INJECTIVE MODULES: PREPARATORY MATERIAL FOR THE SNOWBIRD SUMMER SCHOOL ON COMMUTATIVE ALGEBRA

These notes are intended to give the reader an idea what injective modules are, where they show up, and, to a small extent, what one can do with them. Let R be a commutative Noetherian ring with an identity element. An R-module E is injective if $\operatorname{Hom}_R(-, E)$ is an exact functor. The main messages of these notes are

- Every R-module M has an *injective hull* or *injective envelope*, denoted by $E_R(M)$, which is an injective module containing M, and has the property that any injective module containing M contains an isomorphic copy of $E_R(M)$.
- A nonzero injective module is *indecomposable* if it is not the direct sum of nonzero injective modules. Every injective R-module is a direct sum of indecomposable injective R-modules.
- Indecomposable injective R-modules are in bijective correspondence with the prime ideals of R; in fact every indecomposable injective R-module is isomorphic to an injective hull $E_R(R/\mathfrak{p})$, for some prime ideal $\mathfrak{p}$ of R.
- The number of isomorphic copies of $E_R(R/\mathfrak{p})$ occurring in any direct sum decomposition of a given injective module into indecomposable injectives is independent of the decomposition.
- Let $(R, \mathfrak{m})$ be a complete local ring and $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field of R. The functor $(-)^\vee = \operatorname{Hom}_R(-, E)$ has the following properties, known as *Matlis duality*:
  (1) If M is an R-module which is Noetherian or Artinian, then $M^{\vee\vee} \cong M$.
  (2) If M is Noetherian, then $M^\vee$ is Artinian.
  (3) If M is Artinian, then $M^\vee$ is Noetherian.

Any unexplained terminology or notation can be found in [1] or [3]. Matlis' theory of injective modules was developed in the paper [4], and may also be found in [3, § 18] and [2, § 3].

These notes owe a great deal of intellectual debt to Mel Hochster, whose lecture notes are very popular with the organizers of this workshop. However, the organizers claim intellectual property of all errors here.

1. Injective Modules

Throughout, R is a commutative ring with an identity element $1 \in R$. All R-modules M are assumed to be unitary, i.e., $1 \cdot m = m$ for all $m \in M$.

Definition 1.1. An R-module E is *injective* if for all R-module homomorphisms $\varphi : M \longrightarrow N$ and $\psi : M \longrightarrow E$ where φ is injective, there exists an R-linear homomorphism $\theta : N \longrightarrow E$ such that $\theta \circ \varphi = \psi$.

Exercise 1.2. Show that E is an injective R-module E if and only if $\operatorname{Hom}_R(-, E)$ is an exact functor, i.e., applying $\operatorname{Hom}_R(-, E)$ takes short exact sequences to short exact sequences.

Theorem 1.3 (Baer's Criterion)**.** *An R-module E is injective if and only if every R-module homomorphism $\mathfrak{a} \longrightarrow E$, where $\mathfrak{a}$ is an ideal, extends to a homomorphism $R \longrightarrow E$.*

Proof. One direction is obvious. For the other, if $M \subseteq N$ are R-modules and $\varphi : M \longrightarrow E$, we need to show that φ extends to a homomorphism $N \longrightarrow E$. By Zorn's lemma, there is a module N' with $M \subseteq N' \subseteq N$, which is maximal with respect to the property that φ extends to a homomorphism $\varphi' : N' \longrightarrow E$. If $N' \neq N$, take an element $n \in N \setminus N'$ and consider the ideal $\mathfrak{a} = N' :_R n$. By hypothesis, the composite homomorphism $\mathfrak{a} \xrightarrow{n} N' \xrightarrow{\varphi'} E$ extends to a homomorphism $\psi : R \longrightarrow E$. Define $\varphi'' : N' + Rn \longrightarrow E$ by $\varphi''(n' + rn) = \varphi'(n') + \psi(r)$. This contradicts the maximality of φ', so we must have $N' = N$. □

Exercise 1.4. Let R be an integral domain. An R-module M is *divisible* if $rM = M$ for every nonzero element $r \in R$.

(1) Prove that an injective R-module is divisible.
(2) If R is a principal ideal domain, prove that an R-module is divisible if and only if it is injective.
(3) Conclude that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.
(4) Prove that any nonzero Abelian group has a nonzero homomorphism to $\mathbb{Q}/\mathbb{Z}$.
(5) If $(-)^\vee = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ and M is any $\mathbb{Z}$-module, prove that the natural map $M \longrightarrow M^{\vee\vee}$ is injective.

