Toric rings, CM property, Newton filtration

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1. Setup/Motivation
2. Complications
3. An algorithm
4. Negative examples

Setup

$A \subseteq \mathbb{Z}^d$ finite

- $A = ((a_{i,j})) = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d\times n}$.
- assume $H = \mathbb{N}A$ pointed, set $R = \mathbb{K}\langle\partial\rangle$.
- toric ideal: $I_A = \langle \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \mid A\cdot\mathbf{u} = 0, \mathbf{u} \in \mathbb{Z}^d\rangle \subseteq R$
- $R/I_A = \mathbb{K}[H] = \mathbb{K}[t^{\mathbf{a}_1}, \cdots, t^{\mathbf{a}_n}] \subseteq \mathbb{K}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.
- $\Delta_A = \mathrm{conv}(0, A) \subseteq \mathbb{Q}_+ A$.
- Newton filtration on $\mathbb{K}[H]$, R: for $\mathbf{u} \in \mathbb{N}^n$,

$$\partial^{\mathbf{u}} \in N_\lambda \Leftrightarrow A\cdot\mathbf{u} \in \lambda\Delta_A.$$

Goal:

$$\mathbb{K}[H] \text{ CM} \overset{?}{\Longleftrightarrow} \mathrm{gr}_N(\mathbb{K}[H]) \text{ CM} \overset{?}{\Longleftrightarrow} \text{face rings CM}$$

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Motivation: hypergeometric systems

- Weyl Algebra:

$$D = \mathbb{C}[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle.$$

- Euler fields:

$$E_i = a_{i,1} x_1 \partial_1 + \dots + a_{i,n} x_n \partial_n.$$

Definition (GKZ-system $H_A(\beta)$ to $\beta \in \mathbb{C}^d$)

$$H_A(\beta) = D \cdot I_A + D \cdot \{E_i - \beta_i\}_{i=1}^n, \quad M_A = D/H_A(\beta).$$

- morally:
  - on d-torus, equivariant vector fields $\{t_i \frac{\partial}{\partial t_i}\}_1^d$,
  - $E_i =$ pushforward of $t_i \frac{\partial}{\partial t_i}$,
  - $M_A(\beta)$ "is" pushforward of t^β to $\mathbb{C}^n$.

Motivation II

Occur in:

- Aomoto–Gelfand hypergeometric functions
- Picard–Fuchs equations in mirror symmetry
- recursions in counting problems

- $H_A(\beta)$ is holonomic
- the function $\beta \mapsto \mathrm{rk}(H_A(\beta))$ not constant

Suppose I_A normal.

Theorem (Adolphson(Kouchnirenko))

$\mathbb{K}[H]$ is CM, then so is $\mathrm{gr}_N(\mathbb{K}[H])$,

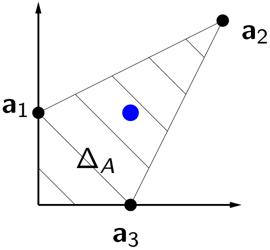
$\Rightarrow$(via s.s.'s)$\Rightarrow$

$\mathrm{rk}(H_A(\beta))$ is independent of β.

G. Okuyama used similar statements in the general case.

An example

- $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$



- For $\mathbf{a} \in \mathbb{Z}^d$, introduce

$$\deg_N(\mathbf{a}) = \min\{\lambda \mid \mathbf{a} \in \lambda\Delta_A\}.$$

- Clearly, $\deg_N(\mathbf{a}_i) = 1$, but $\deg_N((1,1)) = 1/2$.
- In particular, $\mathrm{gr}_N(t^{(1,1)}) \notin R/\,\mathrm{gr}_N(I_A)$.

Shocking! (If one compares to monomial orders.)

Before CM study, need a method to compute $\mathrm{gr}_N(\mathbb{K}[H])$.

Some results by Kouchnirenko

A <u>facet cone</u> is $\mathbb{Q}_+\tau$, $\tau \not\ni 0$ a facet of $\Delta_{A'}$.

