ON NORMALIZED HORN SYSTEMS

CHRISTINE BERKESCH, LAURA FELICIA MATUSEVICH, AND ULI WALTHER

ABSTRACT. We characterize the (regular) holonomicity of Horn systems of differential equations under a hypothesis that captures the most widely studied classical hypergeometric systems.

1. INTRODUCTION

Let $Z = \mathbb{C}^m$ with coordinates $z_1, z_2, \dots, z_m$, and denote by $\partial_{z_1}, \partial_{z_2}, \dots, \partial_{z_m}$ the partial derivative operators $\partial/\partial z_1, \dots, \partial/\partial z_m$. The Weyl algebra D_Z, generated by the z_i and ∂_{z_i}, is the ring of algebraic differential operators on Z.

The goal of this article is to obtain D-module theoretic results about *normalized Horn systems*; in particular, we seek criteria for the following two properties. A (left) D_Z-module M is *holonomic* if $\operatorname{Ext}^j_{D_Z}(M, D_Z) = 0$ whenever $j \neq m$; it is *regular holonomic* if the natural restriction map from formal to analytic solutions of M is an isomorphism in the derived category. We note that if $\mathscr{O}$ is a function space, the space of $\mathscr{O}$-valued solutions of M is $\operatorname{Hom}_{D_Z}(M, \mathscr{O})$. Thus, if $m > 1$, regularity of D_Z-modules involves the derived solutions $\operatorname{Ext}^j_{D_Z}(M, \mathscr{O})$ for $j > 0$, where $\mathscr{O}$ is either the space of formal or analytic solutions of M at any given point of Z.

Definition 1.1. Let B be an $n \times m$ integer matrix of full rank m with rows $B_1, B_2, \dots, B_n$, whose $\mathbb{Z}$-column span contains no nonzero vectors with all nonnegative entries. Let $\kappa \in \mathbb{C}^n$ and $\eta := [z_1\partial_{z_1}, z_2\partial_{z_2}, \dots, z_m\partial_{z_m}]$. Construct the following elements of D_Z:

$$q_k := \prod_{b_{ik}>0} \prod_{\ell=0}^{b_{ik}-1} (B_i \cdot \eta + \kappa_i - \ell) \quad \text{and} \quad p_k := \prod_{b_{ik}<0} \prod_{\ell=0}^{|b_{ik}|-1} (B_i \cdot \eta + \kappa_i - \ell).$$

(1) The *Horn hypergeometric system* associated to B and κ is the left D_Z-ideal

$$\operatorname{Horn}(B, \kappa) := D_Z \cdot \langle q_k - z_k p_k \mid k = 1, 2, \dots m \rangle \subseteq D_Z. \tag{1.1}$$

(2) Assume that B has an $m \times m$ identity submatrix, and assume that the corresponding entries of κ are all zero. The *normalized Horn hypergeometric system* associated to B and κ is the left D_Z-ideal

$$\operatorname{nHorn}(B, \kappa) := D_Z \cdot \left\langle \frac{1}{z_k} q_k - p_k \,\middle|\, k = 1, 2, \dots m \right\rangle \subseteq D_Z. \tag{1.2}$$

$\hexagon$

Normalized Horn systems abound in the mathematical literature, and they include the (generalized) Gauss hypergeometric equation(s), as well as the systems of differential equations corresponding to the Appell series, Horn series in two variables, Lauricella series, and Kampé de Feriét functions,

---

2010 *Mathematics Subject Classification.* Primary: 14L30, 33C70; Secondary: 13N10, 14M25, 32C38.

CB was partially supported by NSF Grants DMS 1440537 and DMS 1661962.
LFM was partially supported by NSF Grants DMS 0703866, DMS 1001763, and a Sloan Research Fellowship.
UW was partially supported by NSF Grant DMS 0901123 and DMS 1401392, and Simons Foundation Collaboration Grant for Mathematicians #580839.

among others. In general, Horn hypergeometric systems have proved resistant to D-module theoretic study; in fact, we are aware of only [Sad02, DS07, DMS05, BMW19], which contain partial results regarding the holonomicity of $\operatorname{Horn}(B,\kappa)$.