Exercise 1.5. Let R be an A-algebra.

(1) Use the adjointness of $\otimes$ and Hom to prove that if E is an injective A-module, and F is a flat R-module, then $\operatorname{Hom}_A(F, E)$ is an injective R-module.
(2) Prove that every R-module can be embedded in an injective R-module. Hint: If M is the R-module, take a free R-module F with a surjection $F \twoheadrightarrow \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Apply $(-)^\vee = \operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$.

Proposition 1.6. *Let $M \neq 0$ and N be R-modules, and let $\theta : M \hookrightarrow N$ be a monomorphism. Then the following are equivalent:*

(1) Every nonzero submodule of N has a nonzero intersection with $\theta(M)$.
(2) Every nonzero element of N has a nonzero multiple in $\theta(M)$.
(3) If $\varphi \circ \theta$ is injective for a homomorphism $\varphi : N \longrightarrow Q$, then φ is injective.

Proof. (1) $\implies$ (2) If n is a nonzero element of N, then the cyclic module Rn has a nonzero intersection with $\theta(M)$.

(2) $\implies$ (3) If not, then $\ker\varphi$ has a nonzero intersection with $\theta(M)$, contradicting the assumption that $\varphi\circ\theta$ is injective.

(3) $\implies$ (1) Let N' be a nonzero submodule of N, and consider the canonical surjection $\varphi : N \longrightarrow N/N'$. Then φ is not injective, hence the composition $\varphi\circ\theta : M \longrightarrow N/N'$ is not injective, i.e., N' contains a nonzero element of $\theta(M)$. ☐

Definition 1.7. If $\theta : M \hookrightarrow N$ satisfies the equivalent conditions of the previous proposition, we say that N is an *essential extension* of M.

Example 1.8. If R is a domain and $\mathrm{Frac}(R)$ is its field of fractions, then $R \subseteq \mathrm{Frac}(R)$ is an essential extension. More generally, if $S \subseteq R$ is the set of nonzerodivisors in R, then $S^{-1}R$ is an essential extension of R.

Example 1.9. Let $(R, \mathfrak{m})$ be a local ring and N be an R-module such that every element of N is killed by a power of $\mathfrak{m}$. The *socle* of N is the submodule $\mathrm{soc}(N) = 0 :_N \mathfrak{m}$. Then $\mathrm{soc}(N) \subseteq N$ is an essential extension: if $n \in N$ is a nonzero element, let t be the smallest integer such that $\mathfrak{m}^t n = 0$. Then $\mathfrak{m}^{t-1}n \subseteq \mathrm{soc}(N)$, and $\mathfrak{m}^{t-1}n$ contains a nonzero multiple of n.

Exercise 1.10. Let I be an index set. Then $M_i \subseteq N_i$ is essential for all $i \in I$ if and only if $\oplus_{i\in I} M_i \subseteq \oplus_{i\in I} N_i$ is essential.

Example 1.11. Let $R = \mathbb{C}[[x]]$ which is a local ring with maximal ideal (x), and take $N = R_x/R$. Every element of N is killed by a power of the maximal ideal, and $\mathrm{soc}(N)$ is the 1-dimensional $\mathbb{C}$-vector space generated by $[1/x]$, i.e., the image of $1/x$ in N. By Example 1.9, $\mathrm{soc}(N) \subseteq N$ is an essential extension. However $\prod_{\mathbb{N}} \mathrm{soc}(N) \subseteq \prod_{\mathbb{N}} N$ is not an essential extension since the element

$$\big([1/x], [1/x^2], [1/x^3], \dots\big) \in \prod_{\mathbb{N}} N$$

does not have a nonzero multiple in $\prod_{\mathbb{N}} \mathrm{soc}(N)$. (Prove!)

