Cohomological properties of A-hypergeometric systems

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joint with

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May 2004

Hypergeometric systems

Example, Roots of a quadric: Let

$$g(x_1, x_2, x_3) = \frac{x_2 \pm \sqrt{x_2^2 - 4x_1x_3}}{2x_1},$$

the zeros of $x_1z^2 + x_2z + x_3$.

Let $D_3 = \mathbb{C}[x_1, x_2, x_3, \partial_1, \partial_2, \partial_3]$, put $\theta_i = x_i\partial_i$.

What in D_3 kills $g(x_1, x_2, x_3)$?

$$\begin{aligned} I \;&=\; \langle 1\theta_1 + 1\theta_2 + 1\theta_3 - 0, \\ &\quad\; 2\theta_1 + 1\theta_2 + 0\theta_3 - (-1)\rangle \\ &\quad + \left\langle \partial_1\partial_3 - \partial_2^2 \right\rangle \end{aligned}$$

Put $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ and $\beta = (0, -1)^T$.

Then I is determined by A and the choice $(0, -1) \in \mathbb{C}^2$.

Using the system of PDE's,

$$g(x_1,x_2,x_3) = \frac{-x_2}{2x_1} \pm \left(\frac{x_2}{2x_1} - \frac{x_3}{x_2} \sum_{t=0}^{\infty} \frac{1}{t+1} \binom{2t}{t} \left(\frac{x_1 x_3}{x_2^2} \right)^t \right)$$

We say that $\sum_{t=1}^{\infty} a(t) x^t$ is

- geometric iff $\frac{a(t+1)}{a(t)}$ is constant,

- hypergeometric iff $\frac{a(t+1)}{a(t)} = q(t) \in \mathbb{Q}(t)$.

So g is hypergeometric (in $x_1 x_3 / x_2^2$).

Definition: $A \in \mathbb{Z}^{d,n}$ of full rank, $\beta \in \mathbb{C}^d$. Put

$$\begin{aligned} E_i &= \sum_{j=1}^{n} a_{i,j}\theta_j, \\ I_A &= \langle \boldsymbol{\partial}^{\boldsymbol{u}} - \boldsymbol{\partial}^{\boldsymbol{v}} : A\boldsymbol{u} = A\boldsymbol{v} \rangle \,. \end{aligned}$$

Then

$$H_A(\beta) := \langle \{E_i - \beta_i\}_{i=1}^n \rangle + I_A \subseteq D_n,$$

the *GKZ-hypergeometric system* to A and β.

Geometry of $H_A(\beta)$:

A gives $(\mathbb{C}^*)^d$-action on $\mathbb{C}^n$, (a_i columns of A):

$$(\boldsymbol{\tau}, \boldsymbol{z}) \mapsto (\boldsymbol{\tau}^{a_1} z_1, \ldots, \boldsymbol{\tau}^{a_n} z_n)$$

E_i are the vector fields tangent to the orbits.

I_A is (toric) orbit ideal of $(1, \ldots, 1) \in \mathbb{C}^n$.

Induced is a monomial map

$$\mathbb{C}[\boldsymbol{\partial}] \to \mathbb{C}[\mathbb{N}A] \subseteq \mathbb{C}[y_1^{\pm 1}, \ldots, y_d^{\pm 1}], \quad \partial_i \to \boldsymbol{y}^{a_i}.$$

We call a_i the degree of ∂_i, β is a character.

This talk is about properties of $H_A(\beta)$.

Some History

Suppose $(1,\ldots,1)$ is in the row span of A.

Let P be n distinct points in $\mathbb{Z}^{d-1}$, put $a_i = (1, p_j)$ in $\mathbb{Z}^d$. P is a polyhedron.

- (GKZ,1987) rk($H_A(\beta)$) equals Vol(A).

- (Sturmfels, Takayama, 1998) Consider P, A as above. There are 5 solutions to $H_A((1,2)^T)$. All other β give 4 solutions.

- (Adolphson, 1990) If $\mathbb{N}A$ is Cohen-Macaulay, then GKZ-proof works.

Definition: $\beta \in \mathbb{C}^d$ is exceptional, $\beta \in \mathcal{E}_A$, iff

$$\text{rk}(H_A(\beta)) \neq \text{Vol}(A)$$

Problem: describe the exceptional set.

- (Adolphson,1994) For fixed A, rk$(H_A(\beta)) = $ Vol(A) for most β. In particular, $-$int $(\mathbb{R}^+ A) \cap \mathcal{E}_A = \emptyset$.

- (Cattani-Dickenstein-d'Andrea,99) $d = 2$. Then $\mathcal{E}_A = ((\mathbb{N}A + \mathbb{Z}a_1) \cap (\mathbb{N}A + \mathbb{Z}a_n)) \backslash \mathbb{N}A$ and

$$\text{rk}(H_A(\beta)) = \begin{cases} \text{Vol}(A) + 1 & : \quad \beta \in \mathcal{E}_A \\ \text{Vol}(A) & : \quad \text{otherwise} \end{cases}$$

- (Saito-Sturmfels-Takayama,1999)

  rk$(H_A(\beta)) \geq$ Vol(A).

- (Matusevich,2001) $\mathcal{E}_A$ is an algebraically constructible set.

