Slopes of hypergeometric D-modules

Uli Walther

Joint work in progress with Mathias Schulze

GKZ systems [Гельфанд, Граев, Капранов, Зелевинский]

$$D = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid [x_i, x_j] = 0 = [\partial_i, \partial_j], [\partial_i, x_j] = \delta_{i,j}\rangle$$

- $A = ((a_{i,j})) = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in \mathbb{Z}^{d\times n}$ integer matrix.
- assume $\mathbb{N}A$ pointed, and $\mathbb{Z}A = \mathbb{Z}^d$; set $R = \mathbb{C}\langle\partial\rangle$
- toric ideal: $I_A = \langle \partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-} \mid A\cdot\mathbf{u} = 0, \mathbf{u} \in \mathbb{Z}^d\rangle \subseteq R$
- $R/I_A = \mathbb{C}[\mathbb{N}A] = \mathbb{C}[t^{\mathbf{a}_1}, \cdots, t^{\mathbf{a}_n}] \subseteq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.
- Euler vector fields: $E_i := \sum_j a_{ij}x_j\partial_j, \quad i = 1, \ldots, d$.
- parameter: $\beta \in \mathbb{C}^d$

Definition (A-hypergeometric system)

$$M_A(\beta) := D/H_A(\beta), \quad H_A(\beta) := \langle E - \beta\rangle + I_A$$

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Regularity in dimension 1

- solutions f of

$$P = p_k(t)\partial_t^k + p_{k-1}(t)\partial_t^{k-1} + \cdots + p_0(t) \in \mathbb{C}\langle t, \partial_t\rangle$$

form k-dimensional vector bundle near $p_k(t_0) \neq 0$.

- near $p_k(t_0) = 0$, P has a singularity: regular or irregular.
- regular = solutions grow polynomially for $t \to t_0$.

$$\text{e.g.} \quad P = t\partial_t + 1, \quad f = 1/t.$$

- irregular = not regular.

$$\text{e.g} \quad P = t^3\partial_t + 2, \quad f = e^{1/t^2}.$$

- When is P regular in 0?
- Fuchs condition: $k - \mathrm{ord}_0(p_k) = \max_i(i - \mathrm{ord}_0(p_i))$

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Fuchs, weights, and slopes in dimension 1

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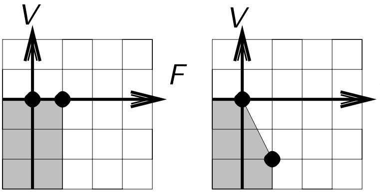
- 

$t\partial_t - 1$ $\qquad$ $t^3\partial_t + 2$

$$F(t^a\partial_t^b) = b$$
$$V(t^a\partial_t^b) = b - a$$

Regular
$\Leftrightarrow$
rightmost=upmost.

- intermediate weights: $t \mapsto -1$, $\partial_t \mapsto 1 + L$, $\infty > L > 0$.
- L-symbol: $\sigma^L(P) = \sum_{i\cdot(1+L)-1\cdot\mathrm{ord}_0(p_i)=max} p_i(t)\partial_t^i$
- L_0 =slope: $\mathrm{Var}(\sigma^L(P))$ changes at $L = L_0 > 0$.

slopes = refined concept of irregularity:

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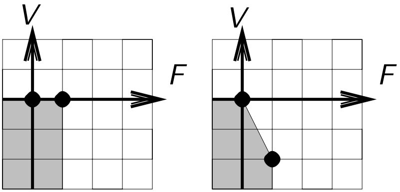
- 

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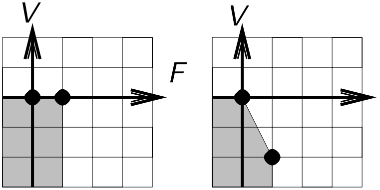
- 

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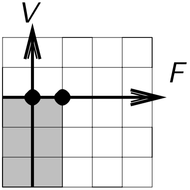
- 

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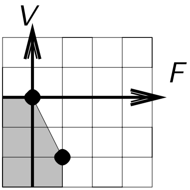