Proposition 1.12. *Let L, M, N be nonzero R-modules.*

1. *$M \subseteq M$ is an essential extension.*
2. *Suppose $L \subseteq M \subseteq N$. Then $L \subseteq N$ is an essential extension if and only if both $L \subseteq M$ and $M \subseteq N$ are essential extensions.*
3. *Suppose $M \subseteq N$ and $M \subseteq N_i \subseteq N$ with $N = \cup_i N_i$. Then $M \subseteq N$ is an essential extension if and only if $M \subseteq N_i$ is an essential extension for every i.*
4. *Suppose $M \subseteq N$. Then there exists a module N' with $M \subseteq N' \subseteq N$, which is maximal with respect to the property that $M \subseteq N'$ is an essential extension.*

Proof. The assertions (1), (2), and (3) elementary. For (4), let

$$\mathcal{F} = \{N' \mid M \subseteq N' \subseteq N \text{ and } N' \text{ is an essential extension of } M\}.$$

Then $M \in \mathcal{F}$ so $\mathcal{F}$ is nonempty. If $N_1' \subseteq N_2' \subseteq N_3' \subseteq \dots$ is a chain in $\mathcal{F}$, then $\cup_i N_i' \in \mathcal{F}$ is an upper bound. By Zorn's Lemma, the set $\mathcal{F}$ has maximal elements. □

Definition 1.13. The module N' in Proposition 1.12 (4) is a *maximal essential extension of M in N*. If $M \subseteq N$ is essential and N has no proper essential extensions, we say that N is a *maximal essential extension of M*.

Proposition 1.14. *Let M be an R-module. The following conditions are equivalent:*

(1) M is injective;
(2) M is a direct summand of every module containing it;
(3) M has no proper essential extensions.

Proof. (1) $\implies$ (2) $\implies$ (3) is left as an exercise, and we prove the implication (3) $\implies$ (2). Consider an embedding $M \hookrightarrow E$ where E is injective. By Zorn's lemma, there exists a submodule $N \subseteq E$ which is maximal with respect to the property that $N \cap M = 0$. This implies that $M \hookrightarrow E/N$ is an essential extension, and hence that it is an isomorphism. But then $E = M + N$ so $E = M \oplus N$. Since M is a direct summand of an injective module, it must be injective. □

Proposition 1.15. *Let M and E be R-modules.*

(1) If E is injective and $M \subseteq E$, then any maximal essential extension of M in E is an injective module, hence is a direct summand of E.
(2) Any two maximal essential extensions of M are isomorphic.

Proof. (1) Let E' be a maximal essential extension of M in E and let $E' \subseteq Q$ be an essential extension. Since E is injective, the identity map $E' \longrightarrow E$ lifts to a homomorphism $\varphi : Q \longrightarrow E$. Since Q is an essential extension of E', it follows that φ must be injective. This gives us $M \subseteq E' \subseteq Q \hookrightarrow E$, and the maximality of E' implies that $Q = E'$. Hence E' has no proper essential extensions, and so it is an injective module by Proposition 1.14.

(2) Let $M \subseteq E$ and $M \subseteq E'$ be maximal essential extensions of M. Then E' is injective, so $M \subseteq E'$ extends to a homomorphism $\varphi : E \longrightarrow E'$. The inclusion $M \subseteq E$ is an essential extension, so φ is injective. But then $\varphi(E)$ is an injective module, and hence a direct summand of E'. Since $M \subseteq \varphi(E) \subseteq E'$ is an essential extension, we must have $\varphi(E) = E'$. □

Definition 1.16. The *injective hull* or *injective envelope* of an R-module M is a maximal essential extension of M, and is denoted by $E_R(M)$.

Definition 1.17. Let M be an R-module. A *minimal injective resolution* of M is a complex

$$0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \dots$$

such that $E^0 = E_R(M)$, $E^1 = E_R(E^0/M)$, and

$$E^{i+1} = E_R(E^i/\operatorname{image}(E^{i-1})) \quad \text{for all } i \geq 2.$$

Note that the modules E^i are injective, and that image$(E^i) \subseteq E^{i+1}$ is an essential extension for all $i \geq 0$.

2. Injectives over a Noetherian Ring

Proposition 2.1 (Bass). *A ring R is Noetherian if and only if every direct sum of injective R-modules is injective.*

Proof. We first show that if M is a finitely generated R-module, then

$$\operatorname{Hom}_R(M, \oplus_i N_i) \cong \oplus_i \operatorname{Hom}_R(M, N_i).$$

Independent of the finite generation of M, there is a natural injective homomorphism $\varphi : \oplus_i \operatorname{Hom}_R(M, N_i) \longrightarrow \operatorname{Hom}_R(M, \oplus_i N_i)$. If M is finitely generated, the image of a homomorphism from M to $\oplus_i N_i$ is contained in the direct sum of finitely many N_i. Since Hom commutes with forming finite direct sums, φ is surjective as well.

