

RESTRICTION OF HYPERGEOMETRIC \mathcal{D} -MODULES WITH RESPECT TO COORDINATE SUBSPACES

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ABSTRACT. We compute the restriction of an A -hypergeometric \mathcal{D} -module with respect to a coordinate subspace under certain genericity conditions on the parameter.

1. INTRODUCTION

Let $A \in \mathbb{Z}^{d \times n}$ be a matrix of rank d with integer entries a_{ij} , $i = 1, \dots, d$, $j = 1, \dots, n$. For $\mathbf{u} \in \mathbb{Z}^n$, let $\mathbf{u}_+, \mathbf{u}_- \in \mathbb{N}^n$ be such that $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$, and write $\square_{\mathbf{u}}$ for the element $\partial^{\mathbf{u}_+} - \partial^{\mathbf{u}_-}$ of $R_A = \mathbb{C}[\partial_1, \dots, \partial_n]$. Here and elsewhere we use multi-index notation: $\partial^{\mathbf{v}} = \prod_{i=1}^n v_i$ for $\mathbf{v} \in \mathbb{N}^n$. The *toric ideal* associated with A is the prime binomial ideal

$$I_A = \langle \square_{\mathbf{u}} : \mathbf{u} \in \mathbb{Z}^n, A\mathbf{u} = 0 \rangle \subseteq R_A.$$

Identifying ∂_j with the partial derivation operator $\frac{\partial}{\partial x_j}$, let $D \supseteq R_A$ be the Weyl algebra of order n , i.e. the \mathbb{C} -algebra of linear partial differential operators with coefficients in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. Following [GGZ87] and [GZK89], the *hypergeometric ideal* associated with a matrix A as above and a vector of complex parameters $\beta \in \mathbb{C}^d$ is the left ideal

$$H_A(\beta) := DI_A + D\langle E_A - \beta \rangle \subseteq D,$$

where $E_A - \beta$ is the sequence of Euler operators

$$E_i - \beta_i = \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i, \quad i = 1, \dots, d.$$

The (global) hypergeometric D -module associated with (A, β) is by definition the quotient

$$M_A(\beta) := D/H_A(\beta).$$

Let \mathcal{D} be the sheaf of linear partial differential operators with holomorphic coefficients in \mathbb{C}^n . To the pair (A, β) one may associate the corresponding analytic hypergeometric \mathcal{D} -module, denoted by $\mathcal{M}_A(\beta)$, which is the quotient of \mathcal{D} modulo the left \mathcal{D} -ideal generated by $H_A(\beta)$.

The restriction functor is a useful tool for the study of the irregularity of a holonomic \mathcal{D} -module \mathcal{M} (see, for example, the Cauchy–Kovalevskaya theorem for Gevrey series [LM02, Cor. 2.2.4]). There is an algorithm due to Oaku–Takayama

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(see [Oak97, OT01]) for the effective computation of the restriction of holonomic D -modules to linear subspaces by means of Gröbner basis calculations in the Weyl algebra. This algorithm is employed in the proof of the explicit restriction formulas in [CJT03, Thm. 4.4] and [FFCJ11, Thm. 4.2] that hold for certain classes of hypergeometric systems. The purpose of our paper is, for sufficiently general parameter $\beta \in \mathbb{C}^d$, to generalize these formulas to the case of arbitrary A by using the Euler–Koszul functor developed in [MMW05]. We also show by example that there are parameters for which our formula does not hold (see Example 2.8).

2. EXPLICIT RESTRICTION FORMULA FOR $\mathcal{M}_A(\beta)$.

Notation 2.1. We denote by $\mathbf{a}_j \in \mathbb{Z}^d$ the j -th column of A , $j = 1, \dots, n$. For any subset $\tau \subseteq \{1, \dots, n\}$ we shall write $x_\tau = (x_i)_{i \in \tau}$, $\partial_\tau = (\partial_i)_{i \in \tau}$, $R_\tau = \mathbb{C}[\partial_\tau]$, and $D_\tau = \mathbb{C}[x_\tau](\partial_\tau)$. If I_τ is the toric ideal associated with the submatrix A_τ consisting of the columns indexed by τ then $R_\tau/I_\tau = S_\tau$ is isomorphic to the semigroup ring $S_\tau = \mathbb{C}[t^{\mathbf{a}_i} : i \in \tau] \subseteq \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$.

