WEIGHT FILTRATIONS ON GKZ-SYSTEMS

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Abstract. Given an integer matrix \( A \in \mathbb{Z}^{d \times n} \), we study the natural mixed Hodge module structure in the sense of Saito on the Gauss–Manin system attached to the monomial map \( \phi: (\mathbb{C}^*)^d \longrightarrow \mathbb{C}^n \) induced by \( A \). We completely determine in the normal case the corresponding weight filtration by computing the intersection complexes with respective multiplicities that constitute the associated graded parts. Our results show that these data are purely combinatorial, and not arithmetic, in the sense that they only depend on the polyhedral structure of the cone of \( A \), but not on the semigroup itself. In particular, we extend results of de Cataldo, Migliorini and Mustata to the setting of torus embeddings and give a closed form for the failure of the Decomposition Theorem.

If \( A \) is homogeneous and if \( \beta \in \mathbb{C}^d \) is an integral but not strongly resonant parameter, we make use of a monodromic Fourier–Laplace transform to carry the mixed Hodge module structure from the Gauss–Manin system to the GKZ-system attached to \( A \) and \( \beta \). In case \( A \) is derived from a normal reflexive Gorenstein polytope \( P \), Batyrev and Stienstra related certain filtrations on the generic fiber of the GKZ-system to the mixed Hodge structure on the cohomology of a generic hyperplane section inside the projective toric variety induced by \( P \). Our formulae, phrased in terms of intersection cohomology groups on induced relative toric varieties, provide the necessary correction terms to globalize their computation. In particular, we document that on the GKZ-system the weight filtration will differ from Batyrev’s filtration-by-faces whenever \( P \) is not a simplex: the intersection complexes contributing to the weight filtration measure the failure of \( P \) to be a simplex.

Irrespective of homogeneity, we obtain a purely combinatorial formula for the length of the Gauss–Manin system, and thus for the corresponding GKZ-system. In dimension up to three, and for simplicial semigroups, we give explicit generators of the weight filtration.

Contents

1. Introduction 2
   1.1. The Decomposition Theorem for proper maps 2
   1.2. Non-proper maps 3
   1.3. Results and techniques 4
   1.4. Consequences, applications, open problems 5

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UW was supported by NSF grant 1401392-DMS and by Simons Foundation Collaboration Grant for Mathematicians #580839.
1.1. The Decomposition Theorem for proper maps. One of the hallmarks of Hodge-theoretic results in algebraic geometry is the Decomposition Theorem. For smooth projective maps between smooth projective varieties this asserts among other things the degeneration of the Leray spectral sequence for \( \mathbb{Q} \)-coefficients on the second page. Decomposition Theorems are refinements and generalizations of the Hard Lefschetz Theorem for projective varieties; the key ingredient is the purity of the Hodge structure on cohomology. In this article we study and quantify an important instance of the failure of purity, and of the Decomposition Theorem. In order to state our results, we give the briefest of historical surveys, and we point to the excellent account \([dCM09]\) for details.

For singular maps and varieties, things can be rescued by replacing usual cohomology with intersection cohomology, and in both instances the statement has a local flavor in the sense that one can restrict to open subsets of the target. The advantage of intersection cohomology is that it has nice formal properties such as Poincaré duality, Lefschetz theorems, and Künneth formula. While it is not a homotopy invariant, there is a natural transformation \( H^i \to \text{IH}^i \) that is an isomorphism on smooth spaces and in general induces a \( H^\bullet \)-module structure on \( \text{IH}^\bullet \). This version of the Decomposition Theorem, conjectured by S. Gel’fand and R. MacPherson, was proved by A. Beilinson, J. Bernstein, P. Deligne and O. Gabber.

The construction allows for generalization of intersection cohomology to coefficients in a local system \( L_U \), defined on a locally closed subset \( U \subseteq Z = \bar{U} \). The intersection complex of such a local system is a constructible complex that extends \( L_U \) as a constructible complex (or the corresponding connection on \( U \) as \( D \)-module). In fact, the best form of the Decomposition Theorem in the projective case is in this language: if \( f: X \to Y \) is a proper map of complex algebraic varieties then \( Rf_* \text{IC}_X \) splits (non-canonically) as
a direct sum of intersection complexes whose supporting sets are induced from a stratification of \( f \).

A particularly interesting case where the decomposition theorem has been well-studied are semismall maps (cf. \([dCM09]\) for a nice survey). These maps arise often in geometric situations:

- the Springer resolution \( f : \tilde{N} \to N \) of the nilpotent cone \( N \) of the Lie algebra to the reductive group \( G \);
- the Hilbert-Chow map between the Hilbert schemes of points \( X = (\mathbb{C}^2)^n \) and the \( n \)-th symmetric product \( Y = (\mathbb{C}^2)^n / S_n \).

The most explicit case is perhaps that of a fibration \( f : X \to Y \) between toric complete varieties: \( H^* \) and \( IH^* \) of a complete toric variety can be written down in purely combinatorial terms, and \([dCMM14]\) spells out how to write \( Rf_*(IC_X) \) as sum of intersection complexes in terms of face numbers.

### 1.2. Non-proper maps.

The moment one moves away from proper maps, direct images of intersection complexes need no more split into sums of such. For example, embedding \( \mathbb{C}^* \) into \( \mathbb{C} = \mathbb{C}^* \sqcup \{pt\} \) leads to a push-forward \( Rf_*\mathcal{O}_{\mathbb{C}^*} \) that naturally contains \( \mathcal{O}_{\mathbb{C}^1} \) but the cokernel \( \mathcal{O}_{pt} \) is not contained in the image. At this point one requires a “weight” filtration on \( Rf_*\mathcal{O}_{\mathbb{C}^*} \) akin to the one that forms part of Deligne’s construction of mixed Hodge structures on the cohomology of complex varieties. In the case \( \mathbb{C}^* \hookrightarrow \mathbb{C} \), level 1 of the weight filtration on \( Rf_*(\mathcal{O}_{\mathbb{C}^*}) \) is \( \mathcal{O}_{\mathbb{C}^1} \); level 2 is the entire image.

The appropriate powerful hybrid of intersection complexes and Deligne’s weights was constructed by M. Saito in his theory of mixed Hodge modules, inspired by the theory of weights for \( \ell \)-adic sheaves \([Sai90]\). The weight filtration, together with a “Hodge filtration” that can be seen as avatar of the usual Hodge filtration on cohomology, form the main ingredients of an object in Saito’s category of mixed Hodge modules. For maps between quasi-projective varieties he introduced a natural geometric filtration on \( Rf_*IC_X \). For proper maps between algebraic varieties, the weight filtration on \( Rf_*IC_X \) is pure and in particular there is a Decomposition Theorem: \( Rf_*IC_X \) splits into intersection complexes and the splitting occurs in the category of mixed Hodge modules.

In several natural situations properness is not available, and this necessitates nontrivial weights. Saito’s theory shows that in general the associated graded pieces of the weight filtration of any push-forward of a mixed Hodge module split as sums of intersection complexes, while for maps to a point the construction agrees with Deligne’s weights.

One is naturally led to a very hard question, crucial to Saito’s theory, on the behavior of pure Hodge modules under open embeddings. In the world of toric varieties, once one gives up on complete fans, the most fundamental situation is the inclusion of an embedded torus into its (likely singular) closure:
**Problem** (Weight Decomposition for Open Tori). Let \( T = (\mathbb{C}^*)^d \) and consider the monomial map

\[
(1.2.1) \quad h: T \longrightarrow \mathbb{C}^n =: \hat{V} \quad \text{where} \quad (t_1, \ldots, t_d) =: t \mapsto t^A := (t^{a_1}, \ldots, t^{a_n})
\]

is an integer \( d \times n \) matrix. Determine the weight filtration \( \{ W_i \}_i \) on \( h^+ (\mathcal{O}_T) \) and for each associated graded quotient \( W_i/W_{i-1} \) indicate the intersection complexes (support and coefficients) that appear as direct summands of this module.

To our knowledge, the only other time where non-proper maps have been studied is the article [CDK16] on certain open subsets of products of Grassmannians.

**1.3. Results and techniques.** Throughout, \( A \) is an integer \( d \times n \) matrix satisfying the three conditions of Notation 3.1 and we consider the induced monomial action of \( T \) on \( \hat{V} = \mathbb{C}^n \) given by

\[
(1.3.1) \quad \mu: T \times \hat{V} \longrightarrow \hat{V}, \quad (t, \eta) \mapsto t^A \cdot \eta.
\]

With \( \sigma = \mathbb{R}_{\geq 0} A \), denote \( X_\sigma \) (or just \( X \)) the closure of the orbit through \( 1 = (1, \ldots, 1) \in \mathbb{C}^n \). There is an orbit decomposition

\[
X = \bigsqcup_{\tau} O_\tau
\]

where the union is over the faces \( \tau \) of \( \sigma \) and \( O_\tau = \mu(T, 1_\tau) \) is the orbit corresponding to \( \tau \) (here, \( 1_\tau \in \mathbb{C}^n \) is defined by \( (1_\tau)_j = 1 \) if \( a_j \in \tau \) and zero otherwise). Denote \( \mathcal{Q}_\tau \) the constant sheaf on the orbit to \( \tau \) and write \( p^* \mathcal{Q}_\tau^H \) for the corresponding (simple, pure) Hodge module. With \( X = X_\sigma = \text{Spec} (\mathbb{C}[\mathcal{N}A]) \) from Section 3, \( h \) factors as

\[
(1.3.2) \quad \xymatrix{T \ar[r]^\varphi & X \ar[r]^i & \mathbb{C}^n =: \hat{V} \ar[r]^h &}
\]

Via Kashiwara equivalence along \( i \), one identifies the mixed Hodge modules on \( X \) with those on \( \hat{V} \) supported on \( X \). The following lists, in brief, our results in Section 3.

1. Since \( \mathcal{O}_T \) is a (strongly) torus equivariant \( \mathcal{D}_T \)-module, the \( T \)-equivariant map \( h \) will produce (strongly) equivariant modules \( R^i h_+ (\mathcal{O}_T) \) (and only the 0-th one is nonzero since \( h \) is affine). The intersection complexes appearing in the decomposition of the weight graded parts of \( Rh_+ (p^* \mathcal{Q}_\sigma^H) \) must be equivariant, supported on orbits. The underlying local systems are constant.

2. We identify two functors on equivariant sheaves with contracting torus actions. Using this identification, we then provide a recursive recipe for the exceptional pullback \( \mathcal{H}^k i^*_\sigma (h_* p^* \mathcal{Q}_\sigma^H / W_i h_* p^* \mathcal{Q}_\sigma^H) \) to an arbitrary orbit of the monomial action from (1.3.1).
(3) We unravel the recursion for $\mathcal{H}^0_{i_\tau} h_\tau p Q^H$, for every $\tau$, to provide an explicit expression for the multiplicity $\mu^\sigma_\tau(e)$ of the constant local system $p Q^H$ in the $(d+e)$-th graded weight part of $h_\tau p Q^H$ in terms of an alternating sum whose constituents are indexed by flags in the face lattice of $\sigma$, see Proposition 3.10. The terms involve intersection cohomology dimensions of the affine toric varieties $X_{\tau/\gamma}$ associated to the semigroup of the cone $\tau/\gamma = (\tau + \mathbb{R} \gamma)/\mathbb{R} \gamma$.

(4) Using some results on intersection cohomology of toric varieties by Stanley, and Braden and MacPherson, we express $\mu^\sigma_\tau(e)$ as a single intersection cohomology rank on the dual affine toric variety $Y_{\sigma/\tau}$, associated to the dual of $\sigma/\tau$, see Theorem 3.17.

There are several noteworthy consequences. First of all, $\mu^\sigma_\tau(e)$ is a relative quantity in the sense that $\mu^\sigma_\tau(e) = \mu^{\sigma/\gamma}_\tau(e)$ for any face $\gamma$ inside $\tau$. Secondly, the arithmetic properties of $\sigma$ are inessential: the only information relevant for $\mu^\sigma_\tau(e)$ is the combinatorics of the polytope obtained from $\sigma/\tau$ by slicing it with a transversal hyperplane that “cuts off the vertex”. This is because the intersection cohomology numbers of $Y_{\sigma/\tau}$ are entirely combinatorial.

### 1.4. Consequences, applications, open problems.

#### 1.4.1. Hodge structures on GKZ-systems.

Some interesting consequences of (1)-(4) above come from applying these results to the Fourier–Laplace transform of $h_+ O_T$, a well-studied $D$-module all by itself.

We briefly recall the notion of an $A$-hypergeometric system in our setup. Let $R_A = \mathbb{C}[\partial_1, \ldots, \partial_n]$ be the polynomial ring, set $D_A = R_A(x_1, \ldots, x_n)$ the Weyl algebra and pick $\beta \in \mathbb{C}^d$. Now consider the left ideal $H_A(\beta)$ of $D_A$ generated by

$$I_A = R_A(\{ \partial^u - \partial^v \mid u, v \in \mathbb{N}^n, A \cdot u = A \cdot v \})$$

and

$$E_i - \beta_i := \sum_{j=1}^n a_{i,j} x_j \partial_j - \beta_i \quad i = 1, \ldots, d.$$  

The module $M^\beta_A := D_A/H_A(\beta)$ is the $A$-hypergeometric system induced by $A$ and $\beta$. These systems were introduced by Gel’fand, Graev, Kapranov and Zelevinsky in the 1980’s; we refer to [SST00] and the current literature for more information on these modules, but highlight some properties.

The strongly resonant quasi-degrees $s\text{Res} (A)$ of $A$ form an infinite discrete hyperplane arrangement in $\mathbb{C}^d$ which was introduced in [SW09] and used to sharpen a result of Gel’fand et al. by showing that $\beta \notin s\text{Res} (A)$ is equivalent to $M^\beta_A$ being the Fourier–Laplace transform of $h_+(O^\beta_T)$ where $O^\beta_T$ is described before Theorem 4.4. In fact, it was Gel’fand and his collaborators that first observed a connection between $A$-hypergeometric systems and intersection complexes in [GKZ90, Prop. 3.2].

If the semigroup ring $S_A := \mathbb{C}[N A] \simeq R_A/I_A$ is normal (or, equivalently, if the semigroup $N A$ is saturated in $ZA$) then 0 is not strongly resonant. In
particular then, the inverse Fourier–Laplace transform of \( M_A^0 \) is the module \( h_+(\mathcal{O}_T) \) from Section 3. Since the Fourier–Laplace transform is an equivalence of categories, our results on \( h_+(\mathcal{O}_T) \) solve for normal \( S_A \) the longstanding problem of determining the composition factors for \( M_A^0 \).

The Fourier–Laplace transform does not necessarily preserve mixed Hodge module structures in general. However, if one assumes that \( I_A \) defines a projective variety, one can use the monodromic Fourier–Laplace transform which produces the same output as the Fourier–Laplace transform on \( h_+(\mathcal{O}_T) \) and does carry mixed Hodge module structures. In particular, this equips \( M_A^0 \) with a natural mixed Hodge module structure inherited from \( h \) (cf. [Rei14]).

There is a filtration-by-faces on a GKZ system, defined via the face filtration on the semigroup ring: the \((d+k)\)-th level of this filtration is the submodule of \( M_A^0 \) generated by all monomials \( \partial^\mu \in S_A \) for which \( A \cdot \mu \) is not contained in a face of dimension \( d - k - 1 \). This filtration was introduced by Batyrev in his study of the Hodge structure on the cohomology of a generic hypersurface constructed from a polytope [Bat93, Sti98]. Adolphson and Sperber, and more recently Fang, also considered the face filtration in [AS, Fan18]. We show that this filtration is bounded above by the weight filtration, and that it really differs from it for all GKZ-systems whose semigroup cone is not the cone over a simplex. On can view the error terms that we find as the necessary “glue” that is required to globalize the result of Batyrev and Stienstra from the generic fiber to the entire GKZ-system. On the other hand, looking at \( h_+(\mathcal{O}_T) \), we show that the corresponding filtration-by-faces always captures the part of the weight filtration that has maximal dimensional support.

1.4.2. Applications. We outline two possible applications of our results; one is concerned with mirror symmetry, the other comes from commutative algebra.

Local cohomology at toric varieties: Let \( R = \mathbb{K}[x_1, \ldots, x_n] \) and suppose \( I \) is an ideal of \( R \) such that \( R/I \) is the semigroup ring \( \mathbb{C}[NA] \) for some matrix \( A \) as above. Let \( J \) denote the variety comprised of the smaller torus orbits of the variety of \( I \). Then there is a natural triangle

\[
\rightarrow \mathbb{R} \Gamma_J(R) \rightarrow \mathbb{R} \Gamma_I(R) \rightarrow \hat{M}_A^0[-c]^{+1} \rightarrow
\]

in the category of mixed Hodge modules where the first morphism is the canonical one and \( \hat{M}_A^0 \) is the Fourier–Laplace transform of \( M_A^0 \) (i.e., \( h_+(\mathcal{O}_T) \) up to shift). In the normal case this degenerates and an inductive procedure can be used to determine from our formulæ for \( \hat{M}_A^0 \) the intersection complexes in the weight filtration of \( H^*_M(R) \). In particular, their vanishing (which at present is an open problem) can be determined. Making this explicit is the topic of a forthcoming work.

Mirror symmetry: Let \( Y_\Sigma \) be a toric variety induced by the fan \( \Sigma \). The secondary fan of \( Y_\Sigma \) induces a toric variety \( M \) and a family of Laurent polynomials over a Zariski open subset of \( M \). This family is known as the Landau–Ginzburg model of \( \Sigma \) and encodes the Gromov–Witten invariants of \( Y_\Sigma \). It turns out that the information relevant to Gromov–Witten invariants
is contained in the smallest weight part of the Gauß–Manin system, compare [Giv96, Giv98, Iri09, RS17, RS15]. It is conjectured that the parts of higher weight describe mirror symmetry for toric degenerations such as flag manifolds [IX16]. Our results here give concrete data on the GKZ side which one should want to match to those toric degenerations.

1.4.3. Open problems. When $S_A$ is normal, the holonomic rank of $M^\beta_A$ (the dimension of the holomorphic solution space in a generic point) equals the volume of the convex hull of the columns of $A$ together with the origin. In particular, this is an arithmetic quantity. In contrast, our results show that the holonomic length is purely combinatorial in that case; it only depends on the cd-index (see [BK91]) of the polytope over which $\sigma$ is the cone. This suggests a new question that deserves study: what is the rank, and more generally the characteristic cycle, of the Fourier–Laplace transformed intersection complex $IC(L_\tau)$? Our results allow for small $d$ direct calculation of the rank of $FL^{-1}(IC(L_\tau))$ to any chosen face. In higher dimension one can write down recursions, but making them explicit is an open question. Further, having a saturated composition chain for a $D$-module informs on the irreducible representations in the monodromy of the solution sheaf. Studying $FL^{-1}(IC(L_\tau))$ would be the first step towards a general understanding of the monodromy of $M^0_A$.

Finally, one should investigate whether one can place mixed Hodge module structures also on $M^\beta_A$ for other $\beta$. Obviously, this is doable in the normal case with $\beta \in \mathbb{N}A$ since [SW09] implies that the corresponding $M^\beta_A$ are isomorphic to $M^0_A$. Similarly, dual ideas reveal that for $\beta$ integral and in the cone roughly opposite to $\mathbb{N}A$, $M^\beta_A$ agrees with the Fourier–Laplace transform of $h_!(\mathcal{O}_T^\beta)$ and hence also inherits a mixed Hodge module structure, dual to the one discussed here. For other integral $\beta$, [Ste17, Ste18] describes $FL^{-1}(M^\beta_A)$ as a composition of a direct and exceptional direct image, which can be used to export a MHM structure. Less clear are non-integral $\beta$: the use of complex Hodge modules allows to equip $M^\beta_A$ with $\beta \in \mathbb{R}^d$ with a MHM structure, see Sabbah’s MHM project [SS]. For certain $\beta$ the Hodge filtration on $M^\beta_A$ is explicitly computed in [RS15].

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2. Functors on $D$-modules

If $K$ is a free Abelian group of finite rank, or a finite dimensional vector space, then we write $K^*$ for the dual group or vector space.