In the late 1980s, Gelfand, Graev, Kapranov, and Zelevinsky introduced a different kind of hypergeometric system, known as *A-hypergeometric*, or *GKZ systems*, that are much more amenable to a D-module theoretic approach [GGZ87, GKZ89]. A modification of these systems led to *lattice basis D-modules*, whose solutions are in one-to-one correspondence with the solutions of Horn systems.

Definition 1.2. Let B and κ be as in Definition 1.1, and set $d = n - m$. Let $A = (a_{ij})$ be a $d \times n$ integer matrix of full rank, whose columns span $\mathbb{Z}^d$ as a lattice, and such that $AB = 0$. Let $X = \mathbb{C}^n$ with coordinates $x_1, x_2, \dots, x_n$ and consider the Weyl algebra D_X generated by the x_i and their corresponding $\partial_{x_i} = \frac{\partial}{\partial x_i}$. Denote $\theta_i = x_i \partial_{x_i}$ for $i = 1, 2, \dots, n$. The polynomial ideal

$$I(B) \coloneqq \left\langle \prod_{(B_j)_i > 0} \partial_{x_i}{}^{(B_j)_i} - \prod_{(B_j)_i < 0} \partial_{x_i}{}^{(B_j)_i} \right\rangle \subset \mathbb{C}[\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}]$$

is called a *lattice basis ideal*. Let $E_i \coloneqq \sum_{j=1}^n a_{ij}\theta_j$, and denote by $E - A\kappa$ the sequence $E_1 - (A\kappa)_1, E_2 - (A\kappa)_2, \dots, E_d - (A\kappa)_d$, which are known as *Euler operators*. The *lattice basis D_X-module associated to B and κ* is the quotient of D_X by the left D_X-ideal $H(B,\kappa)$ generated by $I(B)$ and $E - A\kappa$. ◯

The solutions of Horn hypergeometric systems and lattice basis binomial D-modules are related as follows. Let B and κ be as in Definition 1.1, and denote by $b_1, b_2, \dots, b_m$ the columns of B. Let $\varphi(z) = \varphi(z_1, z_2, \dots, z_m)$ be a germ of a holomorphic function at a point p of Z that is nonsingular for $D_Z/\operatorname{Horn}(B,\kappa)$. Then $\varphi(z)$ is a solution of $D_Z/\operatorname{Horn}(B,\kappa)$ if and only if $x^\kappa g(x^{b_1}, x^{b_2}, \dots, x^{b_m})$ is a solution of $D_X/H(B,\kappa)$ (at a corresponding point $p^B = (p^{B_1}, \dots, p^{B_n})$ in X). Note that this does not imply any relationship among higher derived solutions of the corresponding modules, or about solutions at singular points.

This correspondence between the solutions of the Horn and lattice basis D-modules does not imply that there is a D-module theoretic relationship between the systems. This would be desirable, since lattice basis D-modules are fairly well understood; in particular, there are complete characterizations of their holonomicity and regularity (see Section 3), so one could hope to transfer these results from the lattice basis to the Horn setting. Unfortunately, the following example shows that such a D-module theoretic relationship cannot exist in general.

Example 1.3. The lattice basis D_X-module corresponding to

$$B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \kappa = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is holonomic, but $D_Z/\operatorname{Horn}(B,\kappa)$ is not. This can be tested explicitly using the computer algebra system `Macaulay2` [M2]. ◯

However, for *normalized* Horn systems, the main result in this article provides a relationship between these and their lattice basis counterparts.