Let R be a Noetherian ring, and E_i be injective R-modules. Then for an ideal $\mathfrak{a}$ of R, the natural map $\operatorname{Hom}_R(R, E_i) \longrightarrow \operatorname{Hom}_R(\mathfrak{a}, E_i)$ is surjective. Since $\mathfrak{a}$ is finitely generated, the above isomorphism implies that $\operatorname{Hom}_R(R, \oplus E_i) \longrightarrow \operatorname{Hom}_R(\mathfrak{a}, \oplus E_i)$ is surjective as well. Baer's criterion now implies that $\oplus E_i$ is injective.

If R is not Noetherian, it contains a strictly ascending chain of ideals

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq \dots.$$

Let $\mathfrak{a} = \cup_i \mathfrak{a}_i$. The natural maps $\mathfrak{a} \hookrightarrow R \twoheadrightarrow R/\mathfrak{a}_i \hookrightarrow E_R(R/\mathfrak{a}_i)$ give us a homomorphism $\mathfrak{a} \longrightarrow \prod_i E_R(R/\mathfrak{a}_i)$. The image lies in the submodule $\oplus_i E_R(R/\mathfrak{a}_i)$, (check!) so we have a homomorphism $\varphi : \mathfrak{a} \longrightarrow \oplus_i E_R(R/\mathfrak{a}_i)$. Lastly, check that φ does not extend to homomorphism $R \longrightarrow \oplus_i E_R(R/\mathfrak{a}_i)$. □

Theorem 2.2. *Let E be an injective module over a Noetherian ring R. Then*

$$E \cong \oplus_i E_R(R/\mathfrak{p}_i),$$

where $\mathfrak{p}_i$ are prime ideals of R. Moreover, any such direct sum is an injective R-module.

Proof. The last statement follows from Proposition 2.1. Let E be an injective R-module. By Zorn's Lemma, there exists a maximal family $\{E_i\}$ of injective submodules of E such that $E_i \cong E_R(R/\mathfrak{p}_i)$, and their sum in E is a direct sum. Let $E' = \oplus E_i$, which is an injective module, and hence is a direct summand of E. There exists an R-module E'' such that $E = E' \oplus E''$. If $E'' \neq 0$, pick a nonzero element $x \in E''$. Let $\mathfrak{p}$ be an associated prime of Rx. Then $R/\mathfrak{p} \hookrightarrow Rx \subseteq E''$, so there is a copy of $E_R(R/\mathfrak{p})$ contained in E'' and $E'' = E_R(R/\mathfrak{p}) \oplus E'''$, contradicting the maximality of family $\{E_i\}$. □

Definition 2.3. Let $\mathfrak{a}$ be an ideal of a ring R, and M be an R-module. We say M is $\mathfrak{a}$-*torsion* if every element of M is killed by some power of $\mathfrak{a}$.

Theorem 2.4. *Let $\mathfrak{p}$ be a prime ideal of a Noetherian ring R, and let $E = E_R(R/\mathfrak{p})$ and $\kappa = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, which is the fraction field of $R/\mathfrak{p}$. Then*

(1) if $x \in R \setminus \mathfrak{p}$, then $E \xrightarrow{x} E$ is an isomorphism, and so $E = E_\mathfrak{p}$;
(2) $0 :_E \mathfrak{p} = \kappa$;
(3) $\kappa \subseteq E$ is an essential extension of $R_\mathfrak{p}$-modules and $E = E_{R_\mathfrak{p}}(\kappa)$;
(4) E is $\mathfrak{p}$-torsion and $\operatorname{Ass}(E) = \{\mathfrak{p}\}$;
(5) $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E) = \kappa$ and $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E_R(R/\mathfrak{q})_\mathfrak{p}) = 0$ for primes $\mathfrak{q} \neq \mathfrak{p}$.

Proof. (1) κ is an essential extension of $R/\mathfrak{p}$ by Example 1.8, so E contains a copy of κ and we may assume $R/\mathfrak{p} \subseteq \kappa \subseteq E$. Multiplication by $x \in R \setminus \mathfrak{p}$ is injective on κ, and hence also on its essential extension E. The submodule xE is injective, so it is a direct summand of E. But $\kappa \subseteq xE \subseteq E$ are essential extensions, so $xE = E$.

(2) $0 :_E \mathfrak{p} = 0 :_E \mathfrak{p}R_\mathfrak{p}$ is a vector space over the field κ, and hence the inclusion $\kappa \subseteq 0 :_E \mathfrak{p}$ splits. But $\kappa \subseteq 0 :_E \mathfrak{p} \subseteq E$ is an essential extension, so $0 :_E \mathfrak{p} = \kappa$.

