For, let $\mu : M \to M'$ be an A-homomorphism and $\mathbf{M}'^{\cdot}$ be our chosen injective resolution for M'. Then we build over μ chainmap as before and described in [**9**], 5.12. We would like to show that the diagram

$$\begin{array}{ccc}
T^iM & \xrightarrow{\Omega^T_{M,N}} & Hom_A(Ext^j_M(M, N), T^{i+j}N) \\
{\scriptstyle T^i\mu}\downarrow & & \downarrow {\scriptstyle Hom_A(Ext^j_A(\mu, N), T^{i+j}N)} \\
T^iM' & \xrightarrow[\Omega^T_{M',N}]{} & Hom_A(Ext^j_A(M', N), T^{i+j}N)
\end{array}$$

is commutative. If we take $\overline{m} \in T^i(M)$ and follow it on its way in the diagram, we find

$$\begin{array}{ccc}
\overline{m} & \longrightarrow & ((\overline{f} : M \to N^j) \to T^if(\overline{m})) \\
\downarrow & & \downarrow \\
 & & ((\overline{f}' : M' \to N^j) \to T^i(f'\mu)(\overline{m})) \\
 & & \| \\
(T^i\mu)(\overline{m}) & \longrightarrow & ((\overline{f}' : M' \to N^j) \to (T^i(f')((T^i\mu^0)(\overline{m})))
\end{array}$$

and since T is functor, Ω is natural in M. Consequently, $\Omega^T_{M,N}$ establishes a natural transformation from the covariant functor T^i to the covariant functor $\operatorname{Hom}_A(\operatorname{Ext}^s_A(\cdot, N), T^{s+i}(N))$ for all $i, s \in \mathbb{N}$ and each fixed N.

We will now apply this theory to the functor $\Gamma_{\mathfrak{a}}$ for an ideal $\mathfrak{a}$ of A. We have just established a natural transformation

$$\Omega_{\cdot,N} = \Omega^{\Gamma_{\mathfrak{a}}}_{\cdot,N} : H^i_{\mathfrak{a}}(\cdot) \to \operatorname{Hom}_A(\operatorname{Ext}^s_A(\cdot, N), H^{i+s}_{\mathfrak{a}}(N))$$

for each fixed A-module N and all $i, s \in \mathbb{N}$.

Suppose now, that $(A, \mathfrak{m})$ is local Gorenstein. Then from 5.8 follows, that for all prime ideals $\mathfrak{p}$ in $\mathrm{Spec}(A)$

$$M_i pa = \delta_{i,\mathrm{ht}(\mathfrak{p})}.$$

The local cohomolgy modules of A with respect to $\mathfrak{m}$ are then zero for $i < \dim(A)$ by 4.6 and equal to $E^{\dim(A)}(A)$ for $i = \dim(A)$ by 1.16 and the fact that

$$\mathrm{im}(\Gamma_{\mathfrak{m}}(E^{\dim(A)-1} \to E^{\dim(A)})) = 0.$$

We set now $i + s = \dim(A)$ and have in this case a natural transformation

$$\Omega_{\cdot,A} : H^i_{\mathfrak{m}}(\cdot) \to \mathrm{Hom}_A(\mathrm{Ext}_A^{\dim(A)-i}(\cdot, A), I)$$

for all A-modules N and all $0 \le i \le \dim(A)$ where I denotes $E_A^{dim(A)}(A) = H_{\mathfrak{m}}^{\dim(A)}(A)$. Denote $\dim(A)$ by n. We consider the case $i = n$, $\cdot = A$:

$$\Omega_{A,A} : I = H^n_{\mathfrak{m}}(A) \to \mathrm{Hom}_A(\mathrm{Hom}_A(A, A), I) \cong \mathrm{Hom}_A(A, I) \cong I$$

It would be nice if this were an isomorphism. To answer this question we consider what is going on here. Let $0 \to A \to \mathbf{A}^{\cdot}$ be the injective resolution that belongs according to our agreement to A. In our earlier notation we have $s = 0$. Let α be an element of $H^n_{\mathfrak{m}}(A) = I$. Then Ω is defined by means of α. Take $\mathrm{id}_A \in \mathrm{Hom}_A(A, A) \cong \mathrm{Ext}^0(A, A)$. We have now to ask, how to calculate $\alpha \times \mathrm{id}_A$. By construction of the bijectivity between $\mathfrak{Hom}^s_A(A, \mathbf{A}^{\cdot})$ and $\mathrm{Ext}^s_A(A, A)$ we have to build up a chain map over an A-homomorphism, that makes

$$\begin{array}{ccccc}
0 & \longrightarrow & A & \xrightarrow{\iota} & A^0 \\
 & & \downarrow \mathrm{id}_A & & \begin{array}{c} M_{\mathrm{id}} \\ M_{\mathrm{id}} \\ \downarrow \end{array} \\
0 & \longrightarrow & A & \xrightarrow{\iota} & A^0
\end{array}$$

commute. One easily detects that id_{A^0} is a very suitable map to fill the diagram. Further it is clear, that in every case like

$$\begin{array}{ccccc}
A^{j-1} & \xrightarrow{\alpha^{j-1}} & A^j & \xrightarrow{\alpha^j} & A^{j+1} \\
\downarrow \mathrm{id}_{A^{j-1}} & & \begin{array}{c} M_{\mathrm{id}} \\ M_{\mathrm{id}} \\ \downarrow \end{array} & & \\
A^{j-1} & \xrightarrow{\alpha^{j-1}} & A^j & \xrightarrow{\alpha^j} & A^{j+1}
\end{array}$$

we can use id_{A^j} to complete the diagram commutative for $j \in \mathbb{N}$. It follows, that the induced morphism on the cohomology is the identity map, because $\Gamma_{\mathfrak{m}}$ is restriction of domain when applied to maps. Therefore for all $\alpha \in H^n_{\mathfrak{m}}(A)$

$$\alpha \times id_A = \alpha.$$

So $\Omega_{A,A}$ is injective. Further, all elements of $\mathrm{Hom}_A(A,A)$ are A-multiples of id_A. Since the pairing is linear, $\alpha \times (id_A \cdot a) = a \cdot \alpha$ for $a \in A$. So each map from $\mathrm{Hom}_A(A,A)$ is uniquely described by the image of id_A. From the above line follows then, that $\Omega_{A,A}$ is the identity map. We prove now for all finitely generated modules M that $\Omega_{M,A}$ is actually an isomorphism. To this end let M be finitely generated and

$$F^r \to F^s \to M \to 0$$

be the end of A free resolution for M, where F^s, F^r are the finite direct sum of r and s copies of A respectively. (We have shown in 5.5, that this is possible.) Since $\Gamma_{\mathfrak{m}}$ is A-linear, and hence $H^n_{\mathfrak{m}}$ as well, they carry exact split sequences into split exact sequences, which implies in the current case that by induction on rank $\Omega_{F,A}$ is an isomorphism for finitely generated free F. Then

$$\begin{array}{ccccccccc}
H^n_m(A^s) & \longrightarrow & H^n_m(A^r) & \longrightarrow & H^n_m(M) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow ? & & \downarrow & & \downarrow \\
\mathrm{Hom}_A(\mathrm{Ext}^0_A(A^s,A),I) & \longrightarrow & \mathrm{Hom}_A(\mathrm{Ext}^0_A(A^r,A),I) & \longrightarrow & \mathrm{Hom}_A(\mathrm{Ext}^0_A(M,A),I) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

is a commutative diagram in which all vertical maps except for the middle one are A-isomorphisms. Further, the topline is exact since $n = \dim(A)$ and by 4.9 $H^n_{\mathfrak{m}}(\cdot)$ is right exact. Also, $\mathrm{Ext}^0_A(\cdot, A)$ is naturally isomorphic to $\mathrm{Hom}_A(\cdot, A)$ and the latter is left exact. Since $\mathrm{Hom}_A(\cdot, I)$ is exact, the combination is rightexact. It follows from the cited before five lemma, that $\Omega_{M,A}$ is an isomorphism for all finitely generated modules M.

Since I is injective A-module, the functor $\mathrm{Hom}_A(\cdot, I)$ is exact and so

$$\mathrm{Hom}_A(\mathrm{Ext}^{n-i}_A(\cdot, A), I) \text{ and } \mathcal{L}^{n-i}(\mathrm{Hom}_A(\mathrm{Hom}_A(\cdot, A), I)$$

are naturally eqivalent functors by [**9**], Theorem 5.1 where the $\mathcal{L}$ means 'leftderived'.

We know further, that $\{H^i_{\mathfrak{m}}(\cdot)\}_{i\epsilon\mathcal{N}}$ is a connected sequence of covariant functors and that $H^n_{\mathfrak{m}}(\cdot)$ and $\mathrm{Hom}_A(\mathrm{Ext}^0(\cdot, A), I)$ are naturally equivalent functors on finitely generated modules.

We aim to show for the present that the $\{H^i_{\mathfrak{m}}(\cdot)\}$ are the *left*derived functors of $H^n_{\mathfrak{m}}(\cdot)$ on the subcategory of $\mathcal{C}_A$ consisting only of finitely generated modules. By [**9**], Theorm 6.2 one can calculate these functors by means of a projective resolution consisting of finitely generated modules, if this exists. And it does by 5.5. So by [**9**], Theorem 6.12 and its corollary it is sufficient to show that for all short exact sequences

$$0 \to X \to P \to Y \to 0$$

where X, P, Y are finitely generated and P projective, the sequence

$$0 \to H^{n-i}_{\mathfrak{m}}(Y) \to H^{n-i+1}_{\mathfrak{m}}(X) \to H\mathfrak{m}^{n-i+1}(P)$$

is exact for $i > 0$.