Theorem 1.4. *Suppose that the top m rows of B form an identity matrix and $\kappa_1 = \kappa_2 = \cdots = \kappa_m = 0$. Let r denote the inclusion $r\colon Z \hookrightarrow X$ given by $(z_1, z_2, \dots, z_m) \mapsto (z_1, z_2, \dots, z_m, 1, \dots, 1)$. If r^* is the restriction (inverse image under r) on D_X-modules, then there is an equality*

$$\frac{D_Z}{\mathrm{nHorn}(B,\kappa)} = r^*\left(\frac{D_X}{H(B,\kappa)}\right).$$

This result is inspired by [Beu11b, §§11-13]. In this work, Beukers obtains examples of classical Horn series by setting to one certain variables in the series solutions of associated A-hypergeometric systems. Theorem 1.4 implies this correspondence among series solutions, as well as invariants including characteristic varieties and singular loci.

Corollary 1.5. *Under the hypotheses of Theorem 1.4, the (regular) holonomicity of the modules $D_Z/\mathrm{nHorn}(B,\kappa)$ and $D_X/H(B,\kappa)$ are equivalent.*

Notation. In [BMW19], $\bar{Z}$ and Z are used for $\mathbb{C}^m$ and $(\mathbb{C}^*)^m$, while in this article, we use Z and Z^*, at the suggestion of the referee.

Outline. In §2, we prove Theorem 1.4. In §3, we recall the characterizations for holonomicity and regularity of lattice basis D-modules and prove Corollary 1.5.

Acknowledgements. We are grateful to Frits Beukers, Alicia Dickenstein, Brent Doran, Anton Leykin, Ezra Miller, Christopher O'Neill, Mikael Passare, and Bernd Sturmfels, who have generously shared their insight and expertise with us while we worked on this project. Parts of this work were carried out at the Institut Mittag-Leffler program on Algebraic Geometry with a view towards Applications and the MSRI program on Commutative Algebra. We thank the program organizers and participants for exciting and inspiring research atmospheres.

2. Normalized Horn systems are restrictions

In this section, we prove Theorem 1.4. We use the notation and assumptions introduced in Definitions 1.1 and 1.2.

By [SST00, §5.2], the restriction r^* of a cyclic D_X-module D_X/J is given by

$$r^*\left(\frac{D_X}{J}\right) = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle x_{m+1}-1, x_{m+2}-1, \dots, x_n-1\rangle} \otimes_{\mathbb{C}[x_1,x_2,\dots,x_n]} \frac{D_X}{J}. \tag{2.1}$$

It is a fact that the restriction of a cyclic D_X-module is not necessarily cyclic. Consequently, to establish Theorem 1.4, the first step is to show that $r^*(D_X/H(B,\kappa)) = r^*(D_X/D_X \cdot \langle I(B), E - A\kappa\rangle)$ is cyclic. To do this, we compute the b-function for the restriction, as defined in [SST00, §§5.1-5.2]. The relevant result states that, if the maximal integral root of this b-function is 0, then the restriction is a cyclic module (see [SST00, Algorithm 5.2.8]).

Lemma 2.1. *If the matrix formed by the top m rows of B has rank m, then the b-function $b(s)$ of $H(B,\kappa)$ for restriction to $\{x \in X \mid x_{m+1} = x_{m+2} = \cdots = x_n = 1\}$ divides s.*

Proof. Consider the change of variables $x_j \mapsto x_j + 1$ for $m+1 \leq j \leq m$, and let J denote the D_X-ideal obtained from $H(B,\kappa)$ via this change of variables. We now compute the b-function of J for restriction to $\{x \in X \mid x_{m+1} = x_{m+2} = \cdots = x_n = 1\}$.

