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Algorithmic Determination of the Rational Cohomology of Complex Varieties via Differential Forms

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Abstract. We give algorithms for the computation of the algebraic de Rham cohomology of open and closed algebraic sets inside projective space or other smooth complex toric varieties. The methods, which are based on Gröbner basis computations in rings of differential operators, can also be used to compute the cohomology of intersections of smooth closed and open subsets, and in certain situations the cup-product structure.

We give some examples which were carried out with the help of *Macaulay* 2.

Contents

1. Introduction

The determination of the cohomology of topological spaces has been, and continues to be, a question of interest going back to (at least) Poincaré. This is documented by the beautiful work of Hopf, Leray, Serre, and Milnor, to name just a few.

The advent of reasonably fast computers brought with it a variety Gröbner basis driven algorithms performing a multitude of computations in algebraic and combinatorial settings. This development did not bypass singular cohomology. In the landmark paper [**8**] techniques are presented that compute the dimensions of

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$H^i(U;\mathbb{C})$ where U is the complement of an arbitrary algebraic hypersurface in $\mathbb{C}^n$. These methods were refined in [**12**] in order to deal with a general Zariski-open set $U \subseteq \mathbb{C}^n$. By [**13**] one can also compute the ring structure of $H^*(U;\mathbb{C})$.

In this note we extend the algorithms from [**8, 12, 13**] to the computation of cohomology data on more general types of algebraic sets. These include

1. singular rational cohomology groups of open sets in projective space,
2. singular rational cohomology of projective varieties,
3. compactly supported rational cohomology of locally closed varieties in projective space,
4. singular rational cohomology of open subsets of smooth projective varieties,
5. the ring structure in situation 1.

It follows an overview to the paper.

We shall first give a short review of some known algorithms. The basic idea of these algorithms is the Grothendieck-Deligne isomorphism theorem and work by Hartshorne, which assure that on complex algebraic spaces de Rham cohomology can be computed in the algebraic category, and that the singular theory coincides with the algebraic de Rham theory. At the end of this section we explain some of the bottlenecks of the algorithms.

We next consider the special case of projective space. From there we move on to general open sets in projective space and then, via Alexander duality, to projective varieties. For open sets, the cup product structure can be determined.

Duality can also be used on other spaces, and that gives access to cohomology of open subsets of smooth projective varieties, and, as a corollary, compactly supported cohomology of locally closed sets in projective space.

All the presented methods apply equally well to subvarieties of smooth toric varieties (as opposed to varieties embedded in projective space). Some of these ideas are expanded in the final section.

Notation 1.1. K will be a computable field of characteristic zero contained in $\mathbb{C}$. Although we will work over $\mathbb{C}$, we shall assume (without stating this explicitly every time) that all input data for our algorithms are defined over K. This is to guarantee that we can manipulate the input and recognize vanishing of expressions with the Turing machine.

Whenever a group is pronounced to be "finite dimensional" we will mean it to be a finite dimensional $\mathbb{C}$-vector space. Cosets of elements in a quotient space we usually denote by a bar: $\overline{a}$. We write R_n for the ring of polynomials $\mathbb{C}[x_1,\ldots,x_n]$, and D_n for the Weyl algebra $\mathbb{C}\langle x_1,\partial_1,\ldots,x_n,\partial_n\rangle$. We use multi-index notation: $x^\alpha\partial^\beta$ will mean the monomial $x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}\cdot\partial_1^{\beta_1}\cdot\ldots\cdot\partial_n^{\beta_n}$. Also, $|\alpha|$ denotes in that context $\alpha_1+\ldots+\alpha_n$.

If $f_0,\ldots,f_r \in R_n$ and $I \subseteq \{0,\ldots,r\}$ we write F_I for $\prod_{i\in I} f_i$ and $|I|$ for the cardinality of I.

If $\phi : K^\bullet \to C^\bullet$ is a chain map of two chain complexes of modules over the ring S we write $K^\bullet \cong_S C^\bullet$ if ϕ is a quasi-isomorphism over S.

2. Review of the affine case

The purpose of this section is to review an algorithm that leads to the determination of the singular cohomology groups with rational coefficients and their ring structure for the complement of an affine complex variety.

2.1. Let $Y \subseteq X = \mathrm{Spec}(R_n)$ be defined by the equations $f_0, \ldots, f_r$ in R_n. Then one has associated to $U = X \setminus Y$ a *reduced Čech complex*

(2.1)

$$\check{C}^\bullet = \check{C}^\bullet(f_0, \ldots, f_r) = \left(0 \to \underbrace{\bigoplus_{|I|=1} R_n[F_I{}^{-1}]}_{\text{degree } 0} \to \ldots \to \underbrace{\bigoplus_{|I|=r+1} R_n[F_I{}^{-1}]}_{\text{degree } r} \to 0 \right).$$

If $U = X$, then we set $\check{C}^\bullet$ to be the complex concentrated in degree zero whose entry $\check{C}^0$ is R_n.

One can think of $\check{C}^\bullet$ as the appropriate object for various purposes that replaces the ring of global sections on U if U is not affine.

$\check{C}^\bullet(f_0, \ldots, f_r)$ is a complex of (left) D_n-modules and the maps in the complex are D_n-linear [**5, 7, 11**]. It makes therefore sense to speak of bounded complexes $A^\bullet$ of free finitely generated (left) D_n-modules that are D_n-quasi-isomorphic to the Čech complex.

2.2. For our purposes we will need a special type of resolutions, those that are $\tilde{V}_n$*-strict* [**8, 9, 12**]. This means that the filtration $\tilde{F}^\bullet(A^l)$ induced by the grading on D_n defined by $x_i \to 1, \partial_i \to -1$ for all i is preserved by the maps in the complex, and that the formation of associated graded objects commutes with taking homology in $A^\bullet$:

$$\mathrm{gr}(H^i(A^\bullet[\mathfrak{m}_\bullet]) \cong H^i(\mathrm{gr}(A^\bullet[\mathfrak{m}_\bullet]).$$

It is worth pointing out that this can only be achieved by shifting some of the modules in $A^\bullet$ appropriately, as in the case of graded resolutions over a commutative graded ring. Even with the shifts, $(A^\bullet[\mathfrak{m}_\bullet])$ may not be graded as the Čech complex may not be homogeneous.

It has been shown that for given $f_0, \ldots, f_r$ such a $\tilde{V}_n$-strict resolution of $\check{C}^\bullet = \check{C}^\bullet(f_0, \ldots, f_r)$ is in fact computable. This relies on Gröbner basis techniques, [**9, 10, 12, 13**].

Complexes $A^\bullet[\mathfrak{m}_\bullet] \cong_{D_n} \check{C}^\bullet$ that are $\tilde{V}_n$-strict enjoy a rather stunning property which we describe now. Consider the Euler operator $E = x_1\partial_1 + \ldots + x_n\partial_n$. Then E is $\tilde{V}_n$-homogeneous of degree 0 and hence acts on $\tilde{F}^0(A^i[\mathfrak{m}_i])/\tilde{F}^{-1}(A^i[\mathfrak{m}_i])$. Since the maps in $A^\bullet[\mathfrak{m}_\bullet]$ preserve the filtration, E acts in fact on $\tilde{F}^0(H^i)/\tilde{F}^{-1}(H^i)$ where $H^i = H^i(A^\bullet)$ with the filtration inherited from $A^i[\mathfrak{m}_i]$. The operator $(-E-n)$ has a minimal nonzero polynomial $\tilde{b}_i(s)$ on this quotient. (We remark that this holds not only if $A^\bullet \cong_{D_n} \check{C}^\bullet$ but more generally whenever $A^\bullet$ has *holonomic* cohomology, see [**12**].) We write $\tilde{b}_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ for the least common multiple of all these $\tilde{b}_i(s)$. The polynomial $\tilde{b}_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ is called the *b-function for integration of* $A^\bullet[\mathfrak{m}_\bullet]$ *along* $\partial_1, \ldots, \partial_n$. To describe a certain property of $\tilde{b}_{A^\bullet[\mathfrak{m}_\bullet]}(s)$ we introduce the right module $\Omega = D_n/(\partial_1, \ldots, \partial_n) \cdot D_n$. The functor $\Omega \otimes^L_{D_n} (-)$ is called *integration*.

We say that a cohomology class $\overline{\overline{1} \otimes a}$ in $H^i(\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet])$ *lives in the k-th level of the filtration* if $\overline{\overline{1} \otimes a}$ has a representative in $\overline{1} \otimes F^k(A^i[\mathfrak{m}_i])$ but none in $\overline{1} \otimes F^{k-1}(A^i[\mathfrak{m}_i])$. The amazing fact is that the roots of $\tilde{b}_j(s)$ limit the possible levels of nonzero cohomology classes in $\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$. Namely, a nonzero class living in level k and in cohomological degree i can only occur if $\tilde{b}_j(k) = 0$ for some $j \geq i$.

Note that there are only a finite dimensional vector space of cohomology classes $\overline{1 \otimes a}$ that live in the k-th level of the filtration because all monomials $x^\alpha \partial^\beta$ of $\tilde{V}_n$-degree at most $m-1$ in $D_n[m]$ are right multiples of some ∂_i. From this one can compute the cohomology of $\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$ explicitly because one may simply check all classes of $\tilde{V}_n$-degree at most equal to the largest root of $\tilde{b}_{A^\bullet[\mathfrak{m}_\bullet]}(s)$.

In a nutshell, this gives the following main steps in an algorithm to compute the cohomology of $\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$ ([**8, 9, 12**]):

ALGORITHM 2.1 (Integration of the Čech complex).
INPUT: $f_0, \ldots, f_r \in R_n$, $i \in \mathbb{N}$.
OUTPUT: $\dim_{\mathbb{C}}(H^i(\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]))$ where $A^\bullet \cong_{D_n} \check{C}^\bullet(f_0, \ldots, f_r)$ and $A^\bullet[\mathfrak{m}_\bullet]$ is D_n-free and $\tilde{V}_n$-strict.

1. Compute a $\tilde{V}_n$-strict complex $A^\bullet[\mathfrak{m}_\bullet] \cong_{D_n} \check{C}(f_0, \ldots, f_r)$ ([**9, 10, 12**]).
2. Replace each copy of D_n in $A^\bullet$ by $\Omega \cong \mathbb{C}[x_1, \ldots, x_n]$.
3. Find the b-functions $\tilde{b}_i(s)$ for the integration of $H^i(A^\bullet[\mathfrak{m}_\bullet])$ along $\partial_1, \ldots, \partial_n$, and let k_1 be the largest integral root of their product ([**9**]).
4. Truncate $\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$ to the complex of finite dimensional $\mathbb{C}$-vector spaces $\tilde{F}^{k_1}(\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet])$ with $\mathbb{C}$-linear maps.
5. Take the i-th cohomology and return its dimension.