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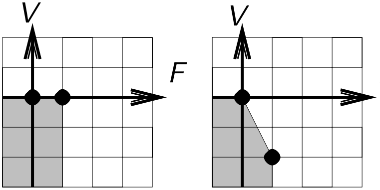
- 

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Filtrations and slopes on D along $\mathrm{Var}(x_1, \ldots, x_k)$

- weights $L = (L_x, L_\partial) \in \mathbb{Q}^{2n}$ give filtration on D if $L_x + L_\partial \geq 0$:

$$x^u \partial^v \in L_k \Leftrightarrow (u, v) \cdot (L_x, L_\partial) \leq k$$

- order filtration: $F = ((0, \ldots, 0), (1, \ldots, 1))$.
- V-filtration along $x_1 = \cdots = x_k = 0$:

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- characteristic cycle of $\mathrm{gr}^L(M)$ in $T^*\mathbb{C}^n$.

Definition (slopes of M)

$$L_0 \text{ is a slope of } M \text{ along } Y \Leftrightarrow \mathrm{cV}^L(M) \text{ changes at } L = L_0$$

Goal:

combinatorial description of slopes for $M_A(\beta)$

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L_0 is a slope of M along $Y \Leftrightarrow \mathrm{cV}^L(M)$ changes at $L = L_0$

Goal:

combinatorial description of slopes for $M_A(\beta)$

In the process, understand geometry of $\mathrm{gr}^L(H_A(\beta))$.

(A, L)-umbrella

$L = (v_1, \ldots, v_n)$ arbitrary weight on $S_A = \mathbb{C}[\partial]/I_A = \mathbb{C}[\mathbb{N}A]$

find min. primes of $gr^L(\mathbb{C}[\mathbb{N}A]) = \mathbb{C}[\partial]/\operatorname{gr}^L(I_A)$ and multiplicities

- $\mathbf{a}_i^L := \mathbf{a}_i / v_i$, $\quad \Delta_A^L := \operatorname{conv}\{0, \mathbf{a}_1^L, \ldots, \mathbf{a}_n^L\}$.
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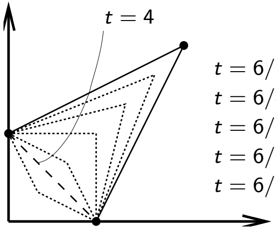
- $$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \qquad L_t = \begin{pmatrix} 1 & 1 & t \end{pmatrix}.$$



- $I_A = \langle \partial_1^2 \partial_2^2 - \partial_3 \rangle = \operatorname{gr}^{L_4}(I_A)$
- $\operatorname{gr}^{L_t}(I_A) = \langle \partial_1^2 \partial_2^2 \rangle$ if $0 < t < 4$, $\quad \operatorname{gr}^{L_t}(I_A) = \langle \partial_3 \rangle$ if $t > 4$.

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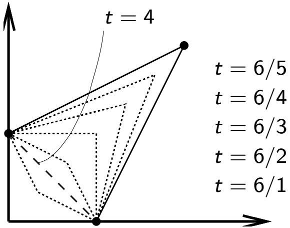

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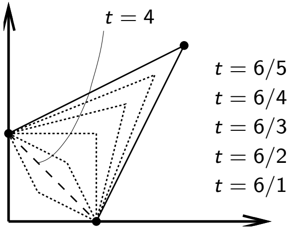

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- $\mathbb{T} = (\mathbb{C}^*)^d$ acts on $\mathrm{Var}(I_A^L)$, orbits indexed by Φ_A^L.
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- Ex: $A = (1)$, $I_A = 0$, $E = x\partial_x$.
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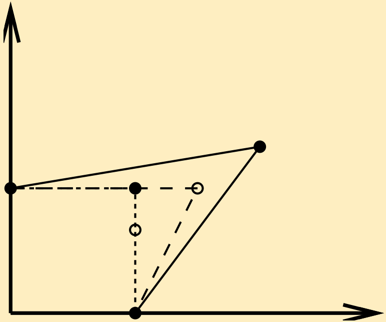
- 

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