We know that the sequence

$$H_{\mathfrak{m}}^{n-i}(P) \to H_{\mathfrak{m}}^{n-i}(Y) \to H_{\mathfrak{m}}^{n-i}(X) \to H_{\mathfrak{m}}^{n-i+1}(P)$$

is exact, because $H_{\mathfrak{m}}^{i}(\cdot)$ are derived functors. It therefore suffices to show, that $H_{\mathfrak{m}}^{i}(P) = 0$ whenever $i < n$. This we do. Since finitely generated projective modules over local rings are finitely generated free modules by [**9**], Theorem 9.12 and hence a finite direct sum of copies of A, it suffices by the linearity of the functors in question, to show that

$$H_{\mathfrak{m}}^{n-i}(A) = 0 \text{ for } i > 0.$$

But this is clear since A is local Gorenstein. So we have shown that for finitely generated modules M, $H_{\mathfrak{m}}^{n-i}(M)$ and $\mathcal{L}^i(H_{\mathfrak{m}}^{n}(M))$ are naturally isomorphic modules.

Together with the above result we get that

$$\{H_{\mathfrak{m}}^{i}(\cdot)\} and \{\mathrm{Hom}_A(\mathrm{Ext}_A^i(\cdot, A), I)\}(i \in \mathbb{N})$$

are naturally equivalent connected sequences of functors on the category of finitely generated A-modules and A-homomorphisms between them. So there is for each finitely generated A-module M a natural isomorphism

$$\mu : H_{\mathfrak{m}}^{i}(M) \to \mathrm{Hom}_A(\mathrm{Ext}_A^{n-i}(M, A), I)$$

and an induced isomorphism

$$\nu : \mathrm{Hom}_A(H_{\mathfrak{m}}^{i}(M), I) \to \mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}_A^{n-i}(M, A), I), I).$$

Let now $\mathfrak{a} = \mathrm{ann}(H_{\mathfrak{m}}^{i}(M))$ and $\mathfrak{b} = \mathrm{ann}(\mathrm{Ext}_A^{n-i}(M, A))$. Then the first isomorphism shows that $\mathfrak{b} \subseteq \mathfrak{a}$, whereas the latter shows

$$\mathfrak{a} \cdot (\mathrm{Hom}_A(\mathrm{Hom}_A(\mathrm{Ext}^{n-i}(M, A), I), I)) = 0.$$

We try to show, that $\mathfrak{a} = \mathfrak{b}$, that is $\mathfrak{a} \cdot \mathrm{Ext}_A^{n-i}(M, A) = 0$. To this end, we show that each A-module X can be embedded in $\mathrm{Hom}_A(\mathrm{Hom}_A(X, I), I)$.

Let $x \in X$. Then the map

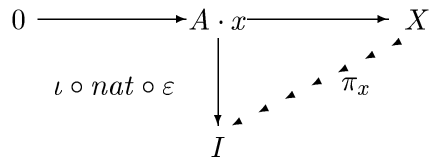
$$\psi : X \to \mathrm{Hom}_A(\mathrm{Hom}_A(X, I), I) \text{ by } x \to (f \to f(x))$$

where $f \in \mathrm{Hom}_A(X, I)$ establishes obviously an A-homomorphism. If then x happens to be mapped into zero, then for each $f \in \mathrm{Hom}_A(X, I)$, $f(x)$ must be zero. So if we can for all $0 \neq x \in X$ show, that there is at least one map f in $\mathrm{Hom}_A(X, I)$ such that $f(x) \neq 0$, we have shown, that ψ is injective and this is what we want.

If $x \in X$ is nonzero, then $A \cdot x$ is a nonzero cyclic module and hence isomorphic to $A/\mathrm{ann}(x)$ where $\mathrm{ann}(x) \neq A$. So $\mathrm{ann}(x) \subseteq \mathfrak{m}$, since A is local, and there is a natural projection $nat : A/\mathrm{ann}(x) \to A/\mathfrak{m}$ carrying $(a + ann(x))$ into $(a + \mathfrak{m})$. Lastly I is the injective hull of $A/\mathfrak{m}$ and so allows a monomorphism $\iota : A/\mathfrak{m} \to I$. The combination of all these homomorphisms is obviously a homomorphism σ_x from $A \cdot x$ into I and x is mapped to

$$x \to 1 + ann(x) \to 1 + \mathfrak{m} \to \iota(1 + \mathfrak{m}) \neq 0.$$

Since I is injective, the diagram



can be completed by a homomorphism π_x under which x is not annihilated. So for each $x \in X$ there is a homomorphism $X \to I$ for which x is not in the kernel. We conclude, that ψ is injective. Hence from

$$\mathfrak{a} \cdot \operatorname{Hom}_A(\operatorname{Hom}_A(\operatorname{Ext}_A^{n-i}(M, A), I), I) = 0$$

follows that

$$\mathfrak{a} \cdot \operatorname{Ext}_A^{n-i}(M, A) = 0.$$

We put together all results in the final statement of this chapter.

THEOREM 5.10.

$$\operatorname{ann}_A(\operatorname{Ext}_A^{n-i}(M, A)) = \operatorname{ann}_A(H_{\mathfrak{m}}^i(M))$$

for all $0 \leq i \leq n = \dim(A)$ *whenever* A *is local Gorenstein and* M *finitely generated.* □

The interested reader might like to consider the skeleton of this proof, which is mainly to be found between Pr. 4.10 and Th. 6.3 of [**4**].

CHAPTER 6

Annihilators

We have set the stage in the last chapter for a theorem, which is known as Faltings' *annihilator theorem*, or, as Faltings himself calls it in a later paper([**19**]) the *weak annihilator theorem*. This theorem will give a partial answer to the question mentioned repeatedly about connections between depths, dimensions and zeros. The result we will prove is up to a corollary the only outstanding point of this chapter and it will be the most involved of this work. However, we have to admit, that we only give a weakened form of the original proof and statement and the reader interested in the *strong annihilator theorem* should study [**20**] rather than what follows. On the other hand, readers who do not know about dualising complexes will fare better with our version.

Although the original proof fills only two small pages, we will need considerably more space and as said before the whole last chapter has been preparation for the proof. We have to make a remark about the notation. In the sequel if I is an ideal of A and $\mathrm{Var}(I) = Z \subseteq \mathrm{Spec}(A)$, we will for the functor H^n_I also use the symbol H^n_Z ,so emphasising the interpretation of Γ given after 1.8.

LEMMA 6.1. *Let B be a commutative Noetherian ring and $\mathfrak{b}$ an ideal of B. Set $A := B/\mathfrak{b}$. Let M be an A-module. Set $Z = \mathrm{Var}(I)$ and $Y = \mathrm{Var}(J)$ for two ideals $I, J \supseteq \mathfrak{b}$ of B. Denote the set $\{\mathfrak{p}/\mathfrak{b} : \mathfrak{p} \in Z\}$ of prime ideals in A by Z' and $\{\mathfrak{p}/\mathfrak{b} : \mathfrak{p} \in Y\}$ by Y'. Suppose, $Z \subseteq Y$. Fix $i \in \mathbb{N}$.Then there exists an ideal $\mathfrak{a}'$ in A with $\mathrm{Var}(\mathfrak{a}') \subseteq Y'$ and $\mathfrak{a} \cdot H^i_{Z'}(M) = 0$ if and only if there exists an ideal $\mathfrak{a} \supseteq \mathfrak{b}$ in B with $\mathrm{Var}(\mathfrak{a}) \subseteq Y$ and $\mathfrak{a}' \cdot H^i_Z(M) = 0$.*

PROOF. Suppose, $\mathfrak{a} \subseteq B$ is an ideal as described in the second part of the claim. Then $\mathfrak{a}' = \mathfrak{a}/\mathfrak{b}$ has variety in Y'. Further, $H^i_Z(M)$ is by independence of the base ring (3.20) when M is considered as B-module isomorphic to $H^i_{Z'}(M)$, this then restricted to a B-module. So $\mathfrak{a}$ annihilates $H^i_{Z'}(M)$. Now, if $(a + \mathfrak{b}) \in \mathfrak{a}'$, then multiplication of an A-module by this yields the same result as multiplication by a alone, because $\mathfrak{b}$ has to be in the annihilator of such a module. It follows, that $\mathfrak{a}' \cdot H^i_{Z'}(M) = 0$.

Conversely, if $\mathfrak{a}'$ is an ideal of A with variety in Y', then $\mathfrak{a}' = \mathfrak{a}/\mathfrak{b}$ for some ideal $\mathfrak{a}$ in B. Of course $\mathrm{Var}(\mathfrak{a}) \subseteq\ Y.Since H^i_{Z'}(M)$ is an A-module, $\mathfrak{b} \cdot H^i_{Z'}(M) = 0$. Then $(a+\mathfrak{b}) \in \mathfrak{a}'$ implies $(a+\mathfrak{b}) \cdot H^i_{Z'} = a \cdot H^i_{Z'}(M)$ and this is zero as we know so that $\mathfrak{a} \cdot H^i_{Z'}(M) = 0$. By 3.20 again $\mathfrak{a} \cdot H^i_Z(M) = 0$. $\square$

We prove two similar lemmata.

LEMMA 6.2. *Let B be a commutative ring. Set $B/\mathfrak{c} = A$ for some ideal $\mathfrak{c}$ in B. Let $\mathfrak{b}$ be a prime ideal in B and set $\mathfrak{a} = \mathfrak{b}/\mathfrak{c}$, a prime ideal in A. Let M be an A-module, such that we may consider it as B-module as well.*

$$\operatorname{depth}_{A_\mathfrak{a}}(M_\mathfrak{a}) = \operatorname{depth}_{B_\mathfrak{b}}(M_\mathfrak{b})$$

for all A-modules M.

PROOF. We have $A_\mathfrak{a} \cong (B/\mathfrak{c})_{\mathfrak{b}/\mathfrak{c}} \cong (B_\mathfrak{b})/(\mathfrak{c}_\mathfrak{b})$ by [**12**], 5.44. Further, an element $m = \frac{\mu}{\sigma} \in M_\mathfrak{b}$ is zero if and only if there is a $\beta \in B_\mathfrak{b} \backslash \mathfrak{b} \cdot B_\mathfrak{b}$ with $\beta \cdot \mu = 0$. This is the case if and only if $(\beta + \mathfrak{c}) \cdot m = 0$ which can happen if and only if $\frac{m}{\sigma + \mathfrak{c}} = 0$ in $M_\mathfrak{a}$.