End.

2.3. Now we explain what such a computation has to do with cohomology of varieties. Let $\Omega^\bullet$ be the Koszul complex on D_n induced by left multiplication by $\partial_1, \ldots, \partial_n$. Then $\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$ and $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet(f_0, \ldots, f_r)$ are naturally quasi-isomorphic up to a cohomological shift by n. This is because $\Omega^\bullet$ is a right D_n-resolution of Ω and computing the Tor-functor can be done by resolving either factor.

Inspection of the tensor product shows now that it computes algebraic de Rham cohomology of U. The start of this "inspection" is the identification of the complex $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ in the case $r = 0$ with the algebraic de Rham functor of [**3**] on X applied to the $\mathcal{O}_X$-module $i_*\mathcal{O}_U = \mathcal{O}_X[f_0^{-1}]$, where $i : U \hookrightarrow X$. This is why we call $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ the *algebraic Čech-de Rham complex of* U. The Grothendieck-Deligne comparison theorem and various other ones imply that $H^i(\Omega^\bullet \otimes_{D_n} \check{C}^\bullet) \cong_{\mathbb{C}} H^{i-n}_{\mathrm{dR}}(U;\mathbb{C}) \cong_{\mathbb{C}} H^{i-n}_{\mathrm{Sing}}(U;\mathbb{C})$, the two latter spaces denoting de Rham and singular cohomology with complex coefficients respectively.

The essence of the above can be summarized in the following theorem.

THEOREM 2.2 (de Rham cohomology in affine space [**8, 12**]). *If $f_0, \ldots, f_r$ are given polynomials in R_n, then there exists an algorithm that produces a finite set of cocycles of differential forms $\{\omega_{i,i'}\}_{i,i'}$ in the algebraic Čech-de Rham complex $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ on $U = \mathbb{C}^n \setminus \mathrm{Var}(f_0, \ldots, f_r)$ such that $\{\omega_{i,i'}\}_{i'}$ span $H^i_{\mathrm{dR}}(U;\mathbb{C})$ for all i.*

PROOF. Use Algorithm 2.1 to obtain a set of generators (over $\mathbb{C}$) for the cohomology of $\Omega \otimes_{D_n} A^\bullet[\mathfrak{m}_\bullet]$. Then use Theorem 2.5 of [**13**] to convert these generators into cohomology generators for $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ whose elements are identified with the cochains in the algebraic Čech-de Rham complex on U. $\square$

Now let us give some bibliographical references. In [**9, 11**] algorithms for the presentation of localizations and more generally the Čech complex are discussed. In [**8, 9, 10, 12**] the $\tilde{V}_n$-filtration is discussed in varying detail. There it is also explained how to construct the complex $A^\bullet[\mathfrak{m}_\bullet]$ from $\check{C}^\bullet$. The article [**9**] gives details to the computation of the b-function and finally [**13**] shows how one translates cohomology classes from $\Omega \otimes_{D_n} A^\bullet$ to classes in $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet$, thus creating actual (algebraic) differential forms.

2.4. It is useful to make some comments about the computational complexity of the constructions that take place in the execution of Algorithm 2.1. The first major computation is to find a presentation of the Čech complex as a complex of free D_n-modules (Step 1). This computation relies on an algorithm by T. Oaku for determining the Bernstein-Sato polynomial of $f_0 \cdot \ldots \cdot f_r$. Computing this polynomial is quite expensive if $\deg(f_0 \cdot \ldots \cdot f_r) > 5$. Faster computers will not be of substantial help here because the complexity of Gröbner basis computations usually grows considerably faster than linearly in the input (the worst possible is doubly exponentially). Thus, in order to make substantial computational progress, better algorithms for the Bernstein-Sato polynomial are needed.

The next potentially hard step in Algorithm 2.1 is to make the complex $\tilde{V}_n$-strict. The author does at this moment not know how big a problem this is.

A true bottleneck however is the computation of the b-functions $\tilde{b}_i(s)$, which appears to be somewhat more complex than the Bernstein-Sato polynomial. But at this time we cannot really make any asymptotic statements. On the positive side, due to the similarity in nature of $\tilde{b}_i(s)$ and Bernstein-Sato polynomials one can hope that progress on one results in progress on the other.

Step 4 consists of (huge problems in) linear algebra. The author thinks that this is the least troublesome part of the algorithm, but whether this is so will much depend on the construction of small $\tilde{V}_n$-strict resolutions.

The algorithms to be described in the sequel use Algorithm 2.1 as a basic building block. None of them involves computations that make a combinatorial explosion likely to occur. Unfortunately, however, the current limitations on what examples can be done with Algorithm 2.1 restrict us to rather small examples to illustrate our algorithms.

3. Chern classes in projective space

In this section we investigate how the cohomology of projective space can be captured by our formalism. It will turn out that it is important to achieve the following.

LEMMA 3.1. *Let $f_0, \ldots, f_r \in R_n$ be given polynomials, and let $\{\omega_{i,i'}\}_{i,i'}$ be given cochains of algebraic differential forms of degree i on $U = X \setminus \operatorname{Var}(f_0, \ldots, f_r)$ (i.e., $\omega_{i,i'} \in (\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^i$). There exists an algorithm that produces a finite dimensional subcomplex $C^\bullet$ of the algebraic Čech-de Rham complex $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ on U such that $C^\bullet \cong_{\mathbb{C}} \Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ and $\omega_{i,i'} \in C^i\ \forall i, i'$.*

PROOF. Let us sketch a proof of the lemma. By Theorem 2.2 it is possible to find a finite dimensional subcomplex of the algebraic de Rham complex that captures all the cohomology (namely, just take all the cohomology generators, with zero differential). However, this may not include the given forms $\omega_{i,i'}$. Thus, as a first approximation $C_1^\bullet$ of the desired complex $C^\bullet$ we take the union of the cohomology generators, the given forms $\omega_{i,i'}$ and their boundaries $d(\omega_{i,i'})$.

This is a complex, but the cohomology may be too big. (It is at least as big as the actual de Rham cohomology but we may have added extra kernel elements.) We must find forms that reduce the cohomology.

By an *exhaustion* of a D_n-module M we mean a sequence of $\mathbb{C}$-subspaces $\{D^k(M)\}_{k\in\mathbb{N}}$ such that $D^k(M) \subseteq D^{k+1}(M)$, $\bigcup_k D^k(M) = M$ and each $D^k(M)$ is finite-dimensional as a $\mathbb{C}$-vector space. If M is finitely generated over D_n then one may produce an exhaustion for M from one for D_n.

Algorithm 3.2.
Input: $C_1^\bullet$, the subcomplex of $\Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ spanned by

- the output of the algorithm of Theorem 2.2,
- forms $\{\omega_{i,i'}\}_{i,i'}$ with $\omega_{i,i'} \in (\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^i$, and their boundaries $\{d(\omega_{i,i'})\}_{i,i'}$.

Output: A finite dimensional complex $C^\bullet \cong_{\mathbb{C}} \Omega^\bullet \otimes_{D_n} \check{C}^\bullet$ containing $C_1^\bullet$ and all $\omega_{i,i'}$.

1. Initialization: set $l = 1$.
2. Let $i_0 = \max\{i : \dim(H^i_{\mathrm{dR}}(U;\mathbb{C})) \neq \dim(H^i(C_l^\bullet))\}$. If $i_0 \leq -1$, return $C_l^\bullet$ and exit.
3. Let $D^k = D^k(D_n)$ be an exhaustion of D_n. For example, let $D^k = \{x^\alpha \partial^\beta : |\alpha + \beta| \leq k\}$.
4. Derive an exhaustion $D^k((\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^{i_0-1})$ of the finitely generated left D_n-module $(\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^{i_0-1}$.
5. For $k = 0, 1, 2, \ldots$ test by trial and error whether there is an element in $D^k((\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^{i_0-1})$ that maps onto a nonzero element in $H^{i_0}(C_l^\bullet)$. As soon as such an element is found, add it to $C_l^{i_0-1}$, call the enlarged complex $C_{l+1}^\bullet$, replace l by $l+1$ and move to Step 6.
6. If $\dim(H^{i_0}_{\mathrm{dR}}(U;\mathbb{C})) \neq \dim(H^{i_0}(C_l^\bullet))$, reenter at Step 5.
7. Reenter at Step 2.

End.

□

Example 3.3. Let $r = 0$, $n = 1$, $x_1 = x$ and $f_0 = x$. We compute $\check{C}^\bullet = (R_1[x^{-1}])$ positioned in cohomological degree 0. Moreover, the de Rham cohomology of $U = \mathbb{C}^1 \setminus \mathrm{Var}(x)$ is generated by 1 in degree 0, and $\frac{dx}{x}$ in degree 1.

Suppose we have the cochain $\omega_{1,1} = x^{-3}\,dx$ which for some reason we would like to be part of our finite dimensional subcomplex $C^\bullet$ of $\Omega^\bullet \otimes_{D_1} \check{C}^\bullet$.

Then $C_1^\bullet$ looks like this:

$$0 \to \mathbb{C}\cdot 1 \to \mathbb{C}\cdot\frac{dx}{x} \oplus \mathbb{C}\cdot\frac{dx}{x^3} \to 0,$$

because $d(x^{-3}dx) = 0$. Clearly $H^1(C_1^\bullet)$ and $H^1_{\mathrm{dR}}(U;\mathbb{C})$ do not agree. Thus $i_0 = 1$ and we have to build an exhaustion for $(\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^0 = \Omega^0 \otimes_{D_n} \check{C}^0 \cong R_1[x^{-1}] = D_1 \bullet \frac{1}{x}$. Take the exhaustion $0 \subset D^0 \subset D^1 \subset \cdots$ on D_1 given by $D^k = \mathbb{C}\cdot\{x^a\partial^b : a + b \leq k\}$. Then D^0 is spanned by $\{1\}$, D^1 by $\{1, x, \partial\}$ and D^2 by $\{\partial^2, \partial, x\partial, 1, x, x^2\}$. So the exhaustion $D^\bullet(\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^0$ on $(\Omega^\bullet \otimes_{D_n} \check{C}^\bullet)^0 = D_1 \bullet \frac{1}{x}$ in level 0 is spanned by $\{\frac{1}{x}\}$, in level 1 by $\{\frac{1}{x^2}, \frac{1}{x}, 1\}$ etc. One easily sees that the complex $C_1^\bullet \cup D^0(\Omega^0 \otimes_{D_n} \check{C}^0) \cup d(D^0(\Omega^0 \otimes_{D_n} \check{C}^0))$ has the same first cohomology as $C_1^\bullet$ while $H^1(C_1^\bullet \cup D^1(\Omega^0 \otimes_{D_n} \check{C}^0) \cup d(D^1(\Omega^0 \otimes_{D_n} \check{C}^0)))$ is one dimensional. The cause for the drop is $\frac{1}{x^2} = \partial \bullet \frac{1}{x}$ with $d(\frac{1}{x^2}) = \frac{-2}{x^3}dx$.

