WEIGHT FILTRATIONS ON GKZ-SYSTEMS

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ABSTRACT. Given an integer matrix $A \in \mathbb{Z}^{d\times n}$, we study the natural mixed Hodge module structure in the sense of Saito on the Gauß–Manin system attached to the monomial map $\phi\colon (\mathbb{C}^*)^d \longrightarrow \mathbb{C}^n$ induced by A. We completely determine in the normal case the corresponding weight filtration by computing the intersection complexes with respective multiplicities that constitute the associated graded parts. Our results show that these data are purely combinatorial, and not arithmetic, in the sense that they only depend on the polyhedral structure of the cone of A, but not on the semigroup itself. In particular, we extend results of de Cataldo, Migliorini and Mustata to the setting of torus embeddings and give a closed form for the failure of the Decomposition Theorem.

If A is homogeneous and if $\beta \in \mathbb{C}^d$ is an integral but not strongly resonant parameter, we make use of a monodromic Fourier–Laplace transform to carry the mixed Hodge module structure from the Gauß–Manin system to the GKZ-system attached to A and β. In case A is derived from a normal reflexive Gorenstein polytope P, Batyrev and Stienstra related certain filtrations on the generic fiber of the GKZ-system to the mixed Hodge structure on the cohomology of a generic hyperplane section inside the projective toric variety induced by P. Our formulae, phrased in terms of intersection cohomology groups on induced relative toric varieties, provide the necessary correction terms to globalize their computation. In particular, we document that on the GKZ-system the weight filtration will differ from Batyrev's filtration-by-faces whenever P is not a simplex: the intersection complexes contributing to the weight filtration measure the failure of P to be a simplex.

Irrespective of homogeneity, we obtain a purely combinatorial formula for the length of the Gauß–Manin system, and thus for the corresponding GKZ-system. In dimension up to three, and for simplicial semigroups, we give explicit generators of the weight filtration.

CONTENTS

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1. Introduction

1.1. **The Decomposition Theorem for proper maps.** One of the hallmarks of Hodge-theoretic results in algebraic geometry is the Decomposition Theorem. For smooth projective maps between smooth projective varieties this asserts among other things the degeneration of the Leray spectral sequence for $\mathbb{Q}$-coefficients on the second page. Decomposition Theorems are refinements and generalizations of the Hard Lefschetz Theorem for projective varieties; the key ingredient is the purity of the Hodge structure on cohomology. In this article we study and quantify an important instance of the failure of purity, and of the Decomposition Theorem. In order to state our results, we give the briefest of historical surveys, and we point to the excellent account [dCM09] for details.

For singular maps and varieties, things can be rescued by replacing usual cohomology with intersection cohomology, and in both instances the statement has a local flavor in the sense that one can restrict to open subsets of the target. The advantage of intersection cohomology is that it has nice formal properties such as Poincaré duality, Lefschetz theorems, and Künneth formula. While it is not a homotopy invariant, there is a natural transformation $H^i \longrightarrow \mathrm{IH}^i$ that is an isomorphism on smooth spaces and in general induces a $H^\bullet$-module structure on $\mathrm{IH}^\bullet$. This version of the Decomposition Theorem, conjectured by S. Gel'fand and R. MacPherson, was proved by A. Beilinson, J. Bernstein, P. Deligne and O. Gabber.

The construction allows for generalization of intersection cohomology to coefficients in a local system L_U, defined on a locally closed subset $U \subseteq Z = \bar{U}$. The intersection complex of such a local system is a constructible complex that extends L_U as a constructible complex (or the corresponding connection on U as D-module). In fact, the best form of the Decomposition Theorem in the projective case is in this language: if $f\colon X \longrightarrow Y$ is a proper map of complex algebraic varieties then $Rf_* \,\mathrm{IC}_X$ splits (non-canonically) as a direct sum of intersection complexes whose supporting sets are induced from a stratification of f.

A particularly interesting case where the decomposition theorem has been well-studied are semismall maps (cf. [dCM09] for a nice survey). These maps arise often in geometric situations:

- the Springer resolution $f\colon \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}$ of the nilpotent cone $\mathcal{N}$ of the Lie algebra to the reductive group G;
- the Hilbert-Chow map between the Hilbert schemes of points $X = (\mathbb{C}^2)^{[n]}$ and the n-th symmetric product $Y = (\mathbb{C}^2)^n/\mathcal{S}_n$.

The most explicit case is perhaps that of a fibration $f\colon X \longrightarrow Y$ between toric complete varieties: $H^\bullet$ and $\mathrm{IH}^\bullet$of a complete toric variety can be written down in purely combinatorial terms, and [dCMM14] spells out how to write $Rf_*(\mathrm{IC}_X)$ as sum of intersection complexes in terms of face numbers.

1.2. **Non-proper maps.** The moment one moves away from proper maps, direct images of intersection complexes need no more split into sums of such. For example, embedding $\mathbb{C}^*$ into $\mathbb{C} = \mathbb{C}^* \sqcup \{pt\}$ leads to a push-forward $Rf_*\mathcal{O}_{\mathbb{C}^*}$ that naturally contains $\mathcal{O}_{\mathbb{C}^1}$ but the cokernel $\mathcal{O}_{pt}$ is not contained in the image. At this point one requires a "weight" filtration on $Rf_*\mathcal{O}_{\mathbb{C}^*}$ akin to the one that forms part of Delignes construction of mixed Hodge structures on the cohomology of complex varieties. In the case $\mathbb{C}^* \hookrightarrow \mathbb{C}$, level 1 of the weight filtration on $Rf_*(\mathcal{O}_{\mathbb{C}^*})$ is $\mathcal{O}_{\mathbb{C}^1}$; level 2 is the entire image.

The appropriate powerful hybrid of intersection complexes and Deligne's weights was constructed by M. Saito in his theory of mixed Hodge modules, inspired by the theory of weights for ℓ-adic sheaves [Sai90]. The weight filtration, together with a "Hodge filtration" that can be seen as avatar of the usual Hodge filtration on cohomology, form the main ingredients of an object in Saito's category of mixed Hodge modules. For maps between quasi-projective varieties he introduced a natural geometric filtration on $Rf_*\,\mathrm{IC}_X$. For proper maps between algebraic varieties, the weight filtration on $Rf_*\,\mathrm{IC}_X$ is pure and in particular there is a Decomposition Theorem: $Rf_*\,\mathrm{IC}_X$ splits into intersection complexes and the splitting occurs in the category of mixed Hodge modules.

In several natural situations properness is not available, and this necessitates nontrivial weights. Saito's theory shows that in general the associated graded pieces of the weight filtration of any push-forward of a mixed Hodge module split as sums of intersection complexes, while for maps to a point the construction agrees with Deligne's weights.

One is naturally led to a very hard question, crucial to Saito's theory, on the behavior of pure Hodge modules under open embeddings. In the world of toric varieties, once one gives up on complete fans, the most fundamental situation is the inclusion of an embedded torus into its (likely singular) closure:

Problem (Weight Decomposition for Open Tori)**.** *Let $T = (\mathbb{C}^*)^d$ and consider the monomial map*

$$(1.2.1)\qquad \begin{array}{rcl} h\colon T & \longrightarrow & \mathbb{C}^n =: \widehat{V} \\ (\mathfrak{t}_1, \dots, \mathfrak{t}_d) =: \mathfrak{t} & \mapsto & \mathfrak{t}^A := (\mathfrak{t}^{\mathbf{a}_1}, \dots, \mathfrak{t}^{\mathbf{a}_n}) \end{array}$$

where

$$A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in (\mathbb{Z}^d)^n$$

is an integer $d \times n$ matrix. Determine the weight filtration $\{W_i\}_i$ on $h_+(\mathcal{O}_T)$ and for each associated graded quotient W_i/W_{i-1} indicate the intersection complexes (support and coefficients) that appear as direct summands of this module.

To our knowledge, the only other time where non-proper maps have been studied is the article [CDK16] on certain open subsets of products of Grassmannians.

1.3. **Results and techniques.** Throughout, A is an integer $d \times n$ matrix satisfying the three conditions of Notation 3.1 and we consider the induced monomial action of T on $\widehat{V} = \mathbb{C}^n$ given by

$$(1.3.1)\qquad \mu\colon T \times \widehat{V} \longrightarrow \widehat{V}, \qquad (\mathfrak{t}, \mathfrak{y}) \mapsto \mathfrak{t}^A \cdot \mathfrak{y}.$$

With $\sigma = \mathbb{R}_{\geq 0} A$, denote X_σ (or just X) the closure of the orbit through $\mathbf{1} = (1, \dots, 1) \in \mathbb{C}^n$. There is an orbit decomposition

$$X = \bigsqcup_\tau O_\tau$$

where the union is over the faces τ of σ and $O_\tau = \mu(T, \mathbf{1}_\tau)$ is the orbit corresponding to τ (here, $\mathbf{1}_\tau \in \mathbb{C}^n$ is defined by $(\mathbf{1}_\tau)_j = 1$ if $\mathbf{a}_j \in \tau$ and zero otherwise). Denote $\mathbb{Q}_\tau$ the constant sheaf on the orbit to τ and write ${}^p\mathbb{Q}_\tau^H$ for the corresponding (simple, pure) Hodge module. With $X = X_\sigma = \operatorname{Spec}(\mathbb{C}[\mathbb{N}A])$ from Section 3, h factors as

$$(1.3.2)\qquad T \xrightarrow{\ \varphi\ } X \xrightarrow{\ i\ } \mathbb{C}^n = \widehat{V}, \qquad h = i \circ \varphi$$

Via Kashiwara equivalence along i, one identifies the mixed Hodge modules on X with those on $\widehat{V}$ supported on X. The following lists, in brief, our results in Section 3.

(1) Since $\mathcal{O}_T$ is a (strongly) torus equivariant $\mathcal{D}_T$-module, the T-equivariant map h will produce (strongly) equivariant modules $R^i h_+(\mathcal{O}_T)$ (and only the 0-th one is nonzero since h is affine). The intersection complexes appearing in the decomposition of the weight graded parts of $Rh_*({}^p\mathbb{Q}_\sigma^H)$ must be equivariant, supported on orbits. The underlying local systems are constant.

(2) We identify two functors on equivariant sheaves with contracting torus actions. Using this identification, we then provide a recursive recipe for the exceptional pullback $\mathcal{H}^k i_\tau^!(h_* {}^p\mathbb{Q}_\sigma^H / W_i h_* {}^p\mathbb{Q}_\sigma^H)$ to an arbitrary orbit of the monomial action from (1.3.1).

(3) We unravel the recursion for $\mathcal{H}^0 i_\tau^! h_* {}^p\mathbb{Q}_\sigma^H$, for every τ, to provide an explicit expression for the multiplicity $\mu_\tau^\sigma(e)$ of the constant local system ${}^p\mathbb{Q}_\tau^H$ in the $(d+e)$-th graded weight part of $h_* {}^p\mathbb{Q}_\sigma^H$ in terms of an alternating sum whose constituents are indexed by flags in the face lattice of σ, see Proposition 3.10. The terms involve intersection cohomology dimensions of the affine toric varieties $X_{\tau/\gamma}$ associated to the semigroup of the cone $\tau/\gamma = (\tau + \mathbb{R}\gamma)/\mathbb{R}\gamma$.

(4) Using some results on intersection cohomology of toric varieties by Stanley, and Braden and MacPherson, we express $\mu_\tau^\sigma(e)$ as a single intersection cohomology rank on the dual affine toric variety $Y_{\sigma/\tau}$, associated to the dual of σ/τ, see Theorem 3.17.

There are several noteworthy consequences. First of all, $\mu_\tau^\sigma(e)$ is a relative quantity in the sense that $\mu_\tau^\sigma(e) = \mu_{\tau/\gamma}^{\sigma/\gamma}(e)$ for any face γ inside τ. Secondly, the arithmetic properties of σ are inessential: the only information relevant for $\mu_\tau^\sigma(e)$ is the combinatorics of the polytope obtained from σ/τ by slicing it with a transversal hyperplane that "cuts off the vertex". This is because the intersection cohomology numbers of $Y_{\sigma/\tau}$ are entirely combinatorial.

1.4. **Consequences, applications, open problems.**

1.4.1. *Hodge structures on GKZ-systems.* Some interesting consequences of (1)-(4) above come from applying these results to the Fourier–Laplace transform of $h_+\mathcal{O}_T$, a well-studied D-module all by itself.

We briefly recall the notion of an A-hypergeometric system in our setup. Let $R_A = \mathbb{C}[\partial_1, \ldots, \partial_n]$ be the polynomial ring, set $D_A = R_A\langle x_1, \ldots, x_n\rangle$ the Weyl algebra and pick $\beta \in \mathbb{C}^d$. Now consider the left ideal $H_A(\beta)$ of D_A generated by

$$I_A = R_A(\{\partial^{\mathbf{u}} - \partial^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^n,\ A \cdot \mathbf{u} = A \cdot \mathbf{v}\})$$

and

$$E_i - \beta_i := \sum_{j=1}^n a_{i,j} x_j \partial_j - \beta_i \qquad i = 1 \ldots, d.$$

The module

$$M_A^\beta := D_A / H_A(\beta)$$

is the A-hypergeometric system induced by A and β. These systems were introduced by Gel'fand, Graev, Kapranov and Zelevinsky in the 1980's; we refer to [SST00] and the current literature for more information on these modules, but highlight some properties.

The *strongly resonant quasi-degrees* $\operatorname{sRes}(A)$ of A form an infinite discrete hyperplane arrangement in $\mathbb{C}^d$ which was introduced in [SW09] and used to sharpen a result of Gel'fand et al. by showing that $\beta \notin \operatorname{sRes}(A)$ is equivalent to M_A^β being the Fourier–Laplace transform of $h_+(\mathcal{O}_T^\beta)$ where $\mathcal{O}_T^\beta$ is described before Theorem 4.4. In fact, it was Gel'fand and his collaborators that first observed a connection between A-hypergeometric systems and intersection complexes in [GKZ90, Prop. 3.2].

If the semigroup ring $S_A := \mathbb{C}[\mathbb{N}A] \simeq R_A/I_A$ is normal (or, equivalently, if the semigroup $\mathbb{N}A$ is saturated in $\mathbb{Z}A$) then 0 is not strongly resonant. In particular then, the inverse Fourier–Laplace transform of M_A^0 is the module $h_+(\mathcal{O}_T)$ from Section 3. Since the Fourier–Laplace transform is an equivalence of categories, our results on $h_+(\mathcal{O}_T)$ solve for normal S_A the long-standing problem of determining the composition factors for M_A^0.

The Fourier–Laplace transform does not necessarily preserve mixed Hodge module structures in general. However, if one assumes that I_A defines a projective variety, one can use the monodromic Fourier–Laplace transform which produces the same output as the Fourier–Laplace transform on $h_+(\mathcal{O}_T)$ and does carry mixed Hodge module structures. In particular, this equips M_A^0 with a natural mixed Hodge module structure inherited from h (cf. [Rei14]).

There is a filtration-by-faces on a GKZ system, defined via the face filtration on the semigroup ring: the $(d+k)$-th level of this fitration is the submodule of M_A^0 generated by all monomials $\partial^{\mathbf{u}} \in S_A$ for which $A \cdot \mathbf{u}$ is not contained in a face of dimension $d-k-1$. This filtration was introduced by Batyrev in his study of the Hodge structure on the cohomology of a generic hypersurface in a toric variety constructed from a polytope [Bat93, Sti98]. Adolphson and Sperber, and more recently Fang, also considered the face filtration in [AS, Fan18]. We show that this filtration is bounded above by the weight filtration, and that it really differs from it for all GKZ-systems whose semigroup cone is not the cone over a simplex. On can view the error terms that we find as the necessary "glue" that is required to globalize the result of Batyrev and Stienstra from the generic fiber to the entire GKZ-system. On the other hand, looking at $h_+(\mathcal{O}_T)$, we show that the corresponding filtration-by-faces always captures the part of the weight filtration that has maximal dimensional support.

1.4.2. *Applications.* We outline two possible applications of our results; one is concerned with mirror symmetry, the other comes from commutative algebra.

Local cohomology at toric varieties: Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and suppose I is an ideal of R such that R/I is the semigroup ring $\mathbb{C}[\mathbb{N}A]$ for some matrix A as above. Let J denote the variety comprised of the smaller torus orbits of the variety of I. Then there is a natural triangle

$$\longrightarrow \mathbb{R}\Gamma_J(R) \longrightarrow \mathbb{R}\Gamma_I(R) \longrightarrow \hat{M}_A^0[-c] \xrightarrow{+1}$$

in the category of mixed Hodge modules where the first morphism is the canonical one and $\hat{M}_A^0$ is the Fourier–Lalace transform of M_A^0 (*i.e.*, $h_+(\mathcal{O}_T)$ up to shift). In the normal case this degenerates and an inductive procedure can be used to determine from our formulæ for $\hat{M}_A^0$ the intersection complexes in the weight filtration of $H_I^\bullet(R)$. In particular, their vanishing (which at present is an open problem) can be determined. Making this explicit is the topic of a forthcoming work.

Mirror symmetry: Let Y_Σ be a toric variety induced by the fan Σ. The secondary fan of Y_Σ induces a toric variety M and a family of Laurent polynomials over a Zariski open subset of M. This family is known as the Landau–Ginzburg model of Σ and encodes the Gromov–Witten invariants of Y_Σ. It turns out that the information relevant to Gromov–Witten invariants is contained in the smallest weight part of the Gauß–Manin system, compare [Giv96, Giv98, Iri09, RS17, RS15] . It is conjectured that the parts of higher weight describe mirror symmetry for toric degenerations such as flag manifolds [IX16]. Our results here give concrete data on the GKZ side which one should want to match to those toric degenerations.

1.4.3. *Open problems.* When S_A is normal, the holonomic rank of M_A^β (the dimension of the holomorphic solution space in a generic point) equals the volume of the convex hull of the columns of A together with the origin. In particular, this is an arithmetic quantity. In contrast, our results show that the holonomic length is purely combinatorial in that case; it only depends on the **cd**-index (see [BK91]) of the polytope over which σ is the cone. This suggests a new question that deserves study: what is the rank, and more generally the characteristic cycle, of the Fourier–Laplace transformed intersection complex $\mathrm{IC}(\mathcal{L}_\tau)$? Our results allow for small d direct calculation of the rank of $\mathrm{FL}^{-1}(\mathrm{IC}(\mathcal{L}_\tau))$ to any chosen face. In higher dimension one can write down recursions, but making them explicit is an open question. Furter, having a saturated composition chain for a D-module informs on the irreducible representations in the monodromy of the solution sheaf. Studying $\mathrm{FL}^{-1}(\mathrm{IC}(\mathcal{L}_\tau))$ would be the first step towards a general understanding of the monodromy of M_A^0.

Finally, one should investigate whether one can place mixed Hodge module structures also on M_A^β for other β. Obviously, this is doable in the normal case with $\beta \in \mathbb{N}A$ since [SW09] implies that the corresponding M_A^β are isomorphic to M_A^0. Similarly, dual ideas reveal that for β integral and in the cone roughly opposite to $\mathbb{N}A$, M_A^β agrees with the Fourier–Laplace transform of $h_!(\mathcal{O}_T^\beta)$ and hence also inherits a mixed Hodge module structure, dual to the one discussed here. For other integral β, [Ste17, Ste18] describes $\mathrm{FL}^{-1}(M_A^\beta)$ as a composition of a direct and exceptional direct image, which can be used to export a MHM structure. Less clear are non-integral β: the use of complex Hodge modules allows to equip M_A^β with $\beta \in \mathbb{R}^d$ with a MHM structure, see Sabbah's MHM project [SS]. For certain β the Hodge filtration on M_A^β is explicitly computed in [RS15].

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2. Functors on $\mathcal{D}$-modules

If K is a free Abelian group of finite rank, or a finite dimensional vector space, then we write K^* for the dual group or vector space.

We introduce the following notation. Let X be a smooth complex algebraic variety of dimension d_X. The Abelian category of algebraic left $\mathcal{D}_X$-modules on X is denoted by $\mathrm{M}(\mathcal{D}_X)$ and the Abelian subcategory of (regular) holonomic $\mathcal{D}_X$-modules by $\mathrm{M}_h(\mathcal{D}_X)$ (resp. $(\mathrm{M}_{rh}(\mathcal{D}_X))$. We abbreviate $\mathrm{D}^b(\mathrm{M}(\mathcal{D}_X))$ to $\mathrm{D}^b(\mathcal{D}_X)$, and denote by $\mathrm{D}^b_h(\mathcal{D}_X)$ (resp. $\mathrm{D}^b_{rh}(\mathcal{D}_X)$)

the full triangulated subcategory in $\mathrm{D}^b(\mathcal{D}_X)$ consisting of objects with holonomic (resp. regular holonomic) cohomology.

Let $f : X \to Y$ be a morphism between smooth algebraic varieties and let $M \in \mathrm{D}^b(\mathcal{D}_X)$ and $N \in \mathrm{D}^b(\mathcal{D}_Y)$. The direct and inverse image functors for $\mathcal{D}$-modules are denoted by

$$f_+ M := \mathrm{R}f_*(\mathcal{D}_{Y \leftarrow X} \overset{L}{\otimes} M) \quad \text{and} \quad f^+ M := \mathcal{D}_{X \to Y} \overset{L}{\otimes} f^{-1} M[d_X - d_Y]$$

respectively. The functors f_+, f^+ preserve (regular) holonomicity (see e.g., [HTT08, Theorem 3.2.3]).

We denote by

$$\mathbb{D} : \mathrm{D}^b_h(\mathcal{D}_X) \to (\mathrm{D}^b_h(\mathcal{D}_X))^{opp}$$

the holonomic duality functor. Recall that for a single holonomic $\mathcal{D}_X$-module M, the holonomic dual is also a single holonomic $\mathcal{D}_X$-module ([HTT08, Proposition 3.2.1]) and that holonomic duality preserves regularity ([HTT08, Theorem 6.1.10]).