Consider for $Y_\tau = \{x_i = 0 : i \notin \tau\}$ the natural inclusion

$$i_\tau : Y_\tau \hookrightarrow X = \mathbb{C}^n.$$

We say that $\beta \in \mathbb{C}^d$ is *generic* if it is outside a hyperplane arrangement depending on A , and that β is *very generic* if it is outside a locally finite arrangement of countably many hyperplanes.

With this notation we will prove:

Theorem 2.2. *Suppose one of the following conditions holds:*

- (i) $\beta \in \mathbb{C}^d$ is generic and $\mathbb{Q}_{\geq 0}A_\tau = \mathbb{Q}_{\geq 0}A$ (i.e., the real positive cones spanned by the columns of A and A_τ respectively agree);
- (ii) $\beta \in \mathbb{C}^d$ is very generic and $\text{rank}(A_\tau) = d$.

Choose a collection $\Omega \subseteq \mathbb{N}A$ of coset representatives for $\mathbb{Z}A/\mathbb{Z}\tau$. Then the (derived) restriction of $\mathcal{M}_A(\beta)$ with respect to Y_τ is given by

$$(2.1) \quad \mathbb{L}^{-k} i_\tau^* \mathcal{M}_A(\beta) \simeq \begin{cases} \bigoplus_{\lambda \in \Omega} \mathcal{M}_{A_\tau}(\beta - \lambda) & \text{if } k = 0; \\ 0 & \text{if } k > 0. \end{cases}$$

Proof. The left-multiplications by x_i , $i \notin \tau$, form commuting endomorphisms of left \mathcal{D}_τ -modules (but not of left \mathcal{D}_A -modules). By [MT04, Prop. 3.1]), $\mathbb{L} i_\tau^* \mathcal{M}_A(\beta)$ is quasi-isomorphic to the Koszul complex $K_\bullet(x_i : i \notin \tau; \mathcal{M}_A(\beta))$ as a complex of left \mathcal{D}_τ -modules. Moreover, because of the flatness of the fibers of \mathcal{D}_A (resp. \mathcal{D}_τ) over their algebraic counterparts and since affine spaces are D -affine, it suffices to prove Theorem 2.2 using the global sections D_A (resp. D_τ) instead of the sheaves \mathcal{D}_A (resp. \mathcal{D}_τ).

Define a \mathbb{Z}^d -grading on $R_A \subseteq D_A$ by

$$\deg(\partial_i) = -\mathbf{a}_i = -\deg(x_i), \quad i = 1, \dots, n.$$

For any \mathbb{Z}^d -graded R_A -module N , the Euler–Koszul complex $\mathcal{K}_\bullet(E_A - \beta, N)$ (see [MMW05, Def. 4.2]) is the Koszul complex of left D_A -modules defined by the sequence $E_A - \beta$ of commuting endomorphisms on the left D_A -module $D_A \otimes_{R_A} N$. This complex is concentrated in homological degrees d to 0 . The i -th Euler–Koszul homology of N is

$$\mathcal{H}_i(E_A - \beta; N) = H_i(\mathcal{K}_\bullet(E_A - \beta, N)).$$

By [MMW05, Theorem 6.6], $\mathcal{K}_\bullet(E_A - \beta, S_A)$ is a resolution of $M_A(\beta)$ for generic β .