Set $C_2^\bullet = C_1^\bullet \cup \{\mathbb{C} \cdot \frac{1}{x^2}\}$. Then $C_2^\bullet \cong_{\mathbb{C}} \Omega \otimes_{D_n} \check{C}^\bullet$ and the algorithm stops.

NOTE 3.4. We need to decide linear dependence of a set of given cochains of differential forms in order to make Algorithm 3.2 run. This can be achieved by clearing denominators for example.

For the remainder of the section we shall consider the case of projective space, which at the same time can be viewed as a warm-up for general open subsets of $\mathbb{P}^n$, and as a necessary step for the computation of the cohomology of (closed) projective varieties.

The major difficulty that arises when going from affine to projective space is that open sets in projective space are usually not open subsets of an affine space. This means there is no uniform Weyl algebra our computations would be done over. Thus we need to do patching work and use the Mayer-Vietoris principle.

Recall that the cone cone(ϕ) of a chain map ϕ is essentially the total complex induced by the chain map ([**14**], page 18).

LEMMA 3.5. *Let $C_1^\bullet$ and $C_2^\bullet$ be two chain complexes of S-modules and let $K_1^\bullet$ and $K_2^\bullet$ be two subcomplexes which are quasi-isomorphic:*

$$K_1^\bullet \overset{\cong_S}{\hookrightarrow} C_1^\bullet, \quad K_2^\bullet \overset{\cong_S}{\hookrightarrow} C_2^\bullet.$$

Let $\phi : C_1^\bullet \to C_2^\bullet$ be a chain map that sends $K_1^\bullet$ into $K_2^\bullet$. Then the cone over ϕ is independent (modulo quasi-isomorphy) of the choice of the complexes:

$$\operatorname{cone}(K_1^\bullet \xrightarrow{\phi} K_2^\bullet) \cong_S \operatorname{cone}(C_1^\bullet \xrightarrow{\phi} C_2^\bullet).$$

PROOF. The inclusions $K_1^\bullet \overset{\cong_S}{\hookrightarrow} C_1^\bullet, K_2^\bullet \overset{\cong_S}{\hookrightarrow} C_2^\bullet$ induce a map $\operatorname{cone}(K_1^\bullet \xrightarrow{\phi} K_2^\bullet) \to \operatorname{cone}(C_1^\bullet \xrightarrow{\phi} C_2^\bullet)$. To see that this is a quasi-isomorphism consider the induced map between the long exact sequences

$$\begin{array}{ccccccc}
H^i(C_2^\bullet) & \longrightarrow & H^i(\operatorname{cone}(C_1^\bullet \xrightarrow{\phi} C_2^\bullet)) & \longrightarrow & H^i(C_1^\bullet[-1]) & \longrightarrow & H^{i+1}(C_2^\bullet) \\
\cong_S \uparrow & & \uparrow & & \cong_S \uparrow & & \cong_S \uparrow \\
H^i(K_2^\bullet) & \longrightarrow & H^i(\operatorname{cone}(K_1^\bullet \xrightarrow{\phi} K_2^\bullet)) & \longrightarrow & H^i(K_1^\bullet[-1]) & \longrightarrow & H^{i+1}(K_2^\bullet)
\end{array}$$

and recall the five-lemma. □

Why do we need this lemma? Let us look at $\mathbb{P}^1$. We cover it by the two open sets $U_1 = \operatorname{Spec} \mathbb{C}[x]$ and $U_2 = \operatorname{Spec} \mathbb{C}[x^{-1}]$ which intersect in $U_{1,2} = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$. By Theorem 2.2 we know how to compute differential forms on each of the three open sets that span the cohomology of the corresponding open set. These would be $\{1_{U_1}\}, \{1_{U_2}\}, \{1_{U_{1,2}}$ and $(\frac{dx}{x})_{U_{1,2}}\}$. The restriction maps on the Čech-de Rham complex level give us restriction maps $\rho_0^0 : 1_{U_1} \to 1_{U_{1,2}}, 1_{U_2} \to -1_{U_{1,2}}$. (The minus sign is owed to the general theme of Mayer-Vietoris type complexes.)

One would like to infer that the cohomology of the projective line is the cohomology of the complex

$$\begin{array}{ccccc} \mathbb{C}\cdot 1_{U_1} & & & & \\ \oplus & & & & \\ \mathbb{C}\cdot 1_{U_2} & \searrow^{\rho_0^0} & & & \\ & & \mathbb{C}\cdot 1_{U_{1,2}} & \xrightarrow{0} & \mathbb{C}\cdot (\frac{dx}{x})_{U_{1,2}} \end{array}$$

which equals $\mathbb{C}\cdot(1_{U_1}, 1_{U_2})$ in degree 0 and $(\frac{dx}{x})_{U_{1,2}}$ in degree 2.

The reason that this is indeed so is Lemma 3.5, which assures us that the cohomology of

$$\begin{array}{ccccc} (\Omega^\bullet_{U_1}\otimes_{D_n}\check{C}^\bullet_{U_1})^0 & & (\Omega^\bullet_{U_1}\otimes_{D_n}\check{C}^\bullet_{U_1})^1 & & \\ \oplus & \longrightarrow & \oplus & & \\ (\Omega^\bullet_{U_2}\otimes_{D_n}\check{C}^\bullet_{U_2})^0 & \searrow^{\rho_0^0} & (\Omega^\bullet_{U_2}\otimes_{D_n}\check{C}^\bullet_{U_2})^1 & \searrow^{\rho_1^0} & \\ & & (\Omega^\bullet_{U_{1,2}}\otimes_{D_n}\check{C}^\bullet_{U_{1,2}})^0 & \longrightarrow & (\Omega^\bullet_{U_{1,2}}\otimes_{D_n}\check{C}^\bullet_{U_{1,2}})^1 \end{array}$$

agrees with the one from the picture above.

Example/Notation 3.6. In this example we investigate projective space $X = \mathbb{P}^n = \operatorname{Proj}(\mathbb{C}[x_0,\ldots,x_n])$. Since X is covered by the $n+1$ open sets $P_j = \operatorname{Spec}(\mathbb{C}[x_0,\ldots,x_n,x_j^{-1}]_0)$, the cohomology and suitable representatives can be computed from the combinatorics of this cover.

We write I for a subset of $\{0,\ldots,n\}$ and set $P_I = \bigcap_{i\in I} P_i$. Then P_I is the set of points in $\mathbb{P}^n$ where $x_I := \prod_{i\in I} x_i$ is nonzero. If $i_0 = \min_{i\in I}\{i\}$ then $P_I \subseteq P_{i_0} \cong \mathbb{A}^n_{\mathbb{C}}$ is a $(|I|-1)$-fold torus, and its cohomology is captured by a complex

(3.1)

$$T_I^\bullet \quad = \quad \left(0\to\mathbb{C}\cdot 1\to \bigoplus_{i_0<i\in I}\mathbb{C}\cdot\frac{d(x_i/x_{i_0})}{x_i/x_{i_0}}\to\cdots\to\mathbb{C}\cdot\prod_{i_0<i\in I}\frac{d(x_i/x_{i_0})}{x_i/x_{i_0}}\to 0\right),$$

where each differential is zero. Here, the term in cohomological degree k is

$$\bigoplus_{\substack{i_0\in J\subseteq I\\ |J|=k+1}}\mathbb{C}\cdot\prod_{i_0<i\in J}\frac{d(x_i/x_{i_0})}{x_i/x_{i_0}}.$$

We note that there are several choices for how to write this complex, because P_I is not only an open subset of P_{i_0} but also of all other P_i with $i\in I$. For computing $H^\bullet_{\mathrm{dR}}(\mathbb{P}^n;\mathbb{C})$ we want to glue all these complexes together, for varying I. Then we need to translate differential forms from the chart $x_j\neq 0$ to those on the chart $x_{j'}\neq 0$.

Fact 3.7. The conversion of differential forms on $P_j\cap P_{j'}$ from the j-chart P_j to the $x_{j'}$-chart $P_{j'}$ is obtained as follows.

- $f(x_0/x_j,\ldots,\widehat{x_j/x_j},\ldots,x_n/x_j) = \frac{f(x_0/x_{j'},\ldots,\widehat{x_{j'}/x_{j'}},\ldots,x_n/x_{j'})}{(x_j/x'_j)^{\deg(f)}}$,
- for all $i\neq j,j'$ we have $d(x_i/x_j) = \frac{d(x_i/x_{j'})}{(x_j/x_{j'})} - \frac{(x_i/x_{j'})d(x_j/x_{j'})}{(x_j/x_{j'})^2}$,
- $d(x_{j'}/x_j) = \frac{-d(x_j/x_{j'})}{(x_j/x_{j'})^2}$.

All these expressions are regular on $P_j \cap P_{j'} = \operatorname{Spec}(K[x_0, \dots, x_n, x_j^{-1}, x_{j'}^{-1}]_0)$.

Note that by Fact 3.7, the space T_I^k is formally invariant under the change of charts $j \to j'$.

Since the de Rham cohomology of $\mathbb{P}^n$ is the cohomology of the total complex of the global sections of the Čech-de Rham complexes on the open tori, we conclude by Lemma 3.5 that the cohomology is captured by the total complex of the complexes (3.1).

For $n = 2$ this looks like this:

$$
\begin{array}{lllll}
\mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot 1 & & & & \\
\quad \searrow \rho_0^0\cdot & & & & \\
 & \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot 1 & \longrightarrow \mathbb{C}\cdot\frac{d(x/y)}{x/y} \oplus \mathbb{C}\cdot\frac{d(x/z)}{x/z} \oplus \mathbb{C}\cdot\frac{d(y/z)}{y/z} & & \\
 & \quad \searrow \rho_1^0\cdot & \quad \searrow \rho_1^1\cdot & & \\
 & & \mathbb{C}\cdot 1 \longrightarrow & \mathbb{C}\cdot\frac{d(x/z)}{x/z} \oplus \mathbb{C}\cdot\frac{d(y/z)}{y/z} & \longrightarrow \mathbb{C}\cdot\frac{d(x/z)\wedge d(y/z)}{(x/z)(y/z)}
\end{array}
$$

Here, the rows correspond to the subcomplexes (3.1) of the algebraic Čech-de Rham complexes on P_1, P_2, P_3 (top block), $P_{1,2}, P_{1,3}, P_{2,3}$ (middle block) and $P_{1,2,3}$ (bottom line). All maps are zero in horizontal direction, and $\rho_0^0 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$, $\rho_1^0 = (1, -1, 1)$ and $\rho_1^1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

One can see (here, as well as in general) that the cohomology of $\mathbb{P}^n$ is one-dimensional in even degree $2k$, generated by the k-cocycle of k-forms

$$
c_k \quad = \sum_{I=\{i_0<\dots<i_k\}\subseteq\{0,\dots,n\}} \prod_{i_0<i\in I} \frac{d(x_i/x_{i_0})}{x_i/x_{i_0}}. \tag{3.2}
$$

where the displayed summand is defined on the $k+1$-fold intersection $P_I = \mathbb{P}^n \setminus \operatorname{Var}(x_{i_0} \cdot \ldots x_{i_k})$.