Let ξ be in $\mathfrak{b} \cdot B_\mathfrak{b}$ and assume, that ξ is a zerodivisor on $M_\mathfrak{b}$. Then there is an $m \in M$ and $\sigma \in B \backslash \mathfrak{b}$, such that $\xi \cdot \frac{m}{\sigma} = 0$. Hence there is $\sigma' \in B \backslash \mathfrak{b}$ with $\xi \cdot m \cdot \sigma' = 0$. Since $\mathfrak{c}$ annihilates M, this implies that

$$(\xi + \mathfrak{c}) \cdot m \cdot (\sigma + \mathfrak{c}) = 0,$$

showing that $(\xi + \mathfrak{c}) \cdot \frac{m}{\sigma + \mathfrak{c}} = 0$ and hence the image of ξ under the natural projection $B_\mathfrak{b} \to A_\mathfrak{a}$ is zerodivisor on $M_\mathfrak{a}$. Similarily is clear that whenever $(\xi + \mathfrak{c}) \in A_\mathfrak{a}$ is zerodivisor on $M_\mathfrak{a}$, that then $\xi \in B_\mathfrak{b}$ is zerodivisor on $M_\mathfrak{b}$.

Since by [**12**], 9.44 for $\xi \in B_\mathfrak{b}$

$$M_\mathfrak{a}/((\xi + \mathfrak{c}) \cdot M_\mathfrak{a}) \cong M_\mathfrak{b}/(((\mathfrak{x}) + \mathfrak{c}) \cdot M_\mathfrak{b},$$

we can in the case that there is a zerodivisor on $M_\mathfrak{a}$ in $\mathfrak{a} \cdot A_\mathfrak{a}$ apply the above argument to the quotient of $M_\mathfrak{a}$. It follows that there is a one-one-correspondence between the $M_\mathfrak{a}$-sequences in $\mathfrak{a} \cdot A_\mathfrak{a}$ and the $M_\mathfrak{b}$-sequences in $\mathfrak{b} \cdot B_\mathfrak{b}$ which preserves length, hence maximal $M_\mathfrak{a}$-sequences have the same length as maximal M_{bB}-sequences. □

LEMMA 6.3. *Let B be a commutative ring and $x, z \in \operatorname{Spec}(B)$. Let $\mathfrak{b} \subseteq x \cap z$ and set $A = B/\mathfrak{b}$, $x' = x/\mathfrak{b}$, $z' - z/\mathfrak{b}$. Then*

$$\dim((B/x)_z) = \dim((A/x')_{z'}).$$

PROOF. If $x \not\subseteq z$, then $x' \not\subseteq z'$ and both dimensions are -1, because the modules are zero. If $x \subseteq z$, then the result follows from 2.46, 3.28 and 5.32 of [**12**]. □

It is admitted, that these three statements are neither difficult nor the proofs very interesting. However, a reader who for the first time comes across these three things in a phrase like 'obviously' and is not given any further comment, might be a bit worried.

We will now develop a functor, that is very related to the torsion functor, but not quite the same. For we make

DEFINITION 6.4. Let A be commutative noetherian. A subset Z of $\operatorname{Spec}(A)$ is said to be *stable under specialisations* if $z \in Z$ implies $\operatorname{Var}(z) \subseteq Z$.

This is to say that Z is the union of closed sets in $\operatorname{Spec}(A)$ where a set is closed, if and only if it is the variety of some ideal of A.

DEFINITION 6.5. Let A be commutative Noetherian and $Z \subseteq \operatorname{Spec}(A)$ be stable under specialisations. Let M be an A-module. We define $\Gamma_Z(M)$ to be the set

$$\Gamma_Z(M) = \{m \in M : \operatorname{supp}(A \cdot m) \subseteq Z\}.$$

Since $A \cdot m$ is finitely generated, this is the same as

$$\{m \in M : \operatorname{Var}(\operatorname{ann}(m)) \subseteq Z\} = \{m \in M : \exists \mathfrak{a} in A : \operatorname{Var}(\mathfrak{a}) \subseteq Z, \mathfrak{a}^n \cdot m = 0\}$$

and it follows from this last description, that $\Gamma_Z(M)$ is a submodule.

Further, given an A-homomorphism $f : M \to N$ it is clear that $f(\Gamma_Z(M))$ consists only of elements which are annihilated by an ideal $\mathfrak{a}$ of A with $\operatorname{Var}(\mathfrak{a}) \subseteq Z$. Since Γ_Z is taking submodules, its restriction to maps is restriction. Therefore combined A-homomorphisms give rise to combined induced homomorphisms, because this Γ too is restriction if applied to maps, and application to the identity map on M leads to the identity map on $\Gamma_Z(M)$. So Γ_Z is a covariant functor. As said before, its application to maps is restriction so Γ_Z is linear.

We come now to statement and proof of the annihilator theorem.

THEOREM 6.6. *Let A be the quotient of a Gorenstein ring of finite dimension. Let $Z \subseteq Y \subseteq \operatorname{Spec}(A)$ be the varieties of two ideals I, J. Suppose, that M is a finitely generated A-module. Then for $s \in \mathbb{N}$*

(i) $\exists \mathfrak{a} \subseteq A$ with $\operatorname{Var}(\mathfrak{a}) \subseteq Y$ such that $\mathfrak{a} \cdot H^i_Z(M) = 0$ for all $0 \leq i \leq s$, is a consequence of

(ii) for all $x \in \operatorname{Spec}(A) \backslash Y$ and for all $z \in \operatorname{Var}(x) \cap Z$ we have

$$\operatorname{depth}_{A_x}(M_x) + \dim(A/x)_{z/x} > s.$$

PROOF. We have shown in 6.1, 6.2 and 6.3 that if condition (i) or (ii) is valid for A, then they are also valid for B with $B/\mathfrak{c} \cong A$ and the other way round. So we may, and do, assume that A itself is a finitedimensional Gorenstein ring.

By theorem 5.8 and 7.11 for a prime $x, d(x) := \operatorname{ht}(x)$ is the unique $i \in \mathbb{N}$ for which

$$\operatorname{Ext}^i_{A_x}(\kappa(x), M_x) \neq 0.$$

We define now a family of subsets of Z by

$$Z_n := \{z \in Z : \operatorname{ht}(z) \geq n\}.$$

These are obviously sets, which are stable under specialisation. Further, $Z_0 = Z$ and $Z_n = 0$ for $n > \dim(A)$. We define now for all $i, n \in \mathbb{N}$ an A-homomorphism

$$\eta^i : H^i_{Z_n}(M) \to \bigoplus_{z \epsilon Z_n \backslash Z_{n+1}} H^i_{z \cdot A_z}(M_z)$$

as follows.

We start with $i = 0$. For $m \in \Gamma_{Z_0}(M)$ define $\eta^0(m)$ to be

$$\bigoplus_{z \epsilon Z_n \setminus Z_{n+1}} (\frac{m}{1})$$

where of course each $\frac{m}{1}$ has a different meaning. Let m be in $\Gamma_{Z_n}(M)$ and suppose, that $z \in \operatorname{supp}(A \cdot m)$. Then $\operatorname{supp}(A \cdot m)$ contains only prime ideals of height at least n and z is in this set. It follows, that z is minimal in $\operatorname{supp}(A \cdot m)$ and by [**12**], 9.39 in $\operatorname{ass}(A \cdot m)$ and a minimal element there. Let $z_1, \dots, z_t$ be the remaining associated primes of $A \cdot m$.

Suppose, $t = 0$. Then z is the unique element of $\operatorname{ass}(A/\operatorname{ann}(m))$ so that $\operatorname{ann}(m)$ is z-primary. It follows, that m is z-torsion and hence $\frac{m}{1}$ is $z \cdot A_z$-torsion.

Suppose now, that $t \neq 0$. By the minimality of z in $\operatorname{ass}(A/\operatorname{ann}(m))$, there is a $\xi \in \bigcup_1^t z_i \setminus z$. 4 Since A is Noetherian, there is an $r \in \mathbb{N}$ such that $\xi^r \cdot z^r \subseteq \operatorname{ann}(m)$. Then

$$z^r \cdot A_z \cdot \frac{m}{1} = z^r \cdot A_z \cdot \frac{\xi^r \cdot m}{\xi^r} = 0$$

and hence $\frac{m}{1}$ is in $\Gamma_{z \cdot A_z}(M_z)$. We have to check, that this is a map. Let $m \in \Gamma_{Z_n}(M)$. All z for which the component of the sum above is nonzero, have to be in $Z_n \setminus Z_{n+1}$ and in $\operatorname{supp}(A \cdot m)$ as well. Because of this all ideals in the support of m have height at least n since $m \in \Gamma_{Z_n}(M)$ and no ideal of height greater than n can occur on the right side. So all z for which the right hand term has nonzero z-component have to be minimal members of $\operatorname{supp}(m)$, hence by [**12**], 9.39 in $\operatorname{ass}(A \cdot m)$. These are finitely many such that η^0 is a map.

The linearity is obvious, because localisation is linear. If $m \in \Gamma_{Z_n}(M)$ is in the kernel of η^0, then no element of support of m is of height n, so that m is in $\Gamma_{Z_{n+1}}(M)$. On the 4 other hand clearly each $m \in \Gamma_{Z_{n+1}}(M)$ is in the kernel of η^0.