We introduce the following notation. Let $X$ be a smooth complex algebraic variety of dimension $d_X$. The Abelian category of algebraic left $D_X$-modules on $X$ is denoted by $M(D_X)$ and the Abelian subcategory of (regular) holonomic $D_X$-modules by $M_h(D_X)$ (resp. $M_{rh}(D_X)$). We abbreviate $D^b(M(D_X))$ to $D^b(D_X)$, and denote by $D^b_h(D_X)$ (resp. $D^b_{rh}(D_X)$)
the full triangulated subcategory in $\mathcal{D}^b(\mathcal{D}_X)$ consisting of objects with holo-
monic (resp. regular holonomic) cohomology.
Let $f : X \to Y$ be a morphism between smooth algebraic varieties and
let $M \in \mathcal{D}^b(\mathcal{D}_X)$ and $N \in \mathcal{D}^b(\mathcal{D}_Y)$. The direct and inverse image func-
tors for $\mathcal{D}$-modules are denoted by
$$f_+M := Rf_*(\mathcal{D}_{Y \to X} L \otimes M) \quad \text{and} \quad f^+M := D_{X \to Y} L \otimes f^{-1}M[dx - dy]$$
respectively. The functors $f_+, f^+$ preserve (regular) holonomicity (see e.g.,
\cite[Theorem 3.2.3]{HTT08}).
We denote by
$$\mathbb{D} : \mathcal{D}_h^b(\mathcal{D}_X) \to (\mathcal{D}_h^b(\mathcal{D}_X))^{opp}$$
the holonomic duality functor. Recall that for a single holonomic $\mathcal{D}_X$-module
$M$, the holonomic dual is also a single holonomic $\mathcal{D}_X$-module ([HTT08, 
Proposition 3.2.1]) and that holonomic duality preserves regularity ([HTT08, 
Theorem 6.1.10]).
For a morphism $f : X \to Y$ between smooth algebraic varieties we addi-
tionally define the functors
$$f^\dagger := \mathbb{D} \circ f_+ \circ \mathbb{D} \quad \text{and} \quad f^\ddagger := \mathbb{D} \circ f^+ \circ \mathbb{D}.$$ 
Let $X$ be an algebraic variety. Denote by $\text{MHM}(X)$ the Abelian category
of algebraic mixed Hodge modules and by $\mathcal{D}^b \text{MHM}(X)$ the corresponding
bounded derived category. If $X$ is smooth the forgetful functor to the bounded derived category of regular holonomic $\mathcal{D}_X$-modules is denoted by
\begin{equation}
D\text{mod} : \mathcal{D}^b \text{MHM}(X) \to \mathcal{D}^b_{\text{reg}}(\mathcal{D}_X).
\end{equation}
For each morphism $f : X \to Y$ between complex algebraic varieties, there
are induced functors
$$f_*, f^* : \mathcal{D}^b \text{MHM}(X) \to \mathcal{D}^b \text{MHM}(Y)$$
and
$$f^!, f_! : \mathcal{D}^b \text{MHM}(Y) \to \mathcal{D}^b \text{MHM}(X),$$
which satisfy $\mathbb{D} \circ f_* = f_! \circ \mathbb{D}$, $\mathbb{D} \circ f^* = f^! \circ \mathbb{D}$, and which lift the analogous
functors $f_+, f^\dagger, f_1, f^\ddagger$ on $\mathcal{D}^b_{\text{reg}}(\mathcal{D}_X)$ in case $X$ is smooth.
Let $\mathcal{Q}^H$ be the trivial Hodge structure $\mathbb{Q}$ of type $(0,0)$, i.e. $\text{gr}^W_i \mathcal{Q}^H = 
\text{gr}^F_i \mathcal{Q}^H_{\text{pt}} = 0$ for $i \neq 0$. Viewing it as a Hodge module on a point $pt$, denote
by $p\mathcal{Q}^H_X := \mathcal{Q}^H_X[d_X]$ the (mixed) Hodge module $(a^*\mathcal{Q}^H)[d_X]$, where
$$a : X \to pt$$
is the unique map to a point. For smooth $X$ the $\mathcal{D}$-module underlying $p\mathcal{Q}^H_X$
is the structure sheaf $\mathcal{O}_X$ in cohomological degree zero with $\text{gr}^W_i \mathcal{O}_X = 0$ for
$i \neq d_X$.
Let $j : U \to X$ be any Zariski dense smooth open subset of $X$ and let
$L$ be a polarizable variation of Hodge structures (i.e., a vector bundle with
a flat connection $\nabla$ such that each fiber carries a Hodge structure, with
$\nabla(F_p) \subseteq F_{p+1}$ for increasing filtrations, and a global polarization pairing)
of weight $w$. Set $pL := L \otimes p\mathcal{Q}^H_U$. We denote by $\text{IC}_X(pL)$ the intersection
cohomology sheaf with coefficients in $pL$; this is a pure Hodge module of
Lemma 2.1. Let \((X, S)\) be an algebraic Whitney stratification of \(X\) with a Zariski dense smooth open stratum \(U\). Denote by \(i_S : S \rightarrow X\) the embedding of the stratum \(S \in S\) in \(X\) and let \(pL\) be as above. The following holds for morphisms in MHM:

1. \(i^!_S\) is left exact for every \(S \in S\) and does not decrease weights. (That is, if \(W_{\leq k}(M) = 0\) then \(W_{\leq k}i^!_S(M) = 0\).)
2. \(H^0i^!_U IC_X(pL) = pL\) and \(H^k i^!_U IC_X(pL) = 0\) for \(k \neq 0\).
3. \(H^0i^!_S IC_X(pL) = 0\) for \(U \neq S\).

Proof. The first statement follows from [KS94, Proposition 10.2.11] and [Sai90, (4.5.2)]. The second statement follows from the fact that \(i^!_U = i^*_U\) is just the restriction to the open subset \(U\) which is exact. The last point follows from the characterization of \(IC_X(pL)\) as \(im(H^0j^!pL \rightarrow H^0j^!pL)\) and [BBD82, 1.4.22 and 1.4.24]. \(\square\)

3. Weight filtration on torus embeddings

3.1. Basic Notions.

Notation 3.1. If \(C\) is a semiring (an additive semigroup closed under multiplication) write \(CA\) for the \(C\)-linear combinations of the columns of the integer \(d \times n\) matrix \(A\). We assume that \(A\) satisfies:

1. \(ZA = \mathbb{Z}^d\);
2. \(A\) is saturated: \(\mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d = NA\);
3. \(A\) is pointed: \(NA \cap (-NA) = \{0_{\mathbb{Z}A}\}\).

We let \(\tau := \mathbb{R}_{\geq 0}A\) be the real cone over \(A\) inside \(\mathbb{R}^d\) and consider the \(d\)-dimensional affine toric variety \(X := X_{\tau} := \text{Spec}(\mathbb{C}[NA]) \subseteq \hat{V}\) together with its open dense torus \(T := T_{\tau} := \text{Spec}(\mathbb{C}[ZA])\). Properties (1)-(3) of \(A\) above imply that \(X\) is \(d\)-dimensional, normal by Hochster’s theorem, and has one \(T\)-fixed point.

Let \(\tau \subseteq \sigma\) be a \(d_{\tau}\)-dimensional face of \(\sigma\). We denote by

\[
\tau_{\mathbb{Z}} := (\tau + (-\tau)) \cap \mathbb{Z}^d, \quad \tau_\mathbb{N} = \tau \cap \mathbb{N}^d \quad \text{and} \quad \tau_\mathbb{R} = \tau + (-\tau) = \text{span}(\tau),
\]

which are the \(\mathbb{Z}\)-, \(\mathbb{N}\)- and \(\mathbb{R}\)-spans of the collection of \(A\)-columns in \(\tau\) (considering that \(NA\) is saturated). We associate to \(\tau\) a \(d_{\tau}\)-dimensional torus orbit

\[
T_{\tau} = \text{Spec}(\mathbb{C}[\tau_{\mathbb{Z}}])
\]

whose closure in \(X_{\tau}\) via the embedding induced by \(NA \rightarrow NF \rightarrow ZF\) is

\[
X_{\tau} = \text{Spec}(\mathbb{C}[\tau_{\mathbb{N}}]).
\]

Saturatedness of \(NA\) implies that \(X_{\tau}\) is normal. The variety

\[
U_{\tau} := \text{Spec}(\mathbb{C}[\sigma_{\mathbb{N}} + \tau_{\mathbb{Z}}])
\]

gives an open neighborhood of \(T_{\tau}\) in \(X\). The affine toric variety

\[
X_{\sigma/\tau} := \text{Spec}(\mathbb{C}[(\sigma_{\mathbb{N}} + \tau_{\mathbb{Z}})/\tau_{\mathbb{Z}}])
\]
with its dense torus
\[ T_{\sigma/\tau} := \text{Spec} (\mathbb{C}[\sigma_\mathbb{Z}/\tau_\mathbb{Z}]) \]
is a normal slice to the stratum \( T_{\tau} \): there is an isomorphism \( U_{\tau} \simeq X_{\sigma/\tau} \times T_{\tau} \) and a (non-canonical) isomorphism
\[ j_\tau : X_{\sigma/\tau} \times T_{\tau} \to X. \]
The inclusions \( T_{\sigma/\tau} \hookrightarrow X_{\sigma/\tau} \hookrightarrow X \) correspond to the (canonical) morphisms
\[ \sigma_\mathbb{N} \to (\sigma_\mathbb{N} + \tau_\mathbb{Z})/\tau_\mathbb{Z} \to \sigma_\mathbb{Z}/\tau_\mathbb{Z}. \]
For any pointed rational polyhedral cone \( \rho \) in (a quotient of) \( \mathbb{R}^d \) we denote by
\[ i_\rho : \{ x_\rho \} \hookrightarrow X_\rho \]
the embedding of the unique torus-fixed point. Then we have the following diagram
\[ \{ x_{\sigma/\tau} \} \times T_{\tau} \xrightarrow{i_{\sigma/\tau} \times \text{id}} X_{\sigma/\tau} \times T_{\tau} \xleftarrow{\varphi_{\sigma/\tau} \times \text{id}} T_{\sigma/\tau} \times T_{\tau} \simeq T \]
of equivariant maps.

**Definition 3.2.** Let \( \mu : \mathbb{G}_m \times Y \to Y \) be a \( \mathbb{G}_m \)-action on the variety \( Y \).
Write \( \text{pr} : \mathbb{G}_m \times Y \to Y \) for the projection. A holonomic \( \mathcal{D}_Y \)-module \( \mathcal{M} \) is called \( \mathbb{G}_m \)-equivariant if \( \mu^+ \mathcal{M} \simeq \text{pr}^+ \mathcal{M} \) as \( \mathcal{D}_{\mathbb{G}_m \times Y} \)-module.

If \( v \in \mathbb{Z}^A \) is in the interior \( \text{Int}(\sigma^\vee) \) of the dual cone \( \sigma^\vee \), then \( v \) defines a 1-parameter subgroup \( \kappa_v : \mathbb{G}_m = \text{Spec} \mathbb{C}[z^\pm] \to T = \text{Spec} \mathbb{C}[\mathbb{Z}^A] \) given by
\[ t^u \mapsto z^{(u,v)} \]
which extends to a map \( \pi_\varphi : \mathbb{A}^1 \to X_\sigma \) with limit point \( x_\sigma \in X_\sigma \).
By adjusting the ambient lattice, similar statements hold for all faces \( \tau \).

**Lemma 3.3.** Let \( i_\tau : \{ x_\tau \} \to X_\tau \) be the inclusion of the torus fixed point and for any space \( X \) denote
\[ a_X : X \to \text{pt} \]
the projection to a point. If \( X = X_\tau \) is one of our orbit closures, identify
\[ a_{X_\tau} \text{ with } a_\tau : X_\tau \to \{ x_\tau \}. \]
Let \( v \in \tau^\vee \) be an integer element in the relative interior of the dual cone \( \tau^\vee \) and consider the induced action of \( \mathbb{G}_m \) on \( X_\tau \). For every \( \mathbb{G}_m \)-equivariant Hodge module \( \mathcal{M} \) on \( X_\tau \) we have the following isomorphisms
\[ a_{\tau\ast} \mathcal{M} \simeq i_{\tau\ast}^! \mathcal{M} \quad \text{and} \quad a_{\tau!} \mathcal{M} \simeq i_{\tau!}^* \mathcal{M} \]

**Proof.** It suffices to consider the case \( \tau = \sigma \). Denote by \( u : X_\sigma \setminus \{ x_\sigma \} \to X_\sigma \) the open embedding of the complement of the fixed point \( x_\sigma \), abbreviate \( i_\sigma \) to \( i \) and denote by \( a \) the map to a point. We have the exact triangle
\[ u_{\ast} u^{-1} \mathcal{M} \to \mathcal{M} \to i_{\ast} i^* \mathcal{M} \xrightarrow{+1} \]
Applying \( a_{\ast} \) we get
\[ a_{\ast} u_{\ast} u^{-1} \mathcal{M} \to a_{\ast} \mathcal{M} \to i_{\ast} i^* \mathcal{M} \xrightarrow{+1} \]
and we will show $a_*u_1u^{-1}\mathcal{M} = 0$. As $v \in \text{Int}(\sigma^\vee)$, we have an action

$$\pi_v : \mathbb{A}^1 \times X_\sigma \to X_\sigma$$

with $\pi_v^{-1}(x) = (\mathbb{A}^1 \times \{x\}) \cup (\{0\} \times X_\sigma)$. This gives the following Cartesian diagram

$$\begin{array}{ccc}
\mathbb{G}_m \times (X_\sigma \setminus \{x\}) & \xrightarrow{u'} & \mathbb{A}^1 \times X_\sigma \\
\pi_v \downarrow \quad & & \downarrow \pi_v \\
X_\sigma \setminus \{x\} & \xrightarrow{u} & X_\sigma
\end{array}$$

where $\pi_v'$ is the restriction and $u'$ is the canonical inclusion. Consider the morphism $g : X_\sigma \to \mathbb{A}^1 \times X_\sigma$ with $g(x) = (1, x)$. The morphism $g$ is a section of $\pi_v$, hence $\pi_v \circ g = \text{id}_{X_\sigma}$. Therefore the composition

$$a_* \to a_*\pi_v^*\pi_v^* = (a_{\mathbb{A}^1 \times X_\sigma})_*\pi_v^* \to (a_{\mathbb{A}^1 \times X_\sigma})_*g_*g^*\pi_v^* = a_*g^*\pi_v^*$$

is the identity transformation. In order to show that $a_*u_1u^{-1}\mathcal{M} = 0$ it is hence enough to prove that the intermediate module $(a_{\mathbb{A}^1 \times X_\sigma})_*\pi_v^*u_1u^{-1}\mathcal{M}$ vanishes.

By base change we get the following isomorphism:

$$(a_{\mathbb{A}^1 \times X_\sigma})_*\pi_v^*u_1u^{-1}\mathcal{M} \simeq (a_{\mathbb{A}^1 \times X_\sigma})_*u_1'(\pi_v^*)^*u^{-1}\mathcal{M}.$$  

Since $u^{-1}\mathcal{M}$ is $\mathbb{G}_m$-equivariant, we have

$$(\pi_v')^*u^{-1}\mathcal{M} \simeq pr^*u^{-1}\mathcal{M} \simeq \mathcal{O}^H_{\mathbb{G}_m} \boxtimes u^{-1}\mathcal{M}.$$  

Therefore we get

$$(a_{\mathbb{A}^1 \times X_\sigma})_*u_1'(\pi_v^*)^*u^{-1}\mathcal{M} \simeq (a_{\mathbb{A}^1 \times X_\sigma})_*u_1'(\mathcal{O}_{\mathbb{G}_m}^H \boxtimes u^{-1}\mathcal{M}) \simeq (a_{\mathbb{A}^1 \times X_\sigma})_*((u_1!\mathcal{O}_{\mathbb{G}_m}^H \boxtimes u^{-1}\mathcal{M}),$$  

where $u_1 : \mathbb{G}_m \to \mathbb{A}^1$ is the canonical inclusion. Since $H^*(\mathbb{A}^1, u_1!\mathcal{O}_{\mathbb{G}_m}^H) = 0$, the K"unneth formula shows that $(a_{\mathbb{A}^1 \times X_\sigma})_*\pi_v^*u_1u^{-1}\mathcal{M} = 0$. This shows the first claim. The second claim follows by dualizing; note that duals of equivariant modules are equivariant.

Recall that $\text{IH}(-)$ (and $\text{IH}_c$) denotes intersection cohomology (with compact support).

**Lemma 3.4.** Let $\gamma$ be a face of $\sigma$, $X_\gamma$ the associated $d_\gamma$-dimensional affine toric variety. The following holds

1. $\text{IH}_c^{d_\gamma+k}(X_\gamma) \simeq (\text{IH}^{d_\gamma-k}(X_\gamma)(d_\gamma))^*$.
2. $\text{IH}^{d_\gamma+k}(X_\gamma) = \text{IH}_c^{d_\gamma-k}(X_\gamma) = 0$ for $k \geq 0$.
3. $\text{IH}^k(X_\gamma) = 0$ for $k$ odd.
4. $\text{IH}^{2k}(X_\gamma)$ and $\text{IH}_c^{2k}(X_\gamma)$ are pure Hodge structures of Hodge–Tate type with weight $2k$, i.e.

$$\text{gr}_i^W \text{IH}^{2k}(X_\gamma) = 0 \quad \text{and} \quad \text{gr}_j^F \text{IH}^{2k}(X_\gamma) = 0 \text{ for } i \neq 2k \text{ and } j \neq -k.$$


Proof. Temporarily, write $a$ for $a_\gamma$ and $i$ for $i_\gamma$. Claim (1) follows from Verdier duality:

\[
\begin{align*}
\mathrm{IH}^d_{\gamma}(X_\gamma) & \simeq H^k a_\gamma \mathrm{IC}_{X_\gamma}(pQ^H_{T_\gamma}) \\
& \simeq H^k \mathcal{D}a_* \mathcal{D} \mathrm{IC}_{X_\gamma}(pQ^H_{T_\gamma}) \\
& \simeq H^k \mathcal{D}a_*(\mathrm{IC}_{X_\gamma}(pQ^H_{T_\gamma}(d_\gamma))) \\
& \simeq \left(H^{-k} a_*(\mathrm{IC}_{X_\gamma}(pQ^H_{T_\gamma}(d_\gamma)))\right)^* \\
& \simeq \left(\mathrm{IH}^{-d-k}(X_\gamma)(d_\gamma)\right)^*.
\end{align*}
\]

From Lemma 3.3 we have the isomorphisms

\[
\begin{align*}
\mathrm{IH}^k(X_\gamma) = H^k a_* \mathrm{IC}_{X_\gamma}(Q^H_{T_\gamma}) & \simeq H^k i^* \mathrm{IC}_{X_\gamma}(Q^H_{T_\gamma}) = H^k \mathrm{IC}_{X_\gamma}(Q^H_{T_\gamma}) ,
\end{align*}
\]

Claim (2) follows from [Fie91, Theorem 1.2], which also implies—in conjunction with Remark ii.) in loc. cit.—Claim (3). Claim (4) follows from [Web04, Corollary 4.12].

Let now $\tau, \gamma$ be faces of $\sigma$ with $\tau \subseteq \gamma$ and set

\[X_{\gamma/\tau} := \mathrm{Spec}(\mathbb{C}[\gamma_N + \tau_Z]/\tau_Z)].\]

The following result discusses (derived) pullbacks of constant variations of Hodge structures to torus orbits.

Lemma 3.5. Let $i_{\tau, \gamma} : T_\tau \to X_\tau \to X_\gamma$ be the torus orbit embedding and let $H$ be a polarizable Hodge structure of weight $w$ (on a point). Then $p\mathcal{L} := H \otimes pQ^H_{T_\gamma}$ is a (constant) variation of polarizable Hodge structures of weight $w + d_\gamma$ on $T_\gamma$. We have the following isomorphisms in $\mathrm{MHM}(T_\gamma)$:

\[
\begin{align*}
\mathcal{H}^k(i^!_{\tau, \gamma} \mathrm{IC}_{X_\gamma}(p\mathcal{L})) & \simeq H \otimes \mathrm{IH}^{d_\gamma-d_\tau+k}(X_{\gamma/\tau}) \otimes pQ^H_{T_\gamma}, \\
\mathcal{H}^k(i^*_{\tau, \gamma} \mathrm{IC}_{X_\gamma}(p\mathcal{L})) & \simeq H \otimes \mathrm{IH}^{d_\gamma-d_\tau+k}(X_{\gamma/\tau}) \otimes pQ^H_{T_\gamma}.
\end{align*}
\]

The weight filtration is given by:

\[
\mathrm{gr}_j^W \mathcal{H}^k(i^!_{\tau, \gamma} \mathrm{IC}_{X_\gamma}(p\mathcal{L})) = \mathrm{gr}_j^W \mathcal{H}^k(i^*_{\tau, \gamma} \mathrm{IC}_{X_\gamma}(p\mathcal{L})) = 0
\]

for $j \neq w + d_\gamma + k$.

