Hypergeometric systems II: GKZ systems

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Outline

The hypergeometric differential equation 1

Our example: $(\partial_t^2 + \frac{c-(a+b+1)t}{t(1-t)}\partial_t - \frac{ab}{t(1-t)})z = 0$.

- Multiply with $t^2(1-t)$, write $\theta = t\partial_t$, to get (standard) form

$$(\theta - 1 + 1)(\theta - 1 + c)z = t \cdot (\theta + a)(\theta + b)z.$$

- General hypergeometric differential equation:

$$\prod_{v_j>0}\prod_{l=0}^{v_j-1}(v_j\theta + c_j - l)z = t \cdot \prod_{v_j<0}\prod_{l=0}^{|v_j|-1}(v_j\theta + c_j - l)z$$

- Power series ansatz: $z = \sum_{k=0}^{\infty} a_k t^k$ shows

$$a_k \prod_{v_j>0}\prod_{l=0}^{v_j-1}(v_j \cdot k + c_j - l) = a_{k-1} \cdot \prod_{v_j<0}\prod_{l=0}^{|v_j|-1}(v_j \cdot (k-1) + c_j - l)$$

since $\theta(t^k) = kt^k$. So

$$a_k/a_{k-1} \in \mathbb{Q}(k).$$

The multivariate case 2

Given

$$\prod_{v_j>0}\prod_{l=0}^{v_j-1}(v_j\theta + c_j - l)z = t\cdot\prod_{v_j<0}\prod_{l=0}^{|v_j|-1}(v_j\theta + c_j - l)z \qquad (1)$$

let $\mathbf{v} = (v_1, \ldots, v_n)$, find $A \in \mathbb{Z}^{n-1,n}$ with $A\cdot\mathbf{v} = 0$, $\beta := A\cdot\mathbf{c}$.

$$\begin{aligned}
\left(\prod_{v_j>0}\partial_j^{v_j} - \prod_{v_j<0}\partial_j^{|v_j|}\right)\bullet\phi &= 0;\\
\left(\sum_{j=1}^{n} a_{1,j}x_j\partial_j\right)\bullet\phi &= \beta_1\phi;\\
&\cdots \qquad (2)\\
\left(\sum_{j=1}^{n} a_{n-1,j}x_j\partial_j\right)\bullet\phi &= \beta_{n-1}\phi.
\end{aligned}$$

The multivariate case II

- For our case, $n = 4$, $\mathbf{v} = (1, 1, -1, -1)$ and $\mathbf{c} = (1, c, a, b)$.
- Pick $A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, then $\beta = (c-1, -a, -b)$, and so

$$\begin{aligned} (\partial_1\partial_2 - \partial_3\partial_4) \bullet \phi &= 0, \\ (-\theta_1 + \theta_2) \bullet \phi &= (c-1)\phi, \\ (\theta_1 + \theta_3) \bullet \phi &= (-a)\phi, \\ (\theta_1 + \theta_4) \bullet \phi &= (-b)\phi. \end{aligned}$$

- Elementary but painful check:

$$[x^\beta \sum_{i\in\mathbb{N}} c_i (x^{\mathbf{v}})^i \in \mathcal{S}\mathrm{ol}(System\ 2)] \Leftrightarrow [\sum_{i\in\mathbb{N}} c_i t^i \in \mathcal{S}\mathrm{ol}(Equation\ 1)].$$

The multivariate case III: GKZ systems

$A \in \mathbb{Z}^{d\times n}$, $\beta \in \mathbb{Z}^d$.

- assume $\mathbb{N}A$ *pointed*, and $\mathbb{Z}A = \mathbb{Z}^d$;
  - $\mathcal{O}_A = \mathbb{C}[x_1, \dots, x_n]$,
  - $D_A = \text{WeylAlgebra}(\mathcal{O}_A) = \mathcal{O}_A\langle\partial\rangle$,
  - $R_A = \mathbb{C}\langle\partial\rangle$.
- toric ideal: $I_A = \langle \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \mid A\cdot\mathbf{u} = 0, \mathbf{u} \in \mathbb{Z}^d\rangle \subseteq R_A$
- Euler operators:

$$E_i = \sum_{j=1}^{n} a_{i,j} x_j \partial_j.$$

- GKZ system $H_A(\beta)$:

$$\begin{aligned} P \bullet \phi &= 0 \qquad \forall P \in I_A; \\ (E_i - \beta_i) \bullet \phi &= 0 \qquad \forall i = 1, \dots, n. \end{aligned}$$

If $d + 1 = n$, $\mathcal{S}\text{ol}(H_A(\beta)) \leftrightarrow \mathcal{S}\text{ol}((\textit{Equation } 1))$.

