Hypergeometric systems III: Studying rank jumps

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Outline

Recall: 1

- $A \in \mathbb{Z}^{d \times n}$, $\mathbb{Z}A = \mathbb{Z}^d$, pointed.
- $R_A = \mathbb{C}[\partial]$, $\mathcal{O}_A = \mathbb{C}[x]$, $D_A = \mathcal{O}\langle\partial\rangle$.
- $I_A = \ker(x_j \to t^{\mathbf{a}_j})$, $R_A/I_A = \mathbb{C}[\mathbb{N}A]$.
- $H_A(\beta) = D_A(I_A, E - \beta)$.

Euler–Koszul technology 2

Exercise: (with $E_i = a_{i,1}x_1\partial_1 + \ldots + a_{i,n}x_n\partial_n$)

$$\begin{aligned} x^{\mathbf{u}}E_i - E_i x^{\mathbf{u}} &= -(A\cdot\mathbf{u})_i x^{\mathbf{u}}, \\ \partial^{\mathbf{u}}E_i - E_i\partial^{\mathbf{u}} &= (A\cdot\mathbf{u})_i\partial^{\mathbf{u}}. \end{aligned}$$

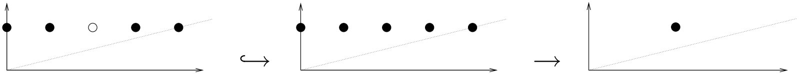
- define $-\deg_A(x_j) = \mathbf{a}_j = \deg_A(\partial_j)$.
- E_i induces D_A-linear endo of $D_A \otimes_{R_A} S_A = D_A/I_A$ by

  $$E_i \circ (P \otimes 1) := PE_i \otimes 1 = (E_i + \deg_i(P))P$$

  where $\deg_A = (\deg_1, \ldots, \deg_d)$.
- $[E_i, E_j] = 0$; can form Koszul complex $\mathcal{K}_\bullet(S_A;\beta)$ on S_A.
- $\rightsquigarrow$ *Euler–Koszul functor*: $\mathbb{Z}^d$-graded R_A-mods $\to$ D_A-mods.
- Key detail: $\mathcal{H}_0(S_A;\beta) = M_A(\beta)$.

Rank $>$ vol I: the setup 3

$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$. Will show: sometimes vol $<$ rk.



S_A $\hookrightarrow$ $\tilde{S}_A$ $\rightarrow$ $C = R_A/\partial$

- Each $N \in \{S_A, \tilde{S}_A, C\}$ gives Euler–Koszul complex

$$\begin{tikzcd}
 & \mathbb{C}[x] \otimes N \arrow[dr, "E_2-\beta_2"] & \\
\mathbb{C}[x] \otimes N \arrow[ur, "E_1-\beta_1"] \arrow[dr, "E_2-\beta_2"'] & \bigoplus & \mathbb{C}[x] \otimes N \\
 & \mathbb{C}[x] \otimes N \arrow[ur, "-E_1+\beta_1"'] &
\end{tikzcd}$$

Rank > vol II: the graded complex 4

- grade D_A by $x \mapsto 1$, $\partial \mapsto 0$. Note: $\mathrm{gr}(D_A) = \mathbb{C}[x, \partial]$.

$$
\mathrm{gr} = \begin{tikzcd}[column sep=small] & \mathbb{C}[x] \otimes N \arrow[dr, "E_2"] & \\ \mathbb{C}[x] \otimes N \arrow[ur, "E_1"] \arrow[dr, "E_2"'] & \bigoplus & \mathbb{C}[x] \otimes N \\ & \mathbb{C}[x] \otimes N \arrow[ur, "-E_1"'] & \end{tikzcd}
$$