Proof. Consider the following diagram:

\[
\begin{array}{cccc}
\{x_{\gamma/\tau}\} & \overset{i_{\gamma/\tau}} \rightarrow & X_{\gamma/\tau} & \overset{\varphi_{\gamma/\tau}} \leftarrow & T_{\gamma/\tau} \\
\downarrow p_1 & & & & \downarrow p_2 \\
\{x_{\gamma/\tau}\} \times T_\tau & \overset{i_{\gamma/\tau} \times \text{id}} \rightarrow & X_{\gamma/\tau} \times T_\tau & \overset{\varphi_{\gamma/\tau} \times \text{id}} \leftarrow & T_{\gamma/\tau} \times T_\tau \simeq T_\gamma \\
\downarrow j_? & & & & \downarrow \varphi_\gamma \\
X_\gamma & & & & .
\end{array}
\]
Since $pL$ is a constant variation of Hodge structures on $T_\gamma \simeq T_\gamma/\tau \times T_\tau$, by (cf. [Sai90, (4.4.2)]),
\[
pL = H \otimes Q_T \simeq (H \otimes Q_T (-d_\tau[-2d_\tau + d_\gamma]) \boxtimes Q_T (d_\tau[2d_\tau])
\]
\[
\simeq p_3^1(H \otimes pQ_T (-d_\tau)[-d_\tau]) \simeq p_3^1(p\hat{\mathcal{L}}(-d_\tau)[-d_\tau])
\]
where we have set $p\hat{\mathcal{L}} = H \otimes pQ_T$. We have the following isomorphisms
\[
\gamma, k \quad IC_{X_\gamma}(p\mathcal{L}) \simeq (i_{\gamma/\tau} \times id)^1 IC_{X_\gamma}(p\mathcal{L})
\]
\[
\simeq (i_{\gamma/\tau} \times id)^1 IC_{X_\gamma/\tau \times T_\tau}(p\mathcal{L})
\]
\[
\simeq (i_{\gamma/\tau} \times id)^1 p_2^1 IC_{X_\gamma/\tau}(p\hat{\mathcal{L}}(-d_\tau)[-d_\tau])
\]
\[
\simeq p_1^1 i_{\gamma/\tau} IC_{X_\gamma/\tau}(p\hat{\mathcal{L}}(-d_\tau)[-d_\tau])
\]
\[
\simeq i_{\gamma/\tau} IC_{X_\gamma/\tau}(p\hat{\mathcal{L}}) \boxtimes Q_T \simeq i_{\gamma/\tau} IC_{X_\gamma/\tau}(p\hat{\mathcal{L}}) \boxtimes Q_T [d_\tau]
\]
\[
\simeq i_{\gamma/\tau} IC_{X_\gamma/\tau}(p\hat{\mathcal{L}}) \boxtimes Q_T \simeq pQ_T.
\]
Since $IC_{X_\gamma/\tau}$ is $T_\gamma/\tau$-equivariant it follows from Lemma 3.3 that
\[
H^k(i_{\gamma/\tau} IC_{X_\gamma/\tau}(p\hat{\mathcal{L}})) \simeq H^k(a_1 IC_{X_\gamma/\tau}(p\hat{\mathcal{L}}))
\]
\[
\simeq H^k(a_1 IC_{X_\gamma/\tau}(H \otimes pQ_T))
\]
\[
\simeq HH^{d_\gamma - d_\tau + k}(X_\gamma/\tau) \otimes H
\]
as mixed Hodge structures. This gives the isomorphism
\[
H^k(i_{\gamma/\tau} IC_{X_\gamma}(\mathcal{L})) \simeq H \boxtimes HH^{d_\gamma - d_\tau + k}(X_\gamma/\tau) \otimes pQ_T.
\]
The weight filtration on the intersection cohomology of $X_\gamma/\tau$ satisfies
\[
gr^W_i HH(X_\gamma/\tau) = 0 \quad \text{if} \quad i \neq k.
\]
Hence we get
\[
gr^W_i HH(i_{\gamma/\tau} IC_{X_\gamma}(\mathcal{L})) = \bigoplus_{i=\gamma} HH^i IC_{X_\gamma}(p\hat{\mathcal{L}}) \bigoplus_{\gamma} HH^i IC_{X_\gamma}(p\hat{\mathcal{L}}) = 0
\]
for $i \neq \gamma + (d_\tau - d_\gamma + k) + d_\gamma = w + d_\gamma + k$.

The statement (3.1.2) follows from a dual proof. \qed

3.2. A recursion. The torus orbits $T_\tau \subseteq X$ equip $X^{an}$ with a Whitney stratification (cf. [Dim92, Proposition 1.14]). Since the morphisms $h, \varphi$ from (1.2.1) are affine, algebraic, and stratified, the perverse sheaf underlying $\varphi^* pQ_T$ is constructible with respect to this stratification. Since $\varphi^* pQ_T$ is a mixed Hodge module its weight graded parts are direct sums of intersection complexes (with possibly twisted coefficients) having support on the orbit closures $X_\tau = \overline{T_\tau}$. We write
\[
(3.2.1) \quad \bigoplus_{\gamma \in \mathcal{P}} IC_{X_\gamma}(p\mathcal{V}_{(\gamma, k)}).
\]
Here the direct sum is understood as a direct sum over all faces $\gamma$ of $\sigma$, and $p\mathcal{V}_{(\gamma, k)}$ is a polarizable variation of Hodge structures of weight $k$ on $T_\gamma$. 

Here and elsewhere, for a mixed Hodge module \( \mathcal{M} \) on \( Y \), we regard as equivalent via Kashiwara equivalence, for \( Y \) closed in \( X \), \( \text{IC}_Y(\mathcal{M}) \) and its direct image on \( X \), without necessarily explicitly referencing \( X \). Moreover, we say that \( \mathcal{M} \) has weight \( \geq k \) if \( \text{gr}^W_{\leq k} \mathcal{M} = 0 \) for \( \ell < k \).

Our first result on (3.2.1) is a recursive formula; we continue to denote \( d_\sigma \) by just \( d \) and \( X_\sigma \) by just \( X \):

**Proposition 3.6.** The weight filtration on the mixed Hodge module \( \varphi_* \mathcal{P}^{\mathcal{Q}}_T \) satisfies the following properties.

1. \( \text{gr}^W_{d+e} \varphi_* \mathcal{P}^{\mathcal{Q}}_T = 0 \) for \( e \neq 0, 1, \ldots, d \).
2. \( \text{supp} \text{gr}^W_{d+e} \varphi_* \mathcal{P}^{\mathcal{Q}}_T \subseteq \bigcup_{d_\gamma \leq d-e} X_\gamma \).
3. \( \text{gr}^W_d \varphi_* \mathcal{P}^{\mathcal{Q}}_T = W_d \varphi_* \mathcal{P}^{\mathcal{Q}}_T = \text{IC}_X(\mathcal{P}^{\mathcal{Q}}_T) \), by which we denote \( \text{IC}_X(\mathcal{P}^{\mathcal{Q}}_T) \).
4. \( \text{gr}^W_{d+1} \varphi_* \mathcal{P}^{\mathcal{Q}}_T = \bigoplus_{\tau} \text{IC}_X(\mathcal{L}^0_{(\tau,d+1)} \otimes \mathcal{P}^{\mathcal{Q}}_{T_\tau}) \) where
   \[
   L^k_{(\tau,d+1)} := \text{IH}^{d_\sigma-d_\tau+k+1}(X_{\sigma/\tau})
   \]
   is the intersection homology group with compact support of \( X_{\sigma/\tau} \).
5. For \( e > 1 \), \( \text{gr}^W_{d+e} \varphi_* \mathcal{P}^{\mathcal{Q}}_T = \bigoplus_{\tau} \text{IC}_X(\mathcal{L}^0_{(\gamma,d+e-1)} \otimes \mathcal{P}^{\mathcal{Q}}_{T_\tau}) \) where
   \[
   L^k_{(\gamma,d+e-1)} \simeq \frac{\bigoplus_{\gamma \geq \tau} \mathcal{L}^0_{(\gamma,d+e-1)} \otimes \text{IH}^{d_\gamma-d_\tau+k+1}(X_{\gamma/\tau})}{L^k_{(\gamma,d+e-1)}}.
   \]
6. For all \( e \), the module
   \[
   \mathcal{P} L^k_{(\tau,d+e)} := \mathcal{H}^{k,d}_{\tau,\sigma}(\varphi_* \mathcal{P}^{\mathcal{Q}}_T/W_d \varphi_* \mathcal{P}^{\mathcal{Q}}_T)
   \]
   is pure of weight \( d + e + k \). It is zero for \( d_\tau \geq d - e + 1 - k \) and in any case isomorphic to a finite sum of copies of \( \mathcal{P}^{\mathcal{Q}}_{T_\tau} \).

**Proof.** In order to ease the notation we denote in this proof by \( \mathcal{Q}_\gamma \) the Hodge module \( \mathcal{P}^{\mathcal{Q}}_T \).

We will proceed by induction on \( e \). Obviously, for \( e \ll 0 \), all parts hold trivially. Assuming Property (6) up to \( e \) as well as Property (5) up to \( e-1 \), we show Property (6) for \( e+1 \) and Property (5) for \( e \). Property (2) is then a direct consequence. Properties (3) and (4) are the induction start and are proved in the same fashion as the induction step, but look less uniform.

Since \( \mathcal{Q}_\sigma \) has weight \( d \) and direct images do not decrease weights, the direct image \( \varphi_* \mathcal{Q}_\sigma = \mathcal{H}^{0,d}_{\varphi_* \mathcal{Q}_\sigma} \) has weight \( \geq d \). Property (1) is hence a consequence of Property (2).

We make the following Ansatz for the part of \( \varphi_* \mathcal{Q}_\sigma \) of weight \( d \):
\[
W_d \varphi_* \mathcal{Q}_\sigma = \bigoplus_{\gamma} \text{IC}_X(\mathcal{P}_\gamma(\mathcal{V}_{(\gamma,d)})),
\]
where \( \mathcal{P}_\gamma(\mathcal{V}_{(\gamma,d)}) \) is a polarizable variation of Hodge structures on \( T_\gamma \) of weight \( d \).
We begin with the lowest weight case \( e = 0 \); then (5) is vacuous. Consider the exact sequence

\[
(3.2.3) \quad 0 \longrightarrow \bigoplus_{\gamma} IC_{X, \gamma} (p_{\gamma,d}) \longrightarrow \varphi_\ast Q_\sigma \longrightarrow \varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma \longrightarrow 0.
\]

Let \( \tau \) be an \( d_\tau \)-dimensional face of \( \sigma \); \( i_{\tau,\sigma} : T_\tau \hookrightarrow X \) is the natural embedding. Apply the functor \( i_{\tau,\sigma}^! \) to (3.2.3), recalling that it is left exact and does not decrease weights (cf. Lemma 2.1.(1)). Because of Lemma 2.1.(2),(3) we get a long exact sequence

\[
0 \longrightarrow p_{\gamma,d} \longrightarrow H^0 i_{\tau,\sigma}^! \varphi_\ast Q_\sigma \longrightarrow H^0 i_{\tau,\sigma}^! (\varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma) \longrightarrow \cdots.
\]

Since \( i_{\tau,\sigma}^! \varphi_\ast Q_\sigma = 0 \) for \( d_\tau < d \) by Lemma 2.1.(3) we obtain Property (6) for \( e = 0 \) and find \( p_{\gamma,d} = p_L^0 (\tau, d) = 0 \) for those \( \tau \). In the case \( d_\tau = d \), we have \( \tau = \sigma \) and \( i_{\sigma,\sigma}^! \varphi_\ast Q_\sigma = Q_\sigma \) and therefore obtain

\[
0 \longrightarrow p_{\gamma,d} \longrightarrow Q_\sigma \longrightarrow H^0 i_{\sigma,\sigma}^! (\varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma).
\]

Since \( \varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma \) has weight \( > d \) and \( i_{\sigma,\sigma}^! \) does not decrease weight, \( p_{\gamma,d} \simeq Q_\sigma \). Altogether we have:

\[
p_{\gamma,d} = \begin{cases} Q_\sigma & \text{for } d_\tau = d, \\ 0 & \text{for } d_\tau < d. \end{cases}
\]

Thus, for all \( \tau \), \( p_{\gamma,d}^0 (\tau, d) = p_{\gamma,d} \) and \( p_L^0 (\tau, d) \) vanishes. This shows Property (3) (and embodies Property (6) for \( e = 0 \)).

We next consider the weight \( d + 1 \) part. To begin, we use the fact that \( W_d \varphi_\ast Q_\sigma = IC_{X,\sigma} \) in order to compute \( H^k (i_{\tau,\sigma}^! (\varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma)) \) for each face \( \tau \) and all \( k \geq 0 \). The exact sequence (3.2.3) becomes

\[
0 \longrightarrow IC_{X,\sigma} \longrightarrow \varphi_\ast Q_\sigma \longrightarrow \varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma \longrightarrow 0.
\]

We apply again \( i_{\tau,\sigma}^! \) to this short exact sequence and obtain

\[
0 \longrightarrow H^0 i_{\tau,\sigma}^! IC_{X,\sigma} \longrightarrow H^0 i_{\tau,\sigma}^! \varphi_\ast Q_\sigma \longrightarrow H^0 i_{\tau,\sigma}^! (\varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma).
\]

Thus, for all \( \tau \), \( p_{\gamma,d}^0 (\tau, d) = p_{\gamma,d} \) and \( p_L^0 (\tau, d) \) vanishes. This shows Property (3) (and embodies Property (6) for \( e = 0 \)).

We next consider the weight \( d + 1 \) part. To begin, we use the fact that \( W_d \varphi_\ast Q_\sigma = IC_{X,\sigma} \) in order to compute \( H^k (i_{\tau,\sigma}^! (\varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma)) \) for each face \( \tau \) and all \( k \geq 0 \). The exact sequence (3.2.3) becomes

\[
0 \longrightarrow IC_{X,\sigma} \longrightarrow \varphi_\ast Q_\sigma \longrightarrow \varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma \longrightarrow 0.
\]

For \( d_\tau = d \) we have \( H^k i_{\tau,\sigma}^! \varphi_\ast Q_\sigma = H^k i_{\tau,\sigma}^! IC_{X,\sigma} = 0 \) for \( k \geq 1 \) (as \( i_{\sigma,\sigma} \) is an open embedding and therefore \( i_{\sigma,\sigma}^! \) is exact) and so

\[
p_L^k (\sigma, d + 1) = H^k i_{\sigma,\sigma}^! (\varphi_\ast Q_\sigma / W_d \varphi_\ast Q_\sigma)
\]

vanishes for all \( k \) (the case \( k = 0 \) follows from \( p_L^0 (\sigma, d) = Q_\sigma \)).
For $d_\tau < d$ we have $H^{k+1}_{\tau,\sigma} \varphi_* Q_\sigma = 0$ for all $k$, hence by Lemma 3.5 we have
\begin{equation}
(3.2.4)

P L^k_{(\tau, d+1)} := H^k_{\tau,\sigma} (\varphi_* Q_\sigma/W d \varphi_* Q_\sigma) \simeq H^{k+1}_{\tau,\sigma} IC X_\sigma \simeq IH^{d_\tau - d_\tau + k + 1}(X_{\sigma/\tau}) \otimes Q_\tau.
\end{equation}

In particular, $P L^k_{(\tau, d+1)} = L^k_{(\tau, d+1)} \otimes Q_\tau$ with $L^k_{(\tau, d+1)} = IH^{d_\sigma - d_\tau + k + 1}(X_{\sigma/\tau})$.

Since $IH^{d_\tau + 1}(X_{\sigma/\tau}) = 0$ for all $k \geq 0$ the formula above makes also sense for $\tau = \sigma$. Notice that $P L^k_{(\tau, d+1)}$ is pure of weight $d + 1 + k$ and since $IH^{d_\tau - d_\tau + k + 1}(X_{\sigma/\tau}) = 0$ for $d_\sigma - d_\tau + k + 1 > 2(d_\sigma - d_\tau)$ we have $P L^k_{(\tau, d+1)} = 0$ for $d_\tau \geq d - k$. This shows Property (6) in the case $e = 1$.

In order to compute the weight $(d + 1)$ part of $\varphi_* Q_\sigma$ we make the Ansatz
\begin{equation}
g^\phi_{d+1} \varphi_* Q_\sigma = \bigoplus IC X_\gamma (P V_{(\gamma, d+1)})
\end{equation}
and consider the exact sequence
\begin{equation}
0 \rightarrow \bigoplus IC X_\gamma (P V_{(\gamma, d+1)}) \rightarrow \varphi_* Q_\sigma/W d \varphi_* Q_\sigma \rightarrow \varphi_* Q_\sigma/W d + 1 \varphi_* Q_\sigma \rightarrow 0.
\end{equation}

The functor $i^!_{\tau,\sigma}$ produces the long exact cohomology sequence
\begin{equation}
0 \rightarrow P V_{(\tau, d+1)} \rightarrow P L^0_{(\tau, d+1)} \rightarrow H^0 i^!_{\tau,\sigma} (\varphi_* Q_\sigma/W d + 1 \varphi_* Q_\sigma) \rightarrow \cdots.
\end{equation}

Since $i^!_{\tau,\sigma}$ does not decrease weight, the third term has weight $d + 2$ or more, and since $P L^0_{(\tau, d+1)}$ is pure of weight $d + 1$, (3.2.4) yields
\begin{equation}
P V_{(\tau, d+1)} = P L^0_{(\tau, d+1)} = IH^{d_\sigma - d_\tau + 1}(X_{\sigma/\tau}) \otimes Q_\tau,
\end{equation}
which shows Property (4).