For a morphism $f : X \to Y$ between smooth algebraic varieties we additionally define the functors

$$f_\dagger := \mathbb{D} \circ f_+ \circ \mathbb{D} \quad \text{and} \quad f^\dagger := \mathbb{D} \circ f^+ \circ \mathbb{D}.$$

Let X be an algebraic variety. Denote by $\mathrm{MHM}(X)$ the Abelian category of algebraic mixed Hodge modules and by $\mathrm{D}^b \mathrm{MHM}(X)$ the corresponding bounded derived category. If X is smooth the forgetful functor to the bounded derived category of regular holonomic $\mathcal{D}_X$-modules is denoted by

$$\mathrm{Dmod} : \mathrm{D}^b \mathrm{MHM}(X) \longrightarrow \mathrm{D}^b_{rh}(\mathcal{D}_X)\,. \tag{2.0.1}$$

For each morphism $f : X \to Y$ between complex algebraic varieties, there are induced functors

$$f_*,\ f_! \colon \mathrm{D}^b \mathrm{MHM}(X) \longrightarrow \mathrm{D}^b \mathrm{MHM}(Y)$$

and

$$f^*,\ f^! \colon \mathrm{D}^b \mathrm{MHM}(Y) \to \mathrm{D}^b \mathrm{MHM}(X)\,,$$

which satisfy $\mathbb{D} \circ f_* = f_! \circ \mathbb{D}$, $\mathbb{D} \circ f^* = f^! \circ \mathbb{D}$, and which lift the analogous functors $f_+, f_\dagger, f^\dagger, f^+$ on $\mathrm{D}^b_{rh}(\mathcal{D}_X)$ in case X is smooth.

Let $\mathbb{Q}^H_{pt}$ be the trivial Hodge structure $\mathbb{Q}$ of type $(0,0)$, i.e. $\mathrm{gr}^W_i \mathbb{Q}^H_{pt} = \mathrm{gr}^F_i \mathbb{Q}^H_{pt} = 0$ for $i \neq 0$. Viewing it as a Hodge module on a point pt, denote by ${}^p\mathbb{Q}^H_X := \mathbb{Q}^H_X[d_X]$ the (mixed) Hodge module $(a^*\mathbb{Q}^H_{pt})[d_X]$, where

$$a : X \to pt$$

is the unique map to a point. For smooth X the $\mathcal{D}$-module underlying ${}^p\mathbb{Q}^H_X$ is the structure sheaf $\mathcal{O}_X$ in cohomological degree zero with $\mathrm{gr}^W_i \mathcal{O}_X = 0$ for $i \neq d_X$.

Let $j : U \to X$ be any Zariski dense smooth open subset of X and let $\mathcal{L}$ be a polarizable variation of Hodge structures (*i.e.*, a vector bundle with a flat connection ∇ such that each fiber carries a Hodge structure, with $\nabla(F_p) \subseteq F_{p+1}$ for increasing filtrations, and a global polarization pairing) of weight w. Set ${}^p\mathcal{L} := \mathcal{L} \otimes {}^p\mathbb{Q}^H_U$. We denote by $\mathrm{IC}_X({}^p\mathcal{L})$ the intersection cohomology sheaf with coefficients in ${}^p\mathcal{L}$; this is a pure Hodge module of weight $w+d_X$ equal to $\operatorname{im}(\mathcal{H}^0 j_! {}^p\mathcal{L} \to \mathcal{H}^0 j_* {}^p\mathcal{L})$. We write IC_X for $\mathrm{IC}_X({}^p\mathbb{Q}^H_U)$; this does not depend on U, see [Dim04, Thm. 5.4.1, p. 156].

Lemma 2.1. *Let $(X,\mathcal{S})$ be an algebraic Whitney stratification of X with a Zariski dense smooth open stratum U. Denote by $i_S : S \to X$ the embedding of the stratum $S \in \mathcal{S}$ in X and let ${}^p\mathcal{L}$ be as above. The following holds for morphisms in* MHM*:*

(1) $i_S^!$ is left exact for every $S \in \mathcal{S}$ and does not decrease weights. (That is, if $W_{\le k}(\mathcal{M}) = 0$ then $W_{\le k} i_S^!(\mathcal{M}) = 0$).
(2) $\mathcal{H}^0 i_U^! \,\mathrm{IC}_X({}^p\mathcal{L}) = {}^p\mathcal{L}$ and $\mathcal{H}^k i_U^! \,\mathrm{IC}_X({}^p\mathcal{L}) = 0$ for $k \neq 0$.
(3) $\mathcal{H}^0 i_S^! \,\mathrm{IC}_X({}^p\mathcal{L}) = 0$ for $U \neq S$.

Proof. The first statement follows from [KS94, Proposition 10.2.11] and [Sai90, (4.5.2)]. The second statement follows from the fact that $i_U^! = i_U^*$ is just the restriction to the open subset U which is exact. The last point follows from the characterization of $\mathrm{IC}_X({}^p\mathcal{L})$ as $\operatorname{im}(\mathcal{H}^0 j_! {}^p\mathcal{L} \to \mathcal{H}^0 j_* {}^p\mathcal{L})$ and [BBD82, 1.4.22 and 1.4.24]. □

3. Weight filtration on torus embeddings

3.1. **Basic Notions.**

Notation 3.1. If C is a semiring (an additive semigroup closed under multiplication) write CA for the C-linear combinations of the columns of the integer $d \times n$ matrix A. We assume that A satisfies:

(1) $\mathbb{Z}A = \mathbb{Z}^d$;
(2) A is *saturated*: $\mathbb{R}_{\ge 0}A \cap \mathbb{Z}^d = \mathbb{N}A$;
(3) A is *pointed*: $\mathbb{N}A \cap (-\mathbb{N}A) = \{0_{\mathbb{Z}A}\}$.

We let $\sigma := \mathbb{R}_{\ge 0}A$ be the real cone over A inside $\mathbb{R}^d$ and consider the d-dimensional affine toric variety $X := X_\sigma := \operatorname{Spec}(\mathbb{C}[\mathbb{N}A]) \subseteq \widehat{V}$ together with its open dense torus $T := T_\sigma := \operatorname{Spec}(\mathbb{C}[\mathbb{Z}A])$. Properties (1)-(3) of A above imply that X is d-dimensional, normal by Hochster's theorem, and has one T-fixed point.

Let $\tau \subseteq \sigma$ be a d_τ-dimensional face of σ. We denote by

$$\tau_{\mathbb{Z}} := (\tau+(-\tau))\cap\mathbb{Z}^d\,, \qquad \tau_{\mathbb{N}} = \tau\cap\mathbb{Z}^d \qquad \text{and} \qquad \tau_{\mathbb{R}} = \tau+(-\tau) = \operatorname{span}(\tau),$$

which are the $\mathbb{Z}$-, $\mathbb{N}$- and $\mathbb{R}$-spans of the collection of A-columns in τ (considering that $\mathbb{N}A$ is saturated). We associate to τ a d_τ-dimensional torus orbit

$$T_\tau = \operatorname{Spec}(\mathbb{C}[\tau_{\mathbb{Z}}])$$

whose closure in X_σ via the embedding induced by $\mathbb{N}A \twoheadrightarrow \mathbb{N}F \longrightarrow \mathbb{Z}F$ is

$$X_\tau = \operatorname{Spec}(\mathbb{C}[\tau_{\mathbb{N}}]).$$

Saturatedness of $\mathbb{N}A$ implies that X_τ is normal. The variety

$$U_\tau := \operatorname{Spec}(\mathbb{C}[\sigma_{\mathbb{N}} + \tau_{\mathbb{Z}}])$$

gives an open neighborhood of T_τ in X. The affine toric variety

$$X_{\sigma/\tau} := \operatorname{Spec}(\mathbb{C}[(\sigma_{\mathbb{N}} + \tau_{\mathbb{Z}})/\tau_{\mathbb{Z}}])$$

with its dense torus

$$T_{\sigma/\tau} := \operatorname{Spec}\left(\mathbb{C}[\sigma_{\mathbb{Z}}/\tau_{\mathbb{Z}}]\right)$$

is a normal slice to the stratum T_τ: there is an isomorphism $U_\tau \simeq X_{\sigma/\tau} \times T_\tau$ and a (non-canonical) isomorphism

$$j_\tau \colon X_{\sigma/\tau} \times T_\tau \to X.$$

The inclusions $T_{\sigma/\tau} \hookrightarrow X_{\sigma/\tau} \hookrightarrow X$ correspond to the (canonical) morphisms $\sigma_{\mathbb{N}} \twoheadrightarrow (\sigma_{\mathbb{N}} + \tau_{\mathbb{Z}})/\tau_{\mathbb{Z}} \longrightarrow \sigma_{\mathbb{Z}}/\tau_{\mathbb{Z}}$. For any pointed rational polyhedral cone ρ in (a quotient of) $\mathbb{R}^d$ we denote by

$$i_\rho \colon \{\mathfrak{x}_\rho\} \hookrightarrow X_\rho$$

the embedding of the unique torus-fixed point. Then we have the following diagram

$$\begin{array}{ccccc}
\{\mathfrak{x}_{\sigma/\tau}\} \times T_\tau & \xrightarrow{i_{\sigma/\tau} \times id} & X_{\sigma/\tau} \times T_\tau & \xleftarrow{\varphi_{\sigma/\tau} \times id} & T_{\sigma/\tau} \times T_\tau \simeq T \\
& \searrow^{\rho_\tau} & \downarrow{\scriptstyle j_\tau} & \swarrow^{\varphi} & \\
& & X & &
\end{array}$$

of equivariant maps. ◇

Definition 3.2. Let $\mu : \mathbb{G}_m \times Y \to Y$ be a $\mathbb{G}_m$-action on the variety Y. Write $\operatorname{pr}\colon \mathbb{G}_m \times Y \longrightarrow Y$ for the projection. A holonomic $\mathcal{D}_Y$-module $\mathcal{M}$ is called $\mathbb{G}_m$-*equivariant* if $\mu^+\mathcal{M} \simeq \operatorname{pr}^+ \mathcal{M}$ as $\mathcal{D}_{\mathbb{G}_m \times Y}$-module. ◇

If $\mathbf{v} \in \mathbb{Z}A$ is in the interior $\operatorname{Int}(\sigma^\vee)$ of the dual cone $\sigma^\vee$, then $\mathbf{v}$ defines a 1-parameter subgroup $\kappa_{\mathbf{v}} \colon \mathbb{G}_m = \operatorname{Spec} \mathbb{C}[z^\pm] \to T = \operatorname{Spec} \mathbb{C}[\mathbb{Z}A]$ given by $t^{\mathbf{u}} \mapsto z^{\langle \mathbf{u}, \mathbf{v}\rangle}$ which extends to a map $\overline{\kappa}_{\mathbf{v}} \colon \mathbb{A}^1 \to X_\sigma$ with limit point $\mathfrak{x}_\sigma \in X_\sigma$. By adjusting the ambient lattice, similar statements hold for all faces τ.

Lemma 3.3. *Let $i_\tau : \{\mathfrak{x}_\tau\} \to X_\tau$ be the inclusion of the torus fixed point and for any space X denote*

$$a_X : X \longrightarrow pt$$

the projection to a point. If $X = X_\tau$ is one of our orbit closures, identify a_{X_τ} with $a_\tau : X_\tau \to \{\mathfrak{x}_\tau\}$.

Let $\mathbf{v} \in \tau^\vee$ be an integer element in the relative interior of the dual cone and consider the induced action of $\mathbb{G}_m$ on X_τ. For every $\mathbb{G}_m$-equivariant Hodge module $\mathcal{M}$ on X_τ we have the following isomorphisms

$$a_{\tau *}\mathcal{M} \simeq i_\tau^* \mathcal{M} \qquad \text{and} \qquad a_{\tau !}\mathcal{M} \simeq i_\tau^! \mathcal{M}$$

Proof. It suffices to consider the case $\tau = \sigma$. Denote by $u : X_\sigma \smallsetminus \{\mathfrak{x}_\sigma\} \to X_\sigma$ the open embedding of the complement of the fixed point $\mathfrak{x}_\sigma$, abbreviate i_σ to i and denote by a the map to a point. We have the exact triangle

$$u_! u^{-1}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow i_* i^* \mathcal{M} \xrightarrow{+1}$$

Applying a_* we get

$$a_* u_! u^{-1}\mathcal{M} \longrightarrow a_*\mathcal{M} \longrightarrow i^* \mathcal{M} \xrightarrow{+1}$$

and we will show $a_* u_! u^{-1}\mathcal{M} = 0$. As $\mathbf{v} \in \mathrm{Int}(\sigma^\vee)$, we have an action $\overline{\kappa}_{\mathbf{v}} \colon \mathbb{A}^1 \times X_\sigma \to X_\sigma$ with $\overline{\kappa}_{\mathbf{v}}^{-1}(\mathfrak{x}_\sigma) = (\mathbb{A}^1 \times \{\mathfrak{x}_\sigma\}) \cup (\{0\} \times X_\sigma)$. This gives the following Cartesian diagram

$$\begin{array}{ccc} \mathbb{G}_m \times (X_\sigma \smallsetminus \{\mathfrak{x}_\sigma\}) & \xrightarrow{\quad u' \quad} & \mathbb{A}^1 \times X_\sigma \\ \Big\downarrow{\scriptstyle \overline{\kappa}'_{\mathbf{v}}} & & \Big\downarrow{\scriptstyle \overline{\kappa}_{\mathbf{v}}} \\ X_\sigma \smallsetminus \{\mathfrak{x}_\sigma\} & \xrightarrow{\quad u \quad} & X_\sigma \end{array}$$

where $\overline{\kappa}'_{\mathbf{v}}$ is the restriction and u' is the canonical inclusion. Consider the morphism $g : X_\sigma \to \mathbb{A}^1 \times X_\sigma$ with $g(x) = (1, x)$. The morphism g is a section of $\overline{\kappa}_{\mathbf{v}}$, hence $\overline{\kappa}_{\mathbf{v}} \circ g = id_{X_\sigma}$. Therefore the composition

$$a_* \to a_* \overline{\kappa}_{\mathbf{v}*} \overline{\kappa}_{\mathbf{v}}^* = (a_{\mathbb{A}^1 \times X_\sigma})_* \overline{\kappa}_{\mathbf{v}}^* \to (a_{\mathbb{A}^1 \times X_\sigma})_* g_* g^* \overline{\kappa}_{\mathbf{v}}^* = a_* g^* \overline{\kappa}_{\mathbf{v}}^*$$

is the identity transformation. In order to show that $a_* u_! u^{-1}\mathcal{M} = 0$ it is hence enough to prove that the intermediate module $(a_{\mathbb{A}^1 \times X_\sigma})_* \overline{\kappa}_{\mathbf{v}}^* u_! u^{-1}\mathcal{M}$ vanishes.

By base change we get the following isomorphism:

$$(a_{\mathbb{A}^1 \times X_\sigma})_* \overline{\kappa}_{\mathbf{v}}^* u_! u^{-1}\mathcal{M} \simeq (a_{\mathbb{A}^1 \times X_\sigma})_* u'_! (\overline{\kappa}'_{\mathbf{v}})^* u^{-1}\mathcal{M}.$$

Since $u^{-1}\mathcal{M}$ is $\mathbb{G}_m$-equivariant, we have

$$(\overline{\kappa}'_{\mathbf{v}})^* u^{-1}\mathcal{M} \simeq pr^* u^{-1}\mathcal{M} \simeq \mathbb{Q}^H_{\mathbb{G}_m} \boxtimes u^{-1}\mathcal{M}.$$

Therefore we get

$$(a_{\mathbb{A}^1 \times X_\sigma})_* u'_! (\overline{\kappa}'_{\mathbf{v}})^* u^{-1}\mathcal{M} \simeq (a_{\mathbb{A}^1 \times X_\sigma})_* u'_! (\mathbb{Q}^H_{\mathbb{G}_m} \boxtimes u^{-1}\mathcal{M}) \simeq (a_{\mathbb{A}^1 \times X_\sigma})_* (u_{1!} \mathbb{Q}^H_{\mathbb{G}_m} \boxtimes u_! u^{-1}\mathcal{M}),$$

where $u_1 : \mathbb{G}_m \to \mathbb{A}^1$ is the canonical inclusion. Since $H^\bullet(\mathbb{A}^1, u_{1!}\mathbb{Q}^H_{\mathbb{G}_m}) = 0$, the Künneth formula shows that $(a_{\mathbb{A}^1 \times X_\sigma})_* \overline{\kappa}_{\mathbf{v}}^* u_! u^{-1}\mathcal{M} = 0$. This shows the first claim. The second claim follows by dualizing; note that duals of equivariant modules are equivariant. □

Recall that $\mathrm{IH}(-)$ (and IH_c) denotes intersection cohomology (with compact support).

Lemma 3.4. *Let γ be a face of σ, X_γ the associated d_γ-dimensional affine toric variety. The following holds*

(1) $\mathrm{IH}_c^{d_\gamma + k}(X_\gamma) \simeq \left(\mathrm{IH}^{d_\gamma - k}(X_\gamma)(d_\gamma)\right)^*$.

(2) $\mathrm{IH}^{d_\gamma + k}(X_\gamma) = \mathrm{IH}_c^{d_\gamma - k}(X_\gamma) = 0$ *for* $k \geq 0$.

(3) $\mathrm{IH}^k(X_\gamma) = 0$ *for k odd.*

(4) $\mathrm{IH}^{2k}(X_\gamma)$ *and* $\mathrm{IH}_c^{2k}(X_\gamma)$ *are pure Hodge structures of Hodge–Tate type with weight $2k$,* i.e.

$$gr_i^W \mathrm{IH}^{2k}(X_\gamma) = 0 \quad \textit{and} \quad gr_j^F \mathrm{IH}^{2k}(X_\gamma) = 0 \textit{ for } i \neq 2k \textit{ and } j \neq -k.$$

Proof. Temporarily, write a for a_γ and i for i_γ. Claim (1) follows from Verdier duality:

$$\begin{aligned}
\mathrm{IH}_c^{d_\gamma+k}(X_\gamma) &\simeq H^k a_! \,\mathrm{IC}_{X_\gamma}({}^p\mathbb{Q}^H_{T_\gamma}) \\
&\simeq H^k \mathbb{D} a_* \mathbb{D}\,\mathrm{IC}_{X_\gamma}({}^p\mathbb{Q}^H_{T_\gamma}) \\
&\simeq H^k \mathbb{D} a_* (\mathrm{IC}_{X_\gamma}({}^p\mathbb{Q}^H_{T_\gamma})(d_\gamma)) \\
&\simeq \Big(H^{-k} a_* (\mathrm{IC}_{X_\gamma}({}^p\mathbb{Q}^H_{T_\gamma})(d_\gamma))\Big)^* \\
&\simeq \Big(\mathrm{IH}^{d-k}(X_\gamma)(d_\gamma)\Big)^*.
\end{aligned}$$

From Lemma 3.3 we have the isomorphisms

$$\mathrm{IH}^k(X_\gamma) = H^k a_* \,\mathrm{IC}_{X_\gamma}(\mathbb{Q}^H_{T_\gamma}) \simeq H^k i^* \,\mathrm{IC}_{X_\gamma}(\mathbb{Q}^H_{T_\gamma}) = H^k \,\mathrm{IC}_{X_\gamma}(\mathbb{Q}^H_{T_\gamma})_{\mathfrak{x}_\gamma}.$$

Claim (2) follows from [Fie91, Theorem 1.2], which also implies—in conjunction with Remark ii.) in loc. cit.—Claim (3). Claim (4) follows from [Web04, Corollary 4.12]. $\square$

Let now τ, γ be faces of σ with $\tau \subseteq \gamma$ and set

$$X_{\gamma/\tau} := \mathrm{Spec}\,(\mathbb{C}[(\gamma_{\mathbb{N}} + \tau_{\mathbb{Z}})/\tau_{\mathbb{Z}}]).$$

The following result discusses (derived) pullbacks of constant variations of Hodge structures to torus orbits.