Set $C_{\bullet, \bullet} = K_\bullet(x_i \cdot : i \notin \tau; \mathcal{K}_\bullet(E_A - \beta, S_A))$, a double complex as follows from [MMW05, Lem. 4.3] and

- $x_i \cdot (D_A \otimes_{R_A} S_A)_\alpha \subseteq (D_A \otimes_{R_A} S_A)_{\alpha + a_i}$,
- $x_i \cdot (E_A - \beta - \alpha) = (E_A - \beta - \alpha - a_i)x_i$, and
- $\mathcal{K}_\bullet(E_A - \beta, S_A) = \bigoplus_{\alpha \in \mathbb{Z}^d} K_\bullet(E_A - \beta - \alpha, (D_A \otimes_{R_A} S_A)_\alpha)$,

showing that all the squares in $C_{\bullet, \bullet}$ are commutative.

Denote by $\bar{\tau}$ the complement $A \setminus \tau$. Let π be the natural projection of $C_{\bullet, \bullet}$ to $K_\bullet(x_i \cdot : i \notin \tau; M_A(\beta))$ and let η be the natural projection of $C_{\bullet, \bullet}$ to $D_A/x_{\bar{\tau}}D_A \otimes_{D_A} \mathcal{K}_\bullet(E_A - \beta, S_A)$. Consider the induced morphisms:

$$(2.2) \quad \text{Tot}(C_{\bullet, \bullet}) \xrightarrow{\pi} \text{Tot}(K_\bullet(x_i \cdot : i \notin \tau; M_A(\beta))) = K_\bullet(x_i \cdot : i \notin \tau; M_A(\beta))$$

and

$$(2.3) \quad \text{Tot}(C_{\bullet, \bullet}) \xrightarrow{\eta} \text{Tot}(D_A/x_{\bar{\tau}}D_A \otimes_{D_A} \mathcal{K}_\bullet(E_A - \beta, S_A))$$

Remark 2.3. By [MMW05, Thm. 6.6], $\mathcal{K}_\bullet(E_A - \beta, S_A)$ is a resolution of $\mathcal{M}_A(\beta)$ if and only if β is not rank-jumping. Thus, for such β , π and η are a quasi-isomorphisms and $H_\bullet \text{Tot}(C_{\bullet, \bullet}) = \text{Tor}_{\bullet}^{D_A}(D_A/x_{\bar{\tau}}D_A, M_A(\beta))$.

Notation 2.4. For a toric S_A -module N we write $\mathcal{K}_\bullet(E_\tau - \beta, N)$ for the Koszul complex of left D_τ -modules defined by the sequence $E_\tau - \beta$ on the (*a fortiori* weakly toric) S_τ -module N .

Using the isomorphism (see [MMW05])

$$(2.4) \quad \begin{aligned} (D_A \otimes_{R_A} S_A) &\longrightarrow \mathbb{C}[x_{\bar{\tau}}] \otimes_{\mathbb{C}} (D_\tau \otimes_{R_\tau} S_A), \\ x_\tau^{\mu_\tau} x_{\bar{\tau}}^{\mu_{\bar{\tau}}} \partial_\tau^{\nu_\tau} \partial_{\bar{\tau}}^{\nu_{\bar{\tau}}} \otimes m &\mapsto x_{\bar{\tau}}^{\mu_{\bar{\tau}}} \otimes (x_\tau^{\mu_\tau} \partial_\tau^{\nu_\tau}) \otimes \partial_{\bar{\tau}}^{\nu_{\bar{\tau}}} m. \end{aligned}$$

one may now identify $D_A/x_{\bar{\tau}}D_A \otimes_{D_A} \mathcal{K}_\bullet(E - \beta, S_A)$ with $\mathcal{K}_\bullet(E_\tau - \beta, S_A)$ as complexes of D_τ -modules. We have thus proved

Lemma 2.5. *If β is not rank-jumping for A and $\text{rank}(A_\tau) = \text{rank}(A)$ then*

$$\mathbb{L}i_\tau^* M_A(\beta) \simeq \mathcal{K}_\bullet(E_\tau - \beta, S_A)$$

as complexes of left D_τ -modules. □

Remark 2.6. For all $\beta \in \mathbb{C}^d$, $i_\tau^* M_A(\beta) \simeq \mathcal{H}_0(E_\tau - \beta, S_A)$ because i_τ^* is right exact.