With this, (3.2.5) becomes now
\begin{equation}
0 \rightarrow \bigoplus IC X_\gamma (P L^0_{(\gamma, d+1)}) \rightarrow \varphi_* Q_\sigma/W d \varphi_* Q_\sigma \rightarrow \varphi_* Q_\sigma/W d + 1 \varphi_* Q_\sigma \rightarrow 0,
\end{equation}
and applying $i^!_{\tau,\sigma}$ we obtain
\begin{equation}
0 \rightarrow \bigoplus \left( L^0_{(\gamma, d+1)} \otimes IH^{d_\sigma - d_\tau}(X_{\gamma/\tau}) \otimes Q_\tau \right) \rightarrow \left( L^0_{(\tau, d+1)} \otimes Q_\tau \right) \rightarrow \left( H^0 i^!_{\tau,\sigma} (\varphi_* Q_\sigma/W d + 1 \varphi_* Q_\sigma) \right)
\end{equation}
\begin{equation}
= \bigoplus \left( L^1_{(\gamma, d+1)} \otimes IH^{d_\sigma - d_\tau + 1}(X_{\gamma/\tau}) \otimes Q_\tau \right) \rightarrow \left( L^1_{(\tau, d+1)} \otimes Q_\tau \right) \rightarrow \left( H^1 i^!_{\tau,\sigma} (\varphi_* Q_\sigma/W d + 1 \varphi_* Q_\sigma) \right)
\end{equation}
\begin{equation}
= \bigoplus \left( L^2_{(\gamma, d+1)} \otimes IH^{d_\sigma - d_\tau + 2}(X_{\gamma/\tau}) \otimes Q_\tau \right) \rightarrow \cdots.
\end{equation}

Here, the first column is owed to Lemma 3.5 and the equality in the first row follows from Lemma 2.1(3) resp. Lemma 3.4 (2).
Since both \( \bigoplus_{\gamma \geq \tau} \left( L^0_{(\gamma,d+1)} \otimes \text{IH}^{d_\gamma - d_\tau + k}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau \right) \) and \( L^k_{(\tau,d+1)} \otimes \mathbb{Q}_\tau \) are pure of weight \( d + 1 + k \), and since \( \mathcal{H}^k i_{\tau,\sigma}^! (\varphi_\sigma \mathcal{Q}_\sigma / W_{d+1} \varphi_\sigma \mathcal{Q}_\sigma) \) has weight \( > (d + 1 + k) \) the long exact sequence splits into sequences

\[
0 \to \mathcal{H}^k i_{\tau,\sigma}^! (\varphi_\sigma \mathcal{Q}_\sigma / W_{d+1} \varphi_\sigma \mathcal{Q}_\sigma) \to \bigoplus_{\gamma \geq \tau} \left( L^0_{(\gamma,d+1)} \otimes \text{IH}^{d_\gamma - d_\tau + k+1}(X_{\gamma/\tau}) \otimes \mathbb{Q}_\tau \right) \to L^k_{(\tau,d+1)} \otimes \mathbb{Q}_\tau \to 0,
\]

pure of weight \( d + 1 + k + 1 \). The category of pure Hodge modules is semisimple and so there is a (non-canonical) splitting which induces an identification

\[
\mathcal{H}^k i_{\tau,\sigma}^! (\varphi_\sigma \mathcal{Q}_\sigma / W_{d+1} \varphi_\sigma \mathcal{Q}_\sigma) \simeq \left( \bigoplus_{\gamma \geq \tau} L^0_{(\gamma,d+1)} \otimes \text{IH}^{d_\gamma - d_\tau + k+1}(X_{\gamma/\tau}) \right) \otimes \mathbb{Q}_\tau
\]

as pure Hodge modules. We now define vector spaces \( \mathcal{P}^k \mathcal{L}_{(\tau,d+2)} \) by

\[
\mathcal{P}^k \mathcal{L}_{(\tau,d+2)} := L^k_{(\tau,d+2)} \otimes \mathbb{Q}_\tau := \mathcal{H}^k i_{\tau,\sigma}^! (\varphi_\sigma \mathcal{Q}_\sigma / W_{d+1} \varphi_\sigma \mathcal{Q}_\sigma),
\]

a pure Hodge module of weight \( d + 2 + k \). Since \( L^0_{(\gamma,d+1)} \) is zero for \( d_\gamma \geq d \) and \( \text{IH}^{d_\gamma - d_\tau + k+1}(X_{\gamma/\tau}) \) is zero for \( d_\gamma - d_\tau + k + 1 > 2(d_\gamma - d_\tau) \), the term \( \mathcal{P}^k \mathcal{L}_{(\tau,d+2)} \) is zero for \( d_\tau \geq d - 1 - k \); this proves Property (6) for \( e = 2 \).

We will now provide the inductive step, much in parallel to the above. Assume that

\[
\mathcal{P}^k \mathcal{L}_{(\tau,d+e)} = L^k_{(\tau,d+e)} \otimes \mathbb{Q}_\tau = \mathcal{H}^k i_{\tau,\sigma}^! (\varphi_\sigma \mathcal{Q}_\sigma / W_{d+e-1} \varphi_\sigma \mathcal{Q}_\sigma)
\]

is pure of weight \( d + e + k \) and \( \mathcal{P}^k \mathcal{L}_{(\tau,d+e)} = 0 \) for \( d_\tau \geq d - e + 1 - k \) (i.e., Property 6 at level \( e \)).

In order to compute the weight \( d + e \) part of \( \varphi_\sigma \mathcal{Q}_\sigma \) we make the Ansatz

\[
g^W_{d+e, \varphi_\sigma \mathcal{Q}_\sigma} = \bigoplus \text{IC}_{\gamma/\tau} (\mathcal{P}^k \mathcal{V}_{(\gamma,d+e)})
\]

and consider the exact sequence

(3.2.6)

\[
0 \to \bigoplus \text{IC}_{\gamma/\tau} (\mathcal{P}^k \mathcal{V}_{(\gamma,d+e)}) \to \varphi_\sigma \mathcal{Q}_\sigma / W_{d+e-1} \varphi_\sigma \mathcal{Q}_\sigma \to \varphi_\sigma \mathcal{Q}_\sigma / W_{d+e} \varphi_\sigma \mathcal{Q}_\sigma \to 0
\]

We apply the functor \( i_{\tau,\sigma}^! \) and get the long exact cohomology sequence

\[
0 \to \mathcal{P}^k \mathcal{V}_{(\tau,d+e)} \to \mathcal{H}^0 i_{\tau,\sigma}^! (\varphi_\sigma \mathcal{Q}_\sigma / W_{d+e} \varphi_\sigma \mathcal{Q}_\sigma) \to \cdots.
\]

As \( i_{\tau,\sigma}^! \) does not decrease weight, the third term has weight greater than \( (d + e) \), and as \( \mathcal{H}^0 \mathcal{L}_{(\tau,d+e)} \) is pure of weight \( d + e \) we find

\[
\mathcal{P}^k \mathcal{V}_{(\tau,d+e)} = \mathcal{P}^k \mathcal{L}_{(\tau,d+e)}.
\]

The exact sequence (3.2.6) now becomes

\[
0 \to \bigoplus \text{IC}_{\gamma/\tau} (\mathcal{P}^k \mathcal{L}_{(\gamma,d+e)}) \to \varphi_\sigma \mathcal{Q}_\sigma / W_{d+e-1} \varphi_\sigma \mathcal{Q}_\sigma \to \varphi_\sigma \mathcal{Q}_\sigma / W_{d+e} \varphi_\sigma \mathcal{Q}_\sigma \to 0,
\]
and \( i_{\tau,\sigma} \) induces

\[
0 \longrightarrow L_{(\tau,d+e)}^0 \otimes Q_\tau \longrightarrow L_{(\tau,d+e)}^{1} \otimes Q_\tau \longrightarrow \mathcal{H}_{i_{\tau,\sigma}}^1 (\varphi^*_e Q_\sigma / W_{d+e} \varphi^*_e Q_\sigma) \longrightarrow L_{(\tau,d+e)}^{2} \otimes Q_\tau \longrightarrow \mathcal{H}_{i_{\tau,\sigma}}^2 (\varphi^*_e Q_\sigma / W_{d+e} \varphi^*_e Q_\sigma) \longrightarrow \cdots.
\]

Since both \( \bigoplus_{\tau \geq \tau} \left( L_{(\gamma,d+e)}^0 \otimes \text{IH}_{c_\tau}^{d_\tau - d_\tau + k} (X_{\gamma/\tau}) \otimes Q_\tau \right) \) and \( L_{(\tau,d+e)}^k \otimes Q_\tau \) are pure of weight \( d + e + k \), and since furthermore \( \mathcal{H}_{i_{\tau,\sigma}}^k (\varphi^*_e Q_\sigma / W_{d+e} \varphi^*_e Q_\sigma) \) has weight greater than \( (d + e + k) \), the long exact sequence splits in \( \text{MHM}(X_\sigma) \) into sequences

\[
0 \to \mathcal{H}_{i_{\tau,\sigma}}^k (\varphi^*_e Q_\sigma / W_{d+e} \varphi^*_e Q_\sigma) \to \bigoplus_{\tau \geq \tau} \left( L_{(\gamma,d+e)}^0 \otimes \text{IH}_{c_\tau}^{d_\tau - d_\tau + k} (X_{\gamma/\tau}) \otimes Q_\tau \right) \to L_{(\tau,d+e)}^{k+1} \otimes Q_\tau \to 0.
\]

The center term is pure of weight \( d + e + k + 1 \); hence the outer terms are as well. Since the category of pure Hodge modules is semisimple, the sequence splits (non-canonically) and there is an identification

\[
\mathcal{H}_{i_{\tau,\sigma}}^k (\varphi^*_e Q_\sigma / W_{d+e} \varphi^*_e Q_\sigma) \simeq \frac{\bigoplus_{\tau \geq \tau} \left( L_{(\gamma,d+e)}^0 \otimes \text{IH}_{c_\tau}^{d_\tau - d_\tau + k} (X_{\gamma/\tau}) \right)}{L_{(\tau,d+e)}^{k+1}} \otimes Q_\tau.
\]

We now define \( \mathcal{L}_{k}^k (\tau,d+e+1) \) and \( L_{(\tau,d+e+1)}^{k} \) by

\[
\mathcal{L}_{k}^k (\tau,d+e+1) := L_{(\tau,d+e+1)}^k \otimes Q_\tau := \mathcal{H}_{i_{\tau,\sigma}}^k (\varphi^*_e Q_\sigma / W_{d+e+1} \varphi^*_e Q_\sigma),
\]

and reiterate that \( \mathcal{L}_{k}^k (\tau,d+e+1) \) is pure of weight \( d + e + 1 + k \). Since \( L_{(\gamma,d+e)}^0 \) is zero for \( d_\gamma \geq d - e + 1 \) and \( \text{IH}_{c_\tau}^{d_\tau - d_\tau + k+1} (X_{\gamma/\tau}) \) is zero for \( d_\tau - d_\tau + k + 1 > 2(d_\gamma - d_\tau) \), the term \( \mathcal{L}_{k}^k (\tau,d+e+1) \) vanishes for \( d_\tau \geq d - e + 1 - k \). This finishes the inductive step for Property (6), establishes (5) in the process, and hence completes the proof. \( \square \)

Remark 3.7. It has been pointed out by A. Lörincz to us that the constancy of the local systems \( \mathcal{L}_{k}^k (\tau,d+e) \) can be also seen as follows: \( \mathcal{L}_{k}^k (\tau,d+e) \) is equivariant, and hence so is \( \varphi^*_e \mathcal{Q}^H_T \). Since all orbit stabilizers are connected, [HTT08, Theorem 11.6.1] shows that each orbit can only support one equivariant local system, the constant one. See [LW19] for more details on equivariant \( D \)-modules.

Example 3.8. We give an explicit description of the vector spaces \( L_{(\tau,d+e)}^{k} \) from Proposition 3.6 in the case \( d = 4 \) for \( e \geq 1 \). Here, the \((k,e)\)-entry for \( L_{(\tau,d+e)}^{k} \) is a sum over all \( \gamma_j \) that arise. For example, \( L_{(\tau,2,6)}^{0} \) is the sum over...
all $\gamma_3$ of dimension 3 with $\tau_2 \subseteq \gamma_3 \subseteq \sigma$ of the terms listed under $k = 0, e = 2$ in Table 3.2.9.

The Hodge-structures $L_{(\tau_0, d + e)}^k$ for the unique $\tau_0 \subseteq \sigma$ with $\dim \tau_0 = 0$:

$$
\begin{array}{cccc}
\text{(3.2.7)} & k = 3 & L_{(\tau_0,5)}^1 = \text{IH}^0_c(X_{\tau_0/5}) & 0 \\
& k = 2 & L_{(\tau_0,6)}^2 = L_{(\tau_0,5)}^1 \otimes \text{IH}^2_c(X_{\tau_0/6}) & L_{(\tau_0,5)}^1 \\
& k = 1 & L_{(\tau_0,5)}^1 = \text{IH}^0_c(X_{\tau_0/5}) & 0 \\
& k = 0 & L_{(\tau_0,6)}^0 = L_{(\tau_0,5)}^1 \otimes \text{IH}^1_c(X_{\tau_0/6}) & L_{(\tau_0,5)}^1 \\
\end{array}
$$

<table>
<thead>
<tr>
<th>$e = 1$</th>
<th>$e = 2$</th>
<th>$e = 3$</th>
<th>$e = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{(\tau_0,6)}^1$</td>
<td>$L_{(\tau_0,5)}^1$</td>
<td>$L_{(\tau_0,5)}^1$</td>
<td>$L_{(\tau_0,5)}^1$</td>
</tr>
</tbody>
</table>

The Hodge-structures $L_{(\tau_1, d + e)}^k$ for all $\tau_1 \subseteq \sigma$ with $\dim \tau_1 = 1$:

$$
\begin{array}{cccc}
\text{(3.2.8)} & k = 3 & L_{(\tau_1,5)}^1 = \text{IH}^0_c(X_{\tau_1/5}) & 0 \\
& k = 2 & L_{(\tau_1,6)}^2 = L_{(\tau_1,5)}^1 \otimes \text{IH}^2_c(X_{\tau_1/6}) & L_{(\tau_1,5)}^1 \\
& k = 1 & L_{(\tau_1,5)}^1 = \text{IH}^0_c(X_{\tau_1/5}) & 0 \\
& k = 0 & L_{(\tau_1,6)}^0 = L_{(\tau_1,5)}^1 \otimes \text{IH}^1_c(X_{\tau_1/6}) & L_{(\tau_1,5)}^1 \\
\end{array}
$$

<table>
<thead>
<tr>
<th>$e = 1$</th>
<th>$e = 2$</th>
<th>$e = 3$</th>
<th>$e = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{(\tau_1,6)}^1$</td>
<td>$L_{(\tau_1,5)}^1$</td>
<td>$L_{(\tau_1,5)}^1$</td>
<td>$L_{(\tau_1,5)}^1$</td>
</tr>
</tbody>
</table>

The Hodge-structures $L_{(\tau_2, d + e)}^k$ for all $\tau_2 \subseteq \sigma$ with $\dim \tau_2 = 2$:

$$
\begin{array}{cccc}
\text{(3.2.9)} & k = 3 & L_{(\tau_2,5)}^1 = \text{IH}^0_c(X_{\tau_2/5}) & 0 \\
& k = 2 & L_{(\tau_2,6)}^2 = L_{(\tau_2,5)}^1 \otimes \text{IH}^2_c(X_{\tau_2/6}) & L_{(\tau_2,5)}^1 \\
& k = 1 & L_{(\tau_2,5)}^1 = \text{IH}^0_c(X_{\tau_2/5}) & 0 \\
& k = 0 & L_{(\tau_2,6)}^0 = L_{(\tau_2,5)}^1 \otimes \text{IH}^1_c(X_{\tau_2/6}) & L_{(\tau_2,5)}^1 \\
\end{array}
$$

<table>
<thead>
<tr>
<th>$e = 1$</th>
<th>$e = 2$</th>
<th>$e = 3$</th>
<th>$e = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{(\tau_2,6)}^1$</td>
<td>$L_{(\tau_2,5)}^1$</td>
<td>$L_{(\tau_2,5)}^1$</td>
<td>$L_{(\tau_2,5)}^1$</td>
</tr>
</tbody>
</table>

The Hodge-structures $L_{(\tau_3, d + e)}^k$ for all $\tau_3 \subseteq \sigma$ with $\dim \tau_3 = 3$:

$$
\begin{array}{cccc}
\text{(3.2.10)} & k = 3 & L_{(\tau_3,5)}^1 = \text{IH}^0_c(X_{\tau_3/5}) & 0 \\
& k = 2 & L_{(\tau_3,6)}^2 = L_{(\tau_3,5)}^1 \otimes \text{IH}^2_c(X_{\tau_3/6}) & L_{(\tau_3,5)}^1 \\
& k = 1 & L_{(\tau_3,5)}^1 = \text{IH}^0_c(X_{\tau_3/5}) & 0 \\
& k = 0 & L_{(\tau_3,6)}^0 = L_{(\tau_3,5)}^1 \otimes \text{IH}^1_c(X_{\tau_3/6}) & L_{(\tau_3,5)}^1 \\
\end{array}
$$

<table>
<thead>
<tr>
<th>$e = 1$</th>
<th>$e = 2$</th>
<th>$e = 3$</th>
<th>$e = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{(\tau_3,6)}^1$</td>
<td>$L_{(\tau_3,5)}^1$</td>
<td>$L_{(\tau_3,5)}^1$</td>
<td>$L_{(\tau_3,5)}^1$</td>
</tr>
</tbody>
</table>

The table for $\sigma = \tau_3$ is determined by Proposition 3.6, Properties (2) and (3); it has only zero entries since $\sigma$ only contributes to weight $d$.  

3.3. **An explicit formula.** If we set

$$
ih^k_c(X_{\gamma/\tau}) := \dim Q \text{IH}^k_c(X_{\gamma/\tau})$$
we can rewrite the dimension of $L^0_{(\gamma,k)}$ in Example 3.8 as follows:

$$\dim_{\mathbb{Q}} L^0_{(\gamma_1, \delta_1)} = \text{ih}_\gamma^e(X_{\gamma_1/\delta_1})$$

$$\dim_{\mathbb{Q}} L^0_{(\gamma_2, \delta_2)} = \text{ih}_\gamma^e(X_{\gamma_2/\delta_2})$$

$$\dim_{\mathbb{Q}} L^0_{(\gamma_3, \delta_3)} = \text{ih}_\gamma^e(X_{\gamma_3/\delta_3})$$

$$\dim_{\mathbb{Q}} L^0_{(\gamma_4, \delta_4)} = \text{ih}_\gamma^e(X_{\gamma_4/\delta_4})$$

$$\dim_{\mathbb{Q}} L^0_{(\gamma_5, \delta_5)} = \text{ih}_\gamma^e(X_{\gamma_5/\delta_5})$$

$$\dim_{\mathbb{Q}} L^0_{(\gamma_6, \delta_6)} = \text{ih}_\gamma^e(X_{\gamma_6/\delta_6})$$

Again, each expression is to be summed over all possible faces $\gamma_i$ of dimension $i$ that satisfy the requisite containment conditions.

The particular structure of the formulas for the dimension of these local systems is not coincidental. Our next task is to turn recursion (3.2.2) for $L^0_{(\gamma,k)}$ into a general explicit combinatorial formula.

We set

$$\mu^\sigma_{\tau}(e) := \dim_{\mathbb{Q}}(L^0_{(\tau,d+e)})$$

for the rank of the (constant!) local system $p\mathcal{L}_{(\tau,d+e)}$ corresponding to the intersection complex $\mathcal{IC}_{\mathcal{X}_\tau}(p\mathcal{L}_{(\tau,d+e)})$ occurring in $\mathcal{gr}^W_{d+e} \mathcal{P}^H_{\tau}$. We further introduce the following abbreviations.

**Notation 3.9.** Let

$$\text{ih}_{\gamma}^e(k) := \dim_{\mathbb{Q}}(\text{IH}_{\mathcal{H}_e}^{d_e-d_{e}+k}(X_{\gamma/\tau}));$$

$$\ell^e_{\gamma}(k,e) := \dim_{\mathbb{Q}}(L^k_{(\gamma,d+e)}).$$

Then

$$\ell^e_{\gamma}(k,1) = \dim_{\mathbb{Q}}(L^k_{\tau,d+1}) = \dim_{\mathbb{Q}}(\text{IH}_{\mathcal{H}_e}^{d_e-d_{e}+k+1}(X_{\sigma/\tau})) = \text{ih}_{\tau}^e(k+1)$$

by Proposition 3.6.(4), while the recursion (3.2.2) yields

$$\ell^e_{\gamma}(k,e) = \left( \sum_{\gamma \subset \tau} \ell^e_{\gamma}(0, e-1) \cdot \text{ih}_{\gamma}^e(k+1) \right) - \ell^e_{\gamma}(k+1, e-1).$$

Let $0 < t \in \mathbb{N}$ and let $\pi = [\pi_1, \ldots, \pi_m] \vdash t$ be a partition\footnote{We always assume that “partition” implies that each $\pi_j$ is nonzero, and that the entries are ordered. The partitions of 3 are [3], [1, 2], [2, 1] and [1, 1, 1].} of $t$ of length $|\pi| = m$. We consider flags $\Gamma = (\gamma_0 \subseteq \gamma_1 \subseteq \ldots \subseteq \gamma_m)$ of faces of $\sigma$, of length $|\Gamma| = m$. Here, $d_i$ is the dimension of $\gamma_{d_i}$. Denote by $\text{ih}_\Gamma(\pi)$ the product

$$\text{ih}_\Gamma(\pi) := \text{ih}_{\gamma_0}^e(\pi_1) \cdot \ldots \cdot \text{ih}_{\gamma_{m-1}}^e(\pi_m).$$

For comparable faces $\gamma \subseteq \gamma'$, set

$$\text{ih}_{\gamma}^e(\pi) = \sum_{|\Gamma'| = |\pi|, \Gamma' = (\gamma, \ldots, \gamma')} \text{ih}_\Gamma(\pi),$$
\[ \text{ih}_\gamma'(t, m) = \sum_{|\pi|=m}^{\pi \vdash t} \text{ih}_\gamma'(\pi). \]

**Proposition 3.10.** The rank of the local system \( p\mathcal{L}_{(\tau, d+e)} \) corresponding to the intersection complex \( IC_{X, p\mathcal{L}_{(\tau, d+e)}} \) occurring in \( gr^W \varphi_* p\mathcal{Q}^H \) is

\[ \mu^\tau_\gamma(e) = \sum_m (-1)^{m+d_e} \text{ih}_\gamma'(e, m). \]

**Proof.** To start, note that, for \( \gamma'' \supseteq \gamma \) one has the “product rule”

\[ \text{ih}_\gamma''(\pi'' \sqcup \pi) = \sum_{\gamma'' \supseteq \gamma' \supseteq \gamma} \text{ih}_\gamma''(\gamma'') \cdot \text{ih}_\gamma'(\gamma) \]

for any two partitions \( \pi'' \vdash t'' \) and \( \pi \vdash t \) and their juxtaposition \( \pi'' \sqcup \pi = [\pi''_1, \ldots, \pi''_{m''}, \pi_1, \ldots, \pi_m] \vdash (t'' + t) \).