With $w = (\mathbf{0}_m, \mathbf{1}_d) \in \mathbb{R}^n$, the vector $(-w,w)$ induces a filtration on D_X, and the b-function we wish to compute is a generator of the principal ideal $\mathrm{gr}^{(-w,w)}(J) \cap \mathbb{C}[s]$, where $s := \theta_{m+1} + \theta_{m+2} + \cdots + \theta_n$. Note that, since the submatrix of B formed by its first m rows has rank m, the submatrix of A consisting of its last $n-m=d$ columns has rank d. Thus there are vectors $\nu^{(m+1)}, \nu^{(m+2)}, \ldots, \nu^{(n)} \in \mathbb{R}^d$ such that $(\nu^{(j)}A)_k = \delta_{jk}$ for $m+1 \leq k \leq n$. For $m+1 \leq j \leq n$, with $\beta = A\kappa$,

$$\sum_{i=1}^{d} \nu_i^{(j)} E_i - \nu^{(j)} \cdot \beta \;=\; \sum_{k=1}^{m} (\nu^{(j)}A)_k \theta_k + \theta_j - \nu^{(j)} \cdot \beta \;\in\; D_X \cdot \langle E - \beta \rangle.$$

Using the change of variables

$$x_j \mapsto \begin{cases} x_j & \text{for } 1 \leq j \leq m \\ x_j + 1 & \text{for } m+1 \leq j \leq n, \end{cases}$$

and then multiplying by x_j, for $m+1 \leq j \leq n$, we obtain

$$\sum_{k=1}^{m} (\nu^{(j)}A)_k x_j \theta_k + x_j^2 \partial_{x_j} + \theta_j - \nu^{(j)} \cdot \beta x_j \;\in\; J.$$

Taking initial terms with respect to $(-w,w)$ of this expression, it follows that $\theta_j \in \mathrm{gr}^{(-w,w)}(J)$ for each $m+1 \leq j \leq n$. Therefore $s = \theta_{m+1} + \theta_{m+2} + \cdots + \theta_n \in \mathrm{gr}^{(-w,w)}(J)$, and the result follows. □

Proof of Theorem 1.4. By Lemma 2.1, $r^*(D_X/H(B,\kappa))$ is of the form D_Z/L. In order to find the ideal L, we must perform the intersection

$$H(B,\kappa) \cap R_m, \text{ where } R_m := \mathbb{C}[x_1, x_2, \ldots, x_n]\langle \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_m} \rangle \subseteq D_X, \tag{2.2}$$

and then set $x_{m+1} = x_{m+1} = \cdots = x_n = 1$. We proceed by systematically producing elements of the intersection (2.2). Using the same argument as in the proof of Lemma 2.1, we see that for $m+1 \leq j \leq n$, each θ_j can be expressed modulo $D_X \cdot \langle E - \beta \rangle$ as a linear combination of $\theta_1, \theta_2, \ldots, \theta_m$ and the parameters κ. By our assumption on B, θ_j can be written explicitly as follows:

$$\theta_j = \kappa_j + \sum_{i=1}^{m} b_{ji} \theta_i \mod D_X \cdot \langle E - \beta \rangle \qquad \text{for } m+1 \leq j \leq n. \tag{2.3}$$

Now if $P \in D_X$, then there is a monomial μ in $x_{m+1}, x_{m+2}, \ldots, x_n$ so that the resulting operator μP can be written in terms of $x_1, x_2, \ldots, x_n, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_m}$, and $\theta_{m+1}, \theta_{m+2}, \ldots, \theta_n$. In addition, working modulo $D_X \cdot \langle E - \beta \rangle$, one can replace θ_j when $j > m$ by the expressions (2.3). Thus μP is an element of R_m modulo $R_m \cdot \langle E - \beta \rangle$. If this procedure is applied to $E_i - \beta_i$, the result is zero. We now apply it to one of the generators $\partial_x^{(b_k)_+} - \partial_x^{(b_k)_-}$ of $I(B)$, where $b_1, b_2, \ldots, b_m$ denote the columns of B. An appropriate monomial in this case is $\mu_k = \prod_{j=m+1}^{n} x_j^{|b_{jk}|}$. Then the fact that