The c_k are, up to a constant, the Chern classes of projective space.

The example gives an indication how more general open sets will be attacked, namely by an open cover, used in conjunction with the translation formulæ from Fact 3.7.

4. Open sets in projective space

Let U be the open set of $\mathbb{P}^n$ defined by the non-vanishing of $f_0, \dots, f_r \in S = K[x_0, \dots, x_n]$. In this section we describe an algorithm to compute the rational cohomology of U using the algebraic Čech-de Rham complex on an open cover by the sets $U_I = U \cap P_I$ where the P_I are the open sets from 3.2 covering $\mathbb{P}^n$.

Let $I \subseteq \{0, \ldots, n\}$ and $j = \min_{i \in I}(i)$. We consider U_I as the open set in $P_j = \operatorname{Spec}(\mathbb{C}[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}])$ whose complement is the variety of $\{\frac{x_I}{x_j^{|I|}} f_i(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j})\}_{i=1}^r$. We denote by D_I the Weyl algebra associated to the ring $\mathbb{C}[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}]$ and by $\Omega_I^\bullet$ the Koszul complex of right D_I-modules induced by $\partial_{x_0/x_j}, \ldots, \partial_{x_n/x_j}$.

Algorithm 2.1 in combination with Theorem 2.2 produces for each such I a finite number of cocycles of differential forms which generate the algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(U_I; \mathbb{C})$.

We can think of these classes for fixed I as a subcomplex of the algebraic Čech-de Rham complex on U_I with zero differential, having the same cohomology as the whole algebraic de Rham complex on U_I. Our goal is to glue these complexes according to the open cover, and compute cohomology.

Unfortunately, the natural maps of differential forms induced by the inclusions $U \cap P_I \cap P_j \hookrightarrow U \cap P_I$ may not be carried by these subcomplexes. Since we need this to happen in order to form a total complex from the subcomplexes and to use Lemma 3.5 we need to enlarge the subcomplexes suitably.

The strategy is to start with the complex on $U \cap P_j$, $0 \leq j \leq n$, and work our way up to higher and higher intersections. What we need to achieve is a set of finite dimensional complexes $C_I^\bullet$ on $U \cap P_I$ such that if $c \in C_I^\bullet$ then its natural image in the algebraic Čech-de Rham complex on $U \cap P_I$ is in $C^\bullet_{I \cup j}$ for all I, j. (This natural image is of course for each differential form given by exactly the same form, considered as a form on an open subset.)

Let us give an outline for how to do one such step. Take $C_I^\bullet$ and $C^\bullet_{I \cup j}$ where the former was obtained from the integration if $|I| = 1$ and from the inductive step otherwise, while the latter comes from Theorem 2.2.

Execute Algorithm 3.2 with the following input and output variables. For $C_1^\bullet$ we take $C^\bullet_{I \cup j}$. The set $\{\omega_{i,i'}\}_{i'}$ is for each i a set of vector space generators for C_I^i. The output $C^\bullet$ is quasi-isomorphic to $C^\bullet_{I \cup j}$, contains $C_I^\bullet$, and replaces the old (input) complex $C^\bullet_{I \cup j}$.

Iterating over $|I|$ from 1 to n we get a collection of finite dimensional complexes $C_I^\bullet$ of differential forms whose k-th cohomology is exactly $H^k_{\mathrm{dR}}(U_I; \mathbb{C})$, and $C_I^\bullet \hookrightarrow \Omega_I^\bullet \otimes_{D_I} \check{C}_I^\bullet \to \Omega^\bullet_{I \cup j} \otimes_{D_{I \cup j}} \check{C}^\bullet_{I \cup j}$ factors through $C^\bullet_{I \cup j} \hookrightarrow \Omega^\bullet_{I \cup j} \otimes_{D_{I \cup j}} \check{C}^\bullet_{I \cup j}$. By Lemma 3.5 the total complex composed of the complexes $C_I^\bullet$ is quasi-isomorphic to the algebraic Čech-de Rham complex on U relative to the cover U_I. We hence have

Algorithm 4.1 (Cohomology of open sets).
Input: Homogeneous polynomials $f_0, \ldots, f_r$ in $K[x_0, \ldots, x_n]$.
Output: The cohomology groups of $U = \mathbb{P}^n_{\mathbb{C}} \setminus \operatorname{Var}(f_0, \ldots, f_r)$.

1. For each $I \subseteq \{0, \ldots, n\}$ compute a finite dimensional subcomplex $C_I^\bullet$ of the Čech-de Rham complex on $U_I = U \cap P_I$ with $C_I^\bullet \cong_{\mathbb{C}} \Omega_I^\bullet \otimes_{D_I} \check{C}_I^\bullet$ (Theorem 2.2).
2. For $k = 1, 2, 3, \ldots, n$ do
   - for all $|I| = k, I \subseteq \{0, \ldots, n\}$ do
     - for all $j \in \{0, \ldots, n\} \setminus I$ do
       run Algorithm 3.2 with
       * Input:
         · $\{\omega_{i,i'}\}_{i'} :=$ a set of vector space generators for C_I^i;
         · $C_1^\bullet := C^\bullet_{I \cup j}$

- $*$ Output $C^\bullet$ replacing $C^\bullet_{I\cup j}$.

3. Set up the total complex $C^\bullet_U$ induced by the maps $C^\bullet_I \hookrightarrow C^\bullet_{I\cup j}$.
4. Compute the cohomology of $C^\bullet_U$ which equals the singular (or de Rham) cohomology of $\mathbb{P}^n_{\mathbb{C}} \setminus \mathrm{Var}(f_0, \dots, f_r)$.

End.

The elements, and hence the cohomology, of $C^\bullet_U$ are Čech cochains of differential forms for the cover $U = \bigcup_{0\le j\le n} U_j$.

We remark that similarly to the affine case the complex $C^\bullet_U$ carries not quite enough information to compute the cup product structure of U, but that also like in the affine case this can be fixed by further enlarging $C^\bullet_U$:

Algorithm 4.2 (Cup products on open sets).
Input: Homogeneous polynomials $f_0, \dots, f_r$ in $K[x_0, \dots, x_n]$.
Output: A multiplication table for $H^\bullet_{\mathrm{dR}}(U; \mathbb{C})$.

1. Run Algorithm 4.1 to get the complex $C^\bullet_U$.
2. Compute explicit generators (cocycles of differential forms) for the cohomology of $C^\bullet_U$.
3. Multiply these forms in the Čech-de Rham complex on U according to the usual rules for multiplying Čech cochains, see for example [**13**], Theorem 4.1.
4. Enlarge $C^\bullet_U$ so that it contains all these products (using Algorithm 3.2).
5. Determine a presentation of the cosets of the products in terms of the chosen representatives for the cohomology of $C^\bullet_U$ to get a multiplication table.

End.

5. An example

In this section we will go through one example in detail: the curve $C = \mathrm{Var}(x^2 + yz)$ in $\mathbb{P}^2$. This is of course a rather specific example, but more interesting examples are too large to be useful for an illustration of the general technique (and, as outlined in the introduction, examples of substantial interest are out of reach at the moment).

On the three coordinate patches of $\mathbb{P}^2$, the complement of C is given by the non-vanishing of $1 + (z/x)(y/x)$, $(x/y)^2 + z/y$ and $(x/z)^2 + y/z$. We shall call U_1, U_2, U_3 the corresponding coordinate patches of $\mathbb{P}^2 \setminus C$, and $U_{1,2}$, $U_{1,3}$, $U_{2,3}$ and $U_{1,2,3}$ their intersections.

First we determine a finite set of (exact) differential forms on each of the U_I such that the inclusion of the complex $C^\bullet_I$ generated by these forms (with trivial differential) into the algebraic Čech-de Rham complex $\Omega^\bullet_I \otimes_{D_I} \check{C}^\bullet_I$ is a quasi-isomorphism. Later we shall consider the natural maps $\Omega^\bullet_I \otimes_{D_I} \check{C}^\bullet_I \to \Omega^\bullet_{I\cup j} \otimes_{D_{I\cup j}} \check{C}^\bullet_{I\cup j}$ obtained from the inclusions $U_{I\cup j} \hookrightarrow U_I$.

With *Macaulay* 2 one computes that the de Rham cohomology groups of the various U_I are generated by the following elements.