(3) The containment $\kappa \subseteq E$ is an essential extension of R-modules, hence also of $R_\mathfrak{p}$-modules. Suppose $E \subseteq M$ is an essential extension of $R_\mathfrak{p}$-modules, pick $m \in M$. Then m has a nonzero multiple $(r/s)m \in E$, where $s \in R \setminus \mathfrak{p}$. But then rm is a nonzero multiple of m in E, so $E \subseteq M$ is an essential extension of R-modules, and therefore $M = E$.

(4) Let $\mathfrak{q} \in \operatorname{Ass}(E)$. Then there exists $x \in E$ such that $Rx \subseteq E$ and $0 :_R x = \mathfrak{q}$. Since $R/\mathfrak{p} \subseteq E$ is essential, x has a nonzero multiple y in $R/\mathfrak{p}$. But then the annihilator of y is $\mathfrak{p}$, so $\mathfrak{q} = \mathfrak{p}$ and $\operatorname{Ass}(E) = \{\mathfrak{p}\}$.

If $\mathfrak{a}$ is the annihilator of a nonzero element of E, then $\mathfrak{p}$ is the only associated prime of $R/\mathfrak{a} \hookrightarrow E$, so E is $\mathfrak{p}$-torsion.

(5) For the first assertion,

$$\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E) = \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, E) \cong 0 :_{\mathfrak{p}R_\mathfrak{p}} E = \kappa.$$

Since elements of $R \setminus \mathfrak{q}$ act invertibly on $E_R(R/\mathfrak{q})$, we see that $E_R(R/\mathfrak{q})_\mathfrak{p} = 0$ if $\mathfrak{q} \not\subseteq \mathfrak{p}$. In the case $\mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E_R(R/\mathfrak{q})_\mathfrak{p}) \cong 0 :_{\mathfrak{p}R_\mathfrak{p}} E_R(R/\mathfrak{q})_\mathfrak{p} = 0 :_{\mathfrak{p}R_\mathfrak{p}} E_R(R/\mathfrak{q}).$$

If this is nonzero, then there is a nonzero element of $E_R(R/\mathfrak{q})$ killed by $\mathfrak{p}$, which forces $\mathfrak{q} = \mathfrak{p}$ since $\operatorname{Ass} E_R(R/\mathfrak{q}) = \{\mathfrak{q}\}$. □

We are now able to strengthen Theorem 2.2 to obtain the following structure theorem for injective modules over Noetherian rings.

Theorem 2.5. *Let E be an injective over a Noetherian ring R. Then*

$$E = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\alpha_\mathfrak{p}},$$

and the numbers $\alpha_\mathfrak{p}$ are independent of the direct sum decomposition.

Proof. Theorem 2.2 implies that a direct sum decomposition exists. By Theorem 2.4 (5), $\alpha_\mathfrak{p}$ is the dimension of the $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$-vector space

$$\operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, E_\mathfrak{p}),$$

which does not depend on the decomposition. □

The following proposition can be proved along the lines of Theorem 2.4, and we leave the proof as an exercise.

Proposition 2.6. *Let $S \subset R$ be a multiplicative set.*

(1) If E is an injective R-module, then $S^{-1}E$ is an injective module over the ring $S^{-1}R$.
(2) If $M \hookrightarrow N$ is an essential extension (or a maximal essential extension) of R-modules, then the same is true for $S^{-1}M \hookrightarrow S^{-1}N$ over $S^{-1}R$.
(3) The indecomposable injectives over $S^{-1}R$ are the modules $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$.

Definition 2.7. Let M be an R-module, and let $E^\bullet$ be a minimal injective resolution of R where

$$E^i = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}.$$

Then $\mu_i(\mathfrak{p}, M)$ is the i-th *Bass number of M with respect to* $\mathfrak{p}$. The following theorem shows that these numbers are well-defined.