$b_{kk} = 1$ for $1 \le k \le m$ and (2.3) together imply that

$$\begin{aligned}
&\mu_k(\partial_x^{(b_k)+} - \partial_x^{(b_k)-}) \\
&\quad = \big(\textstyle\prod_{b_{jk}<0} x_j^{-b_{jk}}\big)\partial_{x_k}{}^{b_{kk}} \prod_{j>m, b_{jk}>0} x_j^{b_{jk}} \partial_{x_j}{}^{b_{jk}} - \prod_{j>m, b_{jk}>0} x_j^{b_{jk}} \prod_{b_{jk}<0} x_j^{-b_{jk}} \partial_{x_j}{}^{-b_{jk}} \\
&\quad = \big(\textstyle\prod_{b_{jk}<0} x_j^{-b_{jk}}\big)\partial_{x_k} \prod_{j>m, b_{jk}>0} \prod_{\ell=0}^{b_{jk}-1} (\kappa_j + \sum_{i=1}^m b_{ji}\theta_i - \ell) \\
&\qquad\qquad - \textstyle\prod_{j>m, b_{jk}>0} x_j^{b_{jk}} \prod_{b_{jk}<0} \prod_{\ell=0}^{-b_{jk}-1} (\kappa_j + \sum_{i=1}^m b_{ji}\theta_i - \ell).
\end{aligned} \tag{2.4}$$

Note that setting $x_{m+1} = x_{m+2} = \cdots = x_n = 1$ in (2.4), we obtain the kth generator of the normalized Horn system $\mathrm{nHorn}(B, \kappa)$, since $b_{jk} < 0$ implies $j > m$. This shows that $\mathrm{nHorn}(B, \kappa)$ is contained in the intersection (2.2) after setting $x_{m+1} = x_{m+2} = \cdots = x_n = 1$.

Now suppose that P is an element of the intersection (2.2). In particular, P belongs to $I(B) + \langle E - \beta\rangle$, so there are $P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_d \in D_X$ such that

$$P = \sum_{k=1}^m P_k(\partial_x^{(b_k)+} - \partial_x^{(b_k)-}) + \sum_{i=1}^d Q_i(E_i - \beta_i).$$

If we multiply P on the left by a monomial in $x_{m+1}, x_{m+2}, \ldots, x_n$ and set $x_{m+1} = x_{m+2} = \cdots = x_n = 1$, the result is the same as if we set $x_{m+1} = x_{m+2} = \cdots = x_n = 1$ on P directly. Thus we choose an appropriate monomial η such that a monomial μ_k as above can be pulled through to the right of each P_k, as follows:

$$\begin{aligned}
\eta P &= \sum_{k=1}^m \eta P_k(\partial_x^{(b_k)+} - \partial_x^{(b_k)-}) + \eta \sum_{i=1}^d Q_i(E_i - \beta_i) \\
&= \sum_{k=1}^m \tilde{P}_k \mu_k(\partial_x^{(b_k)+} - \partial_x^{(b_k)-}) + \eta \sum_{i=1}^d Q_i(E_i - \beta_i)
\end{aligned}$$

for some operators $\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_m$. An appropriate monomial η here is

$$\eta = \prod_{j=m+1}^n x_j^{\omega_j + \sigma},$$

where

$$\begin{aligned}
&\omega_j := \max\{\text{order of } \partial_j \text{ in } P_\ell \mid 1 \le \ell \le m\} \text{ for each } 1 \le j \le m \\
\text{and}\quad &\sigma := \max\{\text{degree of } \mu_k \mid m+1 \le k \le n\} = \max\{|b_{jk}| \mid m+1 \le k \le n\}.
\end{aligned}$$

But now, the result of setting $x_{m+1} = x_{m+2} = \cdots = x_n = 1$ on ηP (the same as if this were done to P) is a combination of the generators of $\mathrm{nHorn}(B, \kappa)$. Thus, we have shown that the intersection (2.2) after setting $x_{m+1} = x_{m+2} = \cdots = x_n = 1$ is contained in $\mathrm{nHorn}(B, \kappa)$. We conclude that $r^*(D_Z/H(B, \kappa)) = D_Z/\mathrm{nHorn}(B, \kappa)$. □