So

$$0 \to \Gamma_{Z_{n+1}}(M) \hookrightarrow \Gamma_{Z_n}(M) \overset{\eta^0}{\to} \bigoplus_{Z_n \setminus Z_{n+1}} \Gamma_{z \cdot A_z}(M_z)$$

is exact. Now for a second module M' and an A-homomorphism $f : M \to M'$ we get the induced diagram

$$\begin{array}{ccccc}
\Gamma_{Z_{n+1}}(M) & \longrightarrow & \Gamma_{Z_n}(M) & \longrightarrow & \bigoplus_{Z_n \setminus Z_{n+1}} M_z \\
\downarrow \Gamma_{Z_{n+1}} & & \downarrow \Gamma_{Z_n} & & \downarrow \bigoplus f_z \\
\Gamma_{Z_{n+1}}(M') & \longrightarrow & \Gamma_{Z_n}(M') & \longrightarrow & \bigoplus_{Z_n \setminus Z_{n+1}} M'_z
\end{array}$$

The left quadrangle commutes, because $\Gamma_{\mathfrak{a}}$ is restriction. In the right quadrangle each $m \in \Gamma_{Z_n}(M)$ is mapped either to

$$m \longrightarrow \bigoplus_{Z_n \setminus Z_{n+1}} (\frac{m}{1}) \longrightarrow \bigoplus_{Z_n \setminus Z_{n+1}} f_z(\frac{m}{1})$$

or

$$m \to f(m) \to \bigoplus_{Z_n \setminus Z_{n+1}} (\frac{f(m)}{1})$$

which is the same by the definition of f_z.

So the exact sequence is natural in M. Let now E be an injective module and $\frac{e}{s}$ be in $\Gamma_{z \cdot A_z}(E_z)$. By 1.13, there exists $e' \in \Gamma_z(E)$ and $s' \in A \setminus z$ such that $\frac{e}{s} = \frac{e'}{s'}$. By 1.16, $\Gamma_z(E)$ is injective. By 7.22, the map μ given by $\mu(\varepsilon) = s' \cdot \varepsilon$ for $\varepsilon \in \Gamma_z(E)$ is an automorphism of $\Gamma_z(E)$. Hence there is an $e" \in \Gamma_z(E)$ with $s' \cdot e" = e'$. So $\frac{e"}{1} = \frac{e'}{s'} = \frac{e}{s}$.

Further, if $z' \in \operatorname{Spec}(A)$ is in the support of $e"$, $A \setminus z'$ cannot contain any element of z since $e"$ is z-torsion. So all such z' contain z. So z is the unique minimal prime ideal in the support of $e"$. Then $\eta^0(e") = (\dots, 0, \frac{e}{s}, 0, \dots)$ and hence η^0 is surjective for injective modules and there is for each injective A-module E an exact sequence

$$0 \to \Gamma_{Z_{n+1}}(E) \hookrightarrow \Gamma_{Z_n}(E) \xrightarrow{\eta^0} \bigoplus_{Z_n \setminus Z_{n+1}} \Gamma_{z \cdot A_z}(E_z) \to 0.$$

It follows, that the application of this exact sequence to an injective resolution of M yields by [**9**], Theorem 4.5, that

$$H^i_{Z_{n+1}}(M) \longrightarrow H^i_{Z_n}(M) \longrightarrow \bigoplus_{Z_n \setminus Z_{n+1}} (H^i_{z \cdot A_z}(M_z)$$

is an exact sequence for all $i \in \mathbb{N}$ and all A-modules M.

The proof of the theorem goes in several stages:

(i) is implied by (iii):

for all $i \leq s$ *and for all* $j \in \mathbb{N}$ *there is an ideal* $\mathfrak{a} \subseteq A$ *with* $\operatorname{Var}(\mathfrak{a}) \subseteq Y$ *such that* $\mathfrak{a} \cdot H^i_{z \cdot A_z}(M_z) = 0$ *for all* $z \in Z_j \setminus Z_{j+1}$.

Proof:

(iii)$\Rightarrow$ (i): Let $\mathfrak{a}^i_j$ be such that $\operatorname{Var}(\mathfrak{a}^i_j) \subseteq Y$ and $\mathfrak{a}^i_j \cdot H^i_{z_j}(M_{z_j})$ for all $i \leq s$, for all $j \in \mathbb{N}$, for all $z_j \in Z_j \setminus Z_{j+1}$. Since A is finitedimensional, $H^i_{Z_{d+r}}(M)$ is zero for $r \in \mathbb{N}^+$ and $d = \dim(A)$ by definition of Z_{d+1}. Then from the exact sequence

$$H^i_{Z_{d+r}}(M) \xrightarrow{\iota} H^i_{Z_{d+r-1}}(M) \to \bigoplus_{Z_{d+1-r} \setminus Z_{d-r}} H^i_{z \cdot A_z}(M_z)$$

and the hypothesis follows that $\mathfrak{a}^i_{d+r-1} \cdot H^i_{Z_{d+r-1}}(M)/im(\iota) = 0$ for all $r \in \mathbb{N}$. It follows, that the annihilator of $\mathfrak{a}^i_d \cdot \mathfrak{a}^i_{d-1} \cdot \ldots \cdot \mathfrak{a}^i_0 \cdot H^i_{Z_0}(M)$ 4 contains the annihilator of $H^i_{Z_{d+1}}(M)$ which is A. Since $H^i_{Z_0}(M) = H^i_Z(M)$ and the product of all these $\mathfrak{a}$'s has variety contained in the

union of the varieties of these ideals, which in turn are all subsets of Y, the product satisfies the conditions of statement (i).
(iii) is equivalent to (iv):

for all $i \leq s$ *and for all* $k \in \mathbb{N} \exists \mathfrak{a} \subseteq A$ *with* $\operatorname{Var}(\mathfrak{a}) \subseteq Y$ *such that* $\mathfrak{a} \cdot \operatorname{Ext}_{A_z}^{d(z)-i}(M_z, A_z) = 0$ *for all* $z \in Z_k \backslash Z_{k+1}$.

Proof: see 5.10.

(iv) is equivalent to (v):

for all $i \leq s$, *for all* $k \in \mathbb{N}$, *for all* $z \in Z$ *with* $d(z) = k$ *there exists an* $\mathfrak{a} \subseteq A$ *with* $\operatorname{Var}(\mathfrak{a}) \subseteq Y$ *and* $\mathfrak{a} \cdot \operatorname{Ext}_{A_z}^{k-i}(M_z, A_z) = 0$.

Proof:

(iv)$\Rightarrow$ (v) is clear.

(v)$\Rightarrow$ (iv): Suppose, $\mathfrak{a} \cdot \operatorname{Ext}_{A_z}^{d(z)-i}(M_z, A_z) = 0$ for some prime ideal z. Then this implies $(\mathfrak{a} \cdot \operatorname{Ext}_A^{d(z)-i}(M, A))_z = 0$ and this is equivalent to $z \notin \operatorname{supp}(\mathfrak{a} \cdot \operatorname{Ext}_A^{d(z)-i}(M, A))$. And since Ext on finitely generated modules is finitely generated, this is equivalent to

$$z \notin \operatorname{Var}(\operatorname{ann}(\mathfrak{a} \cdot \operatorname{Ext}_A^{d(z)-i}(M, A))).$$

So if one $\mathfrak{a}$ works in the sense of the statement (v) for z, it works then for all $z' \in \operatorname{Spec}(A) \backslash \operatorname{Var}(\operatorname{ann}(\mathfrak{a} \cdot \operatorname{Ext}_A^{d(z)-i}(M, A)))$. If we for each fixed i and k take $\mathfrak{a}(z)$ with $d(z) = k$ such that $\mathfrak{a}(z) \cdot \operatorname{Ext}_{A_z}^{k-i}(M_z, A_z) = 0$ which is possible by the hypothesis, then the union of all $\operatorname{Spec}(A) \backslash \operatorname{Var}(\mathfrak{a} \cdot \operatorname{ann}(Ext_A^{k-i}(M, A)))$ contains $Z_k \backslash Z_{k+1}$. But the union of complements of varieties is the complement of the intersection of varieties and the intersection of varietis is the variety of the sum of the corresponding ideals. Since A is Noetherian, there have to be finitely many ideals $\mathfrak{a}_j : 1 \leq j \leq r$ for some $r \in \mathbb{N}$, such that the union of all the corresponding sets $\operatorname{Spec}(A) \backslash \operatorname{Var}(\operatorname{ann}(\mathfrak{a} \cdot \operatorname{Ext}_A^{k-i}(M, A)))$ contains $Z_k \backslash Z_{k+1}$ and each $\mathfrak{a}_j$ is one of the above $\mathfrak{a}(z)$. It follows, that there is for all fixed $k, i \in \mathbb{N}$ a finite set of ideals with variety in Y such that each $z \in Z_k \backslash Z_{k+1}$ may be served by one of them to satisfy condition (iv). But then the product of these works for all 4 $z \in Z_k \backslash Z_{k+1}$ and we are done.

(v) is equivalent to (vi):

for all $0 \leq i \leq s$ *and for all* $z \in Z$ *it is the case that*

$$\operatorname{supp}(\operatorname{Ext}_{A_z}^{d(z)-i}(M_z, A_z)) \subseteq Y.$$

Proof:

Statement (vi) is equivalent to:

for all $0 \leq i \leq s$, for all $k \in \mathbb{N}$ and for all z with $d(z) = k$ is correct that

$$\operatorname{supp}(\operatorname{Ext}_{A_z}^{k-i}(M_z, A_z)) \subseteq Y.$$

This is equivalent to:

for all $0 \leq i \leq s$ and for all $k \in \mathbb{N}$ and for all $z : d(z) = k$ it is the case that

$$\mathrm{Var}(\mathrm{ann}(\mathrm{Ext}^{k-i}_{A_z}(M_z, A_z)) \subseteq Y$$

(since these Ext's are finitely generated), and this is just statement (v) in other words.

(vi) is equivalent to (vii):

for all $x \in \mathrm{Spec}(A)\backslash Y$ *and for all* $z \in \mathrm{Var}(x) \cap Z$ *and for all* $0 \leq i \leq s$

$$(\mathrm{Ext}^{d(z)-i}_{A}(M, A))_x = 0$$

Proof:

(vi)⇒ (vii):

$$(\mathrm{Ext}^{d(z)-i}_{A}(M, A))_x \cong ((\mathrm{Ext}^{d(z)-i}_{A}(M, A))_z)_x \cong (\mathrm{Ext}^{d(z)-i}_{A_z}(M_z, A_z))_x$$

for $x \subseteq z$, and this is zero for all $x \in A\backslash Y$ by (vi).