Lemma 3.5. *Let $i_{\tau,\gamma} : T_\tau \longrightarrow X_\tau \longrightarrow X_\gamma$ be the torus orbit embedding and let H be a polarizable Hodge structure of weight w (on a point). Then ${}^p\mathcal{L} := H \otimes {}^p\mathbb{Q}^H_{T_\gamma}$ is a (constant) variation of polarizable Hodge structures of weight $w + d_\gamma$ on T_γ. We have the following isomorphisms in* $\mathrm{MHM}(T_\tau)$*:*

$$\mathcal{H}^k(i^!_{\tau,\gamma}\; \mathrm{IC}_{X_\gamma}({}^p\mathcal{L})) \simeq H \otimes \mathrm{IH}_c^{d_\gamma - d_\tau + k}(X_{\gamma/\tau}) \otimes {}^p\mathbb{Q}^H_{T_\tau}, \tag{3.1.1}$$

$$\mathcal{H}^k(i^*_{\tau,\gamma}\; \mathrm{IC}_{X_\gamma}({}^p\mathcal{L})) \simeq H \otimes \mathrm{IH}^{d_\gamma - d_\tau + k}(X_{\gamma/\tau}) \otimes {}^p\mathbb{Q}^H_{T_\tau}. \tag{3.1.2}$$

The weight filtration is given by:

$$\mathrm{gr}^W_j\, \mathcal{H}^k(i^!_{\tau,\gamma}\, \mathrm{IC}_{X_\gamma}({}^p\mathcal{L})) = \mathrm{gr}^W_j\, \mathcal{H}^k(i^*_{\tau,\gamma}\, \mathrm{IC}_{X_\gamma}({}^p\mathcal{L})) = 0$$

for $j \neq w + d_\gamma + k$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
\{\mathfrak{x}_{\gamma/\tau}\} & \xrightarrow{\;i_{\gamma/\tau}\;} & X_{\gamma/\tau} & \xleftarrow{\;\varphi_{\gamma/\tau}\;} & T_{\gamma/\tau} \\
\uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle p_2} & & \uparrow{\scriptstyle p_3} \\
\{\mathfrak{x}_{\gamma/\tau}\} \times T_\tau & \xrightarrow{\;i_{\gamma/\tau}\times id\;} & X_{\gamma/\tau} \times T_\tau & \xleftarrow{\;\varphi_{\gamma/\tau}\times id\;} & T_{\gamma/\tau} \times T_\tau \simeq T_\gamma \\
& \searrow{\scriptstyle i_{\tau,\gamma}} & \downarrow{\scriptstyle j^\gamma_\tau} & \swarrow{\scriptstyle \varphi_\gamma} & \\
& & X_\gamma & &
\end{array}$$

Since ${}^p\mathcal{L}$ is a constant variation of Hodge structures on $T_\gamma \simeq T_{\gamma/\tau} \times T_\tau$, by (cf. [Sai90, (4.4.2)]),

$$\begin{aligned} {}^p\mathcal{L} = H \otimes \mathbb{Q}^H_{T_\gamma}[d_\gamma] &\simeq (H \otimes \mathbb{Q}^H_{T_{\gamma/\tau}}(-d_\tau)[-2d_\tau + d_\gamma]) \boxtimes \mathbb{Q}^H_{T_\tau}(d_\tau)[2d_\tau] \\ &\simeq p_3^!(H \otimes {}^p\mathbb{Q}^H_{T_{\gamma/\tau}}(-d_\tau)[-d_\tau]) \simeq p_3^! \, ({}^p\tilde{\mathcal{L}}(-d_\tau)[-d_\tau]) \end{aligned}$$

where we have set ${}^p\tilde{\mathcal{L}} = H \otimes {}^p\mathbb{Q}^H_{T_{\gamma/\tau}}$. We have the following isomorphisms

$$\begin{aligned} i^!_{\tau,\gamma} \operatorname{IC}_{X_\gamma}({}^p\mathcal{L}) &\simeq (i_{\gamma/\tau} \times id)^! (j^\gamma_\tau)^! \operatorname{IC}_{X_\gamma}({}^p\mathcal{L}) \\ &\simeq (i_{\gamma/\tau} \times id)^! \operatorname{IC}_{X_{\gamma/\tau} \times T_\tau}({}^p\mathcal{L}) \\ &\simeq (i_{\gamma/\tau} \times id)^! p_2^! \operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}})(-d_\tau)[-d_\tau] \\ &\simeq p_1^! \, i^!_{\gamma/\tau} \operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}})(-d_\tau)[-d_\tau] \\ &\simeq i^!_{\gamma/\tau} \operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}}) \boxtimes \mathbb{D}\mathbb{Q}^H_{T_\tau}(-d_\tau)[-d_\tau] \\ &\simeq i^!_{\gamma/\tau} \operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}}) \boxtimes \mathbb{Q}^H_{T_\tau}[d_\tau] \\ &\simeq i^!_{\gamma/\tau} \operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}}) \boxtimes {}^p\mathbb{Q}^H_{T_\tau}. \end{aligned}$$

Since $\operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}})$ is $T_{\gamma/\tau}$-equivariant it follows from Lemma 3.3 that

$$\begin{aligned} H^k(i^!_{\gamma/\tau} \operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}})) &\simeq H^k(a_! \operatorname{IC}_{X_{\gamma/\tau}}({}^p\tilde{\mathcal{L}})) \\ &\simeq H^k(a_! \operatorname{IC}_{X_{\gamma/\tau}}(H \otimes {}^p\mathbb{Q}^H_{T_{\gamma/\tau}})) \\ &\simeq \operatorname{IH}_c^{d_\gamma - d_\tau + k}(X_{\gamma/\tau}) \otimes H \end{aligned}$$

as mixed Hodge structures. This gives the isomorphism

$$\mathcal{H}^k(i^!_{\tau,\gamma} \operatorname{IC}_{X_\gamma}({}^p\mathcal{L})) \simeq H \otimes \operatorname{IH}_c^{d_\gamma - d_\tau + k}(X_{\gamma/\tau}) \otimes {}^p\mathbb{Q}^H_{T_\tau} .$$

The weight filtration on the intersection cohomology of $X_{\gamma/\tau}$ satisfies $\operatorname{gr}^W_k \operatorname{IH}^i(X_{\gamma/\tau}) = 0$ if $i \neq k$. Hence we get

$$\operatorname{gr}^W_i \mathcal{H}^k(i^!_{\tau,\gamma} \operatorname{IC}_{X_\gamma}(\mathcal{L})) = \bigoplus_{i = l_1 + l_2 + l_3} \operatorname{gr}^W_{l_1} H \otimes_{\mathbb{C}} \operatorname{gr}^W_{l_2} \operatorname{IH}_c^{d_\gamma - d_\tau + k}(X_{\gamma/\tau}) \otimes_{\mathbb{C}} \operatorname{gr}^W_{l_3} {}^p\mathbb{Q}^H_{T_\tau} = 0$$

for $i \neq w + (d_\gamma - d_\tau + k) + d_\tau = w + d_\gamma + k$.

The statement (3.1.2) follows from a dual proof. □

3.2. **A recursion.** The torus orbits $T_\tau \subseteq X$ equip X^{an} with a Whitney stratification (cf. [Dim92, Proposition 1.14]. Since the morphisms h, φ from (1.2.1) are affine, algebraic, and stratified, the perverse sheaf underlying $\varphi_* {}^p\mathbb{Q}^H_T$ is constructible with respect to this stratification. Since $\varphi_* {}^p\mathbb{Q}^H_T$ is a mixed Hodge module its weight graded parts are direct sums of intersection complexes (with possibly twisted coefficients) having support on the orbit closures $X_\tau = \overline{T}_\tau$. We write

$$\operatorname{gr}^W_k \varphi_* {}^p\mathbb{Q}^H_T = \bigoplus_\gamma \operatorname{IC}_{X_\gamma}({}^p\mathcal{V}_{(\gamma,k)}). \tag{3.2.1}$$

Here the direct sum is understood as a direct sum over all faces γ of σ, and ${}^p\mathcal{V}_{(\gamma,k)}$ is a polarizable variation of Hodge structures of weight k on T_γ.

Here and elsewhere, for a mixed Hodge module $\mathcal{M}$ on Y, we regard as equivalent via Kashiwara equivalence, for Y closed in X, $\mathrm{IC}_Y(\mathcal{M})$ and its direct image on X, without necessarily explicitly referencing X. Moreover, we say that $\mathcal{M}$ *has weight* $\geq k$ if $\mathrm{gr}^W_\ell \mathcal{M} = 0$ for $\ell < k$.

Our first result on (3.2.1) is a recursive formula; we continue to denote d_σ by just d and X_σ by just X:

Proposition 3.6. *The weight filtration on the mixed Hodge module* $\varphi_* {}^p\mathbb{Q}^H_T$ *satisfies the following properties.*

(1) $\mathrm{gr}^W_{d+e} \varphi_* {}^p\mathbb{Q}^H_T = 0$ *for* $e \neq 0, 1, \ldots, d$.
(2) $\mathrm{supp}\, \mathrm{gr}^W_{d+e} \varphi_* {}^p\mathbb{Q}^H_T \subseteq \bigcup_{d_\gamma \leq d-e} X_\gamma$.
(3) $\mathrm{gr}^W_d \varphi_* {}^p\mathbb{Q}^H_T = W_d \varphi_* {}^p\mathbb{Q}^H_T = \mathrm{IC}_{X_\sigma}$, *by which we denote* $\mathrm{IC}_X({}^p\mathbb{Q}^H_T)$.
(4) $\mathrm{gr}^W_{d+1} \varphi_* {}^p\mathbb{Q}^H_T = \bigoplus_\tau \mathrm{IC}_{X_\tau}(L^0_{(\tau,d+1)} \otimes {}^p\mathbb{Q}^H_{T_\tau})$ *where*

$$L^k_{(\tau,d+1)} := \mathrm{IH}_c^{d_\sigma - d_\tau + k + 1}(X_{\sigma/\tau})$$

is the intersection homology group with compact support of $X_{\sigma/\tau}$.
(5) *For* $e > 1$, $\mathrm{gr}^W_{d+e} \varphi_* {}^p\mathbb{Q}^H_T = \bigoplus_\tau \mathrm{IC}_{X_\tau}(L^0_{(\tau,d+e)} \otimes {}^p\mathbb{Q}^H_{T_\tau})$ *where*

$$L^k_{(\tau,d+e)} \simeq \frac{\left(\bigoplus_{\gamma \supseteq \tau} L^0_{(\gamma,d+e-1)} \otimes \mathrm{IH}_c^{d_\gamma - d_\tau + k + 1}(X_{\gamma/\tau}) \right)}{L^{k+1}_{\tau,d+e-1}}. \tag{3.2.2}$$

An essential feature of the situation is that:

(6) *for all* e, *the module*

$${}^p\mathcal{L}^k_{(\tau,d+e)} := \mathcal{H}^k i^!_{\tau,\sigma}(\varphi_* {}^p\mathbb{Q}^H_{T_\sigma} / W_{d+e-1} \varphi_* {}^p\mathbb{Q}^H_{T_\sigma})$$

is pure of weight $d + e + k$. *It is zero for* $d_\tau \geq d - e + 1 - k$ *and in any case isomorphic to a finite sum of copies of* ${}^p\mathbb{Q}^H_{T_\tau}$.

Proof. In order to ease the notation we denote in this proof by $\mathbb{Q}_\gamma$ the Hodge module ${}^p\mathbb{Q}^H_{T_\gamma}$.

We will proceed by induction on e. Obviously, for $e \ll 0$, all parts hold trivially. Assuming Property (6) up to e as well as Property (5) up to $e-1$, we show Property (6) for $e+1$ and Property (5) for e. Property (2) is then a direct consequence. Properties (3) and (4) are the induction start and are proved in the same fashion as the induction step, but look less uniform.

Since $\mathbb{Q}_\sigma$ has weight d and direct images do not decrease weights, the direct image $\varphi_* \mathbb{Q}_\sigma = \mathcal{H}^0 \varphi_* \mathbb{Q}_\sigma$ has weight $\geq d$. Property (1) is hence a consequence of Property (2).

We make the following Ansatz for the part of $\varphi_* \mathbb{Q}_\sigma$ of weight d:

$$W_d \varphi_* \mathbb{Q}_\sigma = \bigoplus_\gamma \mathrm{IC}_{X_\gamma}({}^p\mathcal{V}_{(\gamma,d)}),$$

where ${}^p\mathcal{V}_{(\gamma,d)}$ is a polarizable variation of Hodge structures on T_γ of weight d.

We begin with the lowest weight case $e = 0$; then (5) is vacuous. Consider the exact sequence

$$\text{(3.2.3)} \qquad 0 \longrightarrow \bigoplus_{\gamma} \mathrm{IC}_{X_\gamma}({}^p\mathcal{V}_{(\gamma,d)}) \longrightarrow \varphi_*\mathbb{Q}_\sigma \longrightarrow \varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma \longrightarrow 0.$$

Let τ be an d_τ-dimensional face of σ; $i_{\tau,\sigma} : T_\tau \to X$ is the natural embedding. Apply the functor $i^!_{\tau,\sigma}$ to (3.2.3), recalling that it is left exact and does not decrease weights (*cf.* Lemma 2.1.(1)). Because of Lemma 2.1.(2),(3) we get a long exact sequence

$$0 \longrightarrow {}^p\mathcal{V}_{(\tau,d)} \longrightarrow \mathcal{H}^0 i^!_{\tau,\sigma}\varphi_*\mathbb{Q}_\sigma \longrightarrow \mathcal{H}^0 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma\right) \longrightarrow \cdots .$$

Since $i^!_{\tau,\sigma}\varphi_*\mathbb{Q}_\sigma = 0$ for $d_\tau < d$ by Lemma 2.1.(3) we obtain Property (6) for $e = 0$ and find ${}^p\mathcal{V}_{(\tau,d)} = {}^p\mathcal{L}^k_{(\tau,d)} = 0$ for those τ. In the case $d_\tau = d$, we have $\tau = \sigma$ and $i^!_{\sigma,\sigma}\varphi_*\mathbb{Q}_\sigma = \mathbb{Q}_\sigma$ and therefore obtain

$$0 \longrightarrow {}^p\mathcal{V}_{(\sigma,d)} \longrightarrow \mathbb{Q}_\sigma \longrightarrow \mathcal{H}^0 i^!_{\sigma,\sigma}\left(\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma\right).$$

Since $\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma$ has weight $> d$ and $i^!_{\sigma,\sigma}$ does not decrease weight, ${}^p\mathcal{V}_{(\sigma,d)} \simeq \mathbb{Q}_\sigma$. Altogether we have:

$${}^p\mathcal{V}_{(\tau,d)} = \begin{cases} \mathbb{Q}_\sigma & \text{for } d_\tau = d, \\ 0 & \text{for } d_\tau < d. \end{cases}$$

Thus, for all τ, ${}^p\mathcal{L}^0_{(\tau,d)} = {}^p\mathcal{V}_{(\tau,d)}$ and ${}^p\mathcal{L}^{\neq 0}_{(\tau,d)}$ vanishes. This shows Property (3) (and embodies Property (6) for $e = 0$).

We next consider the weight $d + 1$ part. To begin, we use the fact that $W_d\varphi_*\mathbb{Q}_\sigma = \mathrm{IC}_{X_\sigma}$ in order to compute $\mathcal{H}^k(i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma))$ for each face τ and all $k \geq 0$. The exact sequence (3.2.3) becomes

$$0 \longrightarrow \mathrm{IC}_{X_\sigma} \longrightarrow \varphi_*\mathbb{Q}_\sigma \longrightarrow \varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma \longrightarrow 0.$$

We apply again $i^!_{\tau,\sigma}$ to this short exact sequence and obtain

$$\begin{array}{l} 0 \longrightarrow \mathcal{H}^0 i^!_{\tau,\sigma}\,\mathrm{IC}_{X_\sigma} \longrightarrow \mathcal{H}^0 i^!_{\tau,\sigma}\varphi_*\mathbb{Q}_\sigma \longrightarrow \mathcal{H}^0 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma\right) \longrightarrow \\ \mathcal{H}^1 i^!_{\tau,\sigma}\,\mathrm{IC}_{X_\sigma} \longrightarrow \mathcal{H}^1 i^!_{\tau,\sigma}\varphi_*\mathbb{Q}_\sigma \longrightarrow \mathcal{H}^1 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma\right) \longrightarrow \\ \mathcal{H}^2 i^!_{\tau,\sigma}\,\mathrm{IC}_{X_\sigma} \longrightarrow \cdots \end{array}$$

For $d_\tau = d$ we have $\mathcal{H}^k i^!_{\sigma,\sigma}\varphi_*\mathbb{Q}_\sigma = \mathcal{H}^k i^!_{\sigma,\sigma}\,\mathrm{IC}_{X_\sigma} = 0$ for $k \geq 1$ (as $i_{\sigma,\sigma}$ is an open embedding and therefore $i^!_{\sigma,\sigma}$ is exact) and so

$${}^p\mathcal{L}^k_{(\sigma,d+1)} = \mathcal{H}^k i^!_{\sigma,\sigma}(\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma)$$

vanishes for all k (the case $k = 0$ follows from ${}^p\mathcal{L}^0_{(\sigma,d)} = \mathbb{Q}_\sigma$).

For $d_\tau < d$ we have $\mathcal{H}^k i^!_{\tau,\sigma}\varphi_*\mathbb{Q}_\sigma = 0$ for all k, hence by Lemma 3.5 we have

$$
{}^p\mathcal{L}^k_{(\tau,d+1)} := \mathcal{H}^k i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma / W_d\varphi_*\mathbb{Q}_\sigma\right) \simeq \mathcal{H}^{k+1} i^!_{\tau,\sigma}\,\mathrm{IC}_{X_\sigma} \simeq \mathrm{IH}_c^{d_\sigma - d_\tau + k+1}(X_{\sigma/\tau}) \otimes \mathbb{Q}_\tau. \tag{3.2.4}
$$

In particular, ${}^p\mathcal{L}^k_{(\tau,d+1)} = L^k_{(\tau,d+1)} \otimes \mathbb{Q}_\tau$ with $L^k_{(\tau,d+1)} = \mathrm{IH}_c^{d_\sigma-d_\tau+k+1}(X_{\sigma/\tau})$. Since $\mathrm{IH}_c^{k+1}(X_{\sigma/\sigma}) = 0$ for all $k \geq 0$ the formula above makes also sense for $\tau = \sigma$. Notice that ${}^p\mathcal{L}^k_{(\tau,d+1)}$ is pure of weight $d+1+k$ and since $\mathrm{IH}_c^{d_\sigma-d_\tau+k+1}(X_{\sigma/\tau}) = 0$ for $d_\sigma - d_\tau + k + 1 > 2(d_\sigma - d_\tau)$ we have ${}^p\mathcal{L}^k_{(\tau,d+1)} = 0$ for $d_\tau \geq d - k$. This shows Property (6) in the case $e = 1$.

In order to compute the weight $(d+1)$ part of $\varphi_*\mathbb{Q}_\sigma$ we make the Ansatz

$$
\mathrm{gr}^W_{d+1}\,\varphi_*\mathbb{Q}_\sigma = \bigoplus_\gamma \mathrm{IC}_{X_\gamma}({}^p\mathcal{V}_{(\gamma,d+1)})
$$

and consider the exact sequence

$$
0 \longrightarrow \bigoplus_\gamma \mathrm{IC}_{X_\gamma}({}^p\mathcal{V}_{(\gamma,d+1)}) \longrightarrow \varphi_*\mathbb{Q}_\sigma/W_d\varphi_*\mathbb{Q}_\sigma \longrightarrow \varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma \longrightarrow 0. \tag{3.2.5}
$$

The functor $i^!_{\tau,\sigma}$ produces the long exact cohomology sequence

$$
0 \longrightarrow {}^p\mathcal{V}_{(\tau,d+1)} \longrightarrow {}^p\mathcal{L}^0_{(\tau,d+1)} \longrightarrow \mathcal{H}^0 i^!_{\tau.\sigma}\left(\varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma\right) \longrightarrow \cdots .
$$

Since $i^!_{\tau,\sigma}$ does not decrease weight, the third term has weight $d+2$ or more, and since ${}^p\mathcal{L}^0_{(\tau,d+1)}$ is pure of weight $d+1$, (3.2.4) yields

$$
{}^p\mathcal{V}_{(\tau,d+1)} = {}^p\mathcal{L}^0_{(\tau,d+1)} = \mathrm{IH}_c^{d_\sigma-d_\tau+1}(X_{\sigma/\tau}) \otimes \mathbb{Q}_\tau,
$$

which shows Property (4).

With this, (3.2.5) becomes now

$$
0 \longrightarrow \bigoplus_\gamma \mathrm{IC}_{X_\gamma}({}^p\mathcal{L}^0_{(\gamma,d+1)}) \longrightarrow \varphi_*\mathbb{Q}_\sigma/W_d\varphi_*\mathbb{Q}_\sigma \longrightarrow \varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma \longrightarrow 0,
$$

and applying $i^!_{\tau,\sigma}$ we obtain

$$
\begin{array}{l}
0 \longrightarrow \underbrace{\bigoplus_{\gamma\supseteq\tau}\left(L^0_{(\gamma,d+1)} \otimes \mathrm{IH}_c^{d_\gamma-d_\tau}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau\right)}_{=L^0_{(\tau,d+1)}\otimes\mathbb{Q}_\tau} \longrightarrow L^0_{(\tau,d+1)} \otimes \mathbb{Q}_\tau \longrightarrow \mathcal{H}^0 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma\right) \\
\longrightarrow \bigoplus_{\gamma\supseteq\tau}\left(L^0_{(\gamma,d+1)} \otimes \mathrm{IH}_c^{d_\gamma-d_\tau+1}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau\right) \longrightarrow L^1_{(\tau,d+1)} \otimes \mathbb{Q}_\tau \longrightarrow \mathcal{H}^1 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma\right) \\
\longrightarrow \bigoplus_{\gamma\supseteq\tau}\left(L^0_{(\gamma,d+1)} \otimes \mathrm{IH}_c^{d_\gamma-d_\tau+2}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau\right) \longrightarrow \cdots .
\end{array}
$$

Here, the first column is owed to Lemma 3.5 and the equality in the first row follows from Lemma 2.1(3) resp. Lemma 3.4 (2).

Since both $\bigoplus_{\gamma\supseteq\tau}\left(L^0_{(\gamma,d+1)}\otimes \mathrm{IH}_c^{d_\gamma-d_\tau+k}(X_{\gamma/\tau})\otimes\mathbb{Q}_\tau\right)$ and $L^k_{(\tau,d+1)}\otimes\mathbb{Q}_\tau$ are pure of weight $d+1+k$, and since $\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma)$ has weight $>(d+1+k)$ the long exact sequence splits into sequences

$$0\to\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma)\longrightarrow\bigoplus_{\gamma\supseteq\tau}\left(L^0_{(\gamma,d+1)}\otimes\mathrm{IH}_c^{d_\gamma-d_\tau+k+1}(X_{\gamma/\tau})\otimes\mathbb{Q}_\tau\right)\longrightarrow L^{k+1}_{(\tau,d+1)}\otimes\mathbb{Q}_\tau\to 0,$$

pure of weight $d+1+k+1$. The category of pure Hodge modules is semisimple and so there is a (non-canonical) splitting which induces an identification

$$\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma)\;\simeq\;\frac{\left(\bigoplus_{\gamma\supseteq\tau}L^0_{(\gamma,d+1)}\otimes\mathrm{IH}_c^{d_\gamma-d_\tau+k+1}(X_{\gamma/\tau})\right)}{L^{k+1}_{(\tau,d+1)}}\otimes\mathbb{Q}_\tau$$

as pure Hodge modules. We now define vector spaces ${}^p\mathcal{L}^k_{(\tau,d+2)}$ by

$${}^p\mathcal{L}^k_{(\tau,d+2)}:=L^k_{(\tau,d+2)}\otimes\mathbb{Q}_\tau:=\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma/W_{d+1}\varphi_*\mathbb{Q}_\sigma),$$

a pure Hodge module of weight $d+2+k$. Since $L^0_{(\gamma,d+1)}$ is zero for $d_\gamma\geq d$ and $\mathrm{IH}_c^{d_\gamma-d_\tau+k+1}(X_{\gamma/\tau})$ is zero for $d_\gamma-d_\tau+k+1>2(d_\gamma-d_\tau)$, the term ${}^p\mathcal{L}^k_{(\tau,d+2)}$ is zero for $d_\tau\geq d-1-k$; this proves Property (6) for $e=2$.