3. LATTICE BASIS D-MODULES

The ring D_X is $\mathbb{Z}^d$-graded by setting $\deg(\partial_{x_i}) = -\deg(x_i) = a_i$, where $a_1, \ldots, a_n$ are the columns of the matrix A from Definition 1.2. This grading, which is also inherited by the polynomial ring $\mathbb{C}[\partial_x] := \mathbb{C}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}]$, is known as the *$A$-grading*. An *$A$-homogeneous binomial $\mathbb{C}[\partial_x]$-ideal* I is an ideal generated by A-homogeneous elements of the form $\partial_x^u - \lambda\partial_x^v$. (In this definition, $\lambda = 0$ is allowed; in other words, monomials are admissible generators in a binomial ideal.)

Note that $H(B,\kappa)$ is A-homogeneous, so that the lattice basis binomial D_X-modules are A-graded. It is this grading that can be used to determine the set of parameters κ for which the module $D_X/H(B,\kappa)$ is holonomic (Theorem 3.3). We need the notion of quasidegrees of a module, originally introduced in [MMW05].

Definition 3.1. Let M be an A-graded $\mathbb{C}[\partial_x]$-module. The set of *true degrees of* M is

$$\operatorname{tdeg}(M) = \{\beta \in \mathbb{C}^d \mid M_\beta \neq 0\}.$$

The set of *quasidegrees of* M, denoted $\operatorname{qdeg}(M)$, is the Zariski closure in $\mathbb{C}^d$ of $\operatorname{tdeg}(M)$. ○

Definition 3.2 ([DMM10a, Definition 4.3], [DMM10b, Definitions 1.11 and 6.9])**.** Let A be as in Definition 1.2, and let I be an A-homogeneous binomial $\mathbb{C}[\partial_x]$-ideal. By [ES96], any associated prime of I is of the form $\mathbb{C}[\partial_x] \cdot J + \langle x_j \mid j \notin \sigma \rangle$, where $\sigma \subset \{1, 2, \ldots, n\}$ and $J \subset \mathbb{C}[\partial_{x_i} \mid i \in \sigma]$ is a prime binomial ideal containing no monomials. Such an associated prime is called *toral* if the dimension of $\mathbb{C}[\partial_{x_i} \mid i \in \sigma]/J$ equals the rank of the submatrix of A consisting of the columns indexed by σ. An associated prime of I which is not toral is called *Andean*.

Consider a primary decomposition $I = \bigcap_{\ell=1}^N C_\ell$, where $C_1, C_2, \ldots, C_K$ are the primary components corresponding to Andean associated primes and $C_{K+1}, C_{K+2}, \ldots, C_N$ are the components corresponding to toral associated primes. The *Andean arrangement of* I is

$$\mathscr{Z}_{\text{Andean}}(I) := \bigcup_{\ell=1}^{K} \operatorname{qdeg}\left(\mathbb{C}[\partial_x]/C_\ell\right). \qquad ○$$

The name *Andean* refers to an intuitive picture of the grading of an Andean module (see [DMM10b, Remark 5.3]).

Since Andean primes may be embedded, the definition of the Andean arrangement seems a priori to depend on the specific primary decomposition; however, [DMM10b, Theorem 6.3] shows that this is not the case. We will make use of the following Theorem 3.3, whose first part is a special case of [DMM10b, Theorem 6.3], while its second part is proved in [CF12].

We recall that the *holonomic rank* of a D-module is the dimension of its space of germs of holomorphic solutions at a generic (nonsingular) point.

Theorem 3.3. *Use the notation from Definitions 1.2 and 3.2. The following are equivalent.*

(1) *The D_X-module $D_X/H(B,\kappa)$ has finite holonomic rank.*
(2) *The D_X-module $D_X/H(B,\kappa)$ is holonomic.*
(3) $A\kappa \notin \mathscr{Z}_{\text{Andean}}(H(B,\kappa))$.