It is clear that $\mathbb{Q}_{\geq 0}A = \mathbb{Q}_{\geq 0}A_\tau$ implies that $\text{rank}(A_\tau) = \text{rank}(A) = d$ and this last condition is equivalent to $[\mathbb{Z}A : \mathbb{Z}A_\tau] < +\infty$.

Consider $S_A = \mathbb{C}[\mathbb{N}A]$ and $S_\tau = \mathbb{C}[\mathbb{N}A_\tau]$. Then the assumption $\mathbb{Q}_{\geq 0}A = \mathbb{Q}_{\geq 0}A_\tau$ guarantees that S_A is a finitely generated \mathbb{Z}^d -graded S_τ -module and so it is a toric R_τ -module (see Definition 4.5. and Example 4.7 in [MMW05]). If we don't assume $\mathbb{Q}_{\geq 0}A = \mathbb{Q}_{\geq 0}A_\tau$ but only $\text{rank}(A_\tau) = \text{rank}(A) = d$ then S_A is a weakly toric R_τ -module (see [SW09]).

Choose a collection of coset representatives $\Omega \subseteq \mathbb{N}A$ for $\mathbb{Z}A/\mathbb{Z}A_\tau$. Then, with $S_\tau(\lambda) = t^\lambda S_\tau$, there is a short exact sequence

$$(2.5) \quad 0 \rightarrow \bigoplus_{\lambda \in \Omega} S_\tau(\lambda) \hookrightarrow S_A \rightarrow Q \rightarrow 0,$$

where no shifted copy of $\mathbb{N}\tau$ is contained in $\mathbb{N}A \setminus \bigcup_{\lambda \in \Omega} (\lambda + \mathbb{N}\tau)$. In particular, $\dim(Q) < d$.

Assume from now on that condition (i) is in force. As S_A is then a finite integral extension of S_τ , S_τ -modules are toric over A precisely when they are toric over τ . We consider the long exact sequence of Euler–Koszul homology over D_τ associated with the sequence (2.5).

By [MMW05, Proposition 5.3] vanishing of $\mathcal{H}_i(E_\tau - \beta, Q)$ for all $i \geq 0$ is equivalent to $-\beta \notin \text{qdeg}(Q)$, where $\text{qdeg}(Q)$ is the Zariski closure in \mathbb{C}^d of the set of the \mathbb{Z}^d -degrees of Q . As $\dim(Q) < d$, generic β will be outside $-\text{qdeg}(Q)$. Hence for generic β we have

$$\begin{aligned} \mathcal{H}_i(E_\tau - \beta, S_A) &\simeq \mathcal{H}_i(E_\tau - \beta, \bigoplus_{\lambda \in \Omega} S_\tau(\lambda)) \\ &= \bigoplus_{\lambda \in \Omega} \mathcal{H}_i(E_\tau - \beta, S_\tau(\lambda)) \\ &\simeq \bigoplus_{\lambda \in \Omega} \mathcal{H}_i(E_\tau - \beta + \lambda, S_\tau)(\lambda) \end{aligned}$$

for all $i \geq 0$, and $\mathcal{H}_0(E_\tau - \beta + \lambda, S_\tau) = M_{A_\tau}(\beta - \lambda)$. Furthermore, for generic β , $\beta - \lambda$ is not rank-jumping for A_τ and so $\mathcal{H}_k(E_\tau - \beta + \lambda, S_\tau) = 0$ for all $k > 0$ and for all $\lambda \in \Omega$; this implies that $\mathbb{L}^{-k} i_\tau^* M_A(\beta) = 0$ for all $k > 0$.