We will now provide the inductive step, much in parallel to the above. Assume that

$${}^p\mathcal{L}^k_{(\tau,d+e)}=L^k_{(\tau,d+e)}\otimes\mathbb{Q}_\tau=\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma/W_{d+e-1}\varphi_*\mathbb{Q}_\sigma)$$

is pure of weight $d+e+k$ and ${}^p\mathcal{L}^k_{(\tau,d+e)}=0$ for $d_\tau\geq d-e+1-k$ (*i.e.*, Property 6 at level e).

In order to compute the weight $d+e$ part of $\varphi_*\mathbb{Q}_\sigma$ we make the Ansatz

$$\mathrm{gr}^W_{d+e}\,\varphi_*\mathbb{Q}_\sigma=\bigoplus_\gamma \mathrm{IC}_{X_\gamma}({}^p\mathcal{V}_{(\gamma,d+e)})$$

and consider the exact sequence

(3.2.6)
$$0\longrightarrow\bigoplus_\gamma\mathrm{IC}_{X_\gamma}({}^p\mathcal{V}_{(\gamma,d+e)})\longrightarrow\varphi_*\mathbb{Q}_\sigma/W_{d+e-1}\varphi_*\mathbb{Q}_\sigma\longrightarrow\varphi_*\mathbb{Q}_\sigma/W_{d+e}\varphi_*\mathbb{Q}_\sigma\longrightarrow 0$$

We apply the functor $i^!_{\tau,\sigma}$ and get the long exact cohomology sequence

$$0\longrightarrow{}^p\mathcal{V}_{(\tau,d+e)}\longrightarrow{}^p\mathcal{L}^0_{(\tau,d+e)}\longrightarrow\mathcal{H}^0 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma/W_{d+e}\varphi_*\mathbb{Q}_\sigma\right)\longrightarrow\cdots.$$

As $i^!_{\tau,\sigma}$ does not decrease weight, the third term has weight greater than $(d+e)$, and as ${}^p\mathcal{L}^0_{(\tau,d+e)}$ is pure of weight $d+e$ we find

$${}^p\mathcal{V}_{(\tau,d+e)}={}^p\mathcal{L}^0_{(\tau,d+e)}.$$

The exact sequence (3.2.6) now becomes

$$0\longrightarrow\bigoplus_\gamma\mathrm{IC}_{X_\gamma}({}^p\mathcal{L}^0_{(\gamma,d+e)})\longrightarrow\varphi_*\mathbb{Q}_\sigma/W_{d+e-1}\varphi_*\mathbb{Q}_\sigma\longrightarrow\varphi_*\mathbb{Q}_\sigma/W_{d+e}\varphi_*\mathbb{Q}_\sigma\longrightarrow 0,$$

and $i^!_{\tau,\sigma}$ induces

$$\begin{array}{l}
0 \longrightarrow L^0_{(\tau,d+e)} \otimes \mathbb{Q}_\tau \longrightarrow L^0_{\tau,d+e} \otimes \mathbb{Q}_\tau \longrightarrow \mathcal{H}^0 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma / W_{d+e}\varphi_*\mathbb{Q}_\sigma\right) \\
\longrightarrow \bigoplus_{\gamma\supseteq\tau} \left(L^0_{(\gamma,d+e)} \otimes \mathrm{IH}_c^{d_\gamma - d_\tau + 1}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau\right) \longrightarrow L^1_{\tau,d+e} \otimes \mathbb{Q}_\tau \longrightarrow \mathcal{H}^1 i^!_{\tau,\sigma}\left(\varphi_*\mathbb{Q}_\sigma / W_{d+e}\varphi_*\mathbb{Q}_\sigma\right) \\
\longrightarrow \bigoplus_{\gamma\supseteq\tau} \left(L^0_{(\gamma,d+e)} \otimes \mathrm{IH}_c^{d_\gamma - d_\tau + 2}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau\right) \longrightarrow \cdots .
\end{array}$$

Since both $\bigoplus_{\gamma\supseteq\tau}\left(L^0_{(\gamma,d+e)} \otimes \mathrm{IH}_c^{d_\gamma - d_\tau + k}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau\right)$ and $L^k_{\tau,d+e} \otimes \mathbb{Q}_\tau$ are pure of weight $d+e+k$, and since furthermore $\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma / W_{d+e}\varphi_*\mathbb{Q}_\sigma)$ has weight greater than $(d+e+k)$, the long exact sequence splits in $\mathrm{MHM}(X_\sigma)$ into sequences

$$0 \to \mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma / W_{d+e}\varphi_*\mathbb{Q}_\sigma) \longrightarrow \bigoplus_{\gamma\supseteq\tau}\left(L^0_{(\gamma,d+e)} \otimes \mathrm{IH}_c^{d_\gamma - d_\tau + k+1}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau\right) \longrightarrow L^{k+1}_{(\tau,d+e)} \otimes \mathbb{Q}_\tau \to 0.$$

The center term is pure of weight $d+e+k+1$; hence the outer terms are as well. Since the category of pure Hodge modules is semisimple, the sequence splits (non-canonically) and there is an identification

$$\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma / W_{d+e}\varphi_*\mathbb{Q}_\sigma) \;\simeq\; \frac{\left(\bigoplus_{\gamma\supseteq\tau} L^0_{(\gamma,d+e)} \otimes \mathrm{IH}_c^{d_\gamma - d_\tau + k+1}(X_{\gamma/\tau})\right)}{L^{k+1}_{(\tau,d+e)}} \otimes \mathbb{Q}_\tau.$$

We now define ${}^p\mathcal{L}^k_{(\tau,d+e+1)}$ and $L^k_{(\tau,d+e+1)}$ by

$${}^p\mathcal{L}^k_{(\tau,d+e+1)} := L^k_{(\tau,d+e+1)} \otimes \mathbb{Q}_\tau := \mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma / W_{d+e}\varphi_*\mathbb{Q}_\sigma),$$

and reiterate that ${}^p\mathcal{L}^k_{(\tau,d+e+1)}$ is pure of weight $d+e+1+k$. Since $L^0_{(\gamma,d+e)}$ is zero for $d_\gamma \geq d-e+1$ and $\mathrm{IH}_c^{d_\gamma - d_\tau + k+1}(X_{\gamma/\tau})$ is zero for $d_\gamma - d_\tau + k + 1 > 2(d_\gamma - d_\tau)$, the term ${}^p\mathcal{L}^k_{(\tau,d+e+1)}$ vanishes for $d_\tau \geq d-e+1-k$. This finishes the inductive step for Property (6), establishes (5) in the process, and hence completes the proof. □

Remark 3.7. It has been pointed out by A. Lörincz to us that the constancy of the local systems ${}^p\mathcal{L}_{(\tau,d+e)}$ can be also seen as follows: ${}^p\mathbb{Q}^H_T$ is equivariant, and hence so is $\varphi_*{}^p\mathbb{Q}^H_T$. Since all orbit stabilizers are connected, [HTT08, Theorem 11.6.1] shows that each orbit can only support one equivariant local system, the constant one. See [LW19] for more details on equivariant D-modules.

Example 3.8. We give an explicit description of the vector spaces $L^k_{(\tau,d+e)}$ from Proposition 3.6 in the case $d = 4$ for $e \geq 1$. Here, the (k,e)-entry for $L^k_{(\tau_i,d+e)}$ is a sum over all γ_j that arise. For example, $L^0_{(\tau_2,6)}$ is the sum over all γ_3 of dimension 3 with $\tau_2 \subseteq \gamma_3 \subseteq \sigma$ of the terms listed under $k = 0$, $e = 2$ in Table 3.2.9.

The Hodge-structures $L^k_{(\tau_0,d+e)}$ for the unique $\tau_0 \subseteq \sigma$ with $\dim \tau_0 = 0$:

(3.2.7)

$k=3$	$L^3_{(\tau_0,5)} = \mathrm{IH}^8_c(X_{\sigma/\tau_0})$	0	0	0
$k=2$	0	$L^2_{(\tau_0,6)} = \frac{L^0_{(\gamma_3,5)} \otimes \mathrm{IH}^6_c(X_{\gamma_3/\tau_0})}{L^3_{(\tau_0,5)}}$	0	0
$k=1$	$L^1_{(\tau_0,5)} = \mathrm{IH}^6_c(X_{\sigma/\tau_0})$	0	$L^1_{(\tau_0,7)} = \frac{L^0_{(\gamma_2,6)} \otimes \mathrm{IH}^4_c(\gamma_2/\tau_0)}{L^2_{(\tau_0,6)}}$	0
$k=0$	0	$L^0_{(\tau_0,6)} = \frac{L^0_{(\gamma_3,5)} \otimes \mathrm{IH}^4_c(X_{\gamma_3/\tau_0}) \oplus L^0_{(\gamma_1,5)} \otimes \mathrm{IH}^2_c(X_{\gamma_1/\tau_0})}{L^1_{(\tau_0,5)}}$	0	$L^0_{(\tau_0,8)} = \frac{L^0_{(\gamma_1,7)} \otimes \mathrm{IH}^2_c(X_{\gamma_1/\tau_0})}{L^1_{(\tau_0,7)}}$
	$e=1$	$e=2$	$e=3$	$e=4$

The Hodge-structures $L^k_{(\tau_1,d+e)}$ for all $\tau_1 \subseteq \sigma$ with $\dim \tau_1 = 1$:

(3.2.8)

$k=3$	0	0	0	0
$k=2$	$L^2_{(\tau_1,5)} = \mathrm{IH}^6_c(X_{\sigma/\tau_1})$	0	0	0
$k=1$	0	$L^1_{(\tau_1,6)} = \frac{L^0_{(\gamma_3,5)} \otimes \mathrm{IH}^4_c(X_{\gamma_3/\tau_1})}{L^2_{(\tau_1,5)}}$	0	0
$k=0$	$L^0_{(\tau_1,5)} = \mathrm{IH}^4_c(X_{\sigma/\tau_1})$	0	$L^0_{(\tau_1,7)} = \frac{L^0_{(\gamma_2,6)} \otimes \mathrm{IH}^2_c(\gamma_2/\tau_1)}{L^1_{(\tau_1,6)}}$	0
	$e=1$	$e=2$	$e=3$	$e=4$

The Hodge-structures $L^k_{(\tau_2,d+e)}$ for all $\tau_2 \subseteq \sigma$ with $\dim \tau_2 = 2$:

(3.2.9)

$k=3$	0	0	0	0
$k=2$	0	0	0	0
$k=1$	$L^1_{(\tau_2,5)} = \mathrm{IH}^4_c(X_{\sigma/\tau_2})$	0	0	0
$k=0$	0	$L^0_{(\tau_2,6)} = \frac{L^0_{(\gamma_3,5)} \otimes \mathrm{IH}^2_c(X_{\gamma_3/\tau_2})}{L^1_{(\tau_2,5)}}$	0	0
	$e=1$	$e=2$	$e=3$	$e=4$

The Hodge-structures $L^k_{(\tau_3,d+e)}$ for all $\tau_3 \subseteq \sigma$ with $\dim \tau_3 = 3$:

(3.2.10)

$k=3$	0	0	0	0
$k=2$	0	0	0	0
$k=1$	0	0	0	0
$k=0$	$L^0_{(\tau_3,5)} = \mathrm{IH}^2_c(X_{\sigma/\tau_3})$	0	0	0
	$e=1$	$e=2$	$e=3$	$e=4$

The table for $\sigma = \tau_4$ is determined by Proposition 3.6, Properties (2) and (3); it has only zero entries since σ only contributes to weight d. $\diamond$

3.3. **An explicit formula.** If we set

$$\mathrm{ih}^k_c(X_{\gamma/\tau}) := \dim_{\mathbb{Q}} \mathrm{IH}^k_c(X_{\gamma/\tau})$$

we can rewrite the dimension of $L^0_{(\gamma,k)}$ in Example 3.8 as follows:

$$
\begin{aligned}
\dim_{\mathbb{Q}} L^0_{(\tau_3,5)} &= \mathrm{ih}^2_c(X_{\sigma/\tau_3})\\
\dim_{\mathbb{Q}} L^0_{(\tau_2,6)} &= \mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^2_c(X_{\gamma_3/\tau_2}) - \mathrm{ih}^4_c(X_{\sigma/\tau_2})\\
\dim_{\mathbb{Q}} L^0_{(\tau_1,7)} &= \mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^2_c(X_{\gamma_3/\gamma_2})\,\mathrm{ih}^2_c(X_{\gamma_2/\tau_1}) - \left[\mathrm{ih}^4_c(X_{\sigma/\gamma_2})\,\mathrm{ih}^2_c(X_{\gamma_2/\tau_1}) + \mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^4_c(X_{\gamma_3/\tau_1})\right] + \mathrm{ih}^6_c(X_{\sigma/\tau_1})\\
\dim_{\mathbb{Q}} L^0_{(\tau_0,8)} &= \mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^2_c(X_{\gamma_3/\gamma_2})\,\mathrm{ih}^2_c(X_{\gamma_2/\gamma_1})\,\mathrm{ih}^2_c(X_{\gamma_1/\tau_0})\\
&\quad - \left[\mathrm{ih}^4_c(X_{\sigma/\gamma_2})\,\mathrm{ih}^2_c(X_{\gamma_2/\gamma_1})\,\mathrm{ih}^2_c(X_{\gamma_1/\tau_0}) + \mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^4_c(X_{\gamma_3/\gamma_1})\,\mathrm{ih}^2_c(X_{\gamma_1/\tau_0}) + \mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^2_c(X_{\gamma_3/\gamma_2})\,\mathrm{ih}^4_c(X_{\gamma_2/\tau_0})\right]\\
&\quad + \left[\mathrm{ih}^6_c(X_{\sigma/\gamma_1})\,\mathrm{ih}^2_c(X_{\gamma_1/\tau_0}) + \mathrm{ih}^4_c(X_{\sigma/\gamma_2})\,\mathrm{ih}^4_c(X_{\gamma_2/\tau_0}) + \mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^6_c(X_{\gamma_3/\tau_0})\right] - \mathrm{ih}^8_c(X_{\sigma/\tau_0})\\
\dim_{\mathbb{Q}} L^0_{(\tau_1,5)} &= \mathrm{ih}^4_c(X_{\sigma/\tau_1})\\
\dim_{\mathbb{Q}} L^0_{(\tau_0,6)} &= \left[\mathrm{ih}^2_c(X_{\sigma/\gamma_3})\,\mathrm{ih}^4_c(X_{\gamma_3/\tau_0}) + \mathrm{ih}^4_c(X_{\sigma/\gamma_1})\,\mathrm{ih}^2_c(X_{\gamma_1/\tau_0})\right] - \mathrm{ih}^6_c(X_{\sigma/\tau_0})
\end{aligned}
$$

Again, each expression is to be summed over all possible faces γ_i of dimension i that satisfy the requisite containment conditions.

The particular structure of the formulas for the dimension of these local systems is not coincidental. Our next task is to turn recursion (3.2.2) for $L^0_{(\gamma,k)}$ into a general explicit combinatorial formula.

We set

$$\mu^\sigma_\tau(e) := \dim_{\mathbb{Q}}(L^0_{(\tau,d+e)})$$

for the rank of the (constant!) local system ${}^p\mathcal{L}_{(\tau,d+e)}$ corresponding to the intersection complex $\mathrm{IC}_{X_\tau}({}^p\mathcal{L}_{(\tau,d+e)})$ occurring in $\mathrm{gr}^W_{d+e}\,\varphi_*{}^p\mathbb{Q}^H_T$. We further introduce the following abbreviations.

Notation 3.9. Let

$$\mathrm{ih}^\gamma_\tau(k) := \dim_{\mathbb{Q}}(\mathrm{IH}^{d_\gamma-d_\tau+k}_c(X_{\gamma/\tau})); \tag{3.3.1}$$

$$\ell_\gamma(k,e) := \dim_{\mathbb{Q}}(L^k_{(\gamma,d+e)}). \tag{3.3.2}$$

Then

$$\ell_\tau(k,1) = \dim_{\mathbb{Q}}(L^k_{\tau,d+1}) = \dim_{\mathbb{Q}}(\mathrm{IH}^{d_\sigma-d_\tau+k+1}_c(X_{\sigma/\tau})) = \mathrm{ih}^\sigma_\tau(k+1)$$

by Proposition 3.6.(4), while the recursion (3.2.2) yields

$$\ell_\tau(k,e) = \left(\sum_{\gamma\supseteq\tau}\ell_\gamma(0,e-1)\cdot\mathrm{ih}^\gamma_\tau(k+1)\right) - \ell_\tau(k+1,e-1). \tag{3.3.3}$$

Let $0 < t \in \mathbb{N}$ and let $\pi = [\pi_1,\ldots,\pi_m] \dashv t$ be a *partition*[1] of t of *length* $|\pi| = m$. We consider *flags* $\Gamma = (\gamma_{d_0} \subsetneq \gamma_{d_1} \subsetneq \ldots \subsetneq \gamma_{d_m})$ of faces of σ, of length $|\Gamma| = m$. Here, d_i is the dimension of γ_{d_i}. Denote by $\mathrm{ih}_\Gamma(\pi)$ the product

$$\mathrm{ih}_\Gamma(\pi) := \mathrm{ih}^{\gamma_{d_1}}_{\gamma_{d_0}}(\pi_1)\cdot\ldots\cdot\mathrm{ih}^{\gamma_{d_m}}_{\gamma_{d_{m-1}}}(\pi_m).$$

For comparable faces $\gamma \subsetneq \gamma'$, set

$$\mathrm{ih}^{\gamma'}_\gamma(\pi) = \sum_{\substack{|\Gamma|=|\pi|\\ \Gamma=(\gamma,\ldots,\gamma')}}\mathrm{ih}_\Gamma(\pi),$$

[1]We always assume that "partition" implies that each π_j is nonzero, and that the entries are ordered. The partitions of 3 are [3], [1, 2], [2, 1] and [1, 1, 1].

and

$$\mathrm{ih}_\gamma^{\gamma'}(t,m) = \sum_{\substack{\pi \dashv t\\ |\pi|=m}} \mathrm{ih}_\gamma^{\gamma'}(\pi).$$

◇

Proposition 3.10. *The rank of the local system* ${}^p\mathcal{L}_{(\tau,d+e)}$ *corresponding to the intersection complex* $\mathrm{IC}_{X_\tau}({}^p\mathcal{L}_{(\tau,d+e)})$ *occurring in* $\mathrm{gr}^W_{d+e}\,\varphi_*{}^p\mathbb{Q}_T^H$ *is*

$$\mu_\tau^\sigma(e) = \sum_m (-1)^{m+d_\sigma}\,\mathrm{ih}_\tau^\sigma(e,m).$$

Proof. To start, note that, for $\gamma'' \supseteq \gamma$ one has the "product rule"

$$\mathrm{ih}_\gamma^{\gamma''}(\pi'' \sqcup \pi) = \sum_{\gamma'' \supsetneq \gamma' \supsetneq \gamma} \mathrm{ih}_{\gamma'}^{\gamma''}(\pi'') \cdot \mathrm{ih}_\gamma^{\gamma'}(\pi)$$

for any two partitions $\pi'' \dashv t''$ and $\pi \dashv t$ and their juxtaposition $\pi'' \sqcup \pi = [\pi_1'', \dots, \pi_{m''}'', \pi_1, \dots, \pi_m] \dashv (t'' + t)$.

For each τ, place the numbers $\ell_\tau(k,e)$ on a grid of integer points in the first quadrant of a page associated to τ as follows:

(3.3.4) $((\tau))$:

$\vdots$	$\vdots$	$\vdots$		
$k=2$	$\mathrm{ih}_\tau^\sigma(3) = \ell_\tau(2,1)$	$\ell_\tau(2,2)$	$\ell_\tau(2,3)$	$\cdots$
$k=1$	$\mathrm{ih}_\tau^\sigma(2) = \ell_\tau(1,1)$	$\ell_\tau(1,2)$	$\ell_\tau(1,3)$	$\cdots$
$k=0$	$\mathrm{ih}_\tau^\sigma(1) = \ell_\tau(0,1)$	$\ell_\tau(0,2)$	$\ell_\tau(0,3)$	$\cdots$
	$e=1$	$e=2$	$e=3$	$\cdots$

The column $e=1$ of the τ-page consists of the numbers $\dim_\mathbb{Q} \mathrm{IH}_c^{d_\sigma - d_\tau + k + 1}(X_{\sigma/\tau}) = \mathrm{ih}_\tau^\sigma(k+1) = \ell_\tau(k,1)$. Then (3.3.3) implies that for $e > 1$ the entry in row k and column e of page $((\tau))$ is the difference a) $-$ b) where

a) is the sum over all $\sigma \supsetneq \gamma \supsetneq \tau$ of all products of $\mathrm{ih}_\tau^\gamma(k+1)$ with the entry in row 0 and column $e-1$ on the γ-page;
b) the entry in row $k+1$ and column $e-1$ on the τ-page.

Progressing along increasing column index, all entries on each page can be rewritten as sums of products $\mathrm{ih}_\Gamma(\pi)$ of intersection cohomology dimensions $\mathrm{ih}_{\gamma'}^{\gamma''}(t)$. We call such product $\mathrm{ih}_\Gamma(\pi)$ a "term". It is immediate that each term on page $((\tau))$ arises from a flag that links τ to σ (*i.e.*, $\tau = \gamma_0, \sigma = \gamma_{|\Gamma|}$) with $|\Gamma| = |\pi|$.

The sum in a) contains only terms $\mathrm{ih}_\Gamma(\pi)$ where the initial element of π equals $k+1$. On the other hand, it follows from induction on k that the terms in b) all have the initial element of the corresponding π greater than $k+1$. So, formal cancellation of terms cannot occur in the recursion.

When a term on the τ-page arises through case a) then the length of the term is greater (by one) than the length of the term on the γ-page that gave rise to it. However, that is not the case if it arises from case b) when it simply copied from the appropriate entry on the τ page, and so term length changes if and only if no new factor of -1 is acquired. In particular, the sign of a term is a function of the length of the term, modulo two. The recursion forces the term to the partition $[1, 1, \dots, 1]$ of length d_σ to be positive. Hence all terms $\mathrm{ih}_\Gamma(\pi)$ on each page carry a sign of $(-1)^{|\pi|+d_\sigma}$.