In addition, $D_X/H(B,\kappa)$ is regular holonomic if and only if it is holonomic and the rows of B sum to $\mathbf{0}_m$. □

We need one more result in order to prove Corollary 1.5. Let $Z^* = (\mathbb{C}^*)^n$, and consider its ring of differential operators $D_{Z^*} := \mathbb{C}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \otimes_{\mathbb{C}[z_1, z_2, \ldots, z_m]} D_Z$. The *saturated Horn system corresponding to B and κ* is $\operatorname{sHorn}(B,\kappa) := D_{Z^*} \cdot \operatorname{Horn}(B,\kappa) \cap D_Z$.

Theorem 3.4 ([BMW19, Corollary 7.2])**.** *The D_X-module $D_X/H(B,\kappa)$ is (regular) holonomic if and only if the D_Z-module $D_Z/\operatorname{sHorn}(B,\kappa)$ is (regular) holonomic.* □

Proof of Corollary 1.5. If $D_X/H(B,\kappa)$ is (regular) holonomic, then so is $D_Z/\operatorname{nHorn}(B,\kappa)$ by Theorem 1.4, since restrictions preserve (regular) holonomicity. For the converse, if $D_X/H(B,\kappa)$

is not (regular) holonomic, then neither is $D_Z/\mathrm{sHorn}(B,\kappa)$ by Theorem 3.4. Since $\mathrm{nHorn}(B,\kappa) \subseteq \mathrm{sHorn}(B,\kappa)$, and the category of (regular) holonomic D_Z-modules is closed under quotients of D_Z-modules, $D_Z/\mathrm{nHorn}(B,\kappa)$ also fails to be (regular) holonomic. $\square$

REFERENCES

[BMW19] Christine Berkesch, Laura Felicia Matusevich, and Uli Walther, *Torus equivariant D-modules and hypergeometric systems*. Adv. Math. **350** (2019), 1226–1266. 2, 3, 6

[Beu11b] Frits Beukers, *Notes on A-hypergeometric functions*, Séminaires & Congrès, **23** (2011), 25–61. 3

[CF12] Francisco-Jesús Castro-Jiménez and María-Cruz Fernández-Fernández, *On irregular binomial D-modules*, Math. Z. **272** (2012), no. 3-4, 1321–1337. 6

[DMM10a] Alicia Dickenstein, Laura Felicia Matusevich, and Ezra Miller, *Combinatorics of binomial primary decomposition*, Math Z., **264**, no. 4, (2010) 745–763. 6

[DMM10b] Alicia Dickenstein, Laura Felicia Matusevich, and Ezra Miller, *Binomial D-modules*, Duke Math. J. **151** no. 3 (2010), 385–429. 6

[DMS05] Alicia Dickenstein, Laura Felicia Matusevich, and Timur Sadykov, *Bivariate hypergeometric D-modules*, Advances in Mathematics, **196** (2005), 78–123. 2

[DS07] Alicia Dickenstein and Timur Sadykov, *Bases in the solution space of the Mellin system of equations* (Russian) Mat. Sb. 198 (2007), no. 9, 59–80; translation in Sb. Math. 198 (2007), no. 9-10, 1277–1298. 2

[ES96] David Eisenbud and Bernd Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45. 6

[GGZ87] I. M. Gel′fand, M. I. Graev, and A. V. Zelevinskiĭ, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19. 2

[GKZ89] I. M. Gel′fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26. Correction in ibid, **27** (1993), no. 4, 91. 2

[M2] Daniel R. Grayson and Michael E. Stillman, `Macaulay2`, *a software system for research in algebraic geometry*, available at `http://www2.macaulay2.com/Macaulay2/`. 2

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), no. 4, 919–941. 6

[Sad02] Timur Sadykov, *On the Horn system of partial differential equations and series of hypergeometric type*, Math. Scand. **91** (2002), no. 1, 127–149. 2

[SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer–Verlag, Berlin, 2000. 3

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455.

Email address: `cberkesc@umn.edu`

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843.

Email address: `laura@math.tamu.edu`

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907.

Email address: `walther@math.purdue.edu`