U_1	H^0		1	
	H^1	$e_{1:1}$	$=$	$\frac{(y/x)(z/x)^2\,d(y/x)+(y/x)^2(z/x)\,d(z/x)}{(1+(y/x)(z/x))^2}$
	H^2	$t_{1:1}$	$=$	$\frac{(y/x)(z/x)\,d(y/x)\,d(z/x)}{(1+(y/x)(z/x))^2}$
U_2	H^0		1	
	H^1	$e_{2:1}$	$=$	$\frac{d(z/y)+2(x/y)\,d(x/y)}{(x/y)^2+(z/y)}$
U_3	H^0		1	
	H^1	$e_{3:1}$	$=$	$\frac{d(y/z)+2(x/z)\,d(x/z)}{(x/z)^2+(y/z)}$
$U_{1,2}$	H^0		1	
	H^1	$e_{1,2:1}$	$=$	$\frac{(x/y)(z/y)^2\,d(x/y)-\frac{1}{2}(x/y)^2(z/y)\,d(z/y)}{(((x/y)^2+(z/y))(x/y))^2}$
		$e_{1,2:2}$	$=$	$\frac{d(x/y)}{(x/y)}$
	H^2	$t_{1,2:1}$	$=$	$\frac{(x/y)(z/y)\,d(x/y)\,d(z/y)}{(((x/y)^2+(z/y))(x/y))^2}$
$U_{1,3}$	H^0		1	
	H^1	$e_{1,3:1}$	$=$	$\frac{(x/z)(y/z)^2\,d(x/z)-\frac{1}{2}(x/z)^2(y/z)\,d(y/z)}{(((x/z)^2+(y/z))(x/z))^2}$
		$e_{1,3:2}$	$=$	$\frac{d(x/z)}{(x/z)}$
	H^2	$t_{1,3:1}$	$=$	$\frac{(x/z)(y/z)\,d(x/z)\,d(y/z)}{(((x/z)^2+(y/z))(x/z))^2}$
$U_{3,2}$	H^0		1	
	H^1	$e_{3,2:1}$	$=$	$\frac{-2(y/z)^2(x/z)^3\,d(x/z)+(y/z)(x/z)^4\,d(y/z)}{(((x/z)^2+(y/z))(y/z))^2}$
		$e_{3,2:2}$	$=$	$\frac{d(y/z)}{(y/z)}$
	H^2	$t_{3,2:1}$	$=$	$\frac{(y/z)(x/z)^3\,d(x/z)\,d(y/z)}{(((x/z)^2+(y/z))(y/z))^2}$
$U_{1,2,3}$	H^0		1	
	H^1	$e_{1,2,3:1}$	$=$	$\frac{-2(y/z)^2(x/z)^5\,d(x/z)+(y/z)(x/z)^6}{(((x/z)^2+(y/z))(x/z)(y/z))^2}$
		$e_{1,2,3:2}$	$=$	$\frac{-2(y/z)^4(x/z)\,d(x/z)+(y/z)^3(x/z)^2\,d(y/z)}{(((x/z)^2+(y/z))(x/z)(y/z))^2}$
		$e_{1,2,3:3}$	$=$	$\frac{d(y/z)}{(y/z)}$
	H^2	$t_{1,2,3:1}$	$=$	$\frac{(y/z)(x/z)^5\,d(x/z)\,d(y/z)}{(((x/z)^2+(y/z))(x/z)(y/z))^2}$
		$t_{1,2,3:2}$	$=$	$\frac{(y/z)^3(x/z)\,d(x/z)\,d(y/z)}{(((x/z)^2+(y/z))(x/z)(y/z))^2}$

In this table, $e_{I:k}$ is the k-th generator of $H^1(U_I;\mathbb{C})$ while $t_{I:k}$ is the k-th generator of $H^2(U_I;\mathbb{C})$. For example, the commands for U_3 are

```
load "../m2/Dloadfile.m2"
R=QQ[s,t]    -- s=x/z, t=y/z
f=s^2+t
deRhamAll(f)
```

The first line loads the D-module library [**6**]. From 3.7,

$$
\begin{aligned}
d(x/y) &= (y/z)^{-1}d(x/z)-(x/z)(y/z)^{-2}d(y/z),\\
d(z/y) &= -(y/z)^{-2}d(y/z).
\end{aligned}
$$

This holds of course for any permutation of the variables x,y,z as well. Set $g_{y,z} = \frac{(x/z)^2(y/z)}{((x/z)^2+(y/z))(y/z)}$, $g_{x,z} = \frac{(x/z)(y/z)}{((x/z)^2+(y/z))(x/z)}$, $g_{x,y} = \frac{(x/y)(z/y)}{((x/y)^2+(z/y))(x/y)}$ and note that $g_{x,y} = g_{x,z} = 1 - g_{y,z}$.

With these rules and abbreviations one computes the following identifications representing the maps from 0-cochains to 1-cochains and from 1-cochains to 2-cochains.

$$
\begin{aligned}
e_{3:1} &= -e_{2,3:1} + e_{2,3:2} + d(g_{y,z}) = 2e_{1,3:2} - 2e_{1,3:1} + d(g_{x,z}) \\
e_{2:1} &= -e_{2,3:1} - e_{2,3:2} + d(g_{y,z}) = 2e_{1,2:2} - 2e_{1,2:1} + d(g_{x,y}) \\
e_{1:1} &= -2e_{1,3:1} = -2e_{1,2:1} \\
t_{1:1} &= t_{1,3:1} = -t_{1,2:1} \\
e_{2,3:1} &= e_{1,2,3:1} \\
e_{2,3:2} &= e_{1,2,3:3} \\
t_{2,3:1} &= t_{1,2,3:1} \\
e_{1,3:1} &= -\frac{1}{2}e_{1,2,3:2} \\
e_{1,3:2} &= \frac{1}{2}e_{1,2,3:1} - \frac{1}{2}e_{1,2,3:2} + d(g_{y,z}) \\
t_{1,3:1} &= -t_{1,2,3:2} \\
e_{1,2:1} &= -2e_{1,2,3:2} \\
e_{1,2:2} &= -\frac{1}{2}e_{1,2,3:1} - \frac{1}{2}e_{1,2,3:2} + e_{1,2,3:3} + d(g_{y,z}).
\end{aligned}
$$

Then the following forms generate finite dimensional complexes $C_I^\bullet$ that are quasi-isomorphic by the inclusion to the Čech-de Rham complex on U_I.

U_1	1	$e_{1:1}$	$t_{1:1}$
U_2	1	$e_{2:1}$	
U_3	1	$e_{3:1}$	
$U_{1,2}$	$1, g_{x,y}$	$e_{1,2:1}, e_{1,2:2}, d(g_{x,y})$	$t_{1,2:1}$
$U_{1,3}$	$1, g_{x,z}$	$e_{1,3:1}, e_{1,3:2}, d(g_{x,z})$	$t_{1,3:1}$
$U_{2,3}$	$1, g_{y,z}$	$e_{2,3:1}, e_{2,3:2}, d(g_{y,z})$	$t_{2,3:1}$
$U_{1,2,3}$	$1, g_{y,z}$	$e_{1,2,3:1}, e_{1,2,3:2}, e_{1,2,3:3}, d(g_{y,z})$	$t_{1,2,3:1}, t_{1,2,3:2}$

The forms $g_{x,y}, g_{x,z}$ and $g_{y,z}$ and their boundaries are needed to assure that $C_I^\bullet \subseteq C_{I\cup j}^\bullet$ for all $I \subseteq \{1,2,3\}$ and all $j \in \{1,2,3\} \setminus I$.

By Lemma 3.5 the Čech-de Rham complex on $U = \mathbb{P}^2 \setminus C$ is quasi-isomorphic to the total complex composed of the $C_I^\bullet$.

Let $U_{(1)} = \{U_1, U_2, U_3\}$, $U_{(2)} = \{U_{1,2}, U_{2,3}, U_{1,3}\}$. Then the total complex $C_U^\bullet$ made from the $C_I^\bullet$ has

- three terms in degree zero (the three constants on $U_{(1)}$),
- nine terms in degree 1 (three constants from $U_{(2)}$, three H^1-generators from $U_{(1)}$ and the three 0-forms $g_{x,y}$, $g_{x,z}$, $g_{y,z}$ on $U_{(2)}$),
- twelve terms in degree 2 (one H^2 generator from U_3, six H^1-generators from $U_{(2)}$, the constants from $U_{1,2,3}$, the differentials of the extra 0-forms on $U_{(1)}$ and an additional 0-form on $U_{1,2,3}$),
- seven terms in degree 3 (three H^2-generators from $U_{(2)}$, three H^1-generators from $U_{1,2,3}$ and the differential of the additional 0-form on $U_{1,2,3}$),
- two terms in degree 4 (the H^2-generators on $U_{1,2,3}$).

Since this is a finite-dimensional complex, we can compute its cohomology by linear algebra. This determines Čech-de Rham cochains of differential forms in the Čech-de Rham complex on U that carry the de Rham cohomology of U. With *Macaulay* 2 again one computes the cohomology of this complex to be zero in all degrees but in H^0 where the cohomology is isomorphic to $\mathbb{C}$, generated by the cochain $(1,1,1)$ of 0-forms on $U_{(1)}$.

6. Closed and locally closed subsets

6.1. Closed varieties in $\mathbb{P}^n$. In this subsection we consider what information can be obtained of the cohomology of the closed sets $Y = \mathrm{Var}(f_0,\dots,f_r)$.

We first note that there is a long exact sequence of sheaf cohomology

$$\cdots \to H^k_Y(\mathbb{P}^n;\mathbb{C}) \to H^k(\mathbb{P}^n;\mathbb{C}) \to H^k(U;\mathbb{C}) \to H^{k+1}_Y(\mathbb{P}^n;\mathbb{C}) \to \cdots$$

Here $\mathbb{C}$ denotes the constant sheaf. Furthermore, since $\mathbb{P}^n$ is a manifold of dimension $2n$, we can use Alexander duality [**4**]. Hence

$$H^k_Y(\mathbb{P}^n,\mathbb{C}) \cong H^{2n-k}_c(Y;\mathbb{C})^*,$$

the latter denoting the vector space dual of cohomology with compact supports. Since Y is a compact space however, cohomology with compact supports agrees with the usual cohomology. Considering the structure of the cohomology groups on $\mathbb{P}^n$ (zero in odd degree) there are exact sequences

$$\begin{aligned} 0 \to H^{2k-1}(U;\mathbb{C}) \to H^{2n-2k}(Y;\mathbb{C})^* \to H^{2k}(\mathbb{P}^n;\mathbb{C}) \\ \to H^{2k}(U;\mathbb{C}) \to H^{2n-2k-1}(Y;\mathbb{C})^* \to 0 \end{aligned}$$

for all $i > 0$ and an exact sequence

$$0 \to H^{2n}(Y;\mathbb{C})^* \to H^0(\mathbb{P}^n;\mathbb{C}) \to H^0(U;\mathbb{C}) \to H^{2n-1}(Y;\mathbb{C})^* \to 0$$

where the star denotes the vector space dual. It is not hard to understand the maps $H^k(\mathbb{P}^n;\mathbb{C}) \to H^k(U;\mathbb{C})$ algorithmically. In Section 3 we found generators for the cohomology of $\mathbb{P}^n$. Since the inclusion $U \to \mathbb{P}^n$ induces the maps $H^k(\mathbb{P}^n;\mathbb{C}) \to H^k(U;\mathbb{C})$, the forms $c_k \in H^{2k}(\mathbb{P}^n;\mathbb{C})$ are simply interpreted as forms on U. In order to find the kernel and the cokernel of $H^k(\mathbb{P}^n;\mathbb{C}) \to H^k(U;\mathbb{C})$ it is sufficient to find a subcomplex $C^\bullet \cong_{\mathbb{C}} C^\bullet_U$ of the algebraic Čech-de Rham complex on $U = \bigcup(P_i \cap U)$ that contains each c_k, because of Lemma 3.5. Such a complex can be constructed from Algorithm 3.2.

Hence we have the following algorithm:

Algorithm 6.1 (Cohomology of projective varieties).
Input: Homogeneous polynomials $f_0,\dots,f_r$ in $\mathbb{C}[x_0,\dots,x_n]$.
Output: The cohomology groups of $\mathrm{Var}(f_0,\dots,f_r)$ in $\mathbb{P}^n_{\mathbb{C}}$.