(vii)⇒ (vi):

from the above displayed line follows that whenever the left hand term is zero, the right hand is as well. It follows from statement (vii) that the support of the right hand bracket is contained in Y, what is statement (vi).

(vii) is equivalent to (viii):

for all $x \in \mathrm{Spec}(A)\backslash Y$, *for all* $z \in \mathrm{Var}(x) \cap Z$ *and for all* $0 \leq i \leq s$,

$$H^{i+d(x)-d(z)}_{x \cdot A_x}(M_x) = 0.$$

Proof:

A is Gorenstein, so A_x is local Gorenstein and by 5.10 we can draw the conclusion

$$\begin{aligned} \mathrm{ann}(H^{i+d(x)-d(z)}_{x \cdot A_X}(M_x)) &= \mathrm{ann}(H^{d(x)-(d(z)-i)}_{x \cdot A_x}(M_x)) \\ &= \mathrm{ann}(\mathrm{Ext}^{d(z)-i}_{x \cdot A_x}(M_x, A_x)) \\ &= \mathrm{ann}(\mathrm{Ext}^{d(z)-i}_{A}(M, A))_x \end{aligned}$$

So if one of the two modules is zero, then it has annihilator A and hence the other has to be zero.

(viii) is equivalent to (ii):

Proof:

By 4.6, $H^{i-d(z)+d(x)}_{x \cdot A_X}(M_X) = 0$ for all i smaller than $s+1$ if and only if $\mathrm{depth}(x \cdot A_x) > s - d(z) + d(x)$. The only thing that then remains to show is that $d(x) + \mathrm{ht}(z/x) = d(z)$ and that follows from [**7**], Theorem 17.4. □

The reader will wonder, why in almost all cases we have proved the equivalence of the various statements. The reason is that one can fill the gap (i)⇒ (iii) and hence prove the equivalence of (i) and (ii). This would have been done here, if it could have been achieved by reasonable effort.

Unfortunately the author has found no substitute for this step given by Faltings. And he uses the spectral sequence for combined functors

$$E_2^{p,q} = H^p_{\{z\}}(H^q_Z(M))_z \Rightarrow H^{p+q}_{\{z\}}(M)$$

and this is beyond the scope of this work. However, interested readers may try to read [**11**], theorems 11.38 to 11.42.

The theorem does not give a complete answer to our question, but at least a partial one: if there is no ideal with variety in Y that annihilates $H^i_Z(M)$, then of course this cannot be zero. One can change the appearance of the theorem into the following, which is also due to Faltings and to be found in [**20**].

THEOREM 6.7. *Let A be a quotient of a finite dimensional Gorenstein ring and M a finitely generated A-module. Set further for the ideal $\mathfrak{b}$ of A,* $\mathrm{Var}(\mathfrak{b}) = Z$. *Then*

(i') $H^i_Z(M)$ is finitely generated for $0 \leq i \leq s$ and

(ii) for all $x \in \mathrm{Spec}(A)\backslash Z$ and for all $z \in Z$ with $z \supseteq x$,

$$\mathrm{depth}_{A_x}(M_x) + (\mathrm{ht}(z) - \mathrm{ht}(x) > s$$

are equivalent conditions for a natural number s.

PROOF. We show that condition (i') is equivalent to condition (i) of theorem 6.6:

(i')$\Rightarrow$ (i): If the $H^i_Z(M)$ are finitely generated for $i < s$, then for each generator g there is a natural number $n(g)$ such that $\mathfrak{b}^{n(g)} \cdot g = 0$ because all these modules have to be $\mathfrak{b}$-torsion by definition. It follows that the product of all these annihilators of generators (which are only finitely many for each i) annihilates $H^i_A(M)$ so that the finite product of all these annihilators from $i = 0$ to s annihilates all local cohomology modules with respect to $\mathfrak{b}$ with order at most s.

(i)$\Rightarrow$ (i'): we know that M is finitely generated, hence so is $H^0_Z(M)$. From 1.18 we know that $H^i_Z(M) = H^i_Z(M/\Gamma_Z(M))$ for $i > 0$. So we can, and do, replace M by $M/\Gamma_Z(M) =: N$.

Since $H^0_Z(N) = 0$, it follows, that Z contains no associated prime $\mathfrak{p}$ of N.(Otherwise $(A/\mathfrak{p}) \subseteq N$, a contradiction.) So $\mathrm{Var}(\mathfrak{a}) \subseteq Z \Rightarrow \mathrm{Var}(\mathfrak{a}) \cap \mathrm{Ass}(N) = \emptyset$. So $\mathfrak{a}$ is contained in no associated prime of N. By [**12**], Theorem 3.61 it follows, that $\mathfrak{a}$ is not in the union of the associated primes of N. So we find $x \in \mathfrak{a}\backslash \bigcup_{\mathfrak{p}\in \mathrm{Ass}(N)} \mathfrak{p}$. By [**12**], Theorem 9.36 this x is a nonzerodivisor on N. So

$$0 \to M \xrightarrow{x} M \to M/x \cdot M \to 0$$

is exact and gives rise to

$$H^i_Z(M) \xrightarrow{x} H^i_Z(M) \to H^i_Z(M/M \cdot x) \to H^{i+1}_Z(M) \xrightarrow{x} H^{i+1}_Z(M)$$

because Γ_Z is A-linear. But if we take $\mathfrak{a}$ to be as statement (i) claims and i is smaller than s, then the left and the right map are zero maps. Hence $H^{i+1}_Z(M)$ is a quotient of $H^i_Z(M/x \cdot M)$ and of course $M/x \cdot M$ is finitely generated. 4 It follows further from the displayed sequence, that whenever $\mathfrak{a}$ annihilates the local cohomology modules of M of order less than $s+1$, then it annihilates the local cohomology modules of $M/x \cdot M$ of order less than s. So we can inductively assume that it is proved for all finitely generated modules whose local cohomology modules are annihilated by $\mathfrak{a}$ up to order $r \in \mathbb{N}$, that their local cohomology modules up to order r are finitely generated and by the above argument conclude, that if $r < s$, then $H_Z^{r+1}(M)$ as quotient of a finitely generated module is finitely generated as well. The theorem follows by induction. □

We conclude, as an example, that if one accepts or verifies the proof of (i)⇒(iii), and takes a finitely generated module M of dimension $s \neq 0$ over a local Gorenstein ring A, then there exists a minimal prime of $\operatorname{ann}(M)$, say $\mathfrak{p}$, so that $\operatorname{ht}(\mathfrak{m}/\mathfrak{p}) = s$ and then $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{ht}(\mathfrak{m}) - \operatorname{ht}(\mathfrak{p}) = 0 + s$ by [**7**], 17.4 such that $H^s_\mathfrak{m}(M)$ is not finitely generated.

However, this can be proved without assuming A to be Gorenstein and without spectral sequences. The interested reader is referred to [**22**] where he will meet again secondary representations.

CHAPTER 7

Appendix

This appendix will give some accounts to the structure theory of injective modules, and most of the appendix comes from papers by Eben Matlis and Hyman Bass.

THEOREM 7.1. *Every A-modules M can be embedded in an A-injective module.*

PROOF. See [**9**],Th.5.8.

THEOREM 7.2. *For the Noetherian ring A we have*

- *any direct sum of injective A-modules is injective,*
- *any direct limit over a system of injective A-modules is an injective A-module,.*

PROOF. See [**2**],Ch.1,Ex.8.

In [**18**], Eckmann and Schöpf have shown that every A-modules M can be embedded into an A-injective module $E(M)$ which is in some sense a smallest one. To understand what this means, we introduce the concept of an essential extension:

DEFINITION 7.3. 2.1 Let $M \leq N$ be two A-modules. We call N an *essential extension of M* if and only if each nonzero submodule M' of N intersects M nontrivially. (That is $M \cap M' \neq 0$). We will call an essential extension N of M *proper* whenever $N \neq M$. Further we will call a monomorphism $f : M \to N$ *essential* if N is an essential extension of $f(M)$.

PROPOSITION 7.4. *Let I be an A-module. Then the following are equivalent:*

1. *I is injective,*
2. *I is a direct summand of every extension of itself and*
3. *I has no proper essential extension.*

PROOF. (i)⇒(ii): This is [**9**],Th.5.6.
(ii)⇒(iii): I has to be direct summand of any extension E of itself, so $E = I \bigoplus F$ for some A-module F. For essential extensions this implies $F = 0$ because $I \cap F = 0$ in any case.
(iii)⇒ (i): Let E be injective, $I \subseteq E$. We want to show that

$$0 \to I \to E \to E/I \to 0$$

splits. By Zorn's lemma and since the union of a chain of modules which do not intersect F nontrivially is a module which intersects I only in 0, there is a maximal submodule F of E such that $F \cap I$ is 0. Then $\iota : I \hookrightarrow E$ and $\pi : E \to E/F$ can be combined to a monomorphism $\pi \circ \iota$ because $F \cap I = 0$. Whenever now G/F is a nonzero submodule of E/F, then $F \subset G$ and G must intersect I nontrivially. Then $\pi \circ \iota(I) \cap G/F$ is nonzero and hence E/F is an essential extension of I. By hypothesis, this is improper so $E/F \cong I$. So $E = F + I$ and $F \cap I = 0$ so that $E = F \bigoplus I$. Therefore I is direct summand of E and hence injective.

We will now come to another description of essential extensions, which will be useful in further proofs.

Lemma 7.5. *E is an essential extension of M $\Leftrightarrow$ for all $e \in E$ either $e = 0$ or $\exists a \in A$ such that $e \cdot a \in M$ but $e \cdot a$ is not zero.*

Proof. $\Rightarrow$: If $e \in E$ but $e \neq 0$, then Ae is a nonzero submodule of E, hence meets M nontrivial and each element of the intersection is a multiple of e.
$\Leftarrow$: If F is a nonzero submodule of E then take $f \in F$ with $f \neq 0$ giving an element $a \in A$ with $0 \neq a \cdot f \in M$ and hence $F \cap M \neq 0$ so that the criterion for essential extensions is satisfied.