Theorem 2.8. *Let $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then*

$$\mu_i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}).$$

Proof. Let $E^\bullet$ be a minimal injective resolution of M where the i th module is $E^i = \oplus E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}$. Localizing at $\mathfrak{p}$, Proposition 2.6 implies that $E^\bullet_\mathfrak{p}$ is a minimal injective resolution of $M_\mathfrak{p}$ over the ring $R_\mathfrak{p}$. Moreover, the number of copies of $E_R(R/\mathfrak{p})$ occurring in E^i is the same as the number of copies of $E_R(R/\mathfrak{p})$ in $E^i_\mathfrak{p}$. By definition, $\operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$ is the i-th cohomology module of the complex

$$0 \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^0_\mathfrak{p}) \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^1_\mathfrak{p}) \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^2_\mathfrak{p}) \longrightarrow \dots$$

and we claim all maps in this complex are zero. If $\varphi \in \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^i_\mathfrak{p})$, we need to show that the composition

$$\kappa(\mathfrak{p}) \xrightarrow{\varphi} E^i_\mathfrak{p} \xrightarrow{\delta} E^{i+1}_\mathfrak{p}$$

is the zero map. If $\varphi(x) \neq 0$ for $x \in \kappa(\mathfrak{p})$, then $\varphi(x)$ has a nonzero multiple in $\operatorname{image}(E^{i-1}_\mathfrak{p} \longrightarrow E^i_\mathfrak{p})$. Since $\kappa(\mathfrak{p})$ is a field, it follows that

$$\varphi(\kappa(\mathfrak{p})) \subseteq \operatorname{image}(E^{i-1}_\mathfrak{p} \longrightarrow E^i_\mathfrak{p}),$$

and hence that $\delta \circ \varphi = 0$. By Theorem 2.4 (5)

$$\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^i_\mathfrak{p}) \cong \kappa(\mathfrak{p})^{\mu_i(\mathfrak{p},M)},$$

so $\operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$ is the i-th cohomology module of the complex

$$0 \longrightarrow \kappa(\mathfrak{p})^{\mu_0(\mathfrak{p},M)} \longrightarrow \kappa(\mathfrak{p})^{\mu_1(\mathfrak{p},M)} \longrightarrow \kappa(\mathfrak{p})^{\mu_2(\mathfrak{p},M)} \longrightarrow \dots.$$

where all maps are zero, and the required result follows. □

Remark 2.9. We next want to consider the special case in which $(R, \mathfrak{m}, K)$ is a Noetherian local ring. Recall that we have natural surjections

$$\dots \longrightarrow R/\mathfrak{m}^3 \longrightarrow R/\mathfrak{m}^2 \longrightarrow R/\mathfrak{m} \longrightarrow 0,$$

and that the $\mathfrak{m}$-*adic completion* $\widehat{R}$ of R is the inverse limit of this system, i.e.,

$$\varprojlim_k (R/\mathfrak{m}^k) = \left\{ (r_0, r_1, r_2, \dots) \in \prod_k R/\mathfrak{m}^k \mid r_k - r_{k-1} \in \mathfrak{m}^{k-1} \right\}.$$

Morally, elements of the $\mathfrak{a}$-adic completion of R are power series of elements of R where "higher terms" are those contained in higher powers of the ideal $\mathfrak{a}$. There is no reason to restrict to local rings or maximal ideals—for topological purposes, completions at other ideals can be very interesting; see, for example, [5].

Note that $\widehat{R}/\mathfrak{m}^k\widehat{R} \cong R/\mathfrak{m}^k$. Consequently if M is $\mathfrak{m}$-torsion, then the R-module structure on M makes it an $\widehat{R}$-module. In particular, $E_R(R/\mathfrak{m})$ is an $\widehat{R}$-module.

Theorem 2.10. *Let $(R, \mathfrak{m}, K)$ be a local ring. Then $E_R(K) = E_{\widehat{R}}(K)$.*

Proof. The containment $K \subseteq E_R(K)$ is an essential extension of R-modules, hence also of $\widehat{R}$-modules. If $E_R(K) \subseteq M$ is an essential extension of $\widehat{R}$-modules, then M is $\mathfrak{m}$-torsion. (Prove!) If $m \in M$ is a nonzero element, then $\widehat{R}m \cap E_R(K) \neq 0$. But $\widehat{R}m = Rm$, so $E_R(K) \subseteq M$ is an essential extension of R-modules, which implies $M = E_R(K)$. It follows that $E_R(K)$ is a maximal essential extension of K as an $\widehat{R}$-module. □