Let $U = \mathbb{P}^n \setminus \mathrm{Var}(f_0,\dots,f_r)$.

1. Compute $C^\bullet_U$, a finite dimensional complex of differential forms on $U = \bigcup_{i=0}^r (U \cap P_i)$ from Algorithm 4.1 that computes the de Rham cohomology of U.
2. Use Algorithm 3.2 to enlarge $C^\bullet_U$ so that it contains for all k the cocycles c_k from (3.2).
3. $H^{2n-2k-1}(Y;\mathbb{C})$ is isomorphic to the cokernel of the map
$$H^{2k}(\mathbb{P}^n;\mathbb{C}) \to H^{2k}(U;\mathbb{C}).$$
The dimension of this space agrees with $\dim_{\mathbb{C}} H^{2k}(U;\mathbb{C})$ if c_k represents the zero class in $H^{2k}(U;\mathbb{C})$; else it is $\dim_{\mathbb{C}} H^{2k}(U;\mathbb{C}) - 1$. The vanishing of c_k in $H^{2k}(U;\mathbb{C})$ is equivalent to c_k being an image in $C^\bullet_U$.
4. $\dim_{\mathbb{C}} H^{2n-2k}(Y;\mathbb{C})$ equals $\dim_{\mathbb{C}} H^{2k-1}(U;\mathbb{C})$ if $c_k = 0$ in $H^{2k}(U;\mathbb{C})$ and $\dim_{\mathbb{C}} H^{2k-1}(U;\mathbb{C}) + 1$ otherwise.

End.

REMARK 6.2. Since Chern classes are preserved under pullbacks, if U is some open set in $\mathbb{P}^n$ then the images on U of the generators for $H^{2k}(\mathbb{P}^n;\mathbb{C})$ from (3.1) are the Chern classes of U. This shows how one can determine vanishing of the rational Chern classes on U. Y and U have the same cohomology dimensions except for a difference of 1 or -1. This difference is dictated by the vanishing of the Chern classes of U.

EXAMPLE 6.3. Consider the variety defined by $x^2 + zy$ in $\mathbb{P}^2$. Section 5 shows that the corresponding U has no cohomology but for $H^0(U;\mathbb{C}) \cong \mathbb{C}$. Hence $H^0(Y;\mathbb{C}) \cong \mathbb{C}$ corresponding to the second Chern class on $\mathbb{P}^2$, $H^2(Y;\mathbb{C}) \cong \mathbb{C}$ corresponding to the first Chern class on $\mathbb{P}^2$, and all other cohomology groups of Y vanish.

EXAMPLE 6.4. We consider the curve $C = \mathrm{Var}(x^2y + y^2z + z^2x)$ in $\mathbb{P}^2$. $\mathbb{P}^2$ is covered by $U_1 = \mathrm{Spec}\,\mathbb{C}[y/x, z/x]$, $U_2 = \mathbb{C}[x/y, z/y]$ and $U_3 = \mathbb{C}[x/z, y/z]$. We take as coordinates $s = x/z, t = y/z$ on U_3, $U_{1,3}$, $U_{2,3}$ and $U_{1,2,3}$; $s = y/x, t = z/x$ on U_1 and $U_{1,2}$; $s = x/y, t = z/y$ on U_2. We have the following dehomogenizations for f:

```
R=QQ[s,t]
f1=s+s^2*t+t^2
f2=s^2+t+s*t^2
f3=s^2*t+t^2+s
```

Moreover, on two- and threefold intersections $U \cap P_I$ is defined by the nonvanishing of

```
f12=(s+s^2*t+t^2)*s
f13=(s^2*t+t^2+s)*s
f23=(s^2*t+t^2+s)*t
f123=(s^2*t+t^2+s)*s*t
```

With *Macaulay* 2 one computes with the command

```
deRhamAll(g)
```

the following cohomology generator table where g is one of $f_1, \dots, f_{1,2,3}$.

U_1	H^1	$2st+1$ s^2+2t				
	H^2	1	t	s		
U_2	H^1	$\frac{t^2+2s}{2}$ $\frac{2st+1}{2}$				
	H^2	1	t	s		
U_3	H^1	$2st+1$ s^2+2t				
	H^2	1	t	s		
$U_{1,2}$	H^1	$2t^2+s$ $-s^3-2st$	$\frac{s^2t+t^2+s}{2}$ 0			
	H^2	t	t^2	1	s^2	
$U_{1,3}$	H^1	$-2t^2-s$ s^3+2st	s^2t+t^2+s 0			
	H^2	t	t^2	1	s^2	
$U_{2,3}$	H^1	0 $-s^2t-t^2-s$	$\frac{-2st^2-t}{4}$ $\frac{-t^2+s}{4}$			
	H^2	1	t^2	st	t	
$U_{1,2,3}$	H^1	$t^2s^2-t^3$ s^3t+2st^2	$\frac{2t^3+st}{4}$ $\frac{-st^2+s^2}{4}$	$-\frac{s^2t^2+t^3+st}{2}$ 0		
	H^2	t^2	s	t	s^2t	1

In this table the generators for $H^1(U_I;\mathbb{C})$ correspond to columns where the elements of the top row have to be multiplied with $\frac{ds}{f_I}$ and those of the bottom with $\frac{dt}{f_I}$. So for example $H^1(U_{1,2,3};\mathbb{C})$ has three generators, the first of which is $\frac{(t^2s^2-t^3)ds+(s^3t+2st^2)dt}{f_{1,2,3}}$. Similarly, the polynomials listed next to H^2 are to be multiplied with $\frac{ds\,dt}{f_I}$ and then are generators for $H^2(U_I;\mathbb{C})$. So for example $H^2(U_3;\mathbb{C})$ has three generators the last of which is $\frac{s\,ds\,dt}{f_3}$.

We denote these classes by $e_{I:k}$ (H^1-generator in column k) and $t_{I:k'}$ (H^2-generator in column k') where I is the index of the open set in question (for example, $\{1,3\}$ for $U_{1,3}$). Thus, $t_{1,2,3:2}$ is the class $\frac{s\,ds\,dt}{f_{1,2,3}}$.

As always for a connected set, the group $H^0(U;\mathbb{C})$ is a one-dimensional vector space, and it is here generated by the cocycle $(1,1,1)$ of 0-forms. The 0-forms make no further contribution to the cohomology of U.

Using the transformation rules (for example from U_1 to $U_{1,3}$ they say $s\to s^{-1}t, ds\to -s^{-2}t\,ds+s^{-1}\,dt$ and $t\to s^{-1}, dt\to s^{-2}ds$) one computes that

$$
\begin{aligned}
e_{3:1} &= e_{1,3:1}+2e_{1,3:2} = -e_{2,3:1}-4e_{2,3:2},\\
e_{1:1} &= e_{1,3:1}-e_{1,3:2} = -e_{1,2:1}+4e_{1,2:2},\\
e_{2:1} &= e_{2,3:1}-2e_{2,3:2} = -\frac{1}{2}e_{1,2:1}-e_{1,2:2},\\
e_{1,3:1} &= e_{1,2,3:1}+2e_{1,2,3:3},\\
e_{1,3:2} &= -2e_{1,2,3:3},\\
e_{2,3:1} &= -e_{1,2,3:1}-4e_{1,2,3:2}-2e_{1,2,3:3},\\
e_{2,3:2} &= e_{1,2,3:2}+e_{1,2,3:3},\\
e_{1,2:1} &= e_{1,2,3:1}+8e_{1,2,3:2}+4e_{1,2,3:3},\\
e_{1,2:2} &= \frac{1}{2}e_{1,2,3:1}+2e_{1,2,3:2}+2e_{1,2,3:3}.
\end{aligned}
$$

This shows that the 1-forms form a complex of vector spaces with entries of dimensions $1+1+1$, $2+2+2$ and 3 where the matrices have ranks 3 and 3 respectively. Hence the 1-forms make no contribution to the cohomology of U.

Finally, applying the conversion rules to the 2-forms on the various U_I one obtains a complex that has three entries (on one-, two- and threefold intersections of open sets) with matrices $M_{2:1} : \mathbb{C}^{3\times 3} \to \mathbb{C}^{3\times 4}$ and $M_{2:2} : \mathbb{C}^{4\times 3} \to \mathbb{C}^{1\times 5}$. These matrices turn out to have ranks 7 and 5 respectively. Hence the 2-forms contribute 2 generators of $H^2(U;\mathbb{C})$ and nothing to $H^3(U;\mathbb{C})$ or $H^4(U;\mathbb{C})$.

So U has cohomology only in degrees 0 and 2, and $H^2(U;\mathbb{C})$ is of dimension 2. It is noteworthy that the first Chern class of U must be torsion, because it is the pullback from $\mathbb{P}^2$ of a 1-cocycle of 1-forms and we saw that 1-forms make no contribution to the rational cohomology of U.

From the Alexander duality exact sequence we see that the complementary curve C has either Betti numbers 1,2,1 or 1,1,0. The latter case can occur only if the first Chern class of U is nonzero. Since we know it vanishes, C has a two dimensional H^1 and a one-dimensional H^2. As one can check on a local chart, C is smooth and therefore topologically $S^1 \times S^1$.

6.2. Compact Cohomology. If Y is an affine variety $Y \subseteq X = \mathbb{A}^n$, then one can compute the cohomology groups with compact support from Alexander duality on the affine space.

Example 6.5. Let $f = x^3 + y^3 + z^3$. Then the de Rham cohomology groups on $U = \mathbb{C}^3 \setminus \mathrm{Var}(f)$ have dimensions 1, 1, 2 and 2, which we computed by *Macaulay 2* with

```
R=QQ[x,y,z]
deRhamAll(x^3+y^3+z^3)
```

From Alexander duality one concludes that (since $\mathbb{C}^3$ has real dimension 6 and is contractible)

$$(H^i_c(Y;\mathbb{C}))^* \cong H^{6-i}_Y(X;\mathbb{C}) \cong H^{6-i-1}(U;\mathbb{C})$$

for $i < 5$ and $H^i_c(Y;\mathbb{C}) = 0$ for $i > 4$. Thus the cohomology groups $H^i_c(Y;\mathbb{C})$ with compact support of Y have dimensions 0, 0, 2, 2, 1 for $i = 0,\dots,4$ and are zero otherwise.

One can push the computations a little further in nice situations.