Toric variety/polyhedral structure

- $A = ((a_{i,j})) = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d\times n}$ integer matrix.
- induces $R_A \xrightarrow{\pi_A} \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ with $\partial_j \mapsto t^{\mathbf{a}_j}$.
- $\ker(\pi_A) = I_A = \langle \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \mid A\cdot\mathbf{u} = 0, \mathbf{u} \in \mathbb{Z}^d\rangle \subseteq R_A$
- $S_A = \mathbb{C}[\mathbb{N}A] = \mathbb{C}[t^{\mathbf{a}_1}, \cdots, t^{\mathbf{a}_n}] \subseteq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.
- π_A gives presentation, $S_A = R/I_A$.
- $V_A = \mathrm{Var}(I_A) \subseteq \mathbb{C}^n$
- $\mathbb{R}_+ A$ cvx plhdrl rtl cone, with faces $\{\tau\}$
- $\tau \leftrightarrow R_A(I_A, \{\partial_j\}_{j\notin\tau})$, $\mathbb{Z}^d$-graded primes $\supseteq I_A$.

Examples

Ex. I: $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$,

- $R_A = \mathbb{C}[\partial_1, \partial_2, \partial_3] \supseteq I_A = \langle \partial_1^2 \partial_3^2 - \partial_2 \rangle$,
- $A \leftrightarrow I_A$, $\{1\} \leftrightarrow (\partial_2, \partial_3)$,$\{3\} \leftrightarrow (\partial_1, \partial_2)$, $\emptyset \leftrightarrow (\partial)$.

Ex. II: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$,

- $R_A = \mathbb{C}[\partial_1, \ldots, \partial_4]$,
  $I_A = \langle \partial_1 \partial_4 - \partial_2 \partial_3,\ \ \partial_2^3 - \partial_1^2 \partial_3,\ \ \partial_3^3 - \partial_2 \partial_4^2,\ \ \partial_2^2 \partial_4 - \partial_1 \partial_3^2 \rangle$.
- $A \leftrightarrow I_A$, $\{1\} \leftrightarrow (\partial_2, \partial_3, \partial_4)$,$\{4\} \leftrightarrow (\partial_1, \partial_2, \partial_3)$, $\emptyset \leftrightarrow (\partial)$.

Torus action

- Let $\mathbb{T} = (\mathbb{C}^*)^d$; make map $\mathbb{C}^n \times \mathbb{T} \to \mathbb{C}^n$ via

$$(\xi_1, \ldots, \xi_n) \times (s_1, \ldots, s_d) \mapsto (\mathbf{s}^{\mathbf{a}_1}\xi_1, \ldots, \mathbf{s}^{\mathbf{a}_n}\xi_n),$$

- $(\mathbf{1}_\tau)_j = 1$ if $j \in \tau$ and $= 0$ else.
- $\mathrm{Orb}(\mathbf{1}_\tau) :=$ image of $\mathbf{1}_\tau \times \mathbb{T}$, a torus of dimension $\dim(\tau)$.
- $V_A = \overline{\mathrm{Orb}(\mathbf{1}_A)}^{Zar} = \overline{\mathrm{Orb}(\mathbf{1}_A)}^{cplx} = \bigsqcup_{\tau \text{ face of } \mathbb{R}_+A} \mathrm{Orb}(\mathbf{1}_\tau)$.
- Invariant tangent vectors on $\mathbb{T}$: $\{t_i \frac{\partial}{\partial t_i}\}$.
- Under $\mathbb{T} \to \mathrm{Orb}(\mathbf{1}_A)$, $t_i \frac{\partial}{\partial t_i} \rightsquigarrow \sum_{j=1}^n a_{i,j} x_j \partial_j =: E_i$ (mostly).
- Define:

$$\begin{aligned} H_A(\beta) &= D_A(I_A, \{E_i - \beta_i\}_{i=1}^d) \\ M_A(\beta) &= D_A / H_A(\beta). \end{aligned}$$

Holonomic $\Rightarrow \dim_{\mathbb{C}}(\mathcal{S}ol(H_A(\beta))) < \infty$.