Note that in a) one could allow $\gamma = \tau$ since $\mathrm{ih}^\tau_\tau(k+1) = 0$. Similarly, one can admit $\gamma = \sigma$ since $\ell_\sigma(0, e-1) = 0$ for $e > 1$. The sum in a) involves always all possible choices of γ, $\sigma \supseteq \gamma \supseteq \tau$. Thus, if a partition π occurs at all in an entry on page $((\tau))$ then $\mathrm{ih}_\Gamma(\pi)$ will occur in that entry for all flags Γ with $|\Gamma| = |\pi|$ that start at τ and end with σ. In the following table we tabulate for small k, e the partitions that occur in Figure (3.3.4); here, in each term one should sum over all Γ of the appropriate length that interpolate from τ to σ (we will write $\mathrm{ih}([1,1,2])$ instead of $\mathrm{ih}^\sigma_\tau([1,1,2])$ etc. for ease of readability).

(3.3.5)

$\vdots$	$\vdots$	$\vdots$		
$k=3$	$\mathrm{ih}([4])$	$\mathrm{ih}([1,4]) - \mathrm{ih}([5])$	$\mathrm{ih}([1,1,4]) - \mathrm{ih}([2,4]) - (\mathrm{ih}([1,5]) - \mathrm{ih}([6]))$	$\cdots$
$k=2$	$\mathrm{ih}([3])$	$\mathrm{ih}([1,3]) - \mathrm{ih}([4])$	$\mathrm{ih}([1,1,3]) - \mathrm{ih}([2,3]) - (\mathrm{ih}([1,4]) - \mathrm{ih}([5]))$	$\cdots$
$k=1$	$\mathrm{ih}([2])$	$\mathrm{ih}([1,2]) - \mathrm{ih}([3])$	$\mathrm{ih}([1,1,2]) - \mathrm{ih}([2,2]) - (\mathrm{ih}([1,3]) - \mathrm{ih}([4]))$	$\cdots$
$k=0$	$\mathrm{ih}([1])$	$\mathrm{ih}([1,1]) - \mathrm{ih}([2])$	$\mathrm{ih}([1,1,1]) - \mathrm{ih}([2,1]) - (\mathrm{ih}([1,2]) - \mathrm{ih}([3]))$	$\cdots$
	$e=1$	$e=2$	$e=3$	$\cdots$

It is therefore sufficient to investigate which partitions occur in the (k, e)-entry on page τ. Since the entries in column $e = 1$ come from a unique partition, the entries in column e will come from no more than 2^{e-1} partitions (the variation over all γ in the recursion does not affect the resulting partition π, only the flag Γ). The argument that no cancellation can occur reveals also that no fewer than, and hence exactly, 2^{e-1} partitions occur in each entry of column e.

The partitions π used in the entry (k, e) on page τ have weight $\pi_1 + \ldots + \pi_{|\pi|} = e + k$, again by induction on the column index. But the number of ordered integer partitions of weight e with positive entries is exactly 2^{e-1}. Thus, all 2^{e-1} partitions of weight e actually occur in the entry $(0, e)$, and

- for each partition, each possible flag interpolating from τ to σ contributes, and no other;
- the term $\mathrm{ih}_\Gamma(\pi)$ has sign $(-1)^{|\pi|+d_\sigma}$;

as stated in the proposition. ☐

The recursion as evidenced in Table (3.3.5) leads immediately to the following result.

Corollary 3.11. *The number of copies $\ell_\tau(k, e)$ of $\mathbb{Q}_\tau$ in $\mathcal{H}^k i^!_{\tau,\sigma}(\varphi_*\mathbb{Q}_\sigma / W_{d+e-1}\varphi_*\mathbb{Q}_\sigma) = \bigoplus_{\ell_\tau(k,e)} \mathbb{Q}_\tau$ equals $\sum_m (-1)^{m+d_\sigma}\, \mathrm{ih}^\sigma_\tau(e, m)_{\boldsymbol{k}}$ where the subscript $\boldsymbol{k}$ means each partition $\pi = [\pi_1, \dots, \pi_m] \dashv t$ that contributes to $\mu^\sigma_\tau(e) = \ell_\tau(0, e)$ in Proposition 3.10 is replaced by $[\pi_1, \dots, \pi_{m-1}, \pi_m + k] \dashv (t + k)$.* ☐

3.4. **Dual polytopes.** Our final step in this section is to give a compact value to the formula in Proposition 3.10. In order to carry out this discussion we have to introduce some notions from toric geometry.

Notation 3.12. Let $\tau \subseteq \gamma \subseteq \sigma$ be faces of σ. The *quotient face* of γ by τ is defined as:

$$\gamma/\tau \quad := \quad (\gamma + \tau_{\mathbb{R}})/\tau_{\mathbb{R}} \subseteq \mathbb{R}^d/\tau_{\mathbb{R}}. \tag{3.4.1}$$

We define the dual cone and the annihilator of γ by

$$\gamma^\vee := \{y \in (\mathbb{R}^d)^* \mid y(x) \geq 0 \; \forall x \in \gamma\} \qquad \text{and} \qquad \gamma^\perp := \{y \in (\mathbb{R}^d)^* \mid y(x) = 0 \; \forall x \in \gamma\}.$$

For faces τ and γ of σ, $[\tau \subseteq \gamma \subseteq \sigma] \;\Leftrightarrow\; [\tau^\vee \supseteq \gamma^\vee \supseteq \sigma^\vee]$ and $[\tau \subseteq \gamma \subseteq \sigma] \;\Leftrightarrow\; [\tau^\perp \supseteq \gamma^\perp \supseteq \sigma^\perp]$.

There is an containment-reversing bijection

$$\tau \quad \longleftrightarrow \quad \tau^\star := \tau^\perp \cap \sigma^\vee$$

between faces τ of σ of dimension r and *complementary faces* $\tau^\star$ of $\sigma^\vee$ of dimension $d - r$.

The notions of dual and annihilator as well as complementary face are relative to σ, although we usually suppress it in the notation. ◇

Remark 3.13. We record two properties of μ that will be used later.

(1) The numbers $\mu^\sigma_\tau(e)$ are relative in the sense that they only depend on the quotient variety $X_{\sigma/\tau}$: Proposition 3.10 shows that $\mu^\sigma_\tau(e) = \mu^{\sigma/\tau}_{\tau/\tau}(e)$.

(2) We derive a second recursive formula. Indeed, as an alternating sum of weight e over all flags interpolating from 0 to σ, sorting the terms by their first non-trivial flag entry γ, one obtains

(3.4.2)
$$\mu^\sigma_0(e) = (-1)^{d_\sigma+1} \operatorname{ih}^\sigma_0(e) + \sum_{0 \subsetneq \gamma \subsetneq \sigma} \left((-1)^{d_\gamma - 1} \sum_k \mu^\sigma_\gamma(e-k) \cdot \operatorname{ih}^\gamma_0(k) \right).$$

Here, the first summand corresponds to $\pi = [e]$, the sum collects all others. Moreover, the additional power of -1 in all terms in the sum is owed to the fact that all partitions contributing to $\mu^\sigma_\gamma(e-k)$ are one step shorter than their avatars, the partitions of e.

◇

Define

$$\gamma^\mho := \{y \in (\mathbb{R}^d)^*/\gamma^\perp \mid y(x) \geq 0 \; \forall x \in \gamma\}.$$

Since $(\gamma_\mathbb{R})^* \simeq (\mathbb{R}^d)^*/\gamma^\perp$ naturally, $\gamma^\mho$ is the dual of γ in its own span, hence absolute (independent of σ).

We have the following basic lemma on the dual of the cone γ/τ relative to $\gamma_\mathbb{R}/\tau_\mathbb{R}$.

Lemma 3.14. *Let $\tau \subseteq \gamma$ be faces of σ. Then*

$$(\gamma/\tau)^\mho \simeq \tau^\star/\gamma^\star,$$

the right hand side computed relative to σ.

Proof. We have $(\mathbb{R}^d/\tau_{\mathbb{R}})^\star/(\gamma/\tau)^\perp_{\sigma/\tau} = \tau^\perp_\sigma/\gamma^\perp_\sigma$, computing on the left relative to σ/τ and on the right relative to σ. We have thus:

$$\begin{aligned}
(\gamma/\tau)^{\mho} &= \{y \in (\mathbb{R}^d/\tau_{\mathbb{R}})^\star/(\gamma/\tau)^\perp = \tau^\perp/\gamma^\perp \mid y(x) \geq 0 \ \ \forall\, x \in \gamma/\tau\}\\
&= (\gamma^\vee \cap \tau^\perp)/\gamma^\perp\\
&= ((\sigma^\vee + \gamma^\perp) \cap \tau^\perp)/\gamma^\perp\\
&= (\sigma^\vee \cap \tau^\perp + \gamma^\perp)/\gamma^\perp\\
&\simeq (\sigma^\vee \cap \tau^\perp)/(\sigma^\vee \cap \tau^\perp \cap \gamma^\perp)\\
&= \tau^\star/\gamma^\star
\end{aligned}$$

where the third equality follows from $\gamma^\vee = \sigma^\vee + \gamma^\perp$ (cf. the proof of Proposition 2 on [Ful93, p.13]) and at the end we use the second isomorphism theorem. □

Definition 3.15. If $\tau \subseteq \gamma$ are faces of σ, denote $Y_{\gamma/\tau}$ the spectrum of the semigroup ring induced by the dual cone of σ/τ in its natural lattice. In other words, the cone γ/τ together with its faces defines a fan in $\gamma_{\mathbb{R}}/\tau_{\mathbb{R}}$. The corresponding toric variety is

$$Y_{\gamma/\tau} := X_{\tau^\star/\gamma^\star} = X_{(\gamma/\tau)^{\mho}}.$$

◇

The following lemma compares the intersection cohomology Betti numbers of $Y_{\sigma/\gamma}$ with those of $X_{\gamma/0} = X_\gamma$.

Lemma 3.16. *Let σ be a strongly convex rational polyhedral cone of dimension d as always. Then*

$$\sum_{0 \subseteq \gamma \subseteq \sigma} (-1)^{d_\gamma} \left(\sum_i \mathrm{ih}^{2i}(Y_{\sigma/\gamma})\, t^i\right)\left(\sum_j \mathrm{ih}^{2j}(X_\gamma)\, t^j\right) = 0.$$

Proof. To a cone $\sigma \subseteq \mathbb{R}A = \mathbb{R}^d$ belongs the affine toric variety $X_\sigma = \operatorname{Spec} \mathbb{C}[\sigma \cap \mathbb{Z}^d]$. Here is an overview of the proof. We first explain indpendence of $\mathrm{ih}^\bullet(-)$ of the lattice used to produce X_σ. We then discuss combinatorial intersection homology and how it applies to quotient polytopes and cones. Finally, we put the pieces together, using results of Stanley.

Now let $N \subseteq \mathbb{R}^d$ be another $\mathbb{Z}$-lattice (a free subgroup of rank d whose $\mathbb{Q}$-span is $\mathbb{Q}A$). The affine toric variety $X^N_\sigma := \operatorname{Spec} \mathbb{C}[\sigma \cap N]$ can be different from X_σ, however we have a canonical isomorphism

$$\mathrm{IH}^\bullet(X_\sigma) \simeq \mathrm{IH}^\bullet(X^N_\sigma). \tag{3.4.3}$$

This can be seen as follows: Consider the lattices $N' \supseteq N$ in $\mathbb{R}^d$. It is enough to prove that $\mathrm{IH}^\bullet(X^{N'}_\sigma) \simeq \mathrm{IH}^\bullet(X^N_\sigma)$. The finite group $G := N'/N$ naturally acts on $X^{N'}_\sigma$, and X^N_σ is the quotient of $X^{N'}_\sigma$ under this action (cf. [CLS11, Proposition 1.3.18]). We have the following isomorphism

$$\mathrm{IH}^\bullet(X^N_\sigma) \simeq \mathrm{IH}^\bullet(X^{N'}_\sigma)^G = \mathrm{IH}^\bullet(X^{N'}_\sigma)$$

where $\mathrm{IH}^\bullet(X_\sigma^{N'})^G$ is the G-invariant part. The isomorphism follows from [Kir86, Lemma 2.12] and the equality comes from the fact that the action of G is induced by the action of the open dense $\mathbb{C}$-torus of $X_\sigma^{N'}$ which acts trivially: a $\mathbb{C}$-torus acting continuously on a rational vector space must have a dense subset acting trivially; continuity forces triviality everywhere. Hence when writing $\mathrm{IH}^\bullet(X_\sigma)$ we do not need to worry about the lattice with respect to which X_σ is defined.

Assume that we are given a rational polytope $P \subseteq \mathbb{R}^{d-1}$ of dimension $d-1$. The set of faces of P (including the empty face $\emptyset$), ordered by inclusion, forms a poset. Given such a polytope, Stanley [Sta87] defined polynomials

$$g(P) = \sum g_i(P)t^i \qquad \text{and} \qquad h(P) = \sum h_i(P)t^i \tag{3.4.4}$$

recursively by

- $g(\emptyset) = 1$;
- $h(P) = \sum_{\emptyset \leq F < P} (t-1)^{\dim P - \dim F - 1} g(F)$;
- $g_0(P) = h_0(P)$, $\quad g_i(P) = h_i(P) - h_{i-1}(P)$ for $0 < i \leq \dim P/2$ and $g_i(P) = 0$ for all other i.

Now assume 0 is in the interior $\mathrm{Int}(P)$. From such a polytope we get a fan Σ_P by taking the cones over the faces of P; here the empty face corresponds to the cone $\{0\} \subseteq \mathbb{R}^{d-1}$. This gives a projective toric variety X_P. It was proved independently by Denef and Loeser [DL91] and Fieseler [Fie91] that

$$h_i(P) = \mathrm{ih}^{2i}(X_P).$$

Denote by $\mathrm{cone}(X_P)$ the affine cone of X_P. Then

$$g_i(P) = h_i(P) - h_{i-1}(P) = \mathrm{ih}^{2i}(\mathrm{cone}(X_P)) \qquad \text{for } 0 < i \leq \dim(P)/2.$$

The affine cone of X_P has the following toric description: Consider the embedding of $P \subseteq \mathbb{R}^{d-1}$ in $\mathbb{R}^d$ under the map $i : x \mapsto (1, x)$. Let $\mathrm{Cone}(P)$ be the (rational, polyhedral, strongly convex) cone over $i(P)$ with apex at the origin. Then $\mathrm{cone}(X_P)$ is an affine toric variety given by $\mathrm{cone}(X_P) = X_{\mathrm{Cone}(P)^\vee} = \mathrm{Spec}\, \mathbb{C}[\mathrm{Cone}(P)^\vee \cap (\mathbb{Z}^d)^*]$. Hence we get

$$g_i(P) = \mathrm{ih}^{2i}(X_{\mathrm{Cone}(P)^\vee})\,. \tag{3.4.5}$$

Two polytopes P_1 and P_2 are *combinatorially equivalent* if they have isomorphic face posets, denoted $P_1 \sim P_2$. This is an equivalence relation, and $g(P)$ and $h(P)$ only depend on the equivalence class $[P]$ of P. Similarly, given two strongly convex rational polyhedral cones σ_1 and σ_2 we write $\sigma_1 \sim \sigma_2$ if their face posets are isomorphic. If we have $\sigma_i = \mathrm{Cone}(P_i)$ for $i = 1, 2$ then $[P_1 \sim P_2] \Leftrightarrow [\mathrm{Cone}(P_1) \sim \mathrm{Cone}(P_2)]$.

For a given rational polytope P with $0 \in \mathrm{Int}(P)$, the dual polytope is

$$P^\circ := \{x \in (\mathbb{R}P)^* \mid x(y) \geq -1\ \forall y \in P\},$$

$\mathbb{R}P$ being the affine span of P. There is an order-reversing bijection of the k-dimensional faces F of P and the $(\dim(P) - 1 - k)$-dimensional faces $\{x \in P^\circ \mid x(F) = -1\}$ of P°.

If the origin is not in $\operatorname{Int}(P)$, translate P so that $0 \in \operatorname{Int}(P)$ and then dualize. The combinatorial equivalence class of the dual is then well-defined and we still write P° for this class.

From a k-dimensional face F of the $(d-1)$-dimensional polytope P we construct an equivalence class of $(d-k-2)$-dimensional polytopes P/F as follows. Choose a $(d-k-2)$-dimensional affine subspace L whose intersection with P is a single point of the interior of F. Then a representative of P/F is given by $L' \cap P$ where L' is another $(d-k-2)$-dimensional affine subspace, near L in the appropriate Grassmannians, and such that it meets an interior point of P. (One checks that this representative is well-defined up to projective transformation, hence the combinatorial type is well-defined). One can see easily that the cone over P/F is exactly $\operatorname{Cone}(P)/\operatorname{Cone}(F)$, compare (3.4.1):

$$\operatorname{Cone}(P/F) \sim \operatorname{Cone}(P)/\operatorname{Cone}(F) = (\operatorname{Cone}(P) + \mathbb{R}F)/\mathbb{R}F. \tag{3.4.6}$$

We will prove Lemma 3.16 using the following formula by Stanley [Sta92] (we use here a presentation given by Braden and MacPherson in [BM99, Proposition 8, formula (3)]):

$$\sum_{\emptyset \subseteq F \subseteq P} (-1)^{\dim F} g(F^\circ) g(P/F) = 0 \tag{3.4.7}$$

The dual F° of a rational polytope F is rational in many lattices. Choosing one such lattice yields a rational, polyhedral, strongly convex cone $\operatorname{Cone}(F^\circ)$ for which $\operatorname{Cone}(F^\circ)^\mho$ is well-defined. By (3.4.3), its intersection homology is independent of the lattice choice. It follows that, with γ the cone over F,

$$g_i(F^\circ) = \mathrm{ih}^{2i}(X_{\operatorname{Cone}(F^\circ)^\mho}) \simeq \mathrm{ih}^{2i}(X_{\operatorname{Cone}(F)}) \simeq \mathrm{ih}^{2i}(X_\gamma) \tag{3.4.8}$$

where we used formula (3.4.3) for the last isomorphism. Recalling Definition 3.15 and that $\operatorname{Cone}(P) = \sigma$, we obtain

$$g_i(P/F) = \mathrm{ih}^{2i}(X_{\operatorname{Cone}(P/F)^\mho}) = \mathrm{ih}^{2i}(X_{(\operatorname{Cone}(P)/\operatorname{Cone}(F))^\mho}) = \mathrm{ih}^{2i}(Y_{\operatorname{Cone}(P)/\operatorname{Cone}(F)}) = \mathrm{ih}^{2i}(Y_{\sigma/\gamma}), \tag{3.4.9}$$

where the first equality is (3.4.5), the second equality follows from (3.4.6), the third equality is Definition 3.15, and the last follows from (3.4.3). Plugging (3.4.8) and (3.4.9) into (3.4.7) and multiplying with (-1) we get the statement of the Lemma. □

We are now ready to give our main result about the weight filtration on the inverse Fourier–Laplace transform of the A-hypergeometric system $H_A(0)$:

Theorem 3.17. *The associated graded module to the weight filtration on the mixed Hodge module $h_*({}^p\mathbb{Q}^H_T)$ is for $e = 0, \dots, d$ given by*

$$\mathrm{gr}^W_{d+e}\, \varphi_*({}^p\mathbb{Q}^H_T) \simeq \bigoplus_\tau \mathrm{IC}_{X_\tau}({}^p\mathcal{L}_{(\tau,d+e)}),$$

where ${}^p\mathcal{L}_{(\tau,d+e)} = L^0_{(\tau,d+e)} \otimes {}^p\mathbb{Q}^H_{T_\tau}$ is a constant variation of Hodge structures of weight $d+e$ on T_τ. Here $L^0_{(\tau,d+e)}$ is a Hodge-structure of Hodge–Tate type *of weight $d+e-d_\tau$ of dimension*

$$\mu^\sigma_\tau(e) = \dim_{\mathbb{Q}} L^0_{(\tau,d+e)} = \mathrm{ih}_c^{d_\sigma-d_\tau+e}(Y_{\sigma/\tau}),$$

compare Definition 3.15.

Proof. In light of Proposition 3.10 it only remains to prove that $\mu^\sigma_\tau(e) = \dim_{\mathbb{Q}} L^0_{(\tau,d+e)}$ equals $\mathrm{ih}_c^{d_\sigma-d_\gamma+e}(Y_{\sigma/\tau})$.

An inspection shows that if $\sigma=\tau$ then the theorem is (trivially) correct. We argue by induction on $d_\sigma - d_\tau$. While in principle a Poincaré series only involves non-negative terms there is no harm in allowing negative indices: they just add zero terms.

According to Lemma 3.16 we have

$$\begin{aligned}
0 &= \sum_{0\subseteq\gamma\subseteq\sigma} (-1)^{d_\gamma} \left(\sum_{j=-\infty}^{\infty} t^j \cdot \mathrm{ih}^{2j}(Y_{\sigma/\gamma})\right) \cdot \left(\sum_{i=-\infty}^{\infty} t^i \cdot \mathrm{ih}^{2i}(X_\gamma)\right)\\
&= \sum_{0\subseteq\gamma\subseteq\sigma} (-1)^{d_\gamma} \left(\sum_{j=-\infty}^{\infty} t^j \cdot \mathrm{ih}_c^{2(d_\sigma-d_\gamma-j)}(Y_{\sigma/\gamma})\right) \cdot \left(\sum_{i=-\infty}^{\infty} t^i \cdot \mathrm{ih}_c^{2(d_\gamma-i)}(X_\gamma)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Lemma 3.4)}\\
&= (-1)^{d_\sigma} \cdot \sum_{k=-\infty}^{\infty} t^k \cdot \mathrm{ih}_c^{2(d_\sigma-k)}(X_\sigma)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(from } \gamma=\sigma)\\
&\quad + \sum_{0\subsetneq\gamma\subsetneq\sigma} (-1)^{d_\gamma} \left(\sum_{j=-\infty}^{\infty} t^j \cdot \mathrm{ih}_c^{2(d_\sigma-d_\gamma-j)}(Y_{\sigma/\gamma})\right) \cdot \left(\sum_{j=-\infty}^{\infty} t^i \cdot \mathrm{ih}_c^{2(d_\gamma-i)}(X_\gamma)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(general } \gamma)\\
&\quad + \sum_{k=-\infty}^{\infty} t^k \cdot \mathrm{ih}_c^{2(d_\sigma-k)}(Y_{\sigma/0}).\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(from } \gamma=0)
\end{aligned}$$

where we have used Lemma 3.4 (1) for the second equality.