In case (ii) is in force, the argument is similar, using weakly toric modules, which arise since S_A is not a finite S_τ -module in this case. The main vanishing tool is then [SW09, Thm. 5.4] instead of [MMW05, Prop. 5.3]. This concludes the proof of Theorem 2.2. \square

Remark 2.7. If \mathcal{M} is a holonomic \mathcal{D} -module and Y_τ is non-characteristic for \mathcal{M} then $\mathbb{L}^{-k} i_\tau^* \mathcal{M} = 0$ for all $k > 0$ and the holonomic rank of \mathcal{M} coincides with the holonomic rank of $i_\tau^* \mathcal{M}$ (see for example [MT04]). In our case, Y_τ is non-characteristic for $\mathcal{M}_A(\beta)$ if and only if τ contains all the nonzero vertices of the convex hull Δ_A of $\{0, \mathbf{a}_1, \dots, \mathbf{a}_n\} \subseteq \mathbb{R}^d$.

Let us show that formula (2.1) may fail if β is not sufficiently generic.

Example 2.8. (1) Let $A = (\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_3 \ \mathbf{a}_4)$ with $\mathbf{a}_i = \binom{1}{i}$ for $i = 0, 1, 3, 4$. In this case, by [ST98].

$$\text{rank } M_A(\beta) = \begin{cases} \text{vol}_{\mathbb{Z}A}(\Delta_A) = 4 & \text{if } \beta \in \mathbb{C}^2 \setminus \binom{1}{2}, \\ 5 & \text{if } \beta = \binom{1}{2}. \end{cases}$$

Consider $\tau = \{0, 1, 4\}$ and $Y_\tau = \{x_3 = 0\}$, so $[\mathbb{Z}A : \mathbb{Z}\tau] = 1$ and $\mathbb{Q}_+A = \mathbb{Q}_+\tau$. According to Remark 2.7 the holonomic rank of $i_\tau^* \mathcal{M}_A(\beta)$ is 5 for $\beta = \binom{1}{2}$.

On the other hand, the toric ideal I_τ associated with A_τ is principal and thus Cohen–Macaulay. It follows that the holonomic rank of $\mathcal{M}_{A_\tau}(\beta)$ is $\text{vol}_{\mathbb{Z}\tau}(\Delta_\tau) = 4$ for all $\beta \in \mathbb{C}^2$ (see [MMW05, Corollary 9.2]). This implies that for $\beta = \binom{1}{2}$, $i_\tau^* \mathcal{M}_A(\beta)$ cannot be isomorphic to $\mathcal{M}_{A_\tau}(\beta')$ for any $\beta' \in \mathbb{C}^2$.

(2) Considering $A = (\mathbf{a}_0 \ \mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4)$ with notation as above, and restricting to $x_2 = 0$, one has $\mathbb{Q}_+A = \mathbb{Q}_+\tau$ and $\mathbb{Z}A = \mathbb{Z}\tau$, but the restriction of $\mathcal{M}_A(\binom{1}{2})$ is not $\mathcal{M}_\tau(\binom{1}{2})$ since once again the ranks disagree, this time the original GKZ-system having smaller rank (the rational quartic is arithmetically Cohen–Macaulay). However, by [Sai01, Thm. 6.3], $\mathcal{M}_A(\beta) \simeq \mathcal{M}(\binom{1}{2})$ for $\beta \in \mathbb{N}A$ and restricting $\mathcal{M}(\beta)$ to τ for such β leads to $\mathcal{M}_\tau(\beta)$ unless $\beta = \binom{1}{2}$.

Remark 2.9. We do not know a pair (A, τ) for which the conclusion of our theorem fails on a set that is not a finite subspace arrangement. Indeed, while it seems clear that β not rank-jumping for both A and τ is relevant, we believe that the question whether $\beta \in -\text{qdeg}(Q)$ is a red herring. The situation appears alike to the duality question discussed in [Wal07]

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