We now investigate the behaviour of essential extensions under unions.

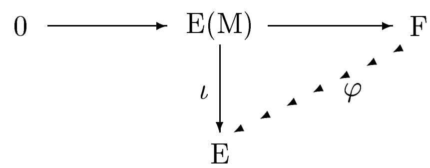
Lemma 7.6. *Let $M \subseteq E$ be two A-modules and let $\{E_i : i \in I\}$ be a family of submodules of E with for $i, j \in I$ either $E_i \subseteq E_j$ or $E_j \subseteq E_i$ such that each E_i is an essential extension of M. Then $\bigcup E_i$ is an essential extension of M.*

Proof. By 7.5 it suffices to show that for all $0 \neq e \in \bigcup E_i$ there is an $a \in A$: $0 \neq a \cdot e \in M$. But this is clear since every element in the union is in some E_i, where this condition is satisfied. □

Proposition 7.7. *For each A-modules M there is an injective module $E(M)$ which is essential over M.*

Proof. By 7.1 there exists for any A-modules M an injective E containing M. We consider the family of essential extensions over M which lie in E. This family is not empty since M is in it and by 7.6 and Zorn's lemma there is a maximal element in this family. We call it $E(M)$. We want to show that $E(M)$ is injective; by 7.4 this is the same as saying that $E(M)$ has no proper essential extension. So suppose the module F contains $E(M)$

and F is essential over $E(M)$. Then we have



which can be completed by a map φ such that the diagram commutes because E is injective.

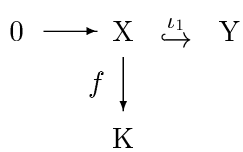
Now $\ker\varphi$ is 0 because otherwise $\ker\varphi \cap M \neq 0$ since F is essential over M, but then $\varphi(m) = 0$ for some $0 \neq m \in M$, which of course is wrong. So $\varphi(F)$ is a carboncopy of F in E containing $E(M)$ and being an essential extension of M. By construction of $E(M)$ we have $\varphi(F) = E(M)$, that is φ is an isomorphism onto $E(M)$ and hence $F = E(M)$ by commutativity of the diagram. So $E(M)$ has no proper extensions and is injective by 7.4. □

Definition 7.8. The module $E(M)$ constructed in 7.7 will be called an injective hull of M and be denoted by $E_A(M)$. If no doubt about the base ring exists, we will omit the index A.

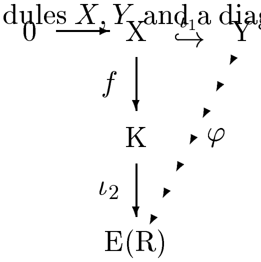
Example 7.9. Let R be an integral domain. Then $E_R(R)$, the injective hull of R considered as module over itself, is precisely its field of fractions $K = (R\backslash\{0\})^{-1}R$.

Proof. Let κ be a submodule of K. Then it is either 0 or contains a nonzero element, say k. In the latter case $k = \frac{p}{q}$ for some elements p, q of R. Then $p \neq 0$. So $k \cdot q = p \in R$ and we have shown that K is essential extension of R.

We show now that K is injective. Since $R \subseteq K, E(R) \subseteq E(K)$. If we form $E(R)$ inside $E(K)$, then $K \subseteq E(R)$: we have just seen, that K is essential extension of R. Whenever there are R-modules X, Y and a diagram



which one can extend to

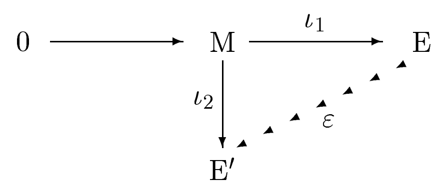


then we can find a φ such that $\iota_2 \circ f = \varphi \circ \iota_1$. Now let $0 \neq y \in Y$ with $\varphi(y)$ not in $\iota_2(K)$. Then there is a $0 \neq r \in R : r \cdot \varphi(y) \in K$ because $E(R)$ is essential over K. Suppose, that $s \in R$ would be a zerodivisor on E. Then there is an $0 \neq e \in E$ such that $s \cdot e = 0$. This implies, that $R \cdot s \cap R \cdot e = 0$. Since R is integral domain, $R \cdot s \cong R$ or $s = 0$. Since $E(R)$ is essential over R and $e \neq 0$, we must have $s = 0$. Hence each nonzeroelement of R is nozerodivisor on E. But we know, that r is nonzero.

Since $r \cdot \varphi(y) = \frac{p}{q}$ for some $p, q \in R$ and r is nonzerodivisor on $E(R)$, $\varphi(y) = \frac{p}{qr}$ and hence $\varphi(y) \in K$. By contradiction $\varphi(Y) \subseteq \iota_2(K)$. Hence φ may be used to complete the left diagram above and so K is injective R-module. □

PROPOSITION 7.10. *2 Given an A-module M and two injective hulls E and E' of M there is an A-isomorphism between them, fixing M.*

PROOF. We have the diagram



which may be completed by ε to a commutative diagram. ε is monic since otherwise its kernel has a nontrivial intersection with M which is impossible. ε is epimorphic too since $\varepsilon(E)$ is injective in E' , therefore a direct summand of E' containing M. So the other summand has to be 0 because E' is essential over M, whence ε is an isomorphism.

That M is fixed under ε is obvious. □

One may therefore speak about *the* essential extension instead of *one*. We turn now to the investigation of the structure of injective hulls. The final goal is to get insight into injective resolutions which is needed thruoghout this work. The main source of references will again be [**23**]. However, the next result is taken from [**17**],(Co.1.3). This can be proved by showing, that localisations are direct limits and direct limits sometimes preserve essential monomorphisms. Since this would cause the reader to switch here to chapter 2, we only give a reference for the interested reader.

PROPOSITION 7.11. *If A is a Noetherian ring and S a multiplicatively closed subset of A, then the exact functor S^{-1} carries essential monomorphisms into essential monomorphisms.*

PROOF. See [**17**],1.3. □

DEFINITION 7.12. Let M be a nonzero A-module. Then M is said to be *indecomposable* iff its only direct summands are 0 and M itself.

PROPOSITION 7.13. *For an A-modules M the following are equivalent:*

1. *$E(M)$ is indecomposable,*
2. *there are no nonzero submodules $S, T of M$ such that $S \cap T = 0$ and*
3. *$E(M)$ is injective hull of every nonzero submodule of itself.*

PROOF. (i)$\Rightarrow$ (ii): By the construction of injective hulls it is clear that $M \supseteq N \Rightarrow E(M) \supseteq E(N)$. Suppose we have $0 \neq S, T \subseteq M$ and $S \cap T = 0$. Then we can think about $E(S)$ and $E(T)$ to be inside $E(M)$. If this is done,

$$S \cap E(T) \neq 0 \Rightarrow (S \cap E(T)) \subseteq E(T) \text{ and nonzero, hence } S \cap T \neq 0$$

so that by contradiction $S \cap E(T) = 0$. Similarily, $T \cap E(S) = 0$.

If we had $E(T) \cap E(S) \neq 0$, then $E(T) \cap S \neq 0$ since E(S) is essential extension of S. The latter is impossible, hence so is the former. Now $E(S) \oplus E(T)$ is injective and submodule of $E(M)$, so it is a direct summand. Hence $E(M) = E(T) \oplus E(S) \oplus I$ for some injective module I. $E(S)$ and $E(T)$ are nonzero, so $E(M)$ is decomposable.

(ii)$\Rightarrow$ (iii): Let $0 \neq F$ be a submodule of $E(M)$. We want to show that $E(M)$ is essential over F. So let G be another submodule of $E(M)$. Suppose, that $G \neq 0$. If $F \cap G = 0$, then $F \cap G \cap M = (F \cap M) \cap (G \cap M) = 0$. Further $F \cap M$ and $G \cap M$ are nonzero because F and G are and $E(M)$ is essential over M. This contradicts statement (ii).

(iii)$\Rightarrow$ (i): suppose $E(M)$ were decomposable, say $E(M) = X \oplus Y$ where X and Y are nonzero. Then X, Y are injective by [**9**],Pr.5.6. If both are nonzero $E(M)$ cannot be the injective hull of the submodule X, contradicting (iii). □

In the sequel we will dwell a great deal on these injective indecomposable modules and we will establish a representation theory for injective modules. Before we show that indecomposable modules may behave in an unexpected way.

EXAMPLE 7.14. It is shown in [**5**], Th.19, that for example the ring of integers possesses an indecomposable module N of rank two (there is one constructed there). It follows, that there are two elements $n, n' \in N$ such that $z \cdot n + z' \cdot n' = 0$ for $z, z' \in \mathbb{Z}$ implies that z, z' are zero. So the injective hull of this module has the same problem. It follows, that $E_{\mathcal{Z}}(N)$ cannot be indecomposable, since by the previous result it has to be the injective hull of each nonzero submodule and this is for $\mathbb{Z} \cdot n$ impossible, because it cannot intersect $\mathbb{Z} \cdot n'$ nontrivially.

The representation we aim to produce is closely related to the ideal structure of A, especially to Spec(A). So we make the

DEFINITION 7.15. Let I be an ideal of A and $J_1, \dots, J_n$ be n further ideals of A such that

$$I = \bigcap_1^n J_i.$$

This we call a *decomposition* of I and whenever we have for $1 \leq i \leq n$ that $J_i \not\supseteq \bigcap_{j \neq i} J_j$ we will call the decomposition *irredundant.*

PROPOSITION 7.16. *Let I be an ideal in A and $I = \bigcap_{i=1}^r J_i$ be an irredundant decomposition of I. If each $E(A/J_j)$ is indecomposable, then the* *natural embedding (via* $\bigoplus_1^n A/J_i$*)*

$$A/I \to \bigoplus_1^n E(A/J_i) =: F$$

can be extended to an isomorphism $E(A/I) \to F$.

