Monodromy of GKZ-systems

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Lisboa, July 2005

Irreducible monodromy

Notation: $\mathcal{O}_n = \mathbb{C}[x_1, \ldots, x_n]$, D_n the Weyl algebra.

For the D_n-module M, its *rank-module* is

$$M(x) = \mathbb{C}(x) \otimes_{\mathcal{O}_n} M.$$

$M(x)$ simple $\Rightarrow$ analytic continuation of any local solution leads to any other local solution. Otherwise, not. (Riemann–Hilbert.)

We say "irreducible monodromy", because

$$\rho : \pi_1 (\mathbb{C}^n \setminus \mathrm{sing}(M), P) \times \mathrm{Sol}(M)_P \to \mathrm{Sol}(M)_P$$

for generic $P \in \mathbb{C}^n$ is irreducible.

Notation: $R = \mathbb{C}[\partial_1, \dots, \partial_n]$, I_A the toric ideal, $S = R/I_A$.

R is graded: $\deg(\partial_i) = -\boldsymbol{a}_i$.

The contiguity map

$$H_A(\beta - \boldsymbol{a}_i) \xrightarrow{\partial_i} H_A(\beta)$$

is induced from

$$S_A(\boldsymbol{a}_i) \xrightarrow{\partial_i} S_A$$

by the Euler–Koszul functor.

In particular, there is a l.e.s. ending with

$$\begin{aligned} \cdots \to H_1(\frac{S_A}{\partial_i S_A}; \beta) &\to \\ H_A(\beta - \boldsymbol{a}_i) \xrightarrow{\cdot\partial_i} H_A(\beta) \to H_0(\frac{S_A}{\partial_i S_A}; \beta) &\to 0. \end{aligned}$$

Definition: A is a *pyramid* if $\exists\partial_i$ such that $A' = \{A \setminus \boldsymbol{a}_i\}$ a face of A.

If A is a pyramid over A',

$$H_A(\beta) = H_{A'}(\widehat{\beta}) \otimes_{\mathbb{C}} \mathbb{C}[x_i, \partial_i] / \left(x_i\partial_i - \beta_i\right).$$

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Definition: For $\beta \in \mathbb{C}^d$ define

$$\begin{aligned} \mathfrak{a}_\beta &= \{P \in R \mid (P \text{ mod } I_A) = 0 \text{ in } H_A(\beta)(x)\} \\ &= \{P \in R \mid \exists 0 \neq f \in \mathcal{O}_n, fP \in D_n \cdot (I_A, E - \beta)\}. \end{aligned}$$

Note that $\mathfrak{a}_\beta$ is an ideal: for all $Q \in R$,

$$f^{\mathrm{deg}(Q)+1} QP \in (fP).$$

Note also that the map $H_A(\beta - \boldsymbol{a}_i) \xrightarrow{\partial_i} H_A(\beta)$ sends $\mathfrak{a}_{\beta-\boldsymbol{a}_i} \xrightarrow{\cdot\partial_i} \mathfrak{a}_\beta$.

Theorem 1: if β is a rank jump then $H_A(\beta)$ has reducible monodromy.

Idea: Consider

$$\begin{array}{l} \qquad\qquad\qquad\qquad \overbrace{0}^{\tilde{S}_A \text{ CM}} \to H_1(\tilde{S}_A/S_A;\beta) \to \\ H_0(S_A;\beta) \xrightarrow{\phi} H_0(\tilde{S}_A;\beta) \to H_0(\tilde{S}_A/S_A;\beta) \to 0. \end{array}$$

Suppose β rank-jumping but irreducible, of minimal dimension.

Then $\phi(x)$ is zero as $\mathrm{rk}(H_A(\beta)) > \mathrm{rk}(H_0(\tilde{S}_A;\beta))$.