- If $N = \tilde{S}_A$, graded complex exact ($\tilde{S}_A$ is Cohen–Macaulay).
- Spectral sequence theorem:

$$\textit{graded complex exact} \Longrightarrow \textit{actual complex exact.}$$

Rank $>$ vol III: hunting for jumps

- L.e.s. of Euler–Koszul homology:

$$
\begin{tikzcd}
 & \mathcal{H}_0(S_A;\beta) \arrow[r] & \mathcal{H}_0(\tilde{S}_A;\beta) \arrow[r] & \mathcal{H}_0(C;\beta) \arrow[r] & 0 \\
 & \mathcal{H}_1(S_A;\beta) \arrow[r] & 0 \arrow[r] & \mathcal{H}_1(C;\beta) \arrow[ull] & \\
0 \arrow[r] & \mathcal{H}_2(S_A;\beta) \arrow[r] & 0 \arrow[r] & \mathcal{H}_2(C;\beta) \arrow[ull] &
\end{tikzcd}
$$

- Rank is additive... $\mathrm{rk}(\mathcal{H}_0(\tilde{S}_A,\beta))$ constant$=\mathrm{vol}(A)$...
- ...investigate $\mathcal{K}_\bullet(C;\beta)$!

Rank > vol IV: getting closer

What is $\mathcal{K}_\bullet(C;\beta)$ like?

- Recall: C sits in degree (1.2).
- $\mathbb{C}[x] \otimes C = D_A/D_A(\partial_1,\ldots,\partial_n) \cong \mathbb{C}[x]$ shifted by $(1,2)$:
- shift affects Euler–Koszul:

  $E_1 \circ P = P(E_1 \boxed{+1}) = P,$

  $E_2 \circ P = P(E_2 \boxed{+2}) = 2P.$
- $(E_1-\beta)\circ P = (1-\beta_1)P, \quad (E_2-\beta)\circ P = (2-\beta_2)P.$
- if $\beta = (1,2)$ both endos are zero, if not one is unit.
- $$\mathcal{H}_\bullet(\beta;C) = \begin{cases} (\mathbb{C}[x], \mathbb{C}[x]\oplus\mathbb{C}[x], \mathbb{C}[x]) & \text{if } \beta=(1,2); \\ (0,0,0) & \text{else.} \end{cases}$$

Rank > vol V: finding jumps

- Feed back into les:
- For $\beta \neq (1,2)$, $\mathrm{rk}(H_A(\beta)) = \mathrm{rk}(\mathcal{H}_0(\tilde{S}_A, \beta)) = \mathrm{vol}(A)$.
- For $\beta = (1,2)$,

$$0 \longrightarrow \underbrace{\mathcal{H}_1(C;\beta)}_{\mathrm{rk}=2} \longrightarrow \underbrace{\mathcal{H}_0(S_A;\beta)}_{\mathrm{rk}=?} \longrightarrow \underbrace{\mathcal{H}_0(\tilde{S}_A;\beta)}_{\mathrm{rk}=4} \longrightarrow \underbrace{\mathcal{H}_0(C;\beta)}_{\mathrm{rk}=1} \longrightarrow 0$$

- Rank is additive in exact sequences.
- $\mathrm{rk}(M_A(\beta)) = 2 + 4 - 1 = 5 > 4$.

Consequences of Euler–Koszul 8

- $\{\beta \mid \mathrm{rk}(H_A(\beta)) > \mathrm{vol}(A)\} =: \mathcal{E}_A$.
- $\mathcal{E}_A$ =subspace arrangement, each parallel to a face of $\mathbb{R}_+ A$
- $\mathcal{E}_A \ni \beta \Leftrightarrow \beta \in \overline{\deg_A(\mathrm{Ext}^i_{R_A}(S_A, R_A))}^{Zar}$ for some $i > n - d$
- $\mathcal{E}_A \ni \beta \Leftrightarrow \mathcal{K}_\bullet(S_A; \beta)$ not resolution
- $\beta \mapsto \mathrm{rk}(H_A(\beta))$ is upper semi-continuous

Open questions

- predict size of jump at $\beta \in \mathcal{E}_A$ by algebraic data
- bound $\sup \frac{\mathrm{rk}(H_A(\beta))}{\mathrm{vol}(A)}$ better than by 2^{2d}