Example 6.6. If $Y = \mathrm{Var}(f) \subset \mathbb{P}^n$ is smooth, then the (usual) cohomology of $\mathrm{Var}(f)\cap P_0$ (an affine chart of $\mathrm{Var}(f)$) can be computed. For example, if $f = x^2+yz$, consider the closed subset Z of Y given by $x = 0$. This is a 2 point set. Let us now compute the cohomology of $Y \cap P_0 = Y \setminus Z = \mathrm{Var}(1+yz) \subset \mathbb{A}^2$. By Alexander duality on Y, $H^{2\cdot 1-i}(Z;\mathbb{C}) = H^i_Z(Y;\mathbb{C})^*$. Set $V = \mathbb{P}^2 \setminus Z$ and $U = \mathbb{P}^2 \setminus Y$. Then the long exact sequence of sheaf cohomology on Y gives (with duality incorporated)

$$\begin{aligned}0 &\to H^2(Z;\mathbb{C})^* \to H^0(Y;\mathbb{C}) \to H^0(V\cap Y;\mathbb{C})\\ &\to H^1(Z;\mathbb{C})^* \to H^1(Y;\mathbb{C}) \to H^1(V\cap Y;\mathbb{C})\\ &\to H^0(Z;\mathbb{C})^* \to H^2(Y;\mathbb{C}) \to H^2(V\cap Y;\mathbb{C}) \to 0.\end{aligned}$$

The map $H^{2-k}(Z;\mathbb{C})^* \to H^k(Y;\mathbb{C}) \cong H^{2-k}(Y;\mathbb{C})^*$ (by Poincaré duality) is induced by Alexander duality and really should be thought of as the dual of the map $H^k(Y;\mathbb{C}) \to H^k(Z;\mathbb{C})$ induced by $Z \hookrightarrow Y$. Alexander duality shows that

$H^{2-k}(Z;\mathbb{C})^* \to H^{2-k}(Y;\mathbb{C})^*$ is equivalent to $H_Z^{4-2+k}(\mathbb{P}^2;\mathbb{C}) \to H_Y^{4-2+k}(\mathbb{P}^2;\mathbb{C})$ (i.e., Alexander duality gives a quasi-isomorphism of the 2-term sequences).

Consider the commutative diagram

$$\begin{array}{ccccccccc}
H^{2n-k}(\mathbb{P}^n;\mathbb{C}) & \longrightarrow & H^{2n-k}(V;\mathbb{C}) & \longrightarrow & H_Z^{2n-k+1}(\mathbb{P}^n;\mathbb{C}) & \longrightarrow & H^{2n-k+1}(\mathbb{P}^n;\mathbb{C}) & \longrightarrow & H^{2n-k+1}(V;\mathbb{C}) \\
\Big\downarrow = & & \Big\downarrow \rho_{U,V}^{2n-k} & & \Big\downarrow & & \Big\downarrow = & & \Big\downarrow \rho_{U,V}^{2n-k+1} \\
H^{2n-k}(\mathbb{P}^n;\mathbb{C}) & \longrightarrow & H^{2n-k}(U;\mathbb{C}) & \longrightarrow & H_Y^{2n-k+1}(\mathbb{P}^n;\mathbb{C}) & \longrightarrow & H^{2n-k+1}(\mathbb{P}^n;\mathbb{C}) & \longrightarrow & H^{2n-k+1}(U;\mathbb{C})
\end{array}$$

Inspection shows that $\ker(H_Z^{k+1}(\mathbb{P}^n;\mathbb{C}) \to H_Y^{k+1}(\mathbb{P}^n;\mathbb{C}))$ is isomorphic to

$$\frac{\ker(H^k(V;\mathbb{C}) \to H^k(U;\mathbb{C}))}{\operatorname{im}(H^k(\mathbb{P}^n;\mathbb{C}) \to H^k(V;\mathbb{C})) \cap \ker(H^k(V;\mathbb{C}) \to H^k(U;\mathbb{C}))}$$

(and zero in the case $k=0$). This dimension can be computed by our methods on the level of differential forms, by repeatedly applying Algorithm 3.2. It follows that we can evaluate the dimensions of the kernel and cokernel of $H_Z^{2-k}(\mathbb{P}^2;\mathbb{C}) \to H_Y^{2-k}(\mathbb{P}^2;\mathbb{C})$ and hence the dimensions of $H^k(V \cap Y;\mathbb{C})$. In our example, U has no nontrivial cohomology as pointed out, $V = \mathbb{P}^2\setminus$ two points. We get the following table of dimensions of cohomology groups:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$H^k(\mathbb{P}^2;\mathbb{C})$	1	0	1	0	1
$H^k(V;\mathbb{C})$	1	0	1	1	0
$H^k(U;\mathbb{C})$	1	0	0	0	0
$H_Z^k(\mathbb{P}^2;\mathbb{C})$	0	0	0	0	2
$H^k(Z;\mathbb{C})$	2	0	0	0	0
$H_Y^k(\mathbb{P}^2;\mathbb{C})$	0	0	1	0	1
$H^k(Y;\mathbb{C})$	1	0	0	0	0
$\ker(H^k(V;\mathbb{C}) \to H^k(U;\mathbb{C}))$	0	0	1	1	0
$\operatorname{im}(H^k(\mathbb{P}^2;\mathbb{C}) \to H^k(V;\mathbb{C}))$	1	0	1	0	0
$H^k(Y \setminus Z;\mathbb{C})$	1	1	0	0	0

EXAMPLE 6.7. We continue Example 6.4 from the previous subsection. There we found that $C = \operatorname{Var}(x^2y + y^2z + z^2x)$ has Betti numbers 1,2 and 1. We consider now the open set V in C defined by the nonvanishing of z. On the open set P_3 of $\mathbb{P}^2$ this set is the cubic curve defined by $s^2t + t^2 + s$. It is easy to see that C meets $z = 0$ in 2 points, $Z = \{(0,1,0),(1,0,0)\}$.

The long exact sequence for the pair (C, Z) gives

$$\begin{aligned}
0 \to H^2(Z;\mathbb{C})^* &\to H^0(C;\mathbb{C}) \to H^0(V;\mathbb{C}) \to \\
\to H^1(Z;\mathbb{C})^* &\to H^1(C;\mathbb{C}) \to H^1(V;\mathbb{C}) \to \\
\to H^0(Z;\mathbb{C})^* &\to H^2(C;\mathbb{C}) \to H^2(V;\mathbb{C}) \to 0
\end{aligned}$$

Of course $H^0(V;\mathbb{C}) \cong \mathbb{C}$ and $H^2(V;\mathbb{C})$ is zero because V is topologically a non-closed surface. Hence the known data imply that $H^1(V;\mathbb{C}) \cong \mathbb{C}^3$.

7. Toric varieties

The principles outlined in the previous sections also apply to open and closed sets within smooth toric varieties. We shall demonstrate this with an example.

EXAMPLE 7.1. Let X be the second Hirzebruch surface F_2 (see [**1**]) defined by the complete fan Δ in the plane whose rays are the vectors $(1,0)$, $(0,1)$, $(-1,2)$, $(0,-1)$. We denote the 4 maximal cones by $A,\ldots,D$, the rays by $AB,\ldots,DA$ and the trivial cone by $ABCD$. We write O_σ for the ring of regular functions on the affine variety defined by the cone σ. One finds easily that $O_A = k[(x/z) = s_A, (yz^2/w) = t_A]$, $O_B = k[(z/x) = s_B, (yz^2/w) = t_B]$, $O_C = k[(z/x) = s_C, (w/yx^2) = t_C]$, $O_D = k[(x/z) = s_D, (w/yz^2) = t_D]$.

Let us first compute the cohomology of X. (This is of course well known from combinatorial methods, see [**1**].) On each maximal cone, a complex quasi-isomorphic to the Čech-de Rham complex is simply given by $(\mathbb{C}\cdot 1)$ concentrated in degree zero. The intersections of neighboring cones lead to spaces isomorphic to $\mathbb{C}\times\mathbb{C}^*$, so they have a Čech-de Rham complex quasi-isomorphic to the complex $\left(\mathbb{C}\cdot 1 \to \mathbb{C}\cdot \frac{df}{f}\right)$ where f is an appropriately chosen divisor (corresponding to the ray of intersection). For example, the intersection of the cones B and C leads to the divisor t_C and a corresponding de Rham cohomology generator $\frac{d(t_C)}{t_C}$.

The intersection of cones A and C, and B and D, and all higher intersections are 2-tori with Čech-de Rham complex quasi-isomorphic to the complex $\mathbb{C}\cdot 1 \to \mathbb{C}\cdot\frac{ds_A}{s_A}\oplus\mathbb{C}\cdot\frac{dt_A}{t_A}\to\mathbb{C}\cdot\frac{ds_A\,dt_A}{s_At_A}$.

We combine the $4+6+4+1$ Čech-de Rham complexes to a complex which computes the cohomology of X in terms of Čech cochains of differential forms.

The de Rham cohomology of X is then generated in degree 0 by the 0-cocycle $(1_A, 1_B, 1_C, 1_D)$ (which means that on each 2-dimensional cone the chosen function is identically 1). The group $H^2(X;\mathbb{C})$ is of rank two and generated by the 1-cocycles of 1-forms $\alpha = (\frac{ds_A}{s_A}, \frac{ds_A}{s_A}, 0, 0, \frac{ds_C}{s_C}, \frac{ds_C}{s_C})$ and $\beta = (\frac{2ds_A}{s_A}, \frac{-2dt_A}{t_A}, \frac{-dt_A}{t_A}, \frac{dt_C}{t_C}, \frac{dt_C}{t_C}, 0)$ where these are the 6 components corresponding to the 6 intersections of the 2-cones, ordered lexicographically. Finally, $H^4(X;\mathbb{C})$ can be seen to be generated by the 2-cocycle of 2-forms $(\frac{ds_A\,dt_A}{s_At_A}, \frac{ds_A\,dt_A}{s_At_A}, \frac{ds_A\,dt_A}{s_At_A}, \frac{ds_A\,dt_A}{s_At_A})$ on the triple intersections. All other cohomology groups are zero.

EXAMPLE 7.2. Now we consider the cohomology of the complement of the divisor $f = w - x^2y + z^2y$ in the surface X of the previous example. To that end we compute generators for the de Rham cohomology for the complement on each affine piece determined by a cone of Δ. This is done by *Macaulay* 2 and we use the following notation.