Cohen-Macaulayness and rank

Put $R_A = \mathbb{C}[\boldsymbol{\partial}]$, $S_A = \frac{R_A}{I_A} = \mathbb{C}[\mathbb{N}A]$, $\mathfrak{m}_A = \langle \boldsymbol{\partial} \rangle$.

S_A is C-M iff $H^i(A) := H^i_{\mathfrak{m}_A}(S_A) = 0 \forall i < d$.

Hochster: $\mathbb{N}A$ saturated $\Rightarrow$ I_A is C-M.

- (Matusevich,2001) When $n = d + 2$ then $[\mathcal{E}_A = \emptyset]$ iff $[I_A$ is C-M$]$.

- (Saito,2002) When P is a simplex then $[\mathcal{E}_A = \emptyset]$ iff $[I_A$ is C-M$]$.

Example: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$.

So $H^0(A) = 0$, $H^1(A) = \mathbb{C} \cdot [\frac{\partial_2^2}{\partial_1} = \frac{\partial_3^2}{\partial_4}]$.

Recall: $(1,2) = \mathcal{E}_A$, and canonically

$$H^1(A) = \mathbb{C} \cdot (((\mathbb{N}A + \mathbb{Z}a_1) \cap (\mathbb{N}A + \mathbb{Z}a_4)) \setminus \mathbb{N}A)$$

in accordance with Cattani-D'Andrea-Dickenstein.

Question:(Sturmfels) $\mathcal{E}_A = \emptyset \Leftrightarrow S_A$ is CM ?
Rank related to local cohomology?

Note: I_A, $\mathfrak{m}_A$ and hence $H^i(A)$ are A-graded.

Definition: Let $E_A = \overline{\bigcup_{i=0}^{d-1} \deg(H^i(A))}^{Zar}$.

Theorem (MMW,2004):

$$\mathcal{E}_A = E_A$$

Generalized GKZ-systems(MMW)

Let $D_A = \mathbb{C}[\boldsymbol{x}, \boldsymbol{\partial}]$.

Basic idea: In

$$D_A/H_A(\beta) = (D_A \otimes_{R_A} S_A)/\{E_i - \beta_i\}_1^d$$

replace S_A by any A-graded S_A-module M.

(GKZ,Adolphson): $E_i - \beta_i$ gives left D_A-endo on $D_A \otimes_{R_A} M$:

$$g \otimes m \mapsto (E_i - \beta_i - \deg_i(g) - \deg_i(m)) g \otimes m$$

and Euler-Koszul complex $\mathcal{K}_\bullet(E - \beta, M)$.

Note: $\mathcal{H}_0(E - \beta, S_A) = D_A/H_A(\beta)$.

Definition:

$$E_A(M) = \overline{\bigcup_{i=0}^{d-1} \mathrm{deg}(H^i_{\mathfrak{m}_A}(M))}^{Zar},$$

witness the failure of M to be C-M of dimension d.

Theorem(MMW):
$\mathcal{K}_\bullet(E-\beta, M)$ is a resolution iff $\beta \notin E_A(M)$.

Basic Tools(MMW):

- a spectral sequence

$$\mathcal{H}_i(E+\beta, \mathrm{Ext}^j_{R_A}(M, \omega_A)) \Rightarrow \mathbb{D}\mathcal{H}_{i+j-n}(E-\beta, M)$$

- $\mathcal{H}_0(E-\beta, M) = 0 \Leftrightarrow \beta \notin \overline{\mathrm{deg}(M)}^{Zar}$

Families of holonomic systems

Let $\mathcal{M}$ be a family of holonomic D_n-modules parameterized by a smooth scheme B.

Theorem(MMW)

- rk$(-)$ is upper semicontinuous on B.
- rk$(\mathcal{M}(\beta_0)) >$ rk(β) iff $\mathrm{Tor}_i^{\mathcal{O}_B}(\kappa_{\beta_0}, \mathcal{M}) \neq 0$ for some $i > 0$.

Consider now an S_A-module M and the holonomic family $\mathcal{M} = \mathcal{H}_0(E - b, M)$ on $\mathbb{C}^n \times \mathbb{C}^d$.

Lemma(MMW) $\mathcal{H}_i(E - \beta, M) = H_i(b - \beta, \mathcal{M})$.

Corollary(MMW) For S_A-modules M,

$$
\begin{aligned}
& \beta \in \mathcal{E}_A(M) \\
\Leftrightarrow\ & \beta \text{ is rank-jumping for } \mathcal{H}_0(E-b, M) \\
\Leftrightarrow\ & \mathcal{M} = \mathcal{H}_0(E-b, M) \text{ is not flat at } \beta \\
\Leftrightarrow\ & H_i(b-\beta, \mathcal{M}) \neq 0 \text{ for some } i > 0 \\
\Leftrightarrow\ & \mathcal{H}_i(E-\beta, M) \neq 0 \text{ for some } i > 0 \\
\Leftrightarrow\ & \beta \in \overline{\bigcup_{i=0}^{d-1} \deg(H^i_{\mathfrak{m}_A}(M))}^{Zar} \\
\Leftrightarrow\ & \beta \in E_A(M).
\end{aligned}
$$

Corollary: E_A is constructible with "face strata", and $E_A(S_A)$ is codimension at least 2.