Induction allows to substitute $\mu^\sigma_\gamma(d_\sigma - d_\gamma - 2j)$ for $\mathrm{ih}_c^{2(d_\sigma-d_\gamma-j)}(Y_{\sigma/\gamma})$ for all $\gamma \neq 0, \sigma$ in the sum "general γ". At the same time we can replace, by definition, $\mathrm{ih}_c^{2(d_\gamma-i)}(X_\gamma)$ by $\mathrm{ih}_0^\gamma(d_\gamma - 2i)$. With these substitutions, collect terms with equal t-power:

$$0=(-1)^{d_\sigma}\cdot\sum_{k=-\infty}^{\infty} t^k\cdot \mathrm{ih}_0^\sigma(d_\sigma-2k) \qquad \text{(from } \gamma=\sigma)$$

$$+\sum_{k=-\infty}^{\infty}\left(\sum_{i+j=k} t^k \sum_{0\subsetneq\gamma\subsetneq\sigma}(-1)^{d_\gamma}\left(\mu_\gamma^\sigma(d_\sigma-d_\gamma-2j)\cdot \mathrm{ih}_0^\gamma(d_\gamma-2i)\right)\right) \qquad \text{(general } \gamma)$$

$$+\sum_{k=-\infty}^{\infty} t^k\cdot \mathrm{ih}_c^{2(d_\sigma-k)}(Y_{\sigma/0}). \qquad \text{(from } \gamma=0)$$

In degree k we have therefore:

(3.4.10)

$$0=(-1)^{d_\sigma}\cdot \mathrm{ih}_0^\sigma(d_\sigma-2k)+\sum_{i+j=k}\left(\sum_{0\subsetneq\gamma\subsetneq\sigma}(-1)^{d_\gamma}\mu_\gamma^\sigma(d_\sigma-d_\gamma-2j)\cdot \mathrm{ih}_0^\gamma(d_\gamma-2i)\right)+\mathrm{ih}_c^{2(d_\sigma-k)}(Y_{\sigma/0}).$$

Since the odd-dimensional intersection homology Betti numbers are zero (cf. Lemma 3.4 (3)), we can include all missing summands $(-1)^{d_\gamma}\mu_\gamma^\sigma(d_\sigma-d_\gamma-j')\cdot \mathrm{ih}_0^\gamma(d_\gamma-i')$ with $i'+j'=2k$ without affecting the value of the sum. Since $\mathrm{ih}_0^\gamma(d_\gamma-i')=\mathrm{ih}_c^{d_\gamma+(d_\gamma-i')}(X_\gamma)$, no summand with $i'':=d_\gamma-i'\le 0$ can contribute (cf. Lemma 3.4 (2)). We can therefore rewrite (3.4.10) to

(3.4.11)

$$0=(-1)^{d_\sigma}\cdot \mathrm{ih}_0^\sigma(d_\sigma-2k)+\sum_{i''}\left(\sum_{0\subsetneq\gamma\subsetneq\sigma}(-1)^{d_\gamma}\mu_\gamma^\sigma(d_\sigma-2k-i'')\cdot \mathrm{ih}_0^\gamma(i'')\right)+\mathrm{ih}_c^{2(d_\sigma-k)}(Y_{\sigma/0}).$$

In light of the recursion (3.4.2), this yields $0=-\mu_0^\sigma(d_\sigma-2k)+\mathrm{ih}_c^{2(d_\sigma-k)}(Y_{\sigma/0})$ and finishes the inductive step. □

4. Weight filtrations on A-hypergeometric systems

In this section we translate the results from the previous section to hypergeometric D-modules on

$$V:=\mathbb{C}^n$$

via the Fourier transform. Part of this is rather mechanical, but identifying the weight filtrations requires some extra hypotheses, see Corollary 4.13.

4.1. Translation of the filtration.

We start this section with various definitions around A-hypergeometric systems. For more details, we refer to (for example) [MMW05, RSW18]. Our terminology is that of [MMW05].

Throughout, we continue Notation 3.1

Definition 4.1. Write $\mathbb{L}_A$ for the $\mathbb{Z}$-module of integer relations among the columns of A and write $\mathcal{D}_{\mathbb{C}^n}$ for the sheaf of rings of differential operators on $V = \mathbb{C}^n$ with coordinates $x_1, \dots, x_n$. Denote ∂_j the operator $\partial/\partial x_j$. For $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{C}^d$ define

$$\mathcal{M}_A^\beta := \mathcal{D}_{\mathbb{C}^n}/\mathcal{I}_A^\beta$$

where $\mathcal{I}_A^\beta$ is the sheaf of left ideals generated by the *toric operators*

$$\Box_{\mathbf{u}} := \prod_{u_j<0} \partial_j^{-u_j} - \prod_{u_j>0} \partial_j^{u_j}$$

for all $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{L}_A$, and the *Euler operators*

$$E_i := \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i.$$

◇

We will write $M_A^\beta := \Gamma(V, \mathcal{M}_A^\beta)$ for the D_A-module of global sections where $D_A = \Gamma(V, \mathcal{D}_V)$. Denote by R_A (resp. O_A) the polynomials rings over $\mathbb{C}$ generated by $\partial_A = \{\partial_j\}_j$ (resp. $x_A = \{x_j\}_j$). and set $S_A := R_A/R_A\{\Box_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{L}_A}$.

We have

$$x^{\mathbf{u}} E_i - E_i x^{\mathbf{u}} = -(A \cdot \mathbf{u})_i x^{\mathbf{u}},$$
$$\partial^{\mathbf{u}} E_i - E_i \partial^{\mathbf{u}} = (A \cdot \mathbf{u})_i \partial^{\mathbf{u}}.$$

Define the A-degree on R_A and D_A as

$$\deg_A(x_j) = \mathbf{a}_j = -\deg_A(\partial_j) \in \mathbb{Z}A$$

and denote by $\deg_{A,i}(-)$ the degree associated to the i-th row of A. This convention agrees with the choices in [MMW05] but is opposite to that in [Rei14]. Then $E_i P = P(E_i + \deg_{A,i}(P))$ for any A-graded $P \in D_A$.

Given a left A-graded D_A-module M we can define commuting D_A-linear endomorphisms E_i via

$$E_i \circ m := (E_i - \deg_{A,i}(m)) \cdot m$$

for A-graded elements of M. If N is an A-graded R_A-module N we get a commuting set of D_A-linear endomorphisms on the left D_A-module $D_A \otimes_{R_A} N$ by

$$E_i \circ (P \otimes Q) := (E_i - \deg_i(P) - \deg_i(Q)) P \otimes Q$$

for any A-graded P, Q. The *Euler–Koszul complex* $K_\bullet(M; E - \beta)$ of the A-graded R_A-module N is the homological Koszul complex induced by $E - \beta := \{(E_i - \beta_i)\circ\}_i$ on $D_A \otimes_{R_A} N$. The terminal module sits in homological degree zero. We denote by $\mathcal{K}_\bullet(N; E - \beta)$ the corresponding complex of quasi-coherent sheaves. The homology objects are $H_\bullet(N; E - \beta)$ and $\mathcal{H}_\bullet(N; E - \beta)$, respectively.

For a finitely generated A-graded R_A-module $N = \bigoplus_\alpha N_\alpha$ write $\deg_A(N) = \{\alpha \in \mathbb{Z}A \mid N_\alpha \neq 0\}$ and then let the *quasi-degrees* of N be

$$\operatorname{qdeg}_A(N) := \overline{\deg_A(N)}^{Zar},$$

the Zariski closure of $\deg_A(N)$ in $\mathbb{C}^d$.

The following subset of parameters $\beta \in \mathbb{C}^d$ will be of importance to us.

Definition 4.2 ([SW09])**.** The set of *strongly resonant parameters* of A is

$$\mathrm{sRes}\,(A) := \bigcup_{j=1}^{d} \mathrm{sRes}_{\,j}(A)$$

where

$$\mathrm{sRes}_{\,j}(A) := \left\{ \beta \in \mathbb{C}^d \mid \beta \in -(\mathbb{N}+1)\mathbf{a}_j - \mathrm{qdeg}\,(S_A/(\partial_j)) \right\}.$$

$\diamond$

Definition 4.3. Let

$$\langle -,- \rangle : \overbrace{\mathbb{C}^n}^{\widehat{V}} \times \overbrace{\mathbb{C}^n}^{V} \to \mathbb{C}, \qquad (\mathfrak{y}_1, \ldots, \mathfrak{y}_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n,) \mapsto \sum_{i=1}^{n} \mathfrak{x}_i \mathfrak{y}_i \,.$$

We define a $\mathcal{D}_{\widehat{V}\times V}$-module by

$$\mathcal{L} := \mathcal{O}_{\widehat{V}\times V} \cdot \exp((-1)\cdot \langle -,- \rangle),$$

and we refer to [KS97, Section 5] for details on these sheaves. Denote by $p_i : \widehat{V} \times V \to \mathbb{C}^n$ for $i = 1,2$ the projection to the first and second factor respectively (identifying the respective factor with the target). The *Fourier–Laplace* transform is defined by

$$\begin{aligned} \mathrm{FL}\colon \mathrm{D}^b_{qc}(\mathcal{D}_{\widehat{V}}) \quad &\longrightarrow \quad \mathrm{D}^b_{qc}(\mathcal{D}_V), \\ \mathcal{M} \quad &\mapsto \quad p_{2+}(p_1^+ \mathcal{M} \overset{L}{\otimes} \mathcal{L})[-n]. \end{aligned}$$

$\diamond$

We denote by $\hat{M}_A^\beta$ the module of global sections to the sheaf

$$\hat{\mathcal{M}}_A^\beta := \mathrm{FL}^{-1}(\mathcal{M}_A^\beta)$$

and define the following twisted structure sheaves on T:

$$\mathcal{O}_T^\beta := \mathcal{D}_T / \mathcal{D}_T \cdot (\partial_t t_1 + \beta_1, \ldots, \partial_{t_d} t_d + \beta_d),$$

where we note that $\mathcal{O}_T^\beta \simeq \mathcal{O}_T^\gamma$ if and only if $\beta - \gamma \in \mathbb{Z}^d$.

Theorem 4.4. *(*[SW09] *Theorem 3.6, Corollary 3.7) Let A be a pointed $(d \times n)$ integer matrix satisfying $\mathbb{Z}A = \mathbb{Z}^d$. Then for the map h in* (1.2.1)*, the following statements are equivalent*

(1) $\beta \notin \mathrm{sRes}\,(A)$*;*
(2) $\hat{\mathcal{M}}_A^\beta \simeq h_+ \mathcal{O}_T^\beta$. $\square$

Theorem 4.4 implies that for $\beta \in \mathbb{Z}^d \smallsetminus \mathrm{sRes}\,(A)$ we have, with notation as in (1.3.2),

$$\hat{\mathcal{M}}_A^\beta \simeq h_+ \mathcal{O}_T \simeq i_+ \varphi_+ \mathcal{O}_T.$$

We now concentrate on integral β. Since $\mathcal{O}_T$ is the underlying left $\mathcal{D}_T$-module of ${}^p\mathbb{Q}_T^H$ this induces the structure of a mixed Hodge module on $\hat{\mathcal{M}}_A^\beta$ from Theorem 3.17. Recalling Definition 2.0.1 and bearing in mind that the functor i_* preserves weight, we infer:

Corollary 4.5. *For $\beta \in \mathbb{Z}^d \smallsetminus \mathrm{sRes}\,(A)$, the module $\hat{\mathcal{M}}_A^\beta = \mathrm{FL}^{-1}(\mathcal{M}_A^\beta)$ carries the structure of a mixed Hodge module ${}^H\hat{\mathcal{M}}_A^\beta$ which is induced by the isomorphism*

$$\hat{\mathcal{M}}_A^\beta \simeq \mathrm{Dmod}(i_* \varphi_* {}^p\mathbb{Q}_T^H).$$

The corresponding weight filtration is given by

$$\mathrm{gr}_{d+e}^W \, {}^H\hat{\mathcal{M}}_A^\beta \simeq \bigoplus_\gamma \bar{i}_{\gamma *} \, \mathrm{IC}_{X_\gamma}({}^p\mathcal{L}_{(\gamma,d+e)})$$

where $\bar{i}_\gamma : X_\gamma \to \mathbb{C}^n$ is the embedding of the closure of the γ-torus, and ${}^p\mathcal{L}_{(\gamma,d+e)} = L^0_{(\gamma,d+e)} \otimes {}^p\mathbb{Q}^H_{T_\gamma}$ is a constant variation of Hodge structures of weight $d+e$. Here $L^0_{(\gamma,d+e)}$ is a Hodge-structure of Hodge–Tate type of weight $d+e-d_\gamma$ of dimension

$$\dim_{\mathbb{Q}} L^0_{(\gamma,d+e)} = \mathrm{ih}_c^{d_\sigma - d_\gamma + e}(Y_{\sigma/\gamma}),$$

with $Y_{\sigma/\gamma}$ as in Definition 3.15. □

As a corollary, we obtain information about the holonomic length of M_A^0. Recall that $\mathcal{M}^{\mathrm{IC}}(X_\gamma) = \mathrm{Dmod}(\mathrm{IC}(X_\gamma))$ is the unique simple T-equivariant $\mathcal{D}$-Module on $\widehat{V}$ with support X_γ.

Corollary 4.6. *Let A be as in Notation 3.1 and choose $\beta \in \mathbb{Z}^d \smallsetminus \mathrm{sRes}\,(A)$. Then $\mathcal{M}_A^\beta$ carries a finite separated exhaustive filtration $\{\hat{W}_\bullet \mathcal{M}_A^\beta\}_{e=0}^d$ given by*

$$\hat{W}_\bullet \mathcal{M}_A^\beta := \mathrm{FL}(W_\bullet {}^H\mathcal{M}_A^\beta).$$

This filtration satisfies

$$\mathrm{gr}_{d+e}^{\hat{W}} \mathcal{M}_A^\beta = \bigoplus_\gamma \bigoplus_{i=1}^{\mu_\gamma^\sigma(e)} C_\gamma.$$

Here, $C_\gamma = \mathrm{FL}\,\mathcal{M}^{\mathrm{IC}}(X_\gamma)$ is a simple equivariant holonomic $\mathcal{D}$-module (that is independent of e and) which occurs in $\mathrm{gr}_{d+e}^{\hat{W}} \mathcal{M}_A^\beta$ with multiplicity $\mu_\gamma^\sigma(e) = \mathrm{ih}_c^{d_\sigma - d_\gamma + e}(Y_{\sigma/\gamma}) = \mathrm{ih}^{d_\sigma - d_\gamma - e}(Y_{\sigma/\gamma})$, the $(d_\sigma - d_\gamma - e)$-th intersection cohomology Betti number of the affine toric variety $Y_{\sigma/\gamma}$. □

4.2. **The homogeneous case: monodromic Fourier–Laplace.** Although the Fourier–Laplace transformation does not preserve regular holonomicity in general, and so $\mathcal{M}_A^\beta$ may not be a mixed Hodge module, it is preserved for the derived category of complexes of $\mathcal{D}$-modules with so-called *monodromic cohomology*. In this case we can express the Fourier–Laplace transformation as a monodromic Fourier transformation (or Fourier–Sato transformation). In order to make this work, we now assume that the matrix A is *homogeneous*, which means that

$$(1, \ldots, 1)^T \in \mathbb{Z}(A^T).$$

Via a suitable coordinate change on the torus T, we can then assume that the top row of A is $(1, \ldots, 1)$.

Denote by

$$\theta : \mathbb{C}^* \times \widehat{V} \to \widehat{V}$$

the standard $\mathbb{C}^*$ action on $\hat{V}$; let z be a coordinate on $\mathbb{C}^*$. We refer to the push-forward $\theta_*(z\partial_z)$ as the *Euler vector field* $\mathfrak{E}$.

Definition 4.7. [Bry86] A regular holonomic $\mathcal{D}_{\widehat{V}}$-module $\mathcal{M}$ is called *monodromic*, if the Euler field $\mathfrak{E}$ acts finitely on the global sections of $\mathcal{M}$: for each global section section v of $\mathcal{M}$ the set $\{\mathfrak{E}^n(v)\}_{n\in\mathbb{N}}$ should generate a finite-dimensional vector space. We denote by $D^b_{mon}(\mathcal{D}_{\widehat{V}})$ the derived category of bounded complexes of $\mathcal{D}_{V'}$-modules with regular holonomic and monodromic cohomology. ◇

Since we assume that A has $(1,\ldots,1)$ as its top row, each $\hat{\mathcal{M}}^\beta_A$ is monodromic.

Theorem 4.8. [Bry86]

(1) FL *preserves complexes with monodromic cohomology.*
(2) In $D^b_{mon}(\mathcal{D}_V)$ *and* $D^b_{mon}(\mathcal{D}_{\widehat{V}})$ *we have*

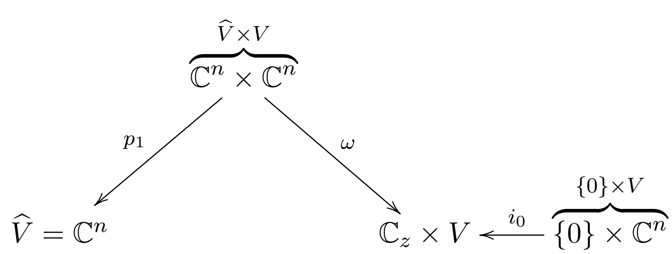
$$\mathrm{FL}\circ\mathrm{FL} \simeq \mathrm{Id} \quad \textit{and} \quad \mathbb{D}\circ\mathrm{FL} \simeq \mathrm{FL}\circ\mathbb{D}\,.$$

(3) FL *is t-exact with respect to the natural t-structures on* $D^b_{mon}(\mathcal{D}_{V'})$ *resp.* $D^b_{mon}(\mathcal{D}_V)$*.*

Proof. The above statements are stated in [Bry86] for constructible monodromic complexes. One has to use the Riemann-Hilbert correspondence, [Bry86, Proposition 7.12, Theorem 7.24] to translate the statements. So the first statement is Corollaire 6.12, the second statement is Proposition 6.13 and the third is Corollaire 7.23 in [Bry86]. □

We will now consider the monodromic Fourier–Laplace transform (or Fourier–Sato transform) which preserves the category of mixed Hodge modules.

Definition 4.9. Consider the diagram



where p_1 is the projection to the first factor, i_0 is the inclusion and the map ω is given by

$$\begin{aligned}\omega\colon \widehat{V}\times V &\longrightarrow \mathbb{C}_z \times V\\ (\mathfrak{y},\mathfrak{x}) &\mapsto (\mathfrak{z} = \sum \mathfrak{x}_i\mathfrak{y}_i, \mathfrak{y})\end{aligned}$$

The *Fourier–Sato transform* or *monodromic Fourier transform* is defined by

$$\begin{aligned} \mathrm{D}^b(\mathrm{MHM}(\widehat{V})) &\longrightarrow \mathrm{D}^b(\mathrm{MHM}(V)) \\ \mathcal{M} &\mapsto \phi_z \omega_* {}^p p_1^! \mathcal{M} \simeq \phi_z \omega_! {}^p p_1^! \mathcal{M} \end{aligned}$$

where ϕ_z is the nearby cycle functor along $z = 0$ and we write ${}^p f^! := f^![d_Y - d_X]$ for a map $f : X \to Y$. The isomorphism follows from [KS94, Proposition 10.3.18]. ◇

Remark 4.10. The original definition of the Fourier–Sato transform is different; we use here an equivalent version (see [KS94, Def. 3.7.8, Prop. 10.3.18]) that is well adapted to mixed Hodge modules. ◇

For a monodromic complex the (usual) Fourier–Laplace transformation and the monodromic Fourier transformation are the same (we use again the equivalent version of the Fourier-Sato version from [KS94]):

Theorem 4.11. [Bry86, Théorème 7.24] *Let $\mathcal{M} \in D^b_{mod}(\mathcal{D}_{\widehat{V}})$ then*

$$DR^{an}(\mathrm{FL}(\mathcal{M})) \simeq \phi_z \omega_* {}^p p_1^! DR^{an}(\mathcal{M}). \qquad \square$$

It follows that the monodromic Fourier transform induces an exact functor

$$\phi_z \omega_* {}^p p_1^! : \mathrm{MHM}(\widehat{V}) \longrightarrow \mathrm{MHM}(V)$$

We next identify a class of modules for which the monodromic Fourier transform has a very simple effect on the weight filtration.