Theorem 2.11. *Let $\varphi : (R, \mathfrak{m}, K) \longrightarrow (S, \mathfrak{n}, L)$ be a homomorphism of local rings such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, the ideal $\varphi(\mathfrak{m})S$ is $\mathfrak{n}$-primary, and S is module-finite over R. Then*

$$\operatorname{Hom}_R(S, E_R(K)) = E_S(L).$$

Proof. By Exercise 1.5, $\operatorname{Hom}_R(S, E_R(K))$ is an injective S-module. Every element of $\operatorname{Hom}_R(S, E_R(K))$ is killed by a power of $\mathfrak{m}$ and hence by a power of $\mathfrak{n}$. It follows that $\operatorname{Hom}_R(S, E_R(K))$ is a direct sum of copies of $E_S(L)$, say $\operatorname{Hom}_R(S, E_R(K)) \cong E_S(L)^\mu$. To determine μ, consider

$$\operatorname{Hom}_S(L, \operatorname{Hom}_R(S, E_R(K))) \cong \operatorname{Hom}_R(L \otimes_S S, E_R(K)) \cong \operatorname{Hom}_R(L, E_R(K)).$$

The image of any element of $\operatorname{Hom}_R(L, E_R(K))$ is killed by $\mathfrak{n}$, hence

$$\operatorname{Hom}_R(L, E_R(K)) \cong \operatorname{Hom}_R(L, K) \cong \operatorname{Hom}_K(L, K)$$

and $L^\mu \cong \operatorname{Hom}_K(L, K)$. Considering vector space dimensions over K, this implies $\mu \dim_K L = \dim_K L$, so $\mu = 1$. ☐

Corollary 2.12. *Let $(R, \mathfrak{m}, K)$ be a local ring and let $S = R/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R. Then the injective hull of the residue field of S is*

$$\operatorname{Hom}_R(R/\mathfrak{a}, E_R(K)) \cong 0 :_{E_R(K)} \mathfrak{a}.$$

Since every element of $E_R(K)$ is killed by a power of $\mathfrak{m}$, we have

$$E_R(K) = \bigcup_{t \in \mathbb{N}} (0 :_{E_R(K)} \mathfrak{m}^t) = \bigcup_{t \in \mathbb{N}} E_{R/\mathfrak{m}^t}(K).$$

This motivates the study of $E_R(K)$ for Artinian local rings.

3. The Artinian case

Recall that the *length* of a module M is the length of a composition series for M, and is denoted $\ell(M)$. The length is additive over short exact sequences. If $(R, \mathfrak{m}, K)$ is an Artinian local ring, then every finitely generated R-module has a composition series with factors isomorphic to $R/\mathfrak{m}$.

Lemma 3.1. *Let $(R, \mathfrak{m}, K)$ be a local ring. Then $(-)^\vee = \operatorname{Hom}_R(-, E_R(K))$ is a faithful functor, and $\ell(M^\vee) = \ell(M)$ for every R-module M of finite length.*

Proof. Note that $(R/\mathfrak{m})^\vee = \operatorname{Hom}_R(R/\mathfrak{m}, E_R(K)) \cong K$. If M is a nonzero R-module, we need to show that $M^\vee$ is nonzero. Taking a nonzero cyclic submodule $R/\mathfrak{a} \hookrightarrow M$, we have $M^\vee \twoheadrightarrow (R/\mathfrak{a})^\vee$, so it suffices to show that $(R/\mathfrak{a})^\vee$ is nonzero. The surjection $R/\mathfrak{a} \twoheadrightarrow R/\mathfrak{m}$ yields $(R/\mathfrak{m})^\vee \hookrightarrow (R/\mathfrak{a})^\vee$, and hence $(-)^\vee$ is faithful.

For M of finite length, we use induction on $\ell(M)$ to prove $\ell(M^\vee) = \ell(M)$. The result is true for modules of length 1 since $(R/\mathfrak{m})^\vee \cong K$. For a module M of finite length, consider $m \in \operatorname{soc}(M)$ and the exact sequence

$$0 \longrightarrow Rm \longrightarrow M \longrightarrow M/Rm \longrightarrow 0.$$

Applying $(-)^\vee$, we obtain an exact sequence

$$0 \longrightarrow (M/Rm)^\vee \longrightarrow M^\vee \longrightarrow (Rm)^\vee \longrightarrow 0.$$

Since $Rm \cong K$ and $\ell(M/Rm) = \ell(M) - 1$, we are done by induction. ☐

Corollary 3.2. *Let $(R, \mathfrak{m}, K)$ be an Artinian local ring. Then $E_R(K)$ is a finite length module and $\ell(E_R(K)) = \ell(R)$.*