For each \( \tau \), place the numbers \( \ell_\tau(k, e) \) on a grid of integer points in the first quadrant of a page associated to \( \tau \) as follows:

\[
(3.3.4) \quad (\langle \tau \rangle) : \\
\begin{array}{cccccc}
\vdots & \vdots & & & & \\
1 & 2 & 3 & \cdots & \\
k = 0 & \text{ih}_\gamma(1) = \ell_\tau(0,1) & \ell_\tau(0,2) & \ell_\tau(0,3) & \cdots & \\
k = 1 & \text{ih}_\gamma(2) = \ell_\tau(1,1) & \ell_\tau(1,2) & \ell_\tau(1,3) & \cdots & \\
k = 2 & \text{ih}_\gamma(3) = \ell_\tau(2,1) & \ell_\tau(2,2) & \ell_\tau(2,3) & \cdots & \\
k = 3 & \vdots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
e = 1 & e = 2 & e = 3 & \cdots & \\
\end{array}
\]

The column \( e = 1 \) of the \( \tau \)-page consists of the numbers \( \dim_Q \Pi_{d'_e - d_e + 1}^{d_e}(X_{\sigma/\tau}) = \text{ih}_\gamma'(k+1) = \ell_\tau(k,1) \).

Then (3.3.3) implies that for \( e > 1 \) the entry in row \( k \) and column \( e \) of page \( (\langle \tau \rangle) \) is the difference \( a) - b) \) where

a) is the sum over all \( \sigma \supseteq \gamma \supseteq \tau \) of all products of \( \text{ih}_\gamma'(k+1) \) with the entry in row 0 and column \( e - 1 \) on the \( \gamma \)-page;

b) the entry in row \( k + 1 \) and column \( e - 1 \) on the \( \tau \)-page.

Progressing along increasing column index, all entries on each page can be rewritten as sums of products \( \text{ih}_\Gamma(\pi) \) of intersection cohomology dimensions \( \text{ih}_\gamma''(t) \). We call such product \( \text{ih}_\Gamma(\pi) \) a “term”. It is immediate that each term on page \( (\langle \tau \rangle) \) arises from a flag that links \( \tau \) to \( \sigma \) (i.e., \( \tau = \gamma_0, \sigma = \gamma_{|\Gamma|} \)) with \( |\Gamma| = |\pi| \).

The sum in a) contains only terms \( \text{ih}_\Gamma(\pi) \) where the initial element of \( \pi \) equals \( k + 1 \). On the other hand, it follows from induction on \( k \) that the terms in b) all have the initial element of the corresponding \( \pi \) greater than \( k + 1 \). So, formal cancellation of terms cannot occur in the recursion.

When a term on the \( \tau \)-page arises through case a) then the length of the term is greater (by one) than the length of the term on the \( \gamma \)-page that gave rise to it. However, that is not the case if it arises from case b) when it simply copied from the appropriate entry on the \( \tau \) page, and so term length changes if and only if no new factor of \( -1 \) is acquired. In particular, the sign of a term is a function of the length of the term, modulo two. The recursion
forces the term to the partition $[1,1,\ldots,1]$ of length $d_\sigma$ to be positive. Hence
all terms $\text{ih}_R(\pi)$ on each page carry a sign of $(-1)^{|\pi|+d_\sigma}$.

Note that in a) one could allow $\gamma = \tau$ since $\text{ih}_R(1) = 0$. Similarly, one
can admit $\gamma = \sigma$ since $\ell_\sigma(0,e-1) = 0$ for $e > 1$. The sum in a) involves
always all possible choices of $\gamma$, $\sigma \supseteq \gamma \supseteq \tau$. Thus, if a partition $\pi$ occurs
at all in an entry on page $(\tau)$ then $\text{ih}_R(\pi)$ will occur in that entry for all
flags $\Gamma$ with $|\Gamma| = |\pi|$ that start at $\tau$ and end with $\sigma$. In the following
we tabulate for small $k,e$ the partitions that occur in Figure (3.3.4);
here, in each term one should sum over all $\Gamma$ of the appropriate length that
interpolate from $\tau$ to $\sigma$ (we will write $\text{ih}([1,1,2])$ instead of $\text{ih}_R([1,1,2])$ etc.
for ease of readability).

(3.3.5)

\begin{align*}
  & k = 3 & & \vdots & & \vdots \\
  & \vdots & & \vdots & & \vdots \\
  & k = 2 & \text{ih}([4]) & \text{ih}([1,4]) - \text{ih}([5]) & \text{ih}([1,1,4]) - \text{ih}([2,4]) - (\text{ih}([1,5]) - \text{ih}([6])) & \cdots \\
  & k = 1 & \text{ih}([3]) & \text{ih}([1,3]) - \text{ih}([4]) & \text{ih}([1,1,3]) - \text{ih}([2,3]) - (\text{ih}([1,4]) - \text{ih}([5])) & \cdots \\
  & \vdots & \text{ih}([2]) & \text{ih}([1,2]) - \text{ih}([3]) & \text{ih}([1,1,2]) - \text{ih}([2,2]) - (\text{ih}([1,3]) - \text{ih}([4])) & \cdots \\
  & k = 0 & \text{ih}([1]) & \text{ih}([1,1]) - \text{ih}([2]) & \text{ih}([1,1,1]) - \text{ih}([2,1]) - (\text{ih}([1,2]) - \text{ih}([3])) & \cdots \\
  & e = 1 & e = 2 & e = 3 & & \cdots 
\end{align*}

It is therefore sufficient to investigate which partitions occur in the $(k,e)$-
entry on page $\tau$. Since the entries in column $e = 1$ come from a unique
partition, the entries in column $e$ will come from no more than $2^{e-1}$ parti-
tions (the variation over all $\gamma$ in the recursion does not affect the resulting
partition $\pi$, only the flag $\Gamma$). The argument that no cancellation can occur
reveals also that no fewer than, and hence exactly, $2^{e-1}$ partitions occur in
each entry of column $e$.

The partitions $\pi$ used in the entry $(k,e)$ on page $\tau$ have weight $\pi_1 + \ldots +
\pi_{|\pi|} = e + k$, again by induction on the column index. But the number of
ordered integer partitions of weight $e$ with positive entries is exactly $2^{e-1}$.
Thus, all $2^{e-1}$ partitions of weight $e$ actually occur in the entry $(0,e)$, and

- for each partition, each possible flag interpolating from $\tau$ to $\sigma$
  contributes, and no other;
- the term $\text{ih}_R(\pi)$ has sign $(-1)^{|\pi|+d_\sigma}$;

as stated in the proposition. \hfill \Box

The recursion as evidenced in Table (3.3.5) leads immediately to the fol-
lowing result.

\textbf{Corollary 3.11.} The number of copies $\ell_\tau(k,e)$ of $\mathbb{Q}_\tau$ in $\mathcal{H}_R \ell_\tau\mathbb{Q}_\tau$ equals $\sum_{m}(\text{ih}_R^\sigma(e,m))_k$ where the subscript $k$ means
each partition $\pi = [\pi_1,\ldots,\pi_m] \vdash t$ that contributes to $\mu^\sigma_\tau(e) = \ell_\tau(0,e)$
in Proposition 3.10 is replaced by $[\pi_1,\ldots,\pi_{m-1},\pi_m+k] \vdash (t+k)$. \hfill \Box

3.4. Dual polytopes. Our final step in this section is to give a compact
value to the formula in Proposition 3.10. In order to carry out this discussion
we have to introduce some notions from toric geometry.

\textbf{Notation 3.12.} Let $\tau \subset \gamma \subset \sigma$ be faces of $\sigma$. The quotient face of $\gamma$ by $\tau$ is
defined as:

\begin{equation}
  \gamma/\tau := (\gamma + \tau_\mathbb{R})/\tau_\mathbb{R} \subset \mathbb{R}^d/\tau_\mathbb{R}.
\end{equation}
We define the dual cone and the annihilator of $\gamma$ by

$$\gamma^\vee := \{ y \in (\mathbb{R}^d)^* \mid y(x) \geq 0 \forall x \in \gamma \} \quad \text{and} \quad \gamma^\perp := \{ y \in (\mathbb{R}^d)^* \mid y(x) = 0 \forall x \in \gamma \}.$$  

For faces $\tau$ and $\gamma$ of $\sigma$, $[\tau \subseteq \gamma \subseteq \sigma] \iff [\tau^\vee \supseteq \gamma^\vee \supseteq \sigma^\vee]$ and $[\tau \subseteq \gamma \subseteq \sigma] \iff [\tau^\perp \supseteq \gamma^\perp \supseteq \sigma^\perp]$.

There is an containment-reversing bijection

$$\tau \iff \tau^* := \tau^\perp \cap \sigma^\vee$$

between faces $\tau$ of $\sigma$ of dimension $r$ and complementary faces $\tau^*$ of $\sigma^\vee$ of dimension $d - r$.

The notions of dual and annihilator as well as complementary face are relative to $\sigma$, although we usually suppress it in the notation.

Remark 3.13. We record two properties of $\mu$ that will be used later.

1. The numbers $\mu^\sigma_\tau(e)$ are relative in the sense that they only depend on the quotient variety $X_{\sigma/\tau}$: Proposition 3.10 shows that $\mu^\sigma_\tau(e) = \mu^\sigma_{\tau/\tau}(e)$.

2. We derive a second recursive formula. Indeed, as an alternating sum of weight $e$ over all flags interpolating from 0 to $\sigma$, sorting the terms by their first non-trivial flag entry $\gamma$, one obtains

$$(3.4.2) \quad \mu^\sigma_0(e) = (-1)^{d_\sigma+1} \text{ih}_0^\sigma(e) + \sum_{0 \leq \gamma \subseteq \sigma} \left( (-1)^{d_\gamma-1} \sum_k \mu^\sigma_\gamma(e - k) \cdot \text{ih}_0^\gamma(k) \right).$$

Here, the first summand corresponds to $\pi = [e]$, the sum collects all others. Moreover, the additional power of $-1$ in all terms in the sum is owed to the fact that all partitions contributing to $\mu^\sigma_\gamma(e - k)$ are one step shorter than their avatars, the partitions of $e$.

Define

$$\gamma^\Omega := \{ y \in (\mathbb{R}^d)^*/\gamma^\perp \mid y(x) \geq 0 \forall x \in \gamma \}.$$  

Since $(\gamma_\mathbb{R})^\ast \simeq (\mathbb{R}^d)^*/\gamma^\perp$ naturally, $\gamma^\Omega$ is the dual of $\gamma$ in its own span, hence absolute (independent of $\sigma$).

We have the following basic lemma on the dual of the cone $\gamma/\tau$ relative to $\gamma_\mathbb{R}/\tau_\mathbb{R}$.

Lemma 3.14. Let $\tau \subseteq \gamma$ be faces of $\sigma$. Then

$$(\gamma/\tau)^\Omega \simeq \tau^*/\gamma^\ast,$$

the right hand side computed relative to $\sigma$.  


Proof. We have \((\R^d/\tau_R)^*/(\gamma/\tau)_{\sigma/\tau}^1 = \tau_\sigma^1/\gamma_\sigma^1\), computing on the left relative to \(\sigma/\tau\) and on the right relative to \(\sigma\). We have thus:

\[
(\gamma/\tau)^{\Omega} = \{y \in (\R^d/\tau_R)^*/(\gamma/\tau)^{\perp} = \tau^{\perp}/\gamma^{\perp} \mid y(x) \geq 0 \ \forall \ x \in \gamma/\tau\}
\]

\[
= (\gamma^{\vee} \cap \tau^{\perp})/\gamma^{\perp}
\]

\[
= ((\sigma^{\vee} + \gamma^{\perp}) \cap \tau^{\perp})/\gamma^{\perp}
\]

\[
= (\sigma^{\vee} \cap \tau^{\perp} + \gamma^{\perp})/\gamma^{\perp}
\]

\[
\simeq (\sigma^{\vee} \cap \tau^{\perp})/(\sigma^{\vee} \cap \tau^{\perp} \cap \gamma^{\perp})
\]

\[
= \tau^{\ast}/\gamma^{\ast}
\]

where the third equality follows from \(\gamma^{\vee} = \sigma^{\vee} + \gamma^{\perp}\) (cf. the proof of Proposition 2 on [Ful93, p.13]) and at the end we use the second isomorphism theorem. \(\square\)

Definition 3.15. If \(\tau \subseteq \gamma\) are faces of \(\sigma\), denote \(Y_{\gamma/\tau}\) the spectrum of the semigroup ring induced by the dual cone of \(\sigma/\tau\) in its natural lattice. In other words, the cone \(\gamma/\tau\) together with its faces defines a fan in \(\gamma_R/\tau_R\). The corresponding toric variety is

\[
Y_{\gamma/\tau} := X_{\tau^{\ast}/\gamma^{\ast}} = X_{(\gamma/\tau)^{\Omega}}.
\]

\(\diamond\)

The following lemma compares the intersection cohomology Betti numbers of \(Y_{\sigma/\gamma}\) with those of \(X_{\gamma/0} = X_\gamma\).

Lemma 3.16. Let \(\sigma\) be a strongly convex rational polyhedral cone of dimension \(d\) as always. Then

\[
\sum_{\emptyset \subseteq \gamma \subseteq \sigma} (-1)^{d_\gamma} \left( \sum_i \text{ih}^{2i}(Y_{\sigma/\gamma}) t^i \right) \left( \sum_j \text{ih}^{2j}(X_\gamma) t^j \right) = 0.
\]

Proof. To a cone \(\sigma \subseteq \R A = \R^d\) belongs the affine toric variety \(X_\sigma = \text{Spec} \ C[\sigma \cap \Z^d]\). Here is an overview of the proof. We first explain independence of \(\text{ih}^\bullet(-)\) of the lattice used to produce \(X_\sigma\). We then discuss combinatorial intersection homology and how it applies to quotient polytopes and cones. Finally, we put the pieces together, using results of Stanley.

Now let \(N \subseteq \R^d\) be another \(\Z\)-lattice (a free subgroup of rank \(d\) whose \(\Q\)-span is \(\Q A\)). The affine toric variety \(X^N_\sigma := \text{Spec} \ C[\sigma \cap N]\) can be different from \(X_\sigma\), however we have a canonical isomorphism

\[
(3.4.3) \quad \text{IH}^\bullet(X_\sigma) \simeq \text{IH}^\bullet(X^N_\sigma).
\]

This can be seen as follows: Consider the lattices \(N' \supseteq N\) in \(\R^d\). It is enough to prove that \(\text{IH}^\bullet(X^{N'}_\sigma) \simeq \text{IH}^\bullet(X^N_\sigma)\). The finite group \(G := N'/N\) naturally acts on \(X_\sigma^{N'}\), and \(X^N_\sigma\) is the quotient of \(X_\sigma^{N'}\) under this action (cf. [CLS11, Proposition 1.3.18]). We have the following isomorphism

\[
\text{IH}^\bullet(X^N_\sigma) \simeq \text{IH}^\bullet(X^{N'}_\sigma)^G = \text{IH}^\bullet(X^{N'}_\sigma)
\]
where $\text{III}^\bullet(X'_\sigma)G$ is the $G$-invariant part. The isomorphism follows from
[Kir86, Lemma 2.12] and the equality comes from the fact that the action
of $G$ is induced by the action of the open dense $\mathbb{C}$-torus of $X'_\sigma$ which acts
trivially: a $\mathbb{C}$-torus acting continuously on a rational vector space must
have a dense subset acting trivially; continuity forces triviality everywhere.
Hence when writing $\text{III}^\bullet(X_\sigma)$ we do not need to worry about the lattice with
respect to which $X_\sigma$ is defined.

Assume that we are given a rational polytope $P \subseteq \mathbb{R}^{d-1}$ of dimension
d$-1$. The set of faces of $P$ (including the empty face $\emptyset$), ordered by inclusion,
forms a poset. Given such a polytope, Stanley [Sta87] defined polynomials
\begin{equation}
(3.4.4) \quad g(P) = \sum g_i(P)t^i \quad \text{and} \quad h(P) = \sum h_i(P)t^i
\end{equation}
recursively by
- $g(\emptyset) = 1$;
- $h(P) = \sum_{\emptyset \subseteq F \subseteq P} (t - 1)^{\dim P - \dim F} g(F)$;
- $g_0(P) = h_0(P)$, $g_i(P) = h_i(P) - h_{i-1}(P)$ for $0 < i \leq \dim P/2$
and $g_i(P) = 0$ for all other $i$.

Now assume $0$ is in the interior $\text{Int}(P)$. From such a polytope we get a fan
$\Sigma_P$ by taking the cones over the faces of $P$; here the empty face corresponds
to the cone $\{0\} \subseteq \mathbb{R}^{d-1}$. This gives a projective toric variety $X_P$. It was
proved independently by Denef and Loeser [DL91] and Fieseler [Fie91] that
$$h_i(P) = \text{ih}^{2i}(X_P).$$

Denote by $\text{cone}(X_P)$ the affine cone of $X_P$. Then
$$g_i(P) = h_i(P) - h_{i-1}(P) = \text{ih}^{2i}(\text{cone}(X_P)) \quad \text{for } 0 < i \leq \dim(P)/2.$$

The affine cone of $X_P$ has the following toric description: Consider the
embedding of $P \subseteq \mathbb{R}^{d-1}$ in $\mathbb{R}^d$ under the map $i : x \mapsto (1, x)$. Let $\text{Cone}(P)$
be the (rational, polyhedral, strongly convex) cone over $i(P)$ with apex at
the origin. Then $\text{cone}(X_P)$ is an affine toric variety given by
$\text{cone}(X_P) = X_{\text{Cone}(P)^\vee} = \text{Spec } \mathbb{C}[\text{Cone}(P)^\vee \cap (\mathbb{Z}^d)^\vee]$. Hence we get
\begin{equation}
(3.4.5) \quad g_i(P) = \text{ih}^{2i}(X_{\text{Cone}(P)^\vee}).
\end{equation}

Two polytopes $P_1$ and $P_2$ are combinatorially equivalent if they have
isomorphic face posets, denoted $P_1 \sim P_2$. This is an equivalence relation,
and $g(P)$ and $h(P)$ only depend on the equivalence class $[P]$ of $P$. Similarly,
given two strongly convex rational polyhedral cones $\sigma_1$ and $\sigma_2$ we write
$\sigma_1 \sim \sigma_2$ if their face posets are isomorphic. If we have $\sigma_i = \text{Cone}(P_i)$ for
$i = 1, 2$ then $[P_1 \sim P_2] \Leftrightarrow [\text{Cone}(P_1) \sim \text{Cone}(P_2)]$.

For a given rational polytope $P$ with $0 \in \text{Int}(P)$, the dual polytope is
$$P^\circ := \{ x \in (\mathbb{R}P)^\ast \mid x(y) \geq -1 \ \forall y \in P \},$$
$\mathbb{R}P$ being the affine span of $P$. There is an order-reversing bijection of
the $k$-dimensional faces $F$ of $P$ and the $(\dim(P) - 1 - k)$-dimensional faces
$\{ x \in P^\circ \mid x(F) = -1 \}$ of $P^\circ$. 

\begin{equation}
(3.4.5) \quad g_i(P) = \text{ih}^{2i}(X_{\text{Cone}(P)^\vee}).
\end{equation}
If the origin is not in \( \text{Int}(P) \), translate \( P \) so that \( 0 \in \text{Int}(P) \) and then dualize. The combinatorial equivalence class of the dual is then well-defined and we still write \( P^\circ \) for this class.