Proposition 4.12. *Let $\pi : \widehat{V} \smallsetminus \{0\} \to \mathbb{P}(\widehat{V})$ be the natural projection and $j_0 : \widehat{V} \smallsetminus \{0\} \to \widehat{V}$ the inclusion. Let $\mathcal{M} \in \mathrm{MHM}(\widehat{V})$ such that $\mathcal{M} \simeq (j_0)_* \pi^! \mathcal{N}$ for some $\mathcal{N} \in \mathrm{D}^b\,\mathrm{MHM}(\mathbb{P}(\widehat{V}))$. Then*

$$W_k \left(\phi_z \omega_* {}^p p_1^! \mathcal{M} \right) \simeq \phi_z \omega_* {}^p p_1^! (W_k \mathcal{M})$$

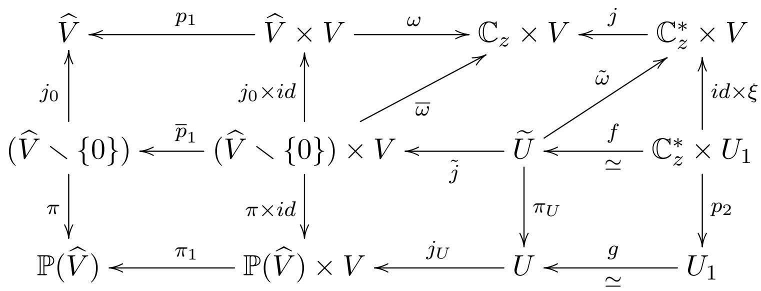
Proof. We first prove that the logarithm of the monodromy N acts trivially on $\phi_z \omega_* p_1^! \mathcal{M}$. Define the subvarieties

$$\begin{aligned} U &:= \{\sum_{i=1}^n \mathfrak{y}_i \mathfrak{x}_i \neq 0\} \subseteq \mathbb{P}(\widehat{V}) \times V \\ \widetilde{U} &:= \{\sum_{i=1}^n \mathfrak{y}_i \mathfrak{x}_i \neq 0\} \subseteq (\widehat{V} \smallsetminus \{0\}) \times V \\ U_1 &:= \{\sum_{i=1}^n \mathfrak{y}_i \mathfrak{x}_i = 1\} \subseteq (\widehat{V} \smallsetminus \{0\}) \times V \end{aligned}$$

with the embeddings $j_U : U \to \mathbb{P}(\widehat{V}) \times V$ and $\widetilde{j} : \widetilde{U} \to (\widehat{V} \smallsetminus \{0\}) \times V$. Notice that we have isomorphisms

$$\begin{aligned} f \colon \mathbb{C}^* \times U_1 &\longrightarrow \widetilde{U} \\ (\mathfrak{z}, \mathfrak{y}, \mathfrak{x}) &\mapsto (\mathfrak{z} \cdot \mathfrak{y}, \mathfrak{x}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} g \colon U_1 &\longrightarrow U \\ (\mathfrak{y}, \mathfrak{x}) &\mapsto ((\mathfrak{y}_1 : \ldots : \mathfrak{y}_n), \mathfrak{x}). \end{aligned}$$

Consider now the following diagram



where $\xi : U_1 \subseteq (\widehat{V} \smallsetminus \{0\}) \times V \to V$ is the projection to the second factor and $\overline{p}_1, \pi_1$ resp. $\overline{\omega}$, $\tilde{\omega}$ are the corresponding restrictions of p_1 resp. ω.

We have the following isomorphisms

$$\begin{aligned}
j^! \omega_* {}^p p_1^! \mathcal{M} &\simeq j^! \omega_* {}^p p_1^! j_{0*} \pi^! \mathcal{N} \\
&\simeq j^! \omega_* (j_0 \times id)_* {}^p \overline{p}_1^! \pi^! \mathcal{N} \\
&\simeq j^! \omega_* (j_0 \times id)_* (\pi \times id)^{!p} \pi_1^! \mathcal{N} \\
&\simeq j^! \overline{\omega}_* (\pi \times id)^{!p} \pi_1^! \mathcal{N} \\
&\simeq \tilde{\omega}_* \tilde{j}^! (\pi \times id)^{!p} \pi_1^! \mathcal{N} \\
&\simeq \tilde{\omega}_* \pi_U^! j_U^! {}^p \pi_1^! \mathcal{N} \\
&\simeq (id \times \pi_2^U)_* (f^{-1})_* \pi_U^! j_U^! {}^p \pi_1^! \mathcal{N} \\
&\simeq (id \times \pi_2^U)_* f^! \pi_U^! j_U^! {}^p \pi_1^! \mathcal{N} \\
&\simeq (id \times \pi_2^U)_* p_2^! g^! j_U^! {}^p \pi_1^! \mathcal{N}
\end{aligned}$$

Set $\mathcal{N}' := g^! j_U^! \pi_1^! \mathcal{N}$. We have $(id \times \xi)_* p_2^! \mathcal{N}' \simeq \tilde{p}_2^! \xi_* \mathcal{N}'$ where $\tilde{p}_2 : \mathbb{C}_z^* \times V \to V$ is the projection to the second factor. This shows that $j^! \omega_* p_1^! \mathcal{M} \simeq \tilde{p}_2^! \xi_* \mathcal{N}'$ is constant in the z-direction. Hence the logarithm of the monodromy N acts trivially on the (unipotent) nearby cycles $\psi_z \omega_* p_1^! \mathcal{M}$ and therefore also on the vanishing cycles $\phi_z \omega_* p_1^! \mathcal{M}$.

Set $L_i \phi_z \omega_* {}^p p_1^! \mathcal{M} := \phi_z W_i \mathcal{H}^0 \omega_* {}^p p_1^! \mathcal{M}$. The weight filtration on $\phi_z \omega_* {}^p p_1^! \mathcal{M} = \phi_z \mathcal{H}^0 \omega_* {}^p p_1^! \mathcal{M}$ is the relative monodromy weight filtration with respect to the filtration L and the nilpotent endomorphism N. In the (current) case $N = 0$ we simply get $W_i \phi_z \omega_* p_1^! \mathcal{M} = L_i \phi_z \omega_* p_1^! \mathcal{M} = \phi_z W_i \mathcal{H}^0 \omega_* p_1^! \mathcal{M}$ (cf. [Sai90, (2.2.7) & Proposition 2.4]).

We now want to prove by decreasing induction on ℓ that

- $W_\ell \phi_z \omega_* {}^p p_1^! \mathcal{M} = \phi_z \omega_* {}^p p_1^! W_\ell \mathcal{M}$
- $\phi_z \omega_* {}^p p_1^! Gr_\ell^W \mathcal{M}$ is pure of weight ℓ.

This is certainly true for $\ell \gg 0$ since in this case $W_\ell \mathcal{M} = \mathcal{M}$. Assume now that the two statements above are true for some ℓ, we prove the two statements for $\ell - 1$. For this consider the exact sequence

$$\phi_z \omega_* {}^p p_1^! W_{\ell-1} \mathcal{M} \longrightarrow \phi_z \omega_* {}^p p_1^! W_\ell \mathcal{M} \longrightarrow \phi_z \omega_* {}^p p_1^! Gr_\ell^W \mathcal{M}. \tag{4.2.1}$$

Since $W_\ell \phi_z \omega_* {}^p p_1^! \mathcal{M} = \phi_z \omega_* {}^p p_1^! W_\ell \mathcal{M}$ and since $\phi_z \omega_* {}^p p_1^! Gr_\ell^W \mathcal{M}$ is pure of weight ℓ we see that

$$\phi_z \omega_* {}^p p_1^! W_{\ell-1} \mathcal{M} \supseteq W_{\ell-1} \phi_z \omega_* {}^p p_1^! \mathcal{M}.$$

To show the other inclusion, we consider the morphism

$$\mathcal{H}^0 \omega_! {}^p p_1^! W_{\ell-1} \mathcal{M} \longrightarrow \mathcal{I}_{\ell-1} \longrightarrow \mathcal{H}^0 \omega_* {}^p p_1^! W_{\ell-1} \mathcal{M},$$

where $\mathcal{I}_{\ell-1}$ is the image of the morphism $\mathcal{H}^0 \omega_! {}^p p_1^! W_{\ell-1} \mathcal{M} \to \mathcal{H}^0 \omega_* {}^p p_1^! W_{\ell-1} \mathcal{M}$. Notice that the map (4.2.1) becomes an isomorphism after applying ϕ_z (cf.[KS94, equation 10.3.32]).

Since ${}^p p_1^! W_{\ell-1} \mathcal{M} = W_{\ell-1} {}^p p_1^! \mathcal{M}$ and the functor $\omega_!$ does not increase weight we have $\mathcal{H}^0 \omega_! {}^p p_1^! W_{\ell-1} \mathcal{M} \subseteq W_{\ell-1} \mathcal{H}^0 \omega_! {}^p p_1^! \mathcal{M}$. Because $\mathcal{I}_{\ell-1}$ is a quotient of $\mathcal{H}^0 \omega_! {}^p p_1^! W_{\ell-1} \mathcal{M}$ we also have $W_{\ell-1} \mathcal{I}_{\ell-1} = \mathcal{I}_{\ell-1}$. Since $\mathcal{I}_{\ell-1}$ is a subobject of $\mathcal{H}^0 \omega_* {}^p p_1^! W_{\ell-1}$ the morphism N acts trivially and therefore $W_{\ell-1} \phi_z \mathcal{I}_{\ell-1} = \phi_z \mathcal{I}_{\ell-1}$. The isomorphism $\phi_z \mathcal{I}_{\ell-1} \simeq \phi_z \omega_* {}^p p_1^! W_{\ell-1} \mathcal{M}$ shows

$$\phi_z \omega_* {}^p p_1^! W_{\ell-1} \mathcal{M} = W_{\ell-1} \phi_z \omega_* {}^p p_1^! W_{\ell-1} \mathcal{M} \subseteq W_{\ell-1} \phi_z \omega_* {}^p p_1^! \mathcal{M}$$

We now want to show that $\phi_z \omega_* {}^p p_1^! Gr_{\ell-1}^W \mathcal{M}$ is pure of weight $\ell - 1$. For this consider the morphisms

$$\mathcal{H}^0 \omega_! {}^p p_1^! Gr_{\ell-1}^W \mathcal{M} \longrightarrow \mathcal{G}_m \longrightarrow \mathcal{H}^0 \omega_* {}^p p_1^! Gr_{\ell-1}^W \mathcal{M}$$

where $\mathcal{G}_{\ell-1}$ is the image of the morphism $\mathcal{H}^0 \omega_! {}^p p_1^! Gr_{\ell-1}^W \mathcal{M} \to \mathcal{H}^0 \omega_* {}^p p_1^! Gr_{\ell-1}^W \mathcal{M}$. Notice again that the map above becomes an isomorphism after applying ϕ_z. Since ${}^p p_1^!$ preserves weight, and since $\omega_!$ does not increase weight and since ω_* does not decrease weight the module $\mathcal{G}_{\ell-1}$ is pure of weight $\ell - 1$. Since $\phi_z \mathcal{G}_{\ell-1}$ is a subobject of $\phi_z \mathcal{H}^0 \omega_* {}^p p_1^! Gr_{\ell-1}^W \mathcal{M}$ and $\phi_z \mathcal{H}^0 \omega_* {}^p p_1^! Gr_{\ell-1}^W \mathcal{M}$ is a quotient of $\phi_z \mathcal{H}^0 \omega_* {}^p p_1^! \mathcal{M}$ the morphism N is trivial on $\phi_z \mathcal{G}_{\ell-1}$. Therefore $\phi_z \mathcal{G}_{\ell-1} \simeq \phi_z \mathcal{H}^0 \omega_* {}^p p_1^! Gr_{\ell-1}^W \mathcal{M}$ is pure of weight $\ell - 1$.

This finishes the proof of the proposition. □

If we endow the GKZ-system $\mathcal{M}_A^0$ with the mixed Hodge module structure coming from the monodromic Fourier transformation we get the following result.

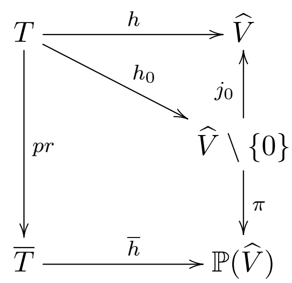
Corollary 4.13. *For homogeneous A in the context of Corollary 4.6 and Definition 4.9, let ${}^H\mathcal{M}_A^0$ be the GKZ-system endowed with the mixed Hodge module structure coming from the isomorphism*

$$\mathcal{M}_A^0 \simeq \mathrm{Dmod}(\phi_z \omega_* {}^p p_1^! \, {}^H\widehat{\mathcal{M}}_A^0)$$

with ${}^H\widehat{\mathcal{M}}_A^0$ as in Corollary 4.5. Then

$$\mathrm{Dmod}(W_k {}^H\mathcal{M}_A^0) = \phi_z \omega_* {}^p p_1^! (W_k {}^H\widehat{\mathcal{M}}_A^0) = \mathrm{FL}(W_k \widehat{\mathcal{M}}_A^0).$$

Proof. It remains to shows that ${}^H\widehat{\mathcal{M}}_A^0$ can be written as $(j_0)_*\pi^!\mathcal{N}$ for some $\mathcal{N} \in \mathrm{D}^b\,\mathrm{MHM}(\mathbb{P}(\widehat{V}))$. Consider the diagram



where $pr : T \to \overline{T}$ is the projection to the last $d-1$ coordinates $h = j_0 \circ h_0$ is the canonical factorization and $\overline{h}$ is the projectivization of h. We have

$$\begin{aligned} {}^H\widehat{\mathcal{M}}_A^0 &\simeq h_*{}^p\mathbb{Q}_T^H \simeq h_* pr^*\,{}^p\mathbb{Q}_{\overline{T}}^H \simeq h_* pr^!\,{}^p\mathbb{Q}_{\overline{T}}^H \simeq h_* pr^!\,{}^p\mathbb{Q}_{\overline{T}}^H(-1)[-1] \\ &\simeq (j_0)_*(h_0)_* pr^!\,{}^p\mathbb{Q}_{\overline{T}}^H(-1)[-1] \simeq (j_0)_*\pi^! \underbrace{\overline{h}_*{}^p\mathbb{Q}_{\overline{T}}^H(-1)[-1]}_{=:\mathcal{N}} \end{aligned}$$

□

5. Explicit weight filtration for $d = 3$

Throughout this section, A is normal but not necessarily homogeneous. Via the Fourier transform FL one can port the weight filtration on the mixed Hodge module $h_*{}^p\mathbb{Q}_T^H$ to the hypergeometric system $\mathcal{M}_A^0$. While the latter may not be a mixed Hodge module, one still obtains in any case a filtration that has semisimple associated graded pieces and which we still denote by $W_\bullet$. If A is homogeneous, then $\mathcal{M}_A^0$ is a mixed Hodge module and, by Corollary 4.13, FL agrees with the functor $\phi_z\omega_*{}^p p_1^!$ and relates the weight filtrations on $\mathcal{M}_A^0$ and $h_*{}^p\mathbb{Q}_T^H$. In this section we consider specifically the cases when either $\mathbb{N}A$ is simplicial, or when $d \leq 3$ and write out an explicit filtration in terms of generators that agrees with $W_\bullet$.

Batyrev proved that in the homogeneous, normal case the weight filtration on the restriction of $\mathcal{M}_A^0$ to the complement of the principal A-discriminant is given by the face filtration on S_A in the sense that (in the localization) $W_{d+k}(\mathcal{M}_A^0)$ is generated by the ∂-monomials whose degree sits in the relative interior of a face of σ whose codimension is at most k; see [Sti98, Thm. 8, p.28]. It has been speculated that this be true even on $\mathcal{M}_A^0$ itself. We show here that this is the case for simplicial homogeneous σ but can fail in the general homogeneous case already in dimension three. We discuss completely in terms of generators the filtration $\mathrm{FL}(W_\bullet h_*(\mathcal{O}_T))$ if $d = 3$ and A is normal (but not necessarily homogeneous). Then σ is the cone over a $(d-1)$-dimensional polygon P with f_0 vertices and P arises as intersection of σ with a generic hyperplane. It is not suggested or required that the columns of A lie on P. It is sufficient to concentrate on the global sections M_A^0.

Notation 5.1. On M_A^0, let $W'_\bullet$ be the filtration of Batyrev:

$$W'_{d+k}(M_A^0) = \text{image of } D_A{\cdot}\{\partial^{\mathbf{u}} \mid A\cdot\mathbf{u}=\mathbf{a}\in \operatorname{Int}(\mathbb{N}\tau), \dim(\tau)\geq \dim(\sigma)-k\} \text{ in } M_A^0$$

for $d\leq k\leq 2d$. In particular, $W'_{<d}(M_A^0)=0$ and $W_{>2d}(M_A^0)=M_A^0$.

For $d=3$ let $W''_\bullet$ be the filtration

$$W''_k(M_A^0) = \begin{cases} W'_k(M_A^0) & \text{if } k\neq 2d-2,\\ W'_k+\sum_r D_A\cdot e_r & \text{if } k=2d-2,\end{cases}$$

where e_r is defined below in (5.0.2).

For ease of notation , we do not repeat "M_A^0" each time we write a filtration piece. We will show that $W''=W$ if $d\leq 3$, and that $W'=W''=W$ if σ is simplicial. For this, consider the toric modules defined as follows.

Notation 5.2. If τ is a face of σ write ∂_τ^+ for the S_A-ideal generated by the ∂-monomials whose degree is interior to τ. Let $S_A^{(k)}$ be the ideal of S_A spanned by the monomials that are interior to a face of codimension k or less, $S_A^{(k)}=\sum_{\dim\tau\geq d-k}\partial_\tau^+$. Then $S_A^{(0)}$ is the interior ideal, $S_A^{(d-1)}$ is the maximal ideal $S_A\partial_A$, and $S_A^{(d)}$ is S_A itself.

We begin with showing that for normal S_A the D_A-module generated by the interior ideal $S_A^{(0)}$ inside M_A^0 is simple and for homogeneous A agrees with W_d so that $W_k=W'_k=W''_k$ for $k\leq d$.

Lemma 5.3. *Suppose A is pointed and saturated, but not necessarily homogeneous. Let $\mathbf{u},\mathbf{v}\in\mathbb{N}^n$ be such that $\mathbf{b}:=A\cdot\mathbf{v}$ is in the interior* $\operatorname{Int}(\mathbb{N}A)$ *of the semigroup (*i.e.*, not on a proper face). Set $\mathbf{a}=A{\cdot}\mathbf{u}$. Then the contiguity map $c_{-\mathbf{a}-\mathbf{b}}\colon M_A^{-\mathbf{a}-\mathbf{b}}\xrightarrow{\cdot\partial^{\mathbf{u}}} M_A^{-\mathbf{b}}$ is an isomorphism.*

In particular, the ideal in M_A^0 generated by $\partial^{\mathbf{b}}$ (the image of the contiguity morphism $c_{-\mathbf{b},0}\colon M_A^{-\mathbf{b}}\longrightarrow M_A^0$) is the same for all $\mathbf{b}=A\cdot\mathbf{v}$ in the interior of A.

Proof. Consider the toric sequence $0\longrightarrow S_A(\mathbf{a})\xrightarrow{\cdot\partial^{\mathbf{u}}} S_A\longrightarrow Q:=S_A/S_A\cdot\partial^{\mathbf{u}}\longrightarrow 0$, and the Euler–Koszul functor attached to $-\mathbf{b}$. By [MMW05, Prop. 5.3], the induced contiguity morphism $c_{-\mathbf{b}-\mathbf{a},-\mathbf{b}}\colon M_A^{-\mathbf{b}-\mathbf{a}}\longrightarrow M_A^{-\mathbf{b}}$ is an isomorphism if and only if $-\mathbf{b}$ is not quasi-degree of Q. The quasi-degrees of $Q=S_A/\partial^{\mathbf{u}}S_A$ are contained in a union of hyperplanes that meet $-\mathbb{N}A$ and are parallel to a face of the cone σ. In particular, these quasi-degrees are disjoint to the interior points $\operatorname{Int}(\mathbb{N}A)$ of $\mathbb{N}A$. It follows that $c_{-\mathbf{b}-\mathbf{a},-\mathbf{b}}$ is an isomorphism for all $\mathbf{b}\in\operatorname{Int}(\mathbb{N}_A)$.

Now consider the composition

$$c_{-\mathbf{b},0}\circ c_{-\mathbf{a}-\mathbf{b},-\mathbf{b}}\colon M_A^{-\mathbf{a}-\mathbf{b}}\longrightarrow M_A^{-\mathbf{b}}\longrightarrow M_A^0$$

with $\mathbf{a},\mathbf{b}\in\operatorname{Int}(\mathbb{N}A)$. The first map is an isomorphism, and so the image of the composition is just the image of $c_{-\mathbf{b},0}$. For any two elements $\mathbf{b},\mathbf{b}'\in\operatorname{Int}(\mathbb{N}A)$, factoring $M_A^{-\mathbf{b}-\mathbf{b}'}\longrightarrow M_A^0$ through $M_A^{-\mathbf{b}}$ or $M_A^{-\mathbf{b}'}$ shows that the images of $c_{-\mathbf{b},0}$ and $c_{-\mathbf{b}',0}$ agree with the image of $c_{-\mathbf{b}-\mathbf{b}',0}$. In particular, they are equal. Since $\partial^{\mathbf{u}}$ is in the image of $c_{-A\cdot\mathbf{u},0}$, the image of $c_{-\mathbf{b},0}$ contains all of $\operatorname{Int}(\mathbb{N}A)$ whenever $\mathbf{b}\in\operatorname{Int}(\mathbb{N}A)$. ∎

It follows that for normal S_A the submodule of M_A^0 generated by any interior monomial of S_A agrees with that submodule generated by $S_A^{(0)}$. If A is homogeneous, so that FL carries the mixed Hodge module structure from $h_* {}^p\mathbb{Q}_T^H$ to $\mathcal{M}_A^0$, the level d part of $W_\bullet$ has the property that $h_* {}^p\mathbb{Q}_T^H / \operatorname{FL}^{-1}(W_d\mathcal{M}_A^0)$ is supported on the boundary tori. Thus, any section of this sheaf is killed by some power of $x_1 \cdots x_n$, so that each element of M_A^0/W_d is killed by some power of $\partial_1 \cdots \partial_n$. That means that $H_A(0) + W_d$ contains (the coset of) an interior monomial of S_A, and hence W_d contains the submodule generated by $S_A^{(0)}$. Since W_d is simple, it cannot strictly contain it, so must be equal to it.

As an aside, note that the Euler–Koszul homology module $H_0^A(S_A^{(0)}; 0)$ associated to the interior ideal is the underlying D_A-module to $\operatorname{FL}(h_\dagger {}^p\mathbb{Q}_T^H) = \operatorname{FL}\mathbb{D}(h_* {}^p\mathbb{Q}_T^H)$. Indeed, it follows from [Wal07] that the dual of M_A^0 is $M_A^{-\gamma}$ for some interior point of $\mathbb{N}A$. Since $S_A^{(0)}$ is the direct limit of all principal ideals generated by interior monomials, $H_0^A(S_A^{(0)}; 0)$ is the direct limit of all $H_0^A(S_A \cdot \partial^{\mathbf{u}}; 0)$ with $\partial^{\mathbf{u}}$ interior to $\mathbb{N}A$. It follows from 5.3 that the structure morphisms in the limit are all isomorphisms. Thus, we can identify the morphisms $H_0^A(\operatorname{Int}(\mathbb{N}A); 0) \longrightarrow M_A^0$ and $\operatorname{FL}\operatorname{Dmod}\left(h_! {}^p\mathbb{Q}_T^H \longrightarrow h_* {}^p\mathbb{Q}_T^H\right)$, and the corresponding statement holds for any face τ with lattice $\tau_\mathbb{Z}$.