Cone	Variables	Ring	Divisor of f
A	$s_A = x/z$, $t_A = yz^2/w$	$k[s_A, t_A]$	$f_A = 1 - s_A^2 t_A + t_A$
B	$s_B = z/x$, $t_B = yx^2/w$	$k[s_B, t_B]$	$f_B = 1 - t_B + s_B^2 t_B$
C	$s_C = z/x$, $t_C = w/yx^2$	$k[s_C, t_C]$	$f_C = t_C - 1 + s_C^2$
D	$s_D = x/z$, $t_D = w/yz^2$	$k[s_D, t_D]$	$f_D = t_D - s_D^2 + 1$
AB	$s_A = x/z$, $t_A = yz^2/w$	$k[s_A, t_A, s_A^{-1}]$	$f_{AB} = (1 - s_A^2 t_A + t_A)s_A$
BC	$s_C = z/x$, $t_C = w/yx^2$	$k[s_C, t_C, t_C^{-1}]$	$f_{BC} = (t_C - 1 + s_C^2)t_C$
CD	$s_C = z/x$, $t_C = w/yx^2$	$k[s_C, t_C, s_C^{-1}]$	$f_{CD} = (t_C - 1 + s_C^2)s_C$
DA	$s_A = x/z$, $t_A = yz^2/w$	$k[s_A, s_A, t_A^{-1}]$	$f_{AD} = (1 - s_A^2 t_A + t_A)t$
all others	$s_A = x/z$, $t_A = yz^2/w$	$k[s_A, t_A, s_A^{-1}, t_A^{-1}]$	$f_{ABCD} = (1 - s_A^2 t_A + t_A)st$

In these local variables, we have the following generators for the cohomology of the various open sets:

	H^0	H^1	H^2
A	1	$\frac{2s_A t_A\, ds_A+(s_A^2-1)dt_A}{f_A} = A_{1,1}$	$\frac{ds_A\, dt_A}{f_A} = A_{2:1}$ $\frac{s_A\, ds_A\, dt_A}{f_A} = A_{2:2}$
B	1	$\frac{2s_B t_B\, ds_B+(s_B^2-1)dt_B}{f_B} = B_{1:1}$	$\frac{ds_B\, dt_B}{f_B} = B_{2:1}$ $\frac{s_B\, ds_B\, dt_B}{f_B} = B_{2:2}$
C	1	$\frac{2s_C\, ds_C+dt_C}{f_C} = C_{1:1}$	
D	1	$\frac{-2s_D\, ds_D+dt_D}{f_D} = D_{1:1}$	
AB	1	$\frac{-(2t_A+2)ds_A-(s_A^3-s_A)dt_A}{f_{AB}} = AB_{1:1}$ $\frac{(s_A^2 t_A-t_A-1)ds_A}{f_{AB}} = AB_{1:2}$	$\frac{ds_A\, dt_A}{f_{AB}} = AB_{2:1}$ $\frac{s_A ds_A}{f_{AB}} = AB_{2:2}$ $\frac{s_A^2 ds_A}{f_{AB}} = AB_{2:3}$
BC	1	$\frac{(s_C^2+t_C-1)dt_C}{f_{BC}} = BC_{1:1}$ $\frac{2s_C t_C\, ds_C+t_C\, dt_C}{f_{BC}} = BC_{1:2}$	$\frac{ds_C\, dt_C}{f_{BC}} = BC_{2:1}$ $\frac{s_C\, ds_C\, dt_C}{f_{BC}} = BC_{2:2}$
CD	1	$\frac{(2-2t_C)ds_C+s_C\, dt_C}{f_{CD}} = CD_{1:1}$ $\frac{(s_C^2+t_C-1)ds_C}{f_{CD}} = CD_{1:2}$	$\frac{ds_C\, dt_C}{f_{CD}} = CD_{2:1}$
AD	1	$\frac{-2s_A t_A^2\, ds_A-dt_A}{f_{AD}} = AD_{1:1}$ $\frac{-2s_A t_A^2\, ds_A+(-s_A^2 t_A+t_A)dt_A}{f_{AD}} = AD_{1:2}$	$\frac{t_A\, ds_A\, dt_A}{f_{AD}} = AD_{2:1}$ $\frac{s_A t_A\, ds_A\, dt_A}{f_{AD}} = AD_{2:2}$
$ABCD$	1	$\frac{(-2t_A^2-2t_A)ds_A-s_A\, dt_A}{f_{ABCD}} = ABCD_{1:1}$ $\frac{2s_A^2 t_A^2\, ds_A+s_A\, dt_A}{f_{ABCD}} = ABCD_{1:2}$ $\frac{2s_A^2 t_A^2\, ds_A+(s_A^3 t_A-s_A t_A)\, dt_A}{f_{ABCD}} = ABCD_{1:3}$	$\frac{ds_A\, dt_A}{f_{ABCD}} = ABCD_{2:1}$ $\frac{t_A\, ds_A\, dt_A}{f_{ABCD}} = ABCD_{2:2}$ $\frac{s_A t_A\, ds_A\, dt_A}{f_{ABCD}} = ABCD_{2:3}$ $\frac{s_A^2 t_A\, ds_A\, dt_A}{f_{ABCD}} = ABCD_{2:4}$

The map of 0-cochains to 1-cochains of 1-forms is given by the following chart.

$$
\begin{array}{lcl}
A_{1,1} & \to & (-AB_{1,1}+2AB_{1,2})+(AC_{1,3})-(AD_{1,2}) \\
B_{1,1} & \to & (-AB_{1,1}+2AB_{1,2})-(BC_{1,1}+BC_{1,2})-(BD_{1,3}) \\
C_{1,1} & \to & (-AC_{1,1})-(BC_{1,2})+(CD_{1,1}+2CD_{1,2}) \\
D_{1,1} & \to & (-AD_{1,1})+(BD_{1,2})-(CD_{1,1}).
\end{array}
$$

On the other hand, the map from 1-cochains to 2-cochains is as follows.

$$
\begin{array}{lcl}
AB_{1,1} & \to & (ABC_{1,1}+ABC_{1,2}-ABC_{1,3})+(ABD_{1,1}+ABD_{1,2}-ABD_{1,3}) \\
AB_{1,2} & \to & (\frac{1}{2}ABC_{1,1}+\frac{1}{2}ABC_{1,2})+(\frac{1}{2}ABD_{1,1}+\frac{1}{2}ABD_{1,2}) \\
AC_{1,1} & \to & (-ABC_{1,1})+(ACD_{1,1}) \\
AC_{1,2} & \to & (-ABC_{1,2})+(ACD_{1,2}) \\
AC_{1,3} & \to & (-ABC_{1,3})+(ACD_{1,3}) \\
AD_{1,1} & \to & (ABD_{1,2})+(ACD_{1,2}) \\
AD_{1,2} & \to & (ABD_{1,3})+(ACD_{1,3}) \\
BC_{1,1} & \to & (ABC_{1,1}+ABC_{1,3})+(BCD_{1,1}+BCD_{1,3}) \\
BC_{1,2} & \to & (ABC_{1,1})+(BCD_{1,1}) \\
BD_{1,1} & \to & (ABD_{1,1})-(BCD_{1,1}) \\
BD_{1,2} & \to & (ABD_{1,2})-(BCD_{1,2}) \\
BD_{1,3} & \to & (ABD_{1,3)})-(BCD_{1,3}) \\
CD_{1,1} & \to & (-ACD_{1,2})-(BCD_{1,2}) \\
CD_{1,2} & \to & (\frac{1}{2}ACD_{1,1}+\frac{1}{2}ACD_{1,2})+(\frac{1}{2}BCD_{1,1}+\frac{1}{2}BCD_{1,2})
\end{array}
$$

Finally, the map to 3-cochains is given by the following table.

	$ABCD_{1,1}$	$ABCD_{1,2}$	$ABCD_{1,3}$
$ABC_{1,1}$	1	0	0
$ABC_{1,2}$	0	1	0
$ABC_{1,3}$	0	0	1
$ABD_{1,1}$	−1	0	0
$ABD_{1,2}$	0	−1	0
$ABD_{1,3}$	0	0	−1
$ACD_{1,1}$	1	0	0
$ACD_{1,2}$	0	1	0
$ACD_{1,3}$	0	0	1
$BCD_{1,1}$	−1	0	0
$BCD_{1,2}$	0	−1	0
$BCD_{1,3}$	0	0	−1

Taking cohomology (in *Macaulay* 2 for example with the commands `ker, image, gens` and `%`) we see that $H^2(U;\mathbb{C})$ is generated by the 1-cocycle of 1-forms $\alpha = (-AB_{1,2}, -\frac{1}{2}AC_{1,1} - \frac{1}{2}AC_{1,2}, 0, 0, \frac{1}{2}BD_{1,1} + \frac{1}{2}BD_{1,2}, CD_{1,2})$ (this is the same α that was a cohomology generator on X, now expressed in terms of the cochains on U). This is the only contribution 1-forms make to the cohomology of U.

On the 2-forms we have the following maps for Čech cocycles from single to double intersections:

(7.1)

$$\begin{aligned} A_{2,1} &\to (AB_{2,2}) + (AC_{2,3}) + (AD_{2,1}) \\ A_{2,2} &\to (AB_{2,3}) + (AC_{2,4}) + (AD_{2,2}) \\ B_{2,1} &\to (AB_{2,2}) - (BC_{2,1}) + (BD_{2,3}) \\ B_{2,2} &\to (AB_{2,1}) - (BC_{2,2}) + (BD_{2,2}) \end{aligned}$$

The matrix for 2-forms from double to triple intersections can be obtained (we omit the obvious maps from AC and BD to any triple intersection) from the equations

$$\begin{gathered} AB_{2,1} = BC_{2,2} = -ABCD_{2,2}, \\ AB_{2,2} = AD_{2,1} = BC_{2,1} = ABCD_{2,3}, \\ AB_{2,3} = AD_{2,2} = ABCD_{2,4}, \\ CD_{2,3} = ABCD_{2,3}. \end{gathered}$$

Linear algebra shows that the 2-forms do not contribute to the cohomology of U. Thus, $H^0(U;\mathbb{C}) \cong H^2(U;\mathbb{C}) \cong \mathbb{C}$ and all other cohomology groups are zero.

It follows from the long exact sequence in 6.1 relating the open set and the variety of f that $\mathrm{Var}(f)$ has its cohomology concentrated in degree 0 and 2 and both are one-dimensional. This is because $H^2(X;\mathbb{C}) \to H^2(U;\mathbb{C})$ is not the zero map since $\alpha \neq 0$ in $H^2(U;\mathbb{C})$. On the other hand, the cocycle $2\alpha - \beta$ that generates cohomology on X is zero on U: as

$$\begin{aligned} \alpha &= (-AB_{1,2}, -\frac{1}{2}AC_{1,1} - \frac{1}{2}AC_{1,2}, 0, 0, \frac{1}{2}BD_{1,1} + \frac{1}{2}BD_{1,2}, CD_{1,2}), \\ \beta &= (-2AB_{1,2}, AC_{1,3} - AC_{1,2}, AD_{1,1} - AD_{1,2}, BC_{1,1}, BD_{1,1} + BD_{1,3}, 0). \end{aligned}$$

one sees that $2\alpha - \beta = d(-A_{1,1} + B_{1,1} + C_{1,1} + D_{1,1})$. Hence $2\alpha - \beta$ is the zero class in $H^2(U;\mathbb{C})$.

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