From a \( k \)-dimensional face \( F \) of the \((d - 1)\)-dimensional polytope \( P \) we construct an equivalence class of \((d - k - 2)\)-dimensional polytopes \( P/F \) as follows. Choose a \((d - k - 2)\)-dimensional affine subspace \( L \) whose intersection with \( P \) is a single point of the interior of \( F \). Then a representative of \( P/F \) is given by \( L' \cap P \) where \( L' \) is another \((d - k - 2)\)-dimensional affine subspace, near \( L \) in the appropriate Grassmannians, and such that it meets an interior point of \( P \). (One checks that this representative is well-defined up to projective transformation, hence the combinatorial type is well-defined).

One can see easily that the cone over \( P/F \) is exactly \( \text{Cone}(P)/\text{Cone}(F) \), compare (3.4.1):

\[
\text{Cone}(P/F) \sim \text{Cone}(P)/\text{Cone}(F) = (\text{Cone}(P) + \mathbb{R}F)/\mathbb{R}F.
\]

We will prove Lemma 3.16 using the following formula by Stanley [Sta92] (we use here a presentation given by Braden and MacPherson in [BM99, Proposition 8, formula (3)]:

\[
\sum_{\emptyset \subseteq F \subseteq P} (-1)^{\dim F} g(F^\circ)g(P/F) = 0
\]

The dual \( F^\circ \) of a rational polytope \( F \) is rational in many lattices. Choosing one such lattice yields a rational, polyhedral, strongly convex cone \( \text{Cone}(F^\circ) \) for which \( \text{Cone}(F^\circ)^\triangledown \) is well-defined. By (3.4.3), its intersection homology is independent of the lattice choice. It follows that, with \( \gamma \) the cone over \( F \),

\[
g_i(F^\circ) = \text{ih}^{2i}(X_{\text{Cone}(F^\circ)^\triangledown}) \simeq \text{ih}^{2i}(X_{\text{Cone}(F)}^\triangledown) \simeq \text{ih}^{2i}(X_\gamma)
\]

where we used formula (3.4.3) for the last isomorphism. Recalling Definition 3.15 and that \( \text{Cone}(P) = \sigma \), we obtain

\[
g_i(P/F) = \text{ih}^{2i}(X_{\text{Cone}(P/F)^\triangledown}) = \text{ih}^{2i}(X(\text{Cone}(P)/\text{Cone}(F))^\triangledown) = \text{ih}^{2i}(Y(\text{Cone}(P)/\text{Cone}(F))) = \text{ih}^{2i}(Y_{\sigma/\gamma}),
\]

where the first equality is (3.4.5), the second equality follows from (3.4.6), the third equality is Definition 3.15, and the last follows from (3.4.3). Plugging (3.4.8) and (3.4.9) into (3.4.7) and multiplying with \((-1)\) we get the statement of the Lemma.

We are now ready to give our main result about the weight filtration on the inverse Fourier–Laplace transform of the \( A \)-hypergeometric system \( H_A(0) \):

**Theorem 3.17.** The associated graded module to the weight filtration on the mixed Hodge module \( h_*^{(p\mathbb{Q}^H)} \) is for \( e = 0, \ldots, d \) given by

\[
g_{d+e}^{\varphi^*(p\mathbb{Q}^H)} \simeq \bigoplus \text{IC}_{X_\tau}(p\mathcal{L}_{(\tau,d+e)}),
\]

where \( p\mathcal{L}_{(\tau,d+e)} = L_0^{\varphi_{(\tau,d+e)}} \otimes p\mathbb{Q}^H_{\tau} \) is a constant variation of Hodge structures of weight \( d + e \) on \( T_\tau \). Here \( L_0^{\varphi_{(\tau,d+e)}} \) is a Hodge-structure of Hodge–Tate type.
of weight \(d + e - d_\tau\) of dimension

\[
\mu_\tau^\sigma(e) = \dim_Q L^0_{(\tau,d+e)} = \ih_c^{d_\sigma - d_\tau + e}(Y_{\sigma/\tau}),
\]

compare Definition 3.15.

**Proof.** In light of Proposition 3.10 it only remains to prove that
\[
\mu_\sigma^\tau(e) = \dim_Q L^0_{(\sigma,d+e)} \text{ equals } \ih_c^{d_\sigma - d_\tau + e}(Y_{\sigma/\tau}).
\]
An inspection shows that if \(\sigma = \tau\) then the theorem is (trivially) correct. We argue by induction on \(d_\sigma - d_\tau\). While in principle a Poincaré series only involves non-negative terms there is no harm in allowing negative indices: they just add zero terms.

According to Lemma 3.16 we have

\[
0 = \sum_{0 \leq \gamma \subseteq \sigma} (-1)^{d_\gamma} \left( \sum_{j = -\infty}^{\infty} t^j \cdot \ih_c^{2j}(Y_{\gamma/\sigma}) \right) \cdot \left( \sum_{i = -\infty}^{\infty} t^i \cdot \ih_c^{2i}(X_\gamma) \right)
\]

\[
= \sum_{0 \leq \gamma \subseteq \sigma} (-1)^{d_\gamma} \left( \sum_{j = -\infty}^{\infty} t^j \cdot \ih_c^{2(d_\sigma - d_\gamma - j)}(Y_{\gamma/\sigma}) \right) \cdot \left( \sum_{i = -\infty}^{\infty} t^i \cdot \ih_c^{2(d_\gamma - i)}(X_\gamma) \right)
\]

\[
= (-1)^{d_\sigma} \cdot \sum_{k = -\infty}^{\infty} t^k \cdot \ih_c^{2(d_\sigma - k)}(X_\sigma)
\]

(Lemma 3.4)

\[
+ \sum_{0 \leq \gamma \subseteq \sigma} (-1)^{d_\gamma} \left( \sum_{j = -\infty}^{\infty} t^j \cdot \ih_c^{2(d_\sigma - d_\gamma - j)}(Y_{\gamma/\sigma}) \right) \cdot \left( \sum_{i = -\infty}^{\infty} t^i \cdot \ih_c^{2(d_\gamma - i)}(X_\gamma) \right)
\]

(general \(\gamma\))

\[
+ \sum_{k = -\infty}^{\infty} t^k \cdot \ih_c^{2(d_\sigma - k)}(Y_{\sigma/0}).
\]

(from \(\gamma = 0\))

where we have used Lemma 3.4 (1) for the second equality.

Induction allows to substitute \(\mu_\gamma^\sigma(d_\sigma - d_\gamma - 2j)\) for \(\ih_c^{2(d_\sigma - d_\gamma - j)}(Y_{\gamma/\sigma})\) for all \(\gamma \neq 0, \sigma\) in the sum “general \(\gamma\)”. At the same time we can replace, by definition, \(\ih_c^{2(d_\gamma - i)}(X_\gamma)\) by \(\ih_c^\gamma(d_\gamma - 2i)\). With these substitutions, collect terms with equal \(t\)-power:
\[ 0 = (-1)^{d_\sigma} \cdot \sum_{k=-\infty}^{\infty} t^k \cdot \text{ih}_0^\gamma(d_\sigma - 2k) \]

(from \( \gamma = \sigma \))

\[ + \sum_{k=-\infty}^{\infty} \left( \sum_{i+j=k} t^k \sum_{0 \leq \gamma \leq \sigma} (-1)^{d_\gamma} \left( \mu_\gamma^\sigma(d_\sigma - d_\gamma - 2j) \cdot \text{ih}_0^{\gamma}(d_\gamma - 2i) \right) \right) \]

(general \( \gamma \))

\[ + \sum_{k=-\infty}^{\infty} t^k \cdot \text{ih}_c^{2(d_\sigma - k)}(Y_{\sigma/0}) \]  

(from \( \gamma = 0 \))

In degree \( k \) we have therefore:

(3.4.10)

\[ 0 = (-1)^{d_\sigma} \cdot \text{ih}_0^\gamma(d_\sigma - 2k) + \sum_{i+j=k} \left( \sum_{0 \leq \gamma \leq \sigma} (-1)^{d_\gamma} \mu_\gamma^\sigma(d_\sigma - d_\gamma - 2j) \cdot \text{ih}_0^{\gamma}(d_\gamma - 2i) \right) + \text{ih}_c^{2(d_\sigma - k)}(Y_{\sigma/0}). \]

Since the odd-dimensional intersection homology Betti numbers are zero (cf. Lemma 3.4 (3)), we can include all missing summands \((-1)^{d_\gamma} \mu_\gamma^\sigma(d_\sigma - d_\gamma - j') \cdot \text{ih}_0^{\gamma}(d_\gamma - i') \) with \( i' + j' = 2k \) without affecting the value of the sum. Since \( \text{ih}_0^{\gamma}(d_\gamma - i') = \text{ih}_c^{d_\gamma+(d_\gamma-i')}(X_\gamma) \), no summand with \( i'' := d_\gamma - i' \leq 0 \) can contribute (cf. Lemma 3.4 (2)). We can therefore rewrite (3.4.10) to

(3.4.11)

\[ 0 = (-1)^{d_\sigma} \cdot \text{ih}_0^\gamma(d_\sigma - 2k) + \sum_{i''} \left( \sum_{0 \leq \gamma \leq \sigma} (-1)^{d_\gamma} \mu_\gamma^\sigma(d_\sigma - 2k - i'') \cdot \text{ih}_0^{\gamma}(i'') \right) + \text{ih}_c^{2(d_\sigma - k)}(Y_{\sigma/0}). \]

In light of the recursion (3.4.2), this yields \( 0 = -\mu_0^\sigma(d_\sigma - 2k) + \text{ih}_c^{2(d_\sigma - k)}(Y_{\sigma/0}) \)

and finishes the inductive step.

4. Weight filtrations on \( A \)-hypergeometric systems

In this section we translate the results from the previous section to hypergeometric \( D \)-modules on

\[ V := \mathbb{C}^n \]

via the Fourier transform. Part of this is rather mechanical, but identifying the weight filtrations requires some extra hypotheses, see Corollary 4.13.

4.1. Translation of the filtration. We start this section with various definitions around \( A \)-hypergeometric systems. For more details, we refer to (for example) [MMW05, RSW18]. Our terminology is that of [MMW05].

Throughout, we continue Notation 3.1

**Definition 4.1.** Write \( L_A \) for the \( \mathbb{Z} \)-module of integer relations among the columns of \( A \) and write \( \mathcal{D}_{C^n} \) for the sheaf of rings of differential operators
on $V = \mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$. Denote $\partial_j$ the operator $\partial/\partial x_j$. For
\[ \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{C}^d \] define
\[ \mathcal{M}_A^\beta := D_{\mathbb{C}^n}/\mathcal{I}_A^\beta \]
where $\mathcal{I}_A^\beta$ is the sheaf of left ideals generated by the toric operators
\[ \square_u := \prod_{u_j < 0} \partial_j^{-u_j} - \prod_{u_j > 0} \partial_j^{u_j} \]
for all $u = (u_1, \ldots, u_n) \in \mathbb{L}_A$, and the Euler operators
\[ E_i := \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i. \]

We will write $M_A^\beta := \Gamma(V, \mathcal{M}_A^\beta)$ for the $D_A$-module of global sections where $D_A = \Gamma(V, D_V)$. Denote by $R_A$ (resp. $O_A$) the polynomials rings over $\mathbb{C}$ generated by $\partial_A = \{\partial_j\}_j$ (resp. $x_A = \{x_j\}_j$). and set $S_A := R_A/R_A[\square_u]_{u \in \mathbb{L}_A}$.

We have
\[ x^u E_i - E_i x^u = -(A \cdot u)_i x^u, \]
\[ \partial^u E_i - E_i \partial^u = (A \cdot u)_i \partial^u. \]

Define the $A$-degree on $R_A$ and $D_A$ as
\[ \deg_A(x_j) = a_j = -\deg_A(\partial_j) \in \mathbb{Z}A \]
and denote by $\deg_{A,i}(-)$ the degree associated to the $i$-th row of $A$. This convention agrees with the choices in [MMW05] but is opposite to that in [Rei14]. Then $E_i P = P(E_i + \deg_{A,i}(P))$ for any $A$-graded $P \in D_A$.

Given a left $A$-graded $D_A$-module $M$ we can define commuting $D_A$-linear endomorphisms $E_i$ via
\[ E_i \circ m := (E_i - \deg_{A,i}(m)) \cdot m \]
for $A$-graded elements of $M$. If $N$ is an $A$-graded $R_A$-module $N$ we get a commuting set of $D_A$-linear endomorphisms on the left $D_A$-module $D_A \otimes_R A$ $N$ by
\[ E_i \circ (P \otimes Q) := (E_i - \deg_i(P) - \deg_i(Q)) P \otimes Q \]
for any $A$-graded $P, Q$. The Euler–Koszul complex $K_i(N; E - \beta)$ of the $A$-graded $R_A$-module $N$ is the homological Koszul complex induced by $E - \beta := \{(E_i - \beta_i)\}_i$ on $D_A \otimes_R A N$. The terminal module sits in homological degree zero. We denote by $K_i(N; E - \beta)$ the corresponding complex of quasi-coherent sheaves. The homology objects are $H_i(N; E - \beta)$ and $\mathcal{H}_i(N; E - \beta)$, respectively.

For a finitely generated $A$-graded $R_A$-module $N = \bigoplus \alpha N_\alpha$ write $\deg_A(N) = \{\alpha \in \mathbb{Z}A \mid N_\alpha \neq 0\}$ and then let the quasi-degrees of $N$ be
\[ q\deg_A(N) := \frac{\deg_A(N)^{\text{var}}}{\deg_A(N)}, \]
the Zariski closure of $\deg_A(N)$ in $\mathbb{C}^d$.

The following subset of parameters $\beta \in \mathbb{C}^d$ will be of importance to us.
Definition 4.2 ([SW09]). The set of strongly resonant parameters of $A$ is

\[ \text{sRes}(A) := \bigcup_{j=1}^{d} \text{sRes}_j(A) \]

where

\[ \text{sRes}_j(A) := \left\{ \beta \in \mathbb{C}^d \mid \beta \in -(N+1)a_j - q\deg(S_A/\partial_j) \right\} . \]

Definition 4.3. Let

\[ \langle -,- \rangle : \hat{V} \times V \to \mathbb{C}, \quad (\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n) \mapsto \sum_{i=1}^{n} \xi_i \eta_i . \]

We define a $\mathcal{D}_{\hat{V} \times V}$-module by

\[ \mathcal{L} := \mathcal{O}_{\hat{V} \times V} \cdot \exp((-1) \cdot \langle -,- \rangle) , \]

and we refer to [KS97, Section 5] for details on these sheaves. Denote by $p_1 : \hat{V} \times V \to \mathbb{C}^n$ for $i = 1, 2$ the projection to the first and second factor respectively (identifying the respective factor with the target). The Fourier–Laplace transform is defined by

\[ \mathcal{F_L} : D_{\mathcal{D}_{\hat{V}}} \to D_{\mathcal{D}_{V}}, \quad \mathcal{O} \mapsto p_2^+(p_1^+ \mathcal{M} \otimes \mathcal{L})[-n] . \]

We define by $\hat{M}_A^\beta$ the module of global sections to the sheaf

\[ \hat{M}_A^\beta := \mathcal{F}_L^{-1}(\mathcal{M}_A^\beta) \]

and define the following twisted structure sheaves on $T$:

\[ \mathcal{O}_T^\beta := \mathcal{D}_T/\mathcal{D}_T \cdot (\partial_t \partial_{t} + \beta_1, \ldots, \partial_t \partial_{d} + \beta_d) , \]

where we note that $\mathcal{O}_T^\beta \simeq \mathcal{O}_T^\gamma$ if and only if $\beta - \gamma \in \mathbb{Z}^d$.

Theorem 4.4. ([SW09] Theorem 3.6, Corollary 3.7) Let $A$ be a pointed $(d \times n)$ integer matrix satisfying $ZA = \mathbb{Z}^d$. Then for the map $h$ in (1.2.1),

the following statements are equivalent

1. $\beta \notin \text{sRes}(A)$;
2. $\hat{M}_A^\beta \simeq h_+ \mathcal{O}_T^\beta$.

\[ \square \]

Theorem 4.4 implies that for $\beta \in \mathbb{Z}^d \setminus \text{sRes}(A)$ we have, with notation as in (1.3.2),

\[ \hat{M}_A^\beta \simeq h_+ \mathcal{O}_T \simeq i_+ \varphi_+ \mathcal{O}_T . \]

We now concentrate on integral $\beta$. Since $\mathcal{O}_T$ is the underlying left $\mathcal{D}_T$-module of $\mathcal{D}_{\mathcal{H}}$ this induces the structure of a mixed Hodge module on $\hat{M}_A^\beta$ from Theorem 3.17. Recalling Definition 2.0.1 and bearing in mind that the functor $i_*$ preserves weight, we infer:
Corollary 4.5. For $\beta \in \mathbb{Z}^d \setminus \operatorname{sRes}(A)$, the module $\hat{M}_A^\beta = \operatorname{FL}^{-1}(M_A^\beta)$ carries the structure of a mixed Hodge module $\hat{H}M_A^\beta$ which is induced by the isomorphism

$$\hat{M}_A^\beta \simeq \mathbb{D} \operatorname{mod}(i_* \varphi_* p\mathbb{Q}^H).$$

The corresponding weight filtration is given by

$$\operatorname{gr}^W_{d+e} \hat{H}M_A^\beta \simeq \bigoplus_{\gamma} \bar{i}_\gamma^* \operatorname{IC}(X_{\gamma}, (p\mathbb{L}_{(\gamma,d+e)}))$$

where $\bar{i}_\gamma : X_{\gamma} \to \mathbb{C}^n$ is the embedding of the closure of the $\gamma$-torus, and $p\mathbb{L}_{(\gamma,d+e)} = L^0_{(\gamma,d+e)} \otimes p\mathbb{Q}^H_{\bar{i}_\gamma}$ is a constant variation of Hodge structures of weight $d+e$. Here $L^0_{(\gamma,d+e)}$ is a Hodge-structure of Hodge–Tate type of weight $d + e - d_\gamma$ of dimension

$$\dim_{\mathbb{Q}} L^0_{(\gamma,d+e)} = i h^d_{c} d_\sigma - d_\gamma + e(Y_{\sigma/\gamma}),$$

with $Y_{\sigma/\gamma}$ as in Definition 3.15.

As a corollary, we obtain information about the holonomic length of $M_A^0$. Recall that $\hat{M}^\text{IC}(X_{\gamma}) = \mathbb{D} \operatorname{mod}(\text{IC}(X_{\gamma}))$ is the unique simple $\mathcal{D}$-equivariant $\mathcal{D}$-module on $\hat{V}$ with support $X_{\gamma}$.

Corollary 4.6. Let $A$ be as in Notation 3.1 and choose $\beta \in \mathbb{Z}^d \setminus \operatorname{sRes}(A)$. Then $\hat{M}_A^\beta$ carries a finite separated exhaustive filtration $\{W_\bullet \hat{M}_A^\beta\}_{e=0}$ given by

$$W_\bullet \hat{M}_A^\beta := \operatorname{FL}(W_\bullet \hat{H}M_A^\beta).$$

This filtration satisfies

$$\operatorname{gr}^W_{d+e} \hat{M}_A^\beta = \bigoplus_{\gamma} \bigoplus_{i=1} C_\gamma^\delta(e).$$

Here, $C_\gamma^\delta = \operatorname{FL} \hat{M}^\text{IC}(X_{\gamma})$ is a simple equivariant holonomic $\mathcal{D}$-module (that is independent of $e$ and) which occurs in $\operatorname{gr}^W_{d+e} \hat{M}_A^\beta$ with multiplicity $\mu_\gamma^\delta(e) = i h^d_{c} d_\sigma - d_\gamma + e(Y_{\sigma/\gamma}) = i h^d_{c} d_\sigma - d_\gamma - e(Y_{\sigma/\gamma})$, the $(d_\sigma - d_\gamma - e)$-th intersection cohomology Betti number of the affine toric variety $Y_{\sigma/\gamma}$. \hfill \Box

4.2. The homogeneous case: monodromic Fourier–Laplace. Although the Fourier–Laplace transformation does not preserve regular holonomicity in general, and so $\hat{M}_A^\beta$ may not be a mixed Hodge module, it is preserved for the derived category of complexes of $\mathcal{D}$-modules with so-called monodromic cohomology. In this case we can express the Fourier–Laplace transformation as a monodromic Fourier transformation (or Fourier–Sato transformation).