It is clear that $S_A^{(k)} \subseteq S_A^{(k+1)}$ and that the quotient $S_A^{(k)}/S_A^{(k-1)}$ is the direct sum of the interior ideals of the face rings S_τ for which $\dim(\tau) = d - k$. It follows that $\operatorname{gr}_k^{W'}(M_A^0)$ surjects onto each $H_0^A(\operatorname{Int}(S_\tau); 0)$, the Euler–Koszul module defined over D_A by the toric module formed by the graded maximal submodule of the toric module $S_\tau = S_A/\{\partial_j \mid j \notin \tau\}$, see [MMW05] for details. It therefore also surjects onto the image of $H_0^A(\operatorname{Int}(S_\tau); 0)$ in $H_0^A(S_\tau; 0)$, the underlying D_A-module corresponding to $\operatorname{IC}_{X_\tau}$ under the monodromic Fourier transform.

If $\mathbb{N}A$ is simplicial, Theorem 3.17 implies that $\operatorname{gr}_{d+k}^W(h_* {}^p\mathbb{Q}_T^H)$ is the sum of intersection complexes $\operatorname{IC}_{X_\tau}$ with $\dim(\tau) + k = d$, and each appears with multiplicity one. Thus, $\operatorname{gr}_k^{W'}(M_A^0)$ surjects onto $\operatorname{gr}_k^W(M_A^0)$ for all k when $\mathbb{N}A$ is simplicial. (We are not asserting that this surjection is induced from a filtered morphism, only that there is one; after all, we don't know W at this point). But M_A^0 is holonomic and by the Jordan–Hölder property this implies that $W' = W$ when $\mathbb{N}A$ is simplicial. This recovers for $\beta \in \mathbb{N}A$ a result of [Fan18].

Now suppose $d = 3$ but don't assume simpliciality.[2] By Theorem 3.17 any composition chain for M_A^0 will (up to Fourier transform) have as composition factors exactly one copy of the intersection complex to τ for $\dim(\tau) > 0$, and $1 + f_0 - d$ copies of IC_0. This means that an epimorphism $\operatorname{gr}^{W''}(M_A^0) \longrightarrow \operatorname{gr}^W(M_A^0)$ alone will not be enough to show $W'' = W$ since the copies of IC_0 need to be shown to live in the right levels.

In any event, $W_6 = M_A^0$ and W_3 is generated by the interior ideal $S_A^{(0)}$. Equivariance and the fact that gr_6^W must equal $\mathbb{C}[x_A]$ shows that W_5 is

[2] If $d \leq 2$, $\mathbb{N}A$ is always simplicial

generated by the maximal ideal $S_A^{(2)}$. It remains to find generators for W_4'', such that there are surjections $\mathrm{gr}_k^{W''}(M_A^0) \longrightarrow \mathrm{gr}_k^W(M_A^0)$ for $k = 4, 5$ such that at least one is an isomorphism.

For arbitrary saturated $\mathbb{N}A$ with $d = 3$, define on $M_A^0 = D_A/(I_A, E)$ a filtration as follows:

- $W_i'' = 0$ for $i < 3$;
- W_3'' is the left ideal generated by ∂_σ^+;
- W_4'' is the left ideal generated by W_3'' and all $\partial_{\tau_2}^+$ where $\dim(\tau_2) = 2$, plus the left ideal generated by all e_r defined below, where e runs through the f_0 vertices of P;
- W_5'' is the left ideal generated by W_4'' and all $\partial_{\tau_1}^+$ where $\dim(\tau_1) = 1$;
- W_6'' is the left ideal generated by $1 \in D$.

We now describe the operators e_r. Choose *distinguished* nonzero columns $\{\mathbf{b}_r\}_1^{f_0}$ of A that correspond to the primitive lattice points on the rays through the vertices of the polygon P which are in A since $\mathbb{N}A$ is saturated).

For each distinguished $\mathbf{b}_r$ define a function F_r on $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ as follows:

$$F_r(\mathbf{a}_j) \quad = \quad \begin{cases} 1 & \text{if } \mathbf{a}_j = \mathbf{b}_r; \\ c_{r,j} & \text{if } \mathbf{a}_j = c_{r,j} \cdot \mathbf{b}_r + c_{r',j} \cdot \mathbf{b}_{r'} \text{ is on the } \sigma\text{-face spanned by } \mathbf{b}_r \text{ and } \mathbf{b}_{r'}; \\ 0 & \text{else.} \end{cases} \tag{5.0.1}$$

We call *invisible from* $\mathbf{b}_r$ any $\mathbf{a} \in \mathbb{Z}A$ for which the ray from $\mathbf{b}_r$ to $\mathbf{a}$ passes through the interior of σ. Then F_r vanishes on all $\mathbf{a}_j$ invisible from $\mathbf{b}_r$, and F_r is piece-wise linear on the 2-faces of σ (which are in bijection with edges of P). Set

$$e_r = \sum_j F_r(\mathbf{a}_j) x_j \partial_j. \tag{5.0.2}$$

We now show that our filtration W'' is indeed the Fourier–Laplace transform of the weight filtration on $h_* {}^p\mathbb{Q}_T^H$. Note first that W_5'' indeed contains W_4'' (specifically, the e_r). We prove now, that each e_r is annihilated by $\mathfrak{m}$ in M_A^0/W_3, and hence they are candidates for the intersection complexes in W_4/W_3 with support in 0.

Let $\mathbf{b}_{r_1}$ and $\mathbf{b}_{r_2}$ be the two distinguished columns that lie on a facet with $\mathbf{b}_r$. Then there is a unique linear function E_r on $\mathbb{R}^3$ whose values agree with those of F_r on $\mathbf{b}_r, \mathbf{b}_{r_1}$ and $\mathbf{b}_{r_2}$. We denote the corresponding Euler operator also by E_r. The linearity of F_r along facets implies that F_r and E_r agree on all $\mathbf{a}_j$ that have $F_r(\mathbf{a}_j)$ nonzero (which are the $\mathbf{a}_j$ not invisible from $\mathbf{b}_r$). Thus, in $M_A^0/W_3'' = D/(I_A, E, \partial_\sigma^+)$, the expression e_r is equivalent to a linear combination $e_{r,0} = e_r - E_r$ of $\{x_j\partial_j\}_j$ for which each $\mathbf{a}_j$ with nonzero coefficient is invisible from $\mathbf{b}_r$. Now, in $D/(I_A, E, \partial_\sigma^+)$, $\partial_j e_r$ is zero for $\mathbf{a}_j$ invisible from $\mathbf{b}_r$, and $\partial_j e_{r,0} = 0$ also for $\mathbf{a}_j$ any integer multiple of $\mathbf{b}_r$ and for $\mathbf{a}_j$ interior to the facets touching $\mathbf{b}_r$. If $\mathbf{a}_j = \mathbf{b}_{r_1}$, consider the Euler operator E_{r_1} that agrees with e_r on $\mathbf{b}_r$ and on $\mathbf{b}_{r_1}$, and which takes value zero on the 2-face of σ containing $\mathbf{b}_{r_1}$ but not $\mathbf{b}_r$. Then $(e_r - E_{r_1})$ has all terms invisible from ∂_j and so $\partial_j(e_r - E_{r_1})$ is zero in W_6''/W_3''. A similar argument works for $\mathbf{a}_j = \mathbf{b}_{r_2}$. Hence every ∂_j annihilates the class of e_r in W_6''/W_3'' and so e_r spans a module in W_6''/W_3'' that is either zero or $D/D\mathfrak{m}$. Note that there are $d = \dim(\sigma) = 3$ linear dependencies between the cosets of the e_r in M_A^β, so the $\{e_r\}_r$ are spanning a module isomorphic to a submodule of $\oplus_1^{f_0-d} D/D\mathfrak{m}$.

Next, let $\mathbf{a}$ be in the relative interior of a facet τ of σ. Since W_3 contains every interior monomial of σ, the coset of $\partial^{\mathbf{a}}$ in M_A^0/W_3'' is ∂_j-torsion for all $j \notin \tau$. Let $h_\tau \colon T_\tau \longrightarrow \mathbb{C}^\tau$ be the toric map, induced by the restriction of A to τ, from the τ-torus to the subspace $\mathbb{C}^\tau$ of $\mathbb{C}^A$ parameterized by the columns of $A \cap \tau$. The submodule generated by $\partial^{\mathbf{a}}$ inside M_A^0/W_3'' is isomorphic to a quotient of the simple module $\mathbb{C}[x_{\tau^c}] \otimes_{\mathbb{C}} \operatorname{FL}\operatorname{im}((h_\tau)_\dagger \longrightarrow (h_\tau)_+)$, where x_{τ^c} are the x_j with $j \notin \tau$.

Now consider an interior monomial $\partial^{\mathbf{a}}$ of a ray τ_1 of σ. Then in M_A^0/W_4'', $\partial^{\mathbf{a}}$ is killed by all ∂_j with $j \notin \tau_1$. Modulo the ∂_j not sitting on any ray, e_r becomes exactly the Euler operator for $M_{\tau_1}^0$ if $\mathbf{b}_r$ sits on τ_1, and hence (after the Fourier transform) the module generated by $\partial^{\mathbf{a}}$ in M_A^0/W_4'' is exactly the intersection complex associated to τ_1 (pushed to V). Hence $W_5''/W_4'' \simeq \bigoplus_{\tau_1} \mathrm{IC}_{\tau_1}$, and so

- $W_k = W_k''$ if $k \leq 3$ and if $k \geq 5$;
- $W_5''/W_4'' \simeq W_5/W_4$;
- hence $W_4''/W_3'' \simeq W_4/W_3$ by Jordan–Hölder.

Since the faces whose intersection complexes appear as summands in W_5/W_4 have dimension one, and those in W_4/W_3 have dimension 0 or 2, W_4'' must equal W_4.

Example 5.4. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, one of the possible matrices whose GKZ-system (with the right β) contains Gauß' hypergeometric ${}_2F_1$ as solution. We have $n = 4$ and $d = 3$, and P is a square in which 1 and 4 are opposite vertices. The Euler space E is spanned by $x_1\partial_1 + x_3\partial_3$, $x_2\partial_2 + x_4\partial_4$ and $x_3\partial_3 + x_4\partial_4$. The four elements e_r are simply $\{x_j\partial_j\}_1^4$. The toric ideal is generated by $\partial_1\partial_4 - \partial_2\partial_3$. The interior ideal of S_A is generated by $\partial_2\partial_3$. The weight filtration on M_A^0 is given by $W_2 = \overline{0}$, $W_3 = \overline{\{E, \partial_1\partial_4, \partial_2\partial_3\}}$, $W_4 = W_3 + \overline{\{\partial_1\partial_2, \partial_2\partial_4, \partial_4\partial_3, \partial_3\partial_1\}} + \overline{\{e_1, e_2, e_3, e_4\}}$, $W_5 = W_4 + \overline{\{\partial_1, \partial_2, \partial_3, \partial_4\}}$, $W_6 = M_A^0$. Here, the bar indicates taking cosets on M_A^0. Note that the three Euler dependencies in $H_A(0)$ imply that the four operators e_r generate only one copy of IC_0 inside W_4/W_3.

6. Concluding remarks and open problems

(1) We assume throughout that S_A is normal, which covers the most significant geometric situations. One obvious challenge is to remove this hypothesis and generalize our results. This would be likely difficult since then arithmetic issues will enter the fray.

(2) In another direction it would be interesting to see what can be done (as mixed Hodge module or otherwise) when $\beta \neq 0$. Recently J. Fang has

posted an article [Fan18] on the arXiv where composition chains for hypergeometric systems are considered that are based on the filtration-by-faces on the semigroup ring, see also [AS] for motivating discussion. These are based on the filtration-by-faces in Notation 5.1 first considered by Batyrev, see [Bat93, Sti98].[3] The hypotheses are somewhat technical, but in the simplicial normal case [Fan18] shows essentially that for $\beta = 0$ the filtration-by-faces gives semisimple composition factors. Comparing with the weight filtration, this corresponds to all non-diagonal terms $\mu_\tau^\sigma(e)$ with $\dim(\tau)+e \neq d$ being zero in Theorem 3.17, the case of trivial combinatorics in the polytope to σ.

(3) By adding all nonzero $\mu_\tau^\sigma(e)$ one obtains the holonomic length of M_A^0. Is there a compact formula? In particular, does it give a better estimate than the general exponential bounds in [SST00]? When P is simplicial, $\ell(M_A^0) = 2^d$, while for $d = 3, 4, 5$ these lengths are for general P as follows, where in generalization of the face numbers f_i of P we denote $f_{i,j}$ the number of all pairs (i-face, j-face) that are contained in one another. For relations between the various $f_{i,j}$ for 4-polytopes, see [Bay87].

d	$\ell(M_A^0)$
3	$1 + f_0 + f_1 + f_2 + (f_0 - 3) = 3f_0 - 1$
4	$1 + f_0 + f_1 + f_2 + f_3 + (f_0 - 4) + (f_{1,0} - 3f_0)$ $= -2f_0 + 4f_1$
5	$1 + f_0 + f_1 + f_2 + f_3 + f_4 + (f_0 - 5) + (f_{1,0} - 4f_0) + (f_{2,1} - 3f_1)$ $+ (f_{2,0} - 3f_2 + f_1 - 4f_0 + 10)$ $= 7 - 5f_0 - f_2 + 2f_{2,0}$

Of course, all these numbers are non-negative. Is there an obviously non-negative representation?

(4) Given A and a face τ, what is the holonomic rank of the Fourier transform of the intersection complex on the orbit to τ? Such formulæ would be very interesting even for normal simplicial A since it interweaves volume-based expressions for rank with combinatorial expressions in the way Pick's theorem talks about polygons. For example, when $d = 2$ and A is normal, one can derive from our results that the rank of $\mathrm{FL}^{-1}(\mathrm{IC}(\mathcal{L}_\tau))$ always differs from the volume of A by one. Induction on d gives recursions, but an explicit formula is unknown.

(5) In Section 5 we explained how to write down explicitly the weight filtration for $d = 3$. For $d = 4$, similar ideas can be used to write out explicit generators. But starting with $d = 5$ this seems a very hard problem. Part of the issue is that writing down such filtration would produce a non-canceling expression for the higher intersection cohomology dimension of polytopes of dimension 4 or greater, which we do not think are known.

[3] We note in passing that the filtration-by-faces is not a natural filtration: typically, if GKZ-systems $M_A^\beta \equiv M_A^\gamma$ are isomorphic under a contiguity morphism, the two face filtrations do not correspond.

List of symbols

- $\mathbb{C}^n = V = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$, the domain of the GKZ system M_A^β,
- $\mathbb{C}^n = \widehat{V} = \operatorname{Spec} [y_1, \ldots, y_n]$, the target of h,
- T the d-torus,
- $h\colon T \longrightarrow \widehat{V}$ the monomial map induced by A,
- X the closure of T in $\widehat{V}$,
- $\varphi\colon T \longrightarrow X$ the restriction of h,
- $\deg(x) = \mathbf{a} = -\deg(\partial)$ the A-degree function on $S_A = \mathbb{C}[\mathbb{N}A]$,
- ϕ_z the vanishing cycle along the function z,
- ψ_z the corresponding nearby cycle,
- $i\colon X \longrightarrow V$ the closed embedding,
- $i_\tau\colon \mathfrak{x}_\tau \hookrightarrow X_\tau$ the embedding of the T-fixed point,
- $\rho_\tau\colon \mathfrak{x}_{\sigma/\tau} \times T_\tau \hookrightarrow X_\sigma$ and $j_\tau\colon X_{\sigma/\tau} \times T_\tau \hookrightarrow X_\sigma$ from $\mathbb{N}A \longrightarrow (\sigma/\tau)_\mathbb{N} \oplus \tau_\mathbb{Z}$,
- the relative version $j_\tau^\gamma\colon X_{\gamma/\tau} \times T_\tau \longrightarrow X_\gamma$ to j_τ,
- $i_{\tau,\gamma}\colon T_\tau \longrightarrow X_\tau \longrightarrow X_\gamma$ from $\gamma_\mathbb{N} \twoheadrightarrow \tau_\mathbb{N} \longrightarrow \tau_\mathbb{Z}$,
- $\kappa_\mathbf{v}\colon \mathbb{G}_m = \operatorname{Spec} \mathbb{C}[z^\pm] \to T = \operatorname{Spec} \mathbb{C}[\mathbb{Z}A]$ the monomial action induced by $\mathbf{v}$,
- $u_\tau\colon X_\tau \smallsetminus \mathfrak{x}_\tau \hookrightarrow X_\tau$,
- $i_\tau\colon \mathfrak{x}_\tau \longrightarrow V$.

References

[AS] Alan Adolphson and Steve Sperber, *Composition series for $M_A(\beta)$*, Conference talk, accessed under `http://www.math.tamu.edu/~laura/pages/organizing/AMS-SLC-2011/adolphson.pdf` on September 21, 2018.

[Bat93] Victor V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. **69** (1993), no. 2, 349–409.

[Bay87] Margaret Bayer, *The extended f-vectors of 4-polytopes*, J. Combin. Theory Ser. A **44** (1987), no. 1, 141–151. MR 871395

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR 751966

[BK91] Margaret M. Bayer and Andrew Klapper, *A new index for polytopes*, Discrete Comput. Geom. **6** (1991), no. 1, 33–47. MR 1073071

[BM99] Tom Braden and Robert MacPherson, *Intersection homology of toric varieties and a conjecture of Kalai*, Comment. Math. Helv. **74** (1999), no. 3, 442–455. MR 1710686

[Bry86] Jean-Luc Brylinski, *Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques*, Astérisque (1986), no. 140-141, 3–134, 251, Géométrie et analyse microlocales.

[CDK16] Sabin Cautis, Christopher Dodd, and Joel Kamnitzer, *Associated graded of Hodge modules and categorical sl_2 actions*, arXiv e-print (2016), no. 1603.07402, 1–30.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[dCM09] Mark Andrea A. de Cataldo and Luca Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633. MR 2525735

[dCMM14] Mark de Cataldo, Luca Migliorini, and Mircea Mustata, *The combinatorics and topology of proper toric maps*, arXiv e-print (2014), no. 1407.3497, 1–30.

[Dim92] Alexandru Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer-Verlag, New York, 1992.

[Dim04] ———, *Sheaves in topology*, Universitext, Springer-Verlag, Berlin, 2004.

[DL91] J. Denef and F. Loeser, *Weights of exponential sums, intersection cohomology, and Newton polyhedra*, Invent. Math. **106** (1991), no. 2, 275–294.

[Fan18] J. Fang, *Composition series for GKZ-systems*, arXiv e-print (2018), no. 1805.05604.

[Fie91] Karl-Heinz Fieseler, *Rational intersection cohomology of projective toric varieties*, J. Reine Angew. Math. **413** (1991), 88–98. MR 1089798

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.

[Giv96] Alexander B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663. MR 1408320

[Giv98] Alexander Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175. MR 1653024

[GKZ90] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), no. 2, 255–271. MR 1080980

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *$\mathcal{D}$-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi.

[Iri09] Hiroshi Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), no. 3, 1016–1079. MR 2553377

[IX16] Hiroshi Iritani and Jifu Xiao, *Extremal transition and quantum cohomology: examples of toric degeneration*, Kyoto J. Math. **56** (2016), no. 4, 873–905. MR 3568645

[Kir86] Frances Kirwan, *Rational intersection cohomology of quotient varieties*, Invent. Math. **86** (1986), no. 3, 471–505. MR 860678

[KS94] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1994, With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original.

[KS97] ———, *Integral transforms with exponential kernels and Laplace transform*, J. Amer. Math. Soc. **10** (1997), no. 4, 939–972. MR 1447834

[LW19] András C. Lőrincz and Uli Walther, *On categories of equivariant $\mathcal{D}$-modules*, Adv. Math. **351** (2019), 429–478. MR 3952575

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), no. 4, 919–941 (electronic).

[Rei14] Thomas Reichelt, *Laurent Polynomials, GKZ-hypergeometric Systems and Mixed Hodge Modules*, Compositio Mathematica **(150)** (2014), 911–941.

[RS15] Thomas Reichelt and Christian Sevenheck, *Hypergeometric Hodge modules*, no. arXiv:1503.01004.

[RS17] ———, *Non-affine Landau-Ginzburg models and intersection cohomology*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 3, 665–753. MR 3665553

[RSW18] Thomas Reichelt, Christian Sevenheck, and Uli Walther, *On the b-Functions of Hypergeometric Systems*, Int. Math. Res. Not. IMRN (2018), no. 21, 6535–6555. MR 3873536

[Sai90] Morihiko Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.

[SS] Claude Sabbah and Christian Schnell, *The MHM-project*.

[SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000. MR 1734566 (2001i:13036)

[Sta87] Richard Stanley, *Generalized H-vectors, intersection cohomology of toric varieties, and related results*, Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 187–213. MR 951205

[Sta92] Richard P. Stanley, *Subdivisions and local h-vectors*, J. Amer. Math. Soc. **5** (1992), no. 4, 805–851. MR 1157293

[Ste17] Avi Steiner, *A-hypergeometric modules and Gauss–Manin systems*, arXiv e-print (2017), no. 1712.00500, 1–35.

[Ste18] ———, *A-hypergeometric modules and Gauss–Manin systems*, arXiv e-print (2018), no. 1805.02727, 1–18.

[Sti98] Jan Stienstra, *Resonant hypergeometric systems and mirror symmetry*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 412–452.

[SW09] Mathias Schulze and Uli Walther, *Hypergeometric $\mathcal{D}$-modules and twisted Gauß-Manin systems*, J. Algebra **322** (2009), no. 9, 3392–3409.

[Wal07] Uli Walther, *Duality and monodromy reducibility of A-hypergeometric systems*, Math. Ann. **338** (2007), no. 1, 55–74.

[Web04] Andrzej Weber, *Weights in the cohomology of toric varieties*, Cent. Eur. J. Math. **2** (2004), no. 3, 478–492 (electronic). MR 2113544

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