- $\deg_N(-)$ is subadditive, but often not additive,
- $\deg_N(t^h) + \deg(t^{h'}) = \deg(t^{h+h'})$ iff h, h' share facet cone,

and

$$\mathbb{K}[H_\tau] \text{ CM } \forall \text{ facet cones} \implies \mathrm{gr}_N(\mathbb{K}[H]) \text{ CM} \implies \mathbb{K}[H] \text{ is CM}.$$

<u>Method:</u>

- $\mathrm{gr}_N(\mathbb{K}[H]) = \bigoplus \mathbb{K}[H_\tau]/(\text{"overlap"})$, sum over facets;
- exact sequence

$$0 \to \mathrm{gr}_N(\mathbb{K}[H]) \to A_d \to A_{d-1} \to \cdots \to A_1 \to 0;$$

- spectral sequence

$$E_1^{p,q} = H^q_{\mathfrak{m}}(A_{r-p}) \implies H^{r-p+q}_{\mathfrak{m}}(\mathrm{gr}_N(\mathbb{K}[H])).$$

Outline of algorithm

Main issue: find $A' \subseteq H$ with $\{\mathrm{gr}_N(t^{A'})\}$ generating $\mathrm{gr}_N(\mathbb{K}[H])$.

- Start with $A' = A$, and search for pairs $\mathbf{a}_i, \mathbf{a}_j$ with

$$\deg(t^{\mathbf{a}_1+\mathbf{a}_2}) < \deg(t^{\mathbf{a}_1}) + \deg(t^{\mathbf{a}_2}).$$

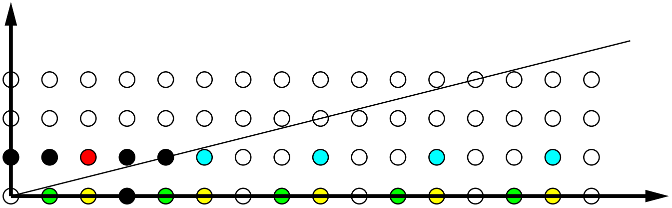
  ($\Rightarrow$ facet cone decomposition, can be done by GB.)

- Now loop:
  - consider $\mathbb{K}[A'] \twoheadrightarrow \mathbb{K}[H]$ with kernel $I_{A'}$.
  - induce monomial order $<_N$ on $\mathbb{K}[A']$ from $\deg_N(-)$.
  - add $\mathbf{a}'_i + \mathbf{a}'_j$ to A' if $t^{\mathbf{a}'_i+\mathbf{a}'_j}$ not leadterm of $I_{A'}$ under $<_N$.

The stable value of A' satisfies $\mathrm{gr}_N(\mathbb{K}[H]) = \mathbb{K}[A']/\,\mathrm{in}_{<_N}(I_{A'})$.

$\mathbb{K}[H]$ is CM $\overset{?}{\Longrightarrow}$ $\mathrm{gr}_N(\mathbb{K}[H])$ is CM

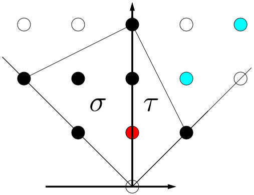
- Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 4 & 3 \end{pmatrix}$; note $\tilde{H} = \mathbb{N}^2$.



- $\mathbb{N}^2/\mathbb{N}A = ((0,1) + \mathbb{N}\mathbf{a}_5) \cup ((0,2) + \mathbb{N}\mathbf{a}_5) \cup ((1,2) + \mathbb{N}\mathbf{a}_5)$.
- So $H^1_{\mathfrak{m}}(\mathbb{K}[H]) = 0$.
- So $\mathbb{K}[H]$ is CM.
- but $\mathrm{gr}_N(\mathbb{K}[H])$ is not since $t^{(1,2)} \in H^1_{\mathfrak{m}}(\mathrm{gr}_N(\mathbb{K}[H]))$..

$\mathrm{gr}_N(\mathbb{K}[H])$ CM $\overset{?}{\Longrightarrow}$ facet cones CM

- Let $A = \begin{pmatrix} 0 & 0 & -1 & -1 & -2 & 1 \\ 2 & 3 & 1 & 2 & 2 & 1 \end{pmatrix}$,



- easy check: $H^1_{\mathfrak{m}_\sigma}(\mathbb{K}[H_\sigma]) = \mathbb{K} \cdot t^{(1,0)}$,

  so H_σ not CM.
- OTOH, $(1,0) + \mathbb{N}\mathbf{a}_6 \cap H = \emptyset$,

  so $t^{(1,0)} \notin H^1_{\mathfrak{m}}(\mathbb{K}[H])$.
- as H_τ, $H_{\sigma\cap\tau}$ are CM, $H^1_{\mathfrak{m}}(\mathrm{gr}_N(\mathbb{K}[H])) = 0$.

  so $\mathrm{gr}_N(\mathbb{K}[H])$ is CM.
- in this case, the spectral sequence is interesting.