In order to make this work, we now assume that the matrix $A$ is homogeneous, which means that

$$(1, \ldots, 1)^T \in \mathbb{Z}(A^T).$$

Via a suitable coordinate change on the torus $T$, we can then assume that the top row of $A$ is $(1, \ldots, 1)$. 

\hfill \Box
Denote by 
\[ \theta : \mathbb{C}^* \times \hat{V} \to \hat{V} \]
the standard \( \mathbb{C}^* \) action on \( \hat{V} \); let \( z \) be a coordinate on \( \mathbb{C}^* \). We refer to the push-forward \( \theta_*(z \partial_z) \) as the Euler vector field \( \mathcal{E} \).

**Definition 4.7.** [Bry86] A regular holonomic \( D\hat{V} \)-module \( M \) is called monodromic, if the Euler field \( \mathcal{E} \) acts finitely on the global sections of \( M \): for each global section section \( v \) of \( M \) the set \( \{ \mathcal{E}^n(v) \}_{n \in \mathbb{N}} \) should generate a finite-dimensional vector space. We denote by \( D^b_{\text{mon}}(D\hat{V}) \) the derived category of bounded complexes of \( D\hat{V} \)-modules with regular holonomic and monodromic cohomology.

Since we assume that \( A \) has \((1, \ldots, 1)\) as its top row, each \( \hat{M}_A^\beta \) is monodromic.

**Theorem 4.8.** [Bry86]

1. \( FL \) preserves complexes with monodromic cohomology.
2. In \( D^b_{\text{mon}}(D\hat{V}) \) and \( D^b_{\text{mon}}(D\hat{V}) \) we have
   \[ FL \circ FL \simeq \text{Id} \quad \text{and} \quad \mathbb{D} \circ FL \simeq FL \circ \mathbb{D}. \]
3. \( FL \) is \( t \)-exact with respect to the natural \( t \)-structures on \( D^b_{\text{mon}}(D\hat{V}) \) resp. \( D^b_{\text{mon}}(D\hat{V}) \).

**Proof.** The above statements are stated in [Bry86] for constructible monodromic complexes. One has to use the Riemann-Hilbert correspondence, [Bry86, Proposition 7.12, Theorem 7.24] to translate the statements. So the first statement is Corollaire 6.12, the second statement is Proposition 6.13 and the third is Corollaire 7.23 in [Bry86].

We will now consider the monodromic Fourier–Laplace transform (or Fourier–Sato transform) which preserves the category of mixed Hodge modules.

**Definition 4.9.** Consider the diagram

\[
\begin{array}{ccc}
\hat{V} \times V & \xrightarrow{\omega} & \hat{V} = \mathbb{C}^n \\
\downarrow \mathbb{C}^n \times \mathbb{C}^n & & \mathbb{C}_z \times V \\
p_1 & \xrightarrow{\text{inclusion}} & \{0\} \times \mathbb{C}^n
\end{array}
\]

where \( p_1 \) is the projection to the first factor, \( i_0 \) is the inclusion and the map \( \omega \) is given by

\[
\omega : \hat{V} \times V \longrightarrow \mathbb{C}_z \times V \\
(\eta, \xi) \mapsto (z = \sum z_i \eta_i, \eta)
\]
The Fourier–Sato transform or monodromic Fourier transform is defined by
\[ D^b(\text{MHM}(\hat{V})) \to D^b(\text{MHM}(V)) \]
\[ \mathcal{M} \mapsto \phi_z \omega_* p_{p_1}^! \mathcal{M} \simeq \phi_z \omega_* p_{p_1}^! \mathcal{M} \]
where \( \phi_z \) is the nearby cycle functor along \( z = 0 \) and we write \( p_{f!} := f^!(dY - dX) \) for a map \( f : X \to Y \). The isomorphism follows from [KS94, Proposition 10.3.18].

Remark 4.10. The original definition of the Fourier–Sato transform is different; we use here an equivalent version (see [KS94, Def. 3.7.8, Prop. 10.3.18]) that is well adapted to mixed Hodge modules.

For a monodromic complex the (usual) Fourier–Laplace transformation and the monodromic Fourier transformation are the same (we use again the equivalent version of the Fourier-Sato version from [KS94]):

Theorem 4.11. ([Bry86, Théorème 7.24] Let \( \mathcal{M} \in D^b_{\text{mod}}(\mathcal{D}_{\hat{V}}) \) then
\[ DR^\text{an}(\text{FL}(\mathcal{M})) \simeq \phi_z \omega_* p_{p_1}^! DR^\text{an}(\mathcal{M}). \]
It follows that the monodromic Fourier transform induces an exact functor
\[ \phi_z \omega_* p_{p_1}^! : \text{MHM}((\hat{V})) \to \text{MHM}(V) \]
We next identify a class of modules for which the monodromic Fourier transform has a very simple effect on the weight filtration.

Proposition 4.12. Let \( \pi : \hat{V} \setminus \{0\} \to \mathbb{P}(\hat{V}) \) be the natural projection and \( j_0 : \hat{V} \setminus \{0\} \to \hat{V} \) the inclusion. Let \( \mathcal{M} \in \text{MHM}(\hat{V}) \) such that \( \mathcal{M} \simeq (j_0)_* \pi^! \mathcal{N} \) for some \( \mathcal{N} \in D^b \text{MHM}(\mathbb{P}(\hat{V})) \). Then
\[ W_k \left( \phi_z \omega_* p_{p_1}^! \mathcal{M} \right) \simeq \phi_z \omega_* p_{p_1}^! \left( W_k \mathcal{M} \right) \]
Proof. We first prove that the logarithm of the monodromy \( N \) acts trivially on \( \phi_z \omega_* p_{p_1}^! \mathcal{M} \). Define the subvarieties
\[ U := \left\{ \sum_{i=1}^n \eta_i x_i 
eq 0 \right\} \subseteq \mathbb{P}(\hat{V}) \times V \]
\[ \tilde{U} := \left\{ \sum_{i=1}^n \eta_i x_i 
eq 0 \right\} \subseteq (\hat{V} \setminus \{0\}) \times V \]
\[ U_1 := \left\{ \sum_{i=1}^n \eta_i x_i = 1 \right\} \subseteq (\hat{V} \setminus \{0\}) \times V \]
with the embeddings \( j_U : U \to \mathbb{P}(\hat{V}) \times V \) and \( j : \tilde{U} \to (\hat{V} \setminus \{0\}) \times V \). Notice that we have isomorphisms
\[ f : \mathbb{C}^* \times U_1 \to \tilde{U} \quad \text{and} \quad g : U_1 \to U \]
\[ (z, \eta, \mathbf{x}) \mapsto (z \cdot \eta, \mathbf{x}) \quad \text{and} \quad (\eta, \mathbf{x}) \mapsto ((\eta_1 : \ldots : \eta_n), \mathbf{x}). \]
Consider now the following diagram

\[
\begin{array}{ccc}
\hat{V} & \overset{p_1}{\longrightarrow} & \hat{V} \times V \\
\downarrow j_0 & & \downarrow j_0 \times id \\
(\hat{V} \setminus \{0\}) & \overset{f}{\longrightarrow} & (\hat{V} \setminus \{0\}) \times V \\
\downarrow \pi & & \downarrow \pi \times id \\
\mathbb{P}(\hat{V}) & \overset{\pi_1}{\longrightarrow} & \mathbb{P}(\hat{V}) \times V \\
\end{array}
\]

where \( \xi : U_1 \subseteq (\hat{V} \setminus \{0\}) \times V \rightarrow V \) is the projection to the second factor and \( \tilde{\mu}, \pi_1 \) resp. \( \tilde{\omega}, \hat{\omega} \) are the corresponding restrictions of \( \mu_1 \) resp. \( \omega \).

We have the following isomorphisms

\[
j^! \omega^* p_1^! \mathcal{M} \simeq j^! \omega^* p_1 j_0^* \pi_1^! N
\]

Set \( N' := g^! j_0^! \pi_1^! N \). We have \((id \times \xi)_* p_2^! N' \simeq \tilde{p}_2^! \xi_* N'\) where \( \tilde{p}_2 : \mathbb{C}^* \times V \rightarrow V \) is the projection to the second factor. This shows that \( j^! \omega^* p_1^! \mathcal{M} \simeq \tilde{p}_2^! \xi_* N' \) is constant in the \( z \)-direction. Hence the logarithm of the monodromy \( N \) acts trivially on the (unipotent) nearby cycles \( \psi_2^* \omega^* p_1^! \mathcal{M} \) and therefore also on the vanishing cycles \( \phi_2^* \omega^* p_1^! \mathcal{M} \).

Set \( L_i \phi_2^* \omega^*_p p_1^! \mathcal{M} := \phi_2^* W_i \mathcal{H}^0 \omega^*_p p_1^! \mathcal{M} \). The weight filtration on \( \phi_2^* \omega^*_p p_1^! \mathcal{M} = \phi_2^* \mathcal{H}^0 \omega^*_p p_1^! \mathcal{M} \) is the relative monodromy weight filtration with respect to the filtration \( L \) and the nilpotent endomorphism \( N \). In the (current) case \( N = 0 \) we simply get \( W_i \phi_2^* \omega^*_p p_1^! \mathcal{M} = L_i \phi_2^* \omega^*_p p_1^! \mathcal{M} = \phi_2^* W_i \mathcal{H}^0 \omega^*_p p_1^! \mathcal{M} \) (cf. [Sai90, (2.27) & Proposition 2.4]).

We now want to prove by decreasing induction on \( \ell \) that

- \( W_\ell \phi_2^* \omega^*_p p_1^! \mathcal{M} = \phi_2^* \omega^*_p p_1^! W_\ell \mathcal{M} \)
- \( \phi_2^* \omega^*_p p_1^! Gr_\ell^W \mathcal{M} \) is pure of weight \( \ell \).

This is certainly true for \( \ell \gg 0 \) since in this case \( W_\ell \mathcal{M} = \mathcal{M} \). Assume now that the two statements above are true for some \( \ell \), we prove the two statements for \( \ell - 1 \). For this consider the exact sequence

\[
(4.2.1) \quad \phi_2^* \omega^*_p p_1^! W_{\ell-1} \mathcal{M} \longrightarrow \phi_2^* \omega^*_p p_1^! W_\ell \mathcal{M} \longrightarrow \phi_2^* \omega^*_p p_1^! Gr_\ell^W \mathcal{M}.
\]
Since \( W_\ell \varphi_2 \omega_s^p p_1^W \mathcal{M} = \varphi_2 \omega_s^p p_1^W \mathcal{M} \) and since \( \varphi_2 \omega_s^p p_1^W \mathcal{M} \) is pure of weight \( \ell \) we see that
\[
\varphi_2 \omega_s^p p_1^W \mathcal{M} \supseteq W_{\ell-1} \varphi_2 \omega_s^p p_1^W \mathcal{M}.
\]

To show the other inclusion, consider the morphism
\[
\mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \to \mathcal{I}_{\ell-1} \to \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M},
\]
where \( \mathcal{I}_{\ell-1} \) is the image of the morphism \( \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \to \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \). Notice that the map \((4.2.1)\) becomes an isomorphism after applying \( \varphi_2 \) (cf. [KS94, equation 10.3.32]).

Since \( p_1^W \mathcal{M} = W_{\ell-1} p_1^W \mathcal{M} \) and the functor \( \omega_1 \) does not increase weight we have \( \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \subseteq W_{\ell-1} \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \). Because \( \mathcal{I}_{\ell-1} \) is a quotient of \( \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \) we also have \( W_{\ell-1} \mathcal{I}_{\ell-1} = \mathcal{I}_{\ell-1} \). Since \( \mathcal{I}_{\ell-1} \) is a subobject of \( \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \) the morphism \( N \) acts trivially and therefore \( W_{\ell-1} \varphi_2 \mathcal{I}_{\ell-1} = \varphi_2 \mathcal{I}_{\ell-1} \). The isomorphism \( \varphi_2 \mathcal{I}_{\ell-1} \simeq \varphi_2 \omega_s^p p_1^W \mathcal{M} \) shows
\[
\varphi_2 \omega_s^p p_1^W \mathcal{M} = W_{\ell-1} \varphi_2 \omega_s^p p_1^W \mathcal{M} \subseteq W_{\ell-1} \varphi_2 \omega_s^p p_1^W \mathcal{M}.
\]

We now want to show that \( \varphi_2 \omega_s^p p_1^W \mathcal{M} \) is pure of weight \( \ell - 1 \). For this consider the morphisms
\[
\mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \to \mathcal{G}_{\ell-1} \to \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M}
\]
where \( \mathcal{G}_{\ell-1} \) is the image of the morphism \( \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \to \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \). Notice again that the map above becomes an isomorphism after applying \( \varphi_2 \).

Since \( p_1^W \mathcal{M} \) preserves weight, and since \( \omega_1 \) does not increase weight and since \( \omega_s \) does not decrease weight the module \( \mathcal{G}_{\ell-1} \) is pure of weight \( \ell - 1 \). Since \( \varphi_2 \mathcal{G}_{\ell-1} \) is a subobject of \( \varphi_2 \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \) and \( \varphi_2 \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \) is a quotient of \( \varphi_2 \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \) the morphism \( N \) is trivial on \( \varphi_2 \mathcal{G}_{\ell-1} \). Therefore \( \varphi_2 \mathcal{G}_{\ell-1} \simeq \varphi_2 \mathcal{H}^0 \omega_s^p p_1^W \mathcal{M} \) is pure of weight \( \ell - 1 \).

This finishes the proof of the proposition. \( \square \)

If we endow the GKZ-system \( \mathcal{M}^0_A \) with the mixed Hodge module structure coming from the monodromic Fourier transformation we get the following result.

**Corollary 4.13.** For homogeneous \( A \) in the context of Corollary 4.6 and Definition 4.9, let \( \tilde{\mathcal{H}}^0 \mathcal{M}^0_A \) be the GKZ-system endowed with the mixed Hodge module structure coming from the isomorphism
\[
\mathcal{M}^0_A \simeq \text{Dmod}(\varphi_2 \omega_s^p p_1^W \tilde{\mathcal{M}}^0_A)
\]
with \( \tilde{\mathcal{M}}^0_A \) as in Corollary 4.5. Then
\[
\text{Dmod}(W_k \tilde{\mathcal{M}}^0_A) = \varphi_2 \omega_s^p p_1^W (W_k \tilde{\mathcal{M}}^0_A) = \text{FL}(W_k \tilde{\mathcal{M}}^0_A).
\]
Proof. It remains to shows that $H^0_{\mathcal{M}_A}$ can be written as $(j_0)_*\pi^!\mathcal{N}$ for some $\mathcal{N} \in D^b \text{MHM}(\mathbb{P}(\hat{V}))$. Consider the diagram

$$
\begin{array}{c}
\begin{array}{ccc}
T & \xrightarrow{h} & \hat{V} \\
\downarrow{h_0} & & \downarrow{j_0} \\
\hat{V}\setminus\{0\} & \xrightarrow{pr} & \hat{V}
\end{array}
\end{array}
\xrightarrow[p]{\pi} \begin{array}{c}
\begin{array}{c}
T \xrightarrow{\overline{h}} \mathbb{P}(\hat{V})
\end{array}
\end{array}
\]

where $pr : T \to \overline{T}$ is the projection to the last $d-1$ coordinates $h = j_0 \circ h_0$ is the canonical factorization and $\overline{h}$ is the projectivization of $h$. We have

\[
H^0_{\mathcal{M}_A} \simeq h_* p^{Q_H} \simeq h_*(j_0)_* \pi^! p^{Q_{\overline{T}}} \simeq (j_0)_*(\overline{h})_* \pi^! p^{Q_{\overline{T}}} (-1)[-1] \\
\simeq (j_0)_*(\overline{h})_* \pi^! p^{Q_{\overline{T}}} \simeq (j_0)_* \pi^! p^{Q_{\overline{T}}} (-1)[-1]
\]

$\mathcal{N}$.

5. Explicit weight filtration for $d = 3$

Throughout this section, $A$ is normal but not necessarily homogeneous. Via the Fourier transform $FL$ one can port the weight filtration on the mixed Hodge module $h_* p^{Q_{\overline{T}}}$ to the hypergeometric system $\mathcal{M}_A^0$. While the latter may not be a mixed Hodge module, one still obtains in any case a filtration that has semisimple associated graded pieces and which we still denote by $W_\bullet$. If $A$ is homogeneous, then $\mathcal{M}_A^0$ is a mixed Hodge module and, by Corollary 4.13, $FL$ agrees with the functor $\phi_2^{\omega_1} p_{\overline{T}}^!$ and relates the weight filtrations on $\mathcal{M}_A^0$ and $h_* p^{Q_{\overline{T}}}$. In this section we consider specifically the cases when either $NA$ is simplicial, or when $d \leq 3$ and write out an explicit filtration in terms of generators that agrees with $W_\bullet$.

Batyrev proved that in the homogeneous, normal case the weight filtration on the restriction of $\mathcal{M}_A^0$ to the complement of the principal $A$-discriminant is given by the face filtration on $S_A$ in the sense that (in the localization) $W_{d+k}(\mathcal{M}_A^0)$ is generated by the $\partial$-monomials whose degree sits in the relative interior of a face of $\sigma$ whose codimension is at most $k$; see [Sti98, Thm. 8, p.28]. It has been speculated that this be true even on $\mathcal{M}_A^0$ itself. We show here that this is the case for simplicial homogeneous $\sigma$ but can fail in the general homogeneous case already in dimension three. We discuss completely in terms of generators the filtration $FL(W_\bullet h_* (O_T))$ if $d = 3$ and $A$ is normal (but not necessarily homogeneous). Then $\sigma$ is the cone over a $(d-1)$-dimensional polygon $P$ with $f_0$ vertices and $P$ arises as intersection of $\sigma$ with a generic hyperplane. It is not suggested or required that the columns of $A$ lie on $P$. It is sufficient to concentrate on the global sections $M_A^0$. 


Notation 5.1. On $M_A^0$, let $W'$ be the filtration of Batyrev:

$$W'_{d+k}(M_A^0) = \text{image of } D_A \{ \partial^u \mid A \cdot u = a \in \text{Int}(\mathbb{N} \tau), \dim(\tau) \geq \dim(\sigma) - k \} \text{ in } M_A^0$$

for $d \leq k \leq 2d$. In particular, $W'_{<d}(M_A^0) = 0$ and $W'_{>d}(M_A^0) = M_A^0$.

For $d = 3$ let $W''$ be the filtration

$$W''_k(M_A^0) = \begin{cases} W'_k(M_A^0) & \text{if } k \neq 2d - 2, \\ W_k + \sum_r D_A \cdot e_r & \text{if } k = 2d - 2, \end{cases}$$

where $e_r$ is defined below in (5.0.2).

For ease of notation, we do not repeat "$M_A^0$" each time we write a filtration piece. We will show that $W'' = W$ if $d = 3$, and that $W'' = W'' = W$ if $\sigma$ is simplicial. For this, consider the toric modules defined as follows.

Notation 5.2. If $\tau$ is a face of $\sigma$ write $\partial_\tau^+$ for the $S_A$-ideal generated by the $\partial$-monomials whose degree is interior to $\tau$. Let $S_A^{(k)}$ be the ideal of $S_A$ spanned by the monomials that are interior to a face of codimension $k$ or less, $S_A^{(k)} = \sum_{\dim \tau \geq d-k} \partial_\tau^+$. Then $S_A^{(0)}$ is the interior ideal, $S_A^{(d-1)}$ is the maximal ideal $S_A \partial_A$, and $S_A^{(d)}$ is $S_A$ itself.

We begin with showing that for normal $S_A$ the $D_A$-module generated by the interior ideal $S_A^{(0)}$ inside $M_A^0$ is simple and for homogeneous $A$ agrees with $W_d$ so that $W_k = W_k' = W_k''$ for $k \leq d$.

Lemma 5.3. Suppose $A$ is pointed and saturated, but not necessarily homogeneous. Let $u, v \in \mathbb{N}^n$ be such that $b := A \cdot v$ is in the interior $\text{Int}(\mathbb{N} A)$ of the semigroup (i.e., not on a proper face). Set $a = A \cdot u$. Then the contiguity map $c_{-a-b} : M_A^{-a-b} \xrightarrow{\partial^u} M_A^{-b}$ is an isomorphism.