Solving polynomials

- Example: $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ and $\beta = (0, -1)$ give

  $H_A(\beta) = D_A(\partial_2^2 - \partial_1\partial_3,\ \ x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\ \ 2x_1\partial_1 + x_2\partial_2 + 1).$

  Then the solutions to $x_1 T^2 + x_2 T^1 + x_3 T^0 = 0$,

  $$\rho_{1,2} = \frac{-x_2 \pm \sqrt{x_2^2 - 4x_1x_3}}{2x_1}$$

  span $\mathcal{S}\mathrm{ol}(H_A(\beta))$. (True more generally.)

Singular locus of $H_A(\beta)$

Rank:

- $\mathrm{rk}(M) := \dim(\mathcal{S}\mathrm{ol}(M))$ in generic pt $= \dim_{\mathbb{C}(x)}(M \otimes \mathbb{C}(x))$.
- A-discriminant (for A with top row $(1, \ldots, 1)$): consider

$$\begin{aligned} x_{2,1} T^{a_{2,1}} + \ldots + x_{2,n} T^{a_{2,n}} &= 0, \\ &\vdots \\ x_{d,1} T^{a_{d,1}} + \ldots + x_{d,n} T^{a_{d,n}} &= 0 \end{aligned}$$

  Thm: $\exists\ \Delta_A(\{x_{i,j}\})$ with $\Delta_A(x) = 0$ iff system has unusual number of torus solutions.
- $\mathrm{rk} = \mathit{const}$ away from $\Delta_A(x) = 0$.
- If $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ then $\Delta_A(x) = x_1 * x_3 * (4x_1x_3 - x_2^2)$.

Nice hypergeometric series, I

Basic idea:

- assume: I_A homogeneous, $\beta = A \cdot \mathbf{v}$, $v_j \notin \{-1, -2, \ldots\}$, $\Lambda = \ker_{\mathbb{Z}}(A)$.
- consider

$$\phi_{\mathbf{v}} = \sum_{\mathbf{u} \in \Lambda} \frac{[\mathbf{v}]_{\mathbf{u}_-}}{[\mathbf{v} + \mathbf{u}]_{\mathbf{u}_+}} \cdot x^{\mathbf{v}+\mathbf{u}} \in \mathbb{C}[[x_1, \ldots, x_n]]$$

where

$$[\mathbf{v}]_{\mathbf{u}} = \prod_j (v_j(v_j - 1) \cdots (v_j - u_j + 1)) \, .$$

- easy check: killed by $H_A(\beta)$.
- problem: not always a function.

Nice hypergeometric series, II

A homog., β generic, L a generic weight on R_A, $\theta_A = \{x_j\partial_j\}_{j=1}^n$.

- $J_A^L = (D_A \cdot \mathrm{gr}^L(I_A)) \cap \mathbb{C}[\theta_A] + (A\cdot\theta_A - \beta)$, a radical ideal.
- $\mathrm{Var}(J_A^L) = \{\mathbf{v}^1, \ldots, \mathbf{v}^{\mathrm{vol}(A)}\}$.

Theorem The following are a basis for $\mathcal{S}\mathrm{ol}(H_A(\beta))$:

$$\left\{ \phi_{\mathbf{v}} = \sum_{\mathbf{u}\in\Lambda} \frac{[\mathbf{v}]_{\mathbf{u}_-}}{[\mathbf{v}+\mathbf{u}]_{\mathbf{u}_+}} \cdot x^{\mathbf{v}+\mathbf{u}} \mid J_A^L(\mathbf{v}) = 0 \right\}$$

Remark L gives *regular triangulation* $\mathcal{T}$ of A;
to each simplex I “belong” $\mathrm{vol}(I)$ many of the $\phi_{\mathbf{v}}$.

- In general: $\mathrm{rk}(H_A(\beta)) \geq \mathrm{vol}(A)$.
  (homog: wiggle β, and then specialize.
  else: ideas from next part.)
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  - sometimes, $J_A^L \neq \mathrm{gr}^L(H_A(\beta)) \cap \mathbb{C}[\theta_A]$
  - (sometimes)^2, more than $\mathrm{vol}(A)$ solutions: coming soon.