PROOF. Whenever we write A/I, we will mean the submodule of F consisting of all elements $(a + J_1, a + J_2, \ldots, a + J_n)$ which is isomorphic to A/I. We know that F is injective from 7.2 such that $E(A/I) \subseteq F$ and it suffices to show that F is an essential extension of A/I. So we will show that for G a nonzero submodule of F, $G \cap A/I \neq 0$.

Let first $G = A/J_1$. Since $J_1 \not\supseteq \bigcap_2^n J_i$, $\exists a \in \bigcap_2^n J_i \backslash J_1$. Then

$$(a + J_1, a + J_2, \ldots, a + J_n) = (a + J_1, 0, \ldots, 0) \neq 0,$$

which is in A/J_1 as well as in A/I. So $A/J_1 \cap A/I \neq 0$. The same argument works for all other $J_2, \ldots, J_n$. Now $E(A/J_i)$ is indecomposable by hypothesis, so by 7.13 $E(A/J_i)$ is the incective hull of $A/I \cap A/J_i$ for all $0 \leq i \leq n$. Now let G be an arbitrary submodule of F. If G is nonzero, $0 \neq x \in G$ exists. Then x is in F and has coordinates $x_1, \ldots, x_n$. Let i_1 be the lowest element of $\{1, \ldots, n\}$ such that $x_j \in A/J_j \cap A/I$ for $j < i_1$ and $x_{i_1} \notin A/J_{i_1} \cap A/I$. If this i_1 does not exist, x is sum of elements in A/I and therefore in A/I. If i_1 exists, there exists $a_{i_1} \in A$ such that $0 \neq a_{i_1} \cdot x$ and $(a_{i_1}x)_j \in A/I \cap A/J_j$ for $j \leq i_1$. We may now repeat this procedure with $a_{i_1}x$ instead of x and find or find not an a_{i_2} such that $i_1 < i_2, 0 \neq (a_{i_2}a_{i_1}x)$ and $(a_{i_2}a_{i_1}x)_j \in A/J_j \cap A/I for j < i_2$ but $(a_{i_2}a_{i_1}x)_{i_2} \notin A/J_{i_2}$. Working in this manner through the components of x, we finally reach a collection $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ such that the product of all a_i's multiplied by x is nonzero and the ith component of the product is contained in $A/J_i \cap A/I$. Hence this product is the sum of elements in A/I and so itself in A/I. This proves that for each nonzero module $G \subseteq F$ we always have $G \cap A/I \neq 0$ and so F is essential over A/I. □

THEOREM 7.17. *A module E over A is indecomposable and injective $\Leftrightarrow E \cong E(A/J)$ for some irreducible ideal J of A. If this is the case, for every $e \in E, e \neq 0, E = E(A/\operatorname{ann}(e))$.*

PROOF. (For irreducible ideals see e.g.[**12**],4.31.)

$\Rightarrow$:If J is irreducible, and F, G are submodules of A/J with $F \cap G = 0$, then $F = K/J$ and $G = L/J$ where $K, L \supseteq J$ are two ideals of A. So $K = J$ or $L = J$ and hence $F = 0$ or $G = 0$. By 7.13, $E(A/J)$ is indecomposable.

$\Leftarrow$: Let E be indecomposable and injective, $0 \neq e \in E$ and $J = \operatorname{ann}(e)$. Then $Ae \neq 0$ and by proposition 7.4 $E = E(Ae)$. Now $Ae \cong A/\operatorname{ann}(e) = A/J$ such that $E = E(A/J)$. Since $E(A/J)$ is indecomposable, we conclude by 7.13 that A/J contains no nonzero submodules S, T such that $S \cap T = 0$. Then an irredundant decomposition $J = K \cap L$ of ideals K, L is impossible, because then $S = K/J$ and $T = L/J$ are submodules of A/J which satisfy $S \cap T = 0$. Hence J is irreducible.

Before we come to the promised structure theorem, let us observe that for a nontrivial A and a maximal ideal $\mathfrak{m}$ in A (which exists by Zorn's lemma: see [**12**],3.9), $E(A/m)$ is indecomposable, simply because maximal ideals are necessarily irreducible.

THEOREM 7.18. *Every injective A-module has A decomposition as a direct sum of indecomposable injective submodules.*

PROOF. Let M be injective. Let $\{E_\omega\}_{\omega\epsilon\Omega}$ be a family of submodules of M which are injective and indecomposable. We will call this family independent, if

$$\Sigma_1^n E_{\omega_i} = \bigoplus_1^n E_{\omega_i}$$

for all finite subsets $\{\omega_1, \dots, \omega_n\}$ of Ω. Then by 7.2 this sum is injective for an independent family and clearly decomposable into the direct sum of injective indecomposable modules. It follows again from 7.2 together with 2.14 that for a chain of such families the union of the corresponding sums is injective and has a decomposition into indecomposable injective modules. By Zorn's lemma and the empty family, there is a maximal submodule N of M which is decomposable into the direct sum of injective indecomposable modules. Suppose $M \neq N$ and look for a contradiction.

N is clearly injective, so direct summand of M. $M = N \oplus P, P \neq 0$. Let $\pi \in P$ and $\pi \neq 0$. Then $\operatorname{ann}(\pi)$ is the intersection of finitely many irreducible ideals since A is Noetherian (see [**12**],4.33). So by 7.16 and 7.17, $E(A/\operatorname{ann}(\pi))$ is a direct sum of finitely many indecomposable injective modules. Further $A\pi \cong A/\operatorname{ann}(\pi)$ so we can build $E(A/\operatorname{ann}(\pi))$ inside $E(P) = P$. But $N \subset N \oplus E(A\pi)$ which contradicts the maximality of N. □

At this stage now, the primes of A start to be important, because we are going to show a highly interesting connection between $E(A/\mathfrak{q})$ and $E(A/\mathfrak{q}')$ where $\mathfrak{q}, \mathfrak{q}'$ are irreducible ideals with equal radical.

THEOREM 7.19. *There is a one-one-correspondence between the prime ideals of A and the injective indecomposable A-modules. It is given by $\mathfrak{p} \leftrightarrow E(A/\mathfrak{p})$ and whenever $\sqrt{\mathfrak{q}} = \mathfrak{p}$ for an irreducible ideal $\mathfrak{q}$, then $E(A/\mathfrak{q}) = E(A/\mathfrak{p})$.*

PROOF. Primes are irreducible and hence by 7.17 $E(A/\mathfrak{p})$ is indecomposable. Suppose $\mathfrak{p}, \mathfrak{p}'$ give $E(A/\mathfrak{p}) \cong E(A/\mathfrak{p}')$. Then, embedding $A/\mathfrak{p}$ and $A/\mathfrak{p}'$ in $E(A/\mathfrak{p})$, their intersection has to be nonzero because $E(A/\mathfrak{p})$ is essential extension of both quotients. Let $0 \neq x$ be in this intersection. Since $\mathfrak{p}, \mathfrak{p}'$ prime, $\sqrt{(}\operatorname{ann}(x)) = \mathfrak{p} = \mathfrak{p}'$. So the map $\mathfrak{p} \to E(A/\mathfrak{p})$ is injective.

Now let E be some indecomposable injective A-module. By 7.17 there exists an irreducible ideal $\mathfrak{q}$ in A such that $E = E(A/\mathfrak{q})$. Suppose, $\mathfrak{q} \neq \sqrt{\mathfrak{q}} =: \mathfrak{p}$. Let n be the smallest integer such that $\mathfrak{p}^n \subseteq \mathfrak{q}$ (see [**12**],8.21). Take $b \in \mathfrak{p}^{n-1} \backslash \mathfrak{q}$. Denote the image of b in $A/\mathfrak{q}$ by $\bar{b}$. Clearly $\operatorname{ann}_A(\bar{b}) \supseteq \mathfrak{p}$.

But on the other hand, if $a \in \operatorname{ann}_A(\overline{b})$, then $a \cdot b \in \mathfrak{q}$ and so $a \in \mathfrak{p}$ because irreducibles are primary (see [**12**],4.34). So $\mathfrak{p} = \operatorname{ann}_A(\overline{b})$. By 7.17 this gives $E(A/\mathfrak{p}) = E(A/\mathfrak{q})$ whence the map $\mathfrak{p} \to E(A/\mathfrak{p})$ is onto. □

THEOREM 7.20. *The decomposition of 7.18 is unique in the sense that given two decompositions of the same injective module, the cardinality of summands equal to $A/\mathfrak{p}$ for a definite $\mathfrak{p}$ are equal.*

PROOF. We will come back to this proof later, as reference is given [**23**],Pr.2.7.

COROLLARY 7.21. *If E is an injective module over A, then $\exists$ cardinals $\mu_p(E)$ ($p \in spec(A)$), such that*

$$E \cong \bigoplus_{p \epsilon spec(A)} (E(A/p)^{\mu_p(E)}$$

where M^n means the direct sum of n copies of M.

PROOF. We need only collect all summands equal to $E(A/\mathfrak{p})$, count them and observe, that the number in each decomposition is invariant by 7.20. □

We have now established a very nice structure theorem which we will use together with the following statement to investigate the behaviour of injective modules under the application of the cohomology functor $\Gamma_{\mathfrak{a}}(\cdot)$.

LEMMA 7.22. *Let $\mathfrak{p}$ be a prime ideal of A and $E = E(E/p)$. Then:*

- *(i) $\mathfrak{q}$ is an irreducible $\mathfrak{p}$-primary ideal if and only if there is an $x \neq 0$ such that* $\operatorname{ann}(x) = q$ *and*
- *(ii) if $a \in A \backslash \mathfrak{p}$, then* $\operatorname{ann}(a \cdot x) = \operatorname{ann}(x)$ *for all $x \in E$ and the endomorphism of E defined by $x \to a \cdot x$ is an automorphism of E.*

PROOF. (i) is an immediate consequence of the fact that for irreducible $\mathfrak{q}$ $E(A/\mathfrak{q}) = E(A/\mathfrak{p})$ (and therefore admits a monomorphism $A/\mathfrak{q} \to E(A/\mathfrak{p})$) and 7.19.