In particular, let $C = \tilde{S}_A/S_A$; then

$$H_1(C;\beta)(x) = H_A(\beta)(x).$$

Technical *Proposition*: There is $\partial^{\boldsymbol{u}}$ such that $A \cdot \boldsymbol{u} = \alpha$,
$\mathfrak{a}_{\beta-\alpha}$ is a $\mathbb{Z}^d$-graded prime, and

$$\partial^{\boldsymbol{u}} : H_A(\beta - \alpha)(x) \to H_A(\beta)(x)$$

is an isomorphism. $\square$

Since $\dim(C) < d$, $\exists \partial^{\boldsymbol{v}}$ with $\partial^{\boldsymbol{v}} C = 0$ and $\mathfrak{a}_{\beta-\alpha} \ni \partial^{\boldsymbol{v}}$ is not zero.

Prime + graded $\Rightarrow$ $\mathfrak{a}_{\beta-\alpha}$ is a face ideal I_A^F.

Then

$$H_A(\beta - \alpha)(x) = H_F(\beta - \alpha)(x_F) \otimes_{\mathbb{C}} \mathbb{C}(x_{A \setminus F})$$

and $H_F(\beta - \alpha)(x_F)$ is irreducible of smaller dimension than $H_A(\beta)$. $\lightning$

Let $\mathcal{IR}_A = \{\beta \in \mathbb{C}^d | H_A(\beta) \text{ irreducible}\}$.

Theorem 2:
$[\beta \in \mathcal{IR}_A] \iff [\gamma \in \mathcal{IR}_A] \quad \forall \beta + \mathbb{Z}^d = \gamma + \mathbb{Z}^d$.

Idea: We may assume that A is not a pyramid.
Let $\beta \in \mathcal{IR}_A$ and $\beta' \in \beta + \mathbb{Z}^d$.

By Technical Proposition,

$$H_A(\beta)(x) = H_A(\beta - \alpha)(x)$$

and $\mathfrak{a}_{\beta-\alpha} = I_A^F$ is some prime ideal.

A not pyramid gives $\mathfrak{a}_{\beta-\alpha} = I_A$, hence $\mathfrak{a}_\beta = I_A$.

In particular, $H_A(\beta - A\boldsymbol{u}) \to H_A(\beta)$ induced by $\partial^{\boldsymbol{u}}$ is an isomorphism for all $\boldsymbol{u}$.

"Irreducibility is backward inherited"

By the above: need: $\gamma' \in (\gamma + \mathbb{N}A) \cap \mathcal{IR}_A$.

Pick $\gamma' \in (\beta + \mathbb{N}A) \cap (\gamma + \mathbb{N}A)$ not a rank jump.

Let $\beta + A\boldsymbol{u} = \gamma'$.

Some homological/commutative algebra on

$$\begin{array}{rcl} \cdots \to H_1(S_A/\left(\partial^{\boldsymbol{u}}\right); \gamma') & \to & \\ H_A(\gamma' - A\boldsymbol{u}) \xrightarrow{\phi} H_A(\gamma') \to H_0(S_A/\left(\partial^{\boldsymbol{u}}\right); \gamma') & \to & 0 \end{array}$$

shows $\phi(x)$ nonzero, hence injective.

But γ' no rank-jump, so

$$H_A(\beta)(x) = H_A(\gamma' - A\boldsymbol{u})(x) \cong H_A(\gamma')(x)$$

is irreducible.

And $\gamma' \in (\gamma + \mathbb{N}A) \cap \mathcal{IR}_A$. $\square$

Theorem 3: $H_A(\beta)$ reducible if $\beta \in \mathbb{Z}^d$ and $\mathrm{rk}(H_A(\beta)) > 1$.

Idea: $\beta = 0$ then $H_A(\beta)(x) \twoheadrightarrow D_n/\left(\partial_1, \ldots, \partial_n\right)(x)$.

By rank hypothesis, has kernel, so is reducible.

For all other $\beta \in \mathbb{Z}^d$ from Theorem 2. $\square$

Corollary: Mellin systems are reducible.

(Because they surject onto GKZ-systems with integer parameter.)