In particular, the ideal in $M_A^0$ generated by $\partial^b$ (the image of the contiguity morphism $c_{-b,0} : M_A^{-b} \to M_A^0$) is the same for all $b = A \cdot v$ in the interior of $A$.

Proof. Consider the toric sequence $0 \to S_A(a) \xrightarrow{\partial^u} S_A \xrightarrow{\tau} \to \text{image of } \partial^u \to 0$, and the Euler–Koszul functor attached to $-b$. By [MMW05, Prop. 5.3], the induced contiguity morphism $c_{-b-a,-b} : M_A^{-b-a} \to M_A^{-b}$ is an isomorphism if and only if $-b$ is not quasi-degree of $Q$. The quasi-degrees of $Q = S_A / \partial^u S_A$ are contained in a union of hyperplanes that meet $-NA$ and are parallel to a face of the cone $\sigma$. In particular, these quasi-degrees are disjoint to the interior points $\text{Int}(\mathbb{N} A)$ of $\mathbb{N} A$. It follows that $c_{-b-a,-b}$ is an isomorphism for all $b \in \text{Int}(\mathbb{N} A)$.

Now consider the composition

$$c_{-b,0} \circ c_{-a-b,-b} : M_A^{-a-b} \xrightarrow{\partial^u} M_A^{-b} \to M_A^0$$

with $a, b \in \text{Int}(\mathbb{N} A)$. The first map is an isomorphism, and so the image of the composition is just the image of $c_{-b,0}$. For any two elements $b, b' \in \text{Int}(\mathbb{N} A)$, factoring $M_A^{-b} \to M_A^0$ through $M_A^{-b}$ or $M_A^{-b'}$ shows that the images of $c_{-b,0}$ and $c_{-b',0}$ agree with the image of $c_{-b,-b',0}$. In particular, they are equal. Since $\partial^u$ is in the image of $c_{-A,u,0}$, the image of $c_{-b,0}$ contains all of $\text{Int}(\mathbb{N} A)$ whenever $b \in \text{Int}(\mathbb{N} A)$. $\Box$
It follows that for normal $S_A$ the submodule of $M_A^0$ generated by any interior monomial of $S_A$ agrees with that submodule generated by $S_A^{(0)}$. If $A$ is homogeneous, so that $FL$ carries the mixed Hodge module structure from $h_*p_Q H$ to $M_A^0$, the level $d$ part of $W_\bullet$ has the property that $h_*p_Q H/FL^{-1}(W_dM_A^0)$ is supported on the boundary tori. Thus, any section of this sheaf is killed by some power of $x_1 \cdots x_n$, so that each element of $M_A^0/W_d$ is killed by some power of $\partial_1 \cdots \partial_n$. That means that $H_A(0) + W_d$ contains (the coset of) an interior monomial of $S_A$, and hence $W_d$ contains the submodule generated by $S_A^{(0)}$. Since $W_d$ is simple, it cannot strictly contain it, so must be equal to it.

As an aside, note that the Euler–Koszul homology module $H_0^A(S_A^{(0)}; 0)$ associated to the interior ideal is the underlying $D_A$-module to $FL(h_*p_Q H) = FL D(h_*p_Q H)$. Indeed, it follows from [Wal07] that the dual of $M_A^0$ is $M_A^{-\gamma}$ for some interior point of $NA$. Since $S_A^{(0)}$ is the direct limit of all principal ideals generated by interior monomials, $H_0^A(S_A^{(0)}; 0)$ is the direct limit of all $H_A^0(S_A, \partial^1; 0)$ with $\partial^1$ interior to $NA$. It follows from 5.3 that the structure morphisms in the limit are all isomorphisms. Thus, we can identify the morphisms $H_0^A(\text{Int}(NA); 0) \rightarrow M_A^0$ and $FL D(h_*p_Q H \rightarrow h_*p_Q H)$, and the corresponding statement holds for any face $\tau$ with lattice $\tau_\mathbb{Z}$.

It is clear that $S_A^{(k)} \subseteq S_A^{(k+1)}$ and that the quotient $S_A^{(k)}/S_A^{(k-1)}$ is the direct sum of the interior ideals of the face rings $S_\tau$ for which $\dim(\tau) = d-k$. It follows that $\text{gr}_W(M_A^0)$ surjects onto each $H_0^A(\text{Int}(S_\tau); 0)$, the Euler–Koszul module defined over $D_A$ by the toric module formed by the graded maximal submodule of the toric module $S_\tau = S_A/\{\partial_j \mid j \notin \tau\}$, see [MMW05] for details. It therefore also surjects onto the image of $H_0^A(\text{Int}(S_\tau); 0)$ in $H_0^A(S_\tau; 0)$, the underlying $D_A$-module corresponding to $IC_{X_\tau}$ under the monodromic Fourier transform.

If $NA$ is simplicial, Theorem 3.17 implies that $\text{gr}_W(M_A^0)$ is the sum of intersection complexes $IC_{X_\tau}$ with $\text{dim}(\tau) + k = d$, and each appears with multiplicity one. Thus, $\text{gr}_W(M_A^0)$ surjects onto $\text{gr}_W(M_A^0)$ for all $k$ when $NA$ is simplicial. (We are not asserting that this surjection is induced from a filtered morphism, only that there is one; after all, we don’t know $W$ at this point.) But $M_A^0$ is holonomic and by the Jordan–Hölder property this implies that $W' = W$ when $NA$ is simplicial. This recovers for $\beta \in NA$ a result of [Fan18].

Now suppose $d = 3$ but don’t assume simpliciality. By Theorem 3.17 any composition chain for $M_A^0$ will (up to Fourier transform) have as composition factors exactly one copy of the intersection complex to $\tau$ for $\dim(\tau) > 0$, and $1 + f_0 - d$ copies of $IC_0$. This means that an epimorphism $\text{gr}_W(M_A^0) \rightarrow \text{gr}_W(M_A^0)$ alone will not be enough to show $W'' = W$ since the copies of $IC_0$ need to be shown to live in the right levels.

In any event, $W_0 = M_A^0$ and $W_3$ is generated by the interior ideal $S_A^{(0)}$. Equivariance and the fact that $\text{gr}_W^0$ must equal $\mathbb{C}[x_A]$ shows that $W_5$ is

\[2\text{If } d \leq 2, NA \text{ is always simplicial.} \]
generated by the maximal ideal $S^{(2)}_A$. It remains to find generators for $W''_4$, such that there are surjections $\operatorname{gr}^W_k(M^0_A) \to \operatorname{gr}^W_k(M^0_A)$ for $k = 4, 5$ such that at least one is an isomorphism.

For arbitrary saturated $\mathcal{NA}$ with $d = 3$, define on $M^0_A = D_A/\langle I_A, E \rangle$ a filtration as follows:

- $W''_i = 0$ for $i < 3$;
- $W''_3$ is the left ideal generated by $\partial_j^+$;
- $W''_4$ is the left ideal generated by $W''_3$ and all $\partial_{r_2}^+$ where $\dim(\tau_2) = 2$, plus the left ideal generated by all $e_r$ defined below, where $e$ runs through the $f_0$ vertices of $P$;
- $W''_5$ is the left ideal generated by $W''_4$ and all $\partial_{r_1}^+$ where $\dim(\tau_1) = 1$;
- $W''_6$ is the left ideal generated by $1 \in D$.

We now describe the operators $e_r$. Choose distinguished nonzero columns of $A$ that correspond to the primitive lattice points on the rays through the vertices of the polygon $P$ which are in $A$ since $\mathcal{NA}$ is saturated.

For each distinguished $\mathbf{b}_r$ define a function $F_r$ on $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ as follows:

$$F_r(\mathbf{a}_j) = \begin{cases} 1 & \text{if } \mathbf{a}_j = \mathbf{b}_r; \\ 5 \mathbf{b}_r & \text{if } \mathbf{a}_j = c_{r,j} \cdot \mathbf{b}_r + c_{r'',j} \cdot \mathbf{b}_{r''}; \\ 0 & \text{else}. \end{cases}$$

We call invisible from $\mathbf{b}_r$ any $\mathbf{a} \in \mathbb{Z}A$ for which the ray from $\mathbf{b}_r$ to $\mathbf{a}$ passes through the interior of $\sigma$. Then $F_r$ vanishes on all $\mathbf{a}_j$ invisible from $\mathbf{b}_r$, and $F_r$ is piece-wise linear on the 2-faces of $\sigma$ (which are in bijection with edges of $P$). Set

$$(5.0.2) \quad e_r = \sum_j F_r(\mathbf{a}_j)x_j \partial_j.$$

We now show that our filtration $W''_r$ is indeed the Fourier–Laplace transform of the weight filtration on $h^{p,0}_W$. Note first that $W''_3$ indeed contains $W''_4$ (specifically, the $e_r$). We prove now, that each $e_r$ is annihilated by $m$ in $M^0_A/W_3$, and hence they are candidates for the intersection complexes in $W_4/W_3$ with support in $0$.

Let $\mathbf{b}_{r_1}$ and $\mathbf{b}_{r_2}$ be the two distinguished columns that lie on a facet with $\mathbf{b}_r$. Then there is a unique linear function $E_{r_1}$ on $\mathbb{R}^3$ whose values agree with those of $F_r$ on $\mathbf{b}_r, \mathbf{b}_{r_1}$ and $\mathbf{b}_{r_2}$. We denote the corresponding Euler operator also by $E_{r_1}$. The linearity of $F_r$ along facets implies that $F_r$ and $E_{r_1}$ agree on all $\mathbf{a}_j$ that have $F_r(\mathbf{a}_j)$ nonzero (which are the $\mathbf{a}_j$ not invisible from $\mathbf{b}_r$). Thus, in $M^0_A/W''_3 = D/(I_A, E, \partial^+_\sigma)$, the expression $e_r$ is equivalent to a linear combination $e_{r,0} = e_r - E_{r_1}$ of $\{x_j \partial_j\}_{j}$ for which each $\mathbf{a}_j$, with nonzero coefficient is invisible from $\mathbf{b}_r$. Now, in $D/(I_A, E, \partial^+_{\sigma})$, $\partial_j e_{r,0} = 0$ for $\mathbf{a}_j$ invisible from $\mathbf{b}_r$, and $\partial_j e_{r,0} = 0$ for $\mathbf{a}_j$ any integer multiple of $\mathbf{b}_r$ and for $\mathbf{a}_j$ interior to the facets touching $\mathbf{b}_r$. If $\mathbf{a}_j = \mathbf{b}_{r_1}$, consider the Euler operator $E_{r_1}$ that agrees with $e_{r_1}$ on $\mathbf{b}_r$ and on $\mathbf{b}_{r_1}$, and which takes value zero on the 2-face of $\sigma$ containing $\mathbf{b}_{r_1}$ but not $\mathbf{b}_r$. Then $(e_r - E_{r_1})$ has all terms invisible from $\partial_j$ and so $\partial_j(e_r - E_{r_1})$ is zero in $W''_6/W''_3$. A
similar argument works for $a_j = b_{r_2}$. Hence every $\partial_j$ annihilates the class of $e_r$ in $W''_6/W'_3$ and so $e_r$ spans a module in $W''_6/W'_3$ that is either zero or $D/Dm$. Note that there are $d = \dim(\sigma) = 3$ linear dependencies between the cosets of the $e_r$ in $M_A^0$, so the $\{e_r\}_r$ are spanning a module isomorphic to a submodule of $\oplus_{j \neq d} D/Dm$.

Next, let $a$ be in the relative interior of a facet $\tau$ of $\sigma$. Since $W_3$ contains every interior monomial of $\sigma$, the coset of $\partial^a$ in $M_A^0/W''_3$ is $\partial_j$-torsion for all $j \notin \tau$. Let $h_\tau : T_r \rightarrow \mathbb{C}^r$ be the toric map, induced by the restriction of $A$ to $\tau$, from the $\tau$-torus to the subspace $\mathbb{C}^r$ of $\mathbb{C}^A$ parameterized by the columns of $A \cap \tau$. The submodule generated by $\partial^a$ inside $M_A^0/W''_3$ is isomorphic to a quotient of the simple module $\mathbb{C}[x_\tau] \otimes \mathbb{C} \text{FLim}(h_\tau)_1 \rightarrow (h_\tau)_+$, where $x_\tau$ are the $x_j$ with $j \notin \tau$.

Now consider an interior monomial $\partial^a$ of a ray $\tau_1$ of $\sigma$. Then in $M_A^0/W''_4$, $\partial^a$ is killed by all $\partial_j$ with $j \notin \tau_1$. Modulo the $\partial_j$ not sitting on any ray, $e_r$ becomes exactly the Euler operator for $M_A^0$ if $b_r$ sits on $\tau_1$, and hence (after the Fourier transform) the module generated by $\partial^a$ in $M_A^0/W''_4$ is exactly the intersection complex associated to $\tau_1$ (pushed to $V$). Hence $W''_5/W''_4 \simeq \bigoplus_{\tau_1} IC_{\tau_1}$, and so

- $W_k = W_k'$ if $k \leq 3$ and if $k \geq 5$;
- $W_5''/W_4'' \simeq W_5/W_4$;
- hence $W_4''/W_3'' \simeq W_4/W_3$ by Jordan–Hölder.

Since the faces whose intersection complexes appear as summands in $W_5/W_3$ have dimension one, and those in $W_4/W_3$ have dimension 0 or 2, $W''_4$ must equal $W_4$.

**Example 5.4.** Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, one of the possible matrices whose GKZ-system (with the right $\beta$) contains Gauss' hypergeometric $2F_1$ as solution. We have $n = 4$ and $d = 3$, and $P$ is a square in which 1 and 4 are opposite vertices. The Euler space $E$ is spanned by $x_1 \partial_1 + x_3 \partial_3$, $x_2 \partial_2 + x_4 \partial_4$ and $x_3 \partial_3 + x_4 \partial_4$. The four elements $e_r$ are simply $\{x_j \partial_j\}_{j=1}^4$.

The toric ideal is generated by $\partial_1 \partial_4 - \partial_2 \partial_3$. The interior ideal of $S_A$ is generated by $\partial_2 \partial_3$. The weight filtration on $M_A^0$ is given by $W_2 = 0$,

$W_3 = \{E, \partial_1 \partial_4, \partial_2 \partial_3\}$, $W_4 = W_3 + \{\partial_1 \partial_2, \partial_2 \partial_4, \partial_4 \partial_3, \partial_3 \partial_1\} + \{e_1, e_2, e_3, e_4\}$,

$W_5 = W_4 + \{\partial_1, \partial_2, \partial_3, \partial_4\}$, $W_6 = M_A^0$. Here, the bar indicates taking cosets on $M_A^0$. Note that the three Euler dependencies in $H_A(0)$ imply that the four operators $e_r$ generate only one copy of $IC_0$ inside $W_4/W_3$.

6. Concluding remarks and open problems

(1) We assume throughout that $S_A$ is normal, which covers the most significant geometric situations. One obvious challenge is to remove this hypothesis and generalize our results. This would be likely difficult since then arithmetic issues will enter the fray.

(2) In another direction it would be interesting to see what can be done (as mixed Hodge module or otherwise) when $\beta \neq 0$. Recently J. Fang has
posted an article [Fan18] on the arXiv where composition chains for hyper-

gleometric systems are considered that are based on the filtration-by-faces

on the semigroup ring, see also [AS] for motivating discussion. These are

based on the filtration-by-faces in Notation 5.1 first considered by Batyrev,

see [Bat93, Sti98]. The hypotheses are somewhat technical, but in the sim-

cplicial normal case [Fan18] shows essentially that for $\beta = 0$ the filtration-by-

faces gives semisimple composition factors. Comparing with the weight fil-

tration, this corresponds to all non-diagonal terms $\mu^\sigma_\tau(e)$ with $\dim(\tau) + e \neq d$

being zero in Theorem 3.17, the case of trivial combinatorics in the polytope
to $\sigma$.

(3) By adding all nonzero $\mu^\sigma_\tau(e)$ one obtains the holonomic length of $M^0_A$.

Is there a compact formula? In particular, does it give a better estimate

than the general exponential bounds in [SST00]? When $P$ is simplicial,

$\ell(M^0_A) = 2^d$, while for $d = 3, 4, 5$ these lengths are for general $P$ as follows,

where in generalization of the face numbers $f_i$ of $P$ we denote $f_{i,j}$ the number

of all pairs ($i$-face, $j$-face) that are contained in one another. For relations

between the various $f_{i,j}$ for 4-polytopes, see [Bay87].

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\ell(M^0_A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$1 + f_0 + f_1 + f_2 + (f_0 - 3) = 3f_0 - 1$</td>
</tr>
<tr>
<td>4</td>
<td>$1 + f_0 + f_1 + f_2 + f_3 + (f_0 - 4) + (f_{1,0} - 3f_0)$</td>
</tr>
<tr>
<td></td>
<td>$= -2f_0 + 4f_1$</td>
</tr>
<tr>
<td>5</td>
<td>$1 + f_0 + f_1 + f_2 + f_3 + f_4 + (f_0 - 5) + (f_{1,0} - 4f_0) + (f_{2,1} - 3f_1)$</td>
</tr>
<tr>
<td></td>
<td>$= (f_{2,0} - 3f_2 + f_1 - 4f_0 + 10)$</td>
</tr>
<tr>
<td></td>
<td>$= 7 - 5f_0 - f_2 + 2f_{2,0}$</td>
</tr>
</tbody>
</table>

Of course, all these numbers are non-negative. Is there an obviously non-
negative representation?

(4) Given $A$ and a face $\tau$, what is the holonomic rank of the Fourier

transform of the intersection complex on the orbit to $\tau$? Such formulae

would be very interesting even for normal simplicial $A$ since it interweaves

volume-based expressions for rank with combinatorial expressions in the

way Pick’s theorem talks about polygons. For example, when $d = 2$ and $A$
is normal, one can derive from our results that the rank of $\text{FL}^{-1}(\text{IC}(L_\tau))$
always differs from the volume of $A$ by one. Induction on $d$ gives recursions,

but an explicit formula is unknown.

(5) In Section 5 we explained how to write down explicitly the weight

filtration for $d = 3$. For $d = 4$, similar ideas can be used to write out

explicit generators. But starting with $d = 5$ this seems a very hard problem.

Part of the issue is that writing down such filtration would produce a non-
canceling expression for the higher intersection cohomology dimension of

polytopes of dimension 4 or greater, which we do not think are known.

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3We note in passing that the filtration-by-faces is not a natural filtration: typically,

if GKZ-systems $M^\beta_A \equiv M^\gamma_A$ are isomorphic under a contiguity morphism, the two face

filtrations do not correspond.
List of symbols

- $\mathbb{C}^n = V = \text{Spec } \mathbb{C}[x_1, \ldots, x_n]$, the domain of the GKZ system $M_A^\beta$.
- $\mathbb{C}^n = \hat{V} = \text{Spec } [y_1, \ldots, y_n]$, the target of $h$.
- $T$ the $d$-torus,
- $h : T \to \hat{V}$ the monomial map induced by $A$,
- $X$ the closure of $T$ in $\hat{V}$,
- $\varphi : T \to X$ the restriction of $h$,
- $\deg(x) = a = - \deg(\varphi)$ the $A$-degree function on $S_A = \mathbb{C}[NA]$,
- $\phi_x$ the vanishing cycle along the function $z$,
- $\psi_x$ the corresponding nearby cycle,
- $i : X \to V$ the closed embedding,
- $i_r : \tau_r \to X_r$ the embedding of the $T$-fixed point,
- $\rho : \tau_{\sigma/\tau} \times T_r \to X_\sigma$ and $j_r : X_{\sigma/\tau} \times T_r \hookrightarrow X_\sigma$ from $NA \to (\sigma/\tau)_\mathbb{N} \oplus \tau_z$,
- the relative version $j^2 : X_{\gamma/\tau} \times T_r \to X_\gamma$ to $j_r$,
- $i_{\tau,\gamma} : T_r \to X_{\tau} \to X_\gamma$ from $(\gamma, \tau) \to \tau \to \tau_z$,
- $\kappa_v : C_m = \text{Spec } \mathbb{C}[z^2] \to T = \text{Spec } \mathbb{C}[ZA]$ the monomial action
induced by $v$,
- $u_r : X_r \setminus \tau_r \to X_r$,
- $i_r : \tau_r \to V$.

References