(ii) If $a \in A \backslash \mathfrak{p}$, the map $\xi : E \to E$ by $x \to a \cdot x$ has kernel 0 because otherwise a should be in $\mathfrak{p}$ by (i). Therefore the image of E under this map is injective in E. Now $E(A/\mathfrak{p})$ is indecomposable and hence either $\xi(E(A/\mathfrak{p}))$ is the zero module or $E(A/\mathfrak{p})$ itself. Since $A/\mathfrak{p}$ is nonzero and $\ker(\xi) \neq 0$, 0 as image is not possible. So this multiplication by a provides a surjective monomorphism. □

PROPOSITION 7.23. *(See [**16**]2.7) With the notation of above we have*

$$\mu^i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} Ext^i_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), M) = \dim_{\kappa(\mathfrak{p})} (\operatorname{Ext}^i_A(A/\mathfrak{p}, M))_\mathfrak{p}$$

for all $i \in \mathbb{N}, all p \in \operatorname{Spec}(A)$, all A-modules M.

PROOF. If we take a minimal injective resolution for M and localize at $\mathfrak{p}$, then by 7.11 and 7.20

$$\mu^i(\mathfrak{p}, M) = \mu^i(\mathfrak{p} \cdot A_\mathfrak{p}, M_\mathfrak{p}).$$

So it suffices to show that

$$\mu^i(\mathfrak{p}\cdot A_{\mathfrak{p}}, M_{\mathfrak{p}}) = \dim_{\kappa(\mathfrak{p})} Ext^i_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}).$$

So we may assume, that $\mathfrak{p}$ is maximal in A and A is local. This we do. Let

$$0 \to\ M \to\ E^1 \to\ E^2 \to \dots$$

be a minimal injective resolution for M. Now define T_i to be the submodule of E^i which consists of the elements of E^i which are annihilated by $\mathfrak{p}$. Then each element of T_i belongs by 7.22 and the fact that $\mathfrak{p}$ is maximal to a component of E^i isomorphic to $E(A/\mathfrak{p})$. We investigate these T_i. Suppose for a moment, that $\mu^i(\mathfrak{p}, M) = 1$. Then $T^i \supseteq A/\mathfrak{p}$ of course. If there were $t \in T^i \backslash A/\mathfrak{p}$, then $A\cdot t \cap A/\mathfrak{p} \neq 0$ by the fact that E^i is essential extension of $A/\mathfrak{p}$. So there are $a, \alpha \in A$ such that $a\cdot t = \alpha + \mathfrak{p} \neq 0$. It follows $a \notin \mathfrak{p}$ since $t \in T^i$. Now A is local, hence a is unit. It follows, $t = a^{-1}\alpha + \mathfrak{p} \in A/\mathfrak{p}$, a contradiction. Hence in this case $T^i = A/\mathfrak{p}$. We drop now the assumption, that $\mu^i(\mathfrak{p}, M) = 1$. It follows that $T^i \cong (A/\mathfrak{p})^{\mu^i(\mathfrak{p},M)}$. Since $(A, \mathfrak{p})$ is supposed to be local, $\kappa(\mathfrak{p}) \cong A/\mathfrak{p}$ such that $T^i \cong (\kappa(\mathfrak{p}))^{\mu^i(\mathfrak{p},M)}$. Since $\kappa(\mathfrak{p})$ is field, $\mathrm{Hom}_{\kappa(\mathfrak{p})}(\kappa(\mathfrak{p}), T^i) \cong \mathrm{Hom}_A(\kappa(\mathfrak{p}), T^i) \cong (\kappa(\mathfrak{p}))^{\mu^i(\mathfrak{p},M)}$. Hence

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_A(\kappa(\mathfrak{p}), E^i) \geq \mu^i(\mathfrak{p}, M).$$

Now $>$ can only occur if there are homomorphisms from $\kappa(\mathfrak{p})$ to E^i the image of which is not contained in T^i. But since $\mathfrak{p}\cdot\kappa(\mathfrak{p}) = 0$, $\mathfrak{p}\cdot f = 0$ for all $f \in \mathrm{Hom}_A(\kappa(\mathfrak{p}), E^i)$. So f$(\kappa(\mathfrak{p})) \subseteq T^i$.

Further if $t \in T^i$, $A\cdot t \cap d^{i-1}(E^{i-1}) \neq 0$ since by definition E^i is injective hull of $d^{i-1}(E^{i-1})$. So $a\cdot t = d^{i-1}(e^{i-1}) \neq\ 0$ for some $a \in A$ and $e^{i-1} \in E^{i-1}$ and it follows that $a \notin \mathfrak{p}$ so that $t = d^{i-1}(a^{-1}\cdot e^{i-1})$. So $T^i \subseteq ker d^{i-1}$. Therefore

$$0 \to \mathrm{Hom}_A(\kappa, E^0) \overset{\mathrm{Hom}_A(\kappa, d^0)}{\longrightarrow} \mathrm{Hom}_A(\kappa, E^1) \overset{\mathrm{Hom}_A(\kappa, d^1)}{\longrightarrow} \dots$$

is a complex with zero differentiation: $f^i \in \mathrm{Hom}_A(\kappa, E^i)$ finishes in T^i and that is killed under d^i. It follows, that $\mathrm{im}(\mathrm{Hom}_A(\kappa, d^{i-1}) = 0$ such that $\mathrm{Hom}_A(\kappa, E^i) = \mathrm{Ext}^i_A(\kappa, M)$. The result follows. $\square$.

We want to remark, that since the Ext-functors are independent from the injective resolution chosen, this statement shows the correctness of 7.20

Further Reading

There are many possibilities the reader might be interested in after finishing this dissertation and we will try to match some of these:

- for geometrical interest an easy (compared with other texts!) introduction is I. G. Macdonald's "Introduction to Algebraic Geometry" ([**6**]), afterwards one may proceed with [**3**]. Another possibility is "Introduction to Algebraic Geometry" by D. Mumford ([**8**]).
- For the more algebraic interested reader, a script by M. P. Brodman should be mentioned, in it is investigated the "ideal transform" $\varinjlim \operatorname{Ext}^n_A(\mathfrak{a}^i, M)$
- One might have found interest in duality; in this case [**4**] is a book one can proceed with.

In all, but especially the last, one needs a bit more than a basic background in sheaf theory. To this topic Tennison's "Sheaf theory" ([**14**]) is a good introduction or the book by Swan ([**13**]).

However in all cases the reader might look forward to the first edition of [**1**].

Bibliography

[1] M. P. BRODMAN and R. Y. SHARP, *Local cohomology* (to appear)
[2] H. CARTAN and S. EILENBERG, *Homological algebra* (Princeton, University Press Princeton, New Jersey, 1956)
[3] R. HARTSHORNE, *Local Cohomology* (Lecture notes in mathematics, **41**, Springer-Verlag, 1967).
[4] R. HARTSHORNE, *Residues and Duality*, (Lecture notes on mathematics, **20**, Springer-Verlag, 1966).
[5] I. KAPLANSKY, *Infinite Abelian groups* (University of Michigan Press, Ann Arbor, 1954).
[6] I. G. MACDONALD, *Introduction to algebraic geometry* (Lecture Notes from the University of Sussex, 1964/65.)
[7] H. MATSUMURA, *Commutative ring theory* (Cambridge University Press, 1986)
[8] D. MUMFORD, *Introduction to algebraic geometry* (mimeographed notes from Harvard University.)
[9] D. G. NORTHCOTT, *Introduction to homological algebra* (Cambridge University Press 1960).
[10] D. G. NORTHCOTT, *Lessons on rings, modules and multiplicities* (Cambridge University Press, 1968).
[11] J. ROTMAN, *Homological algebra*(Academic Press London, 1979)
[12] R. Y. SHARP, *Steps in commutative algebra* (Cambridge University Press, 1990).
[13] R. G. SWAN, *The theory of sheaves*, (Chicago University Press, 1964.)
[14] B. TENNISON, *Sheaf theory* , (Cambridge University Press, **20**, 1975.)
[15] M. AUSLANDER and O. GOLDMAN, Maximal orders, *Trans. Amer. Math. Soc.* **97** (1960), 1-24.
[16] H. BASS, On the ubiquity of Gorenstein rings,*Math. Zeitschr.* **82**, 8-28 (1963).
[17] H. BASS, Injective dimensions in Noetherian rings, *Trans. Amer. Math. Soc.*, **102** (1962), 18-29.
[18] B. ECKMANN and A. SCHÖPF, Über injektive Moduln, *Arch. Math.*, **4** (1953), 75-78.
[19] G. FALTINGS, Der Endlichkeitssatz in der lokalen Kohomoloie, *Math. Annalen*, **255** (1981), 45-56.
[20] G. FALTINGS, Über die Anullatoren lokaler Kohomologiegruppen, *Arch. Math.*, **Vol. 30** (1978), 473-476.
[21] I. G. MACDONALD, A note on local cohomology, *J. London Math. Soc.* (2) **10** (1975), 263-264.
[22] I. G. MACDONALD and R. Y. SHARP, An elementary proof of the non-vanishing of certain local cohomology modules *Quart. J. Oxford Ser.* (2), 23 (1972), 197-204.
[23] E. MATLIS, Injective modules over Noetherian rings, *Pac. J. Math.*, **8** (1958).
[24] R. Y. SHARP, Local cohomology theory in commutative algebra, *Quart. J. Math. Oxford* (2) **21** (1970), 425-434.
[25] R. Y. SHARP, Secondary representations for injective modules over Noetherian rings, *Proc. Edinburgh Math. Soc.*, **20** (2) (1976), 143-51.

[26] R. Y. SHARP, Some results on the vanishing of local cohomology modules, *Proc. London Math. Soc.* (3) , **30** (1975), 177-195.