Theorem 3.3. *Let $(R, \mathfrak{m}, K)$ be a Artinian local ring and $E = E_R(K)$. Then the map $R \longrightarrow \operatorname{Hom}_R(E, E)$, which takes a ring element r to the homomorphism "multiplication by r," is an isomorphism.*

Proof. By the previous results, $\ell(R) = \ell(E) = \ell(E^\vee)$, so R and $\operatorname{Hom}_R(E, E)$ have the same length, and it suffices to show the map is injective. If $rE = 0$, then $E = E_{R/Rr}(K)$ so $\ell(R) = \ell(R/Rr)$, forcing $r = 0$. ☐

Theorem 3.4. *Let $(R, \mathfrak{m}, K)$ be a local ring. Then R is an injective R-module if and only if the following two conditions are satisfied:*

(1) R is Artinian, and

(2) $\operatorname{soc}(R)$ is 1-dimensional vector space over K.

Proof. If $R = M \oplus N$ then $K \cong (M \otimes_R K) \oplus (N \otimes_R K)$, so one of the two summands must be zero, say $M \otimes_R K = 0$. But then Nakayama's lemma implies that $M = 0$. It follows that a local ring in indecomposable as a module over itself. Hence if R is injective, then $R \cong E_R(R/\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} R$. This implies R that is $\mathfrak{p}$-torsion and it follows that $\mathfrak{p}$ is the only prime ideal of R and hence that R is Artinian. Furthermore, $\operatorname{soc}(R)$ is isomorphic to $\operatorname{soc}(E_R(K))$, which is 1-dimensional.

Conversely, if R is Artinian with $\operatorname{soc}(R) = K$, then R is an essential extension of its socle. The essential extension $K \subseteq R$ can be enlarged to a maximal essential extension $K \subseteq E_R(K)$. Since $\ell(E_R(K)) = \ell(R)$, we must have $E_R(K) = R$. □

4. Matlis duality

Theorem 4.1. *Let $(R, \mathfrak{m}, K)$ be a local ring and let $E = E_R(K)$. Then E is also an $\widehat{R}$-module, and the map $\widehat{R} \longrightarrow \operatorname{Hom}_R(E, E)$, which takes an element $r \in \widehat{R}$ to the homomorphism "multiplication by r," is an isomorphism.*

Proof. Since $E = E_{\widehat{R}}(K)$, there is no loss of generality in assuming that R is complete. For integers $t \geq 1$, consider the rings $R_t = R/\mathfrak{m}^t$. Then $E_t = 0 :_E \mathfrak{m}^t$ is the injective hull of the residue field of R_t. If $\varphi \in \operatorname{Hom}_R(E, E)$, then $\varphi(E_t) \subseteq E_t$, so φ restricts to an element of $\operatorname{Hom}_{R_t}(E_t, E_t)$, which equals R_t by Theorem 3.3. The homomorphism φ, when restricted to E_t, is multiplication by an element $r_t \in R_t$. Moreover $E = \cup_t E_t$ and the elements r_t are compatible under restriction, i.e., $r_{t+1} - r_t \in \mathfrak{m}^t$. Thus φ is precisely multiplication by the element $(r_1 - r_2) + (r_2 - r_3) + \cdots \in R$. □

Corollary 4.2. *For a local ring $(R, \mathfrak{m}, K)$, the module $E_R(K)$ satisfies the descending chain condition (DCC).*

Proof. Consider a descending chain of submodules

$$E_R(K) = E \supseteq E_1 \supseteq E_2 \supseteq \dots .$$

Applying the functor $(-)^\vee = \operatorname{Hom}_R(-, E)$ gives us surjections

$$\widehat{R} \cong E^\vee \twoheadrightarrow E_1^\vee \twoheadrightarrow E_2^\vee \twoheadrightarrow \dots .$$

Since $\widehat{R}$ is Noetherian, the ideals $\ker(\widehat{R} \twoheadrightarrow E_t^\vee)$ stabilize for large t, and hence $E_t^\vee \twoheadrightarrow E_{t+1}^\vee$ is an isomorphism for $t \gg 0$. Since $(-)^\vee$ is faithful, it follows that $E_t = E_{t+1}$ for $t \gg 0$. □

Theorem 4.3. *Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring. The following conditions are equivalent for an R-module M.*

(1) M is $\mathfrak{m}$-torsion and $\operatorname{soc}(M)$ is a finite-dimensional K-vector space;

(2) M is an essential extension of a finite-dimensional K-vector space;
(3) M can be embedded in a direct sum of finitely many copies of $E_R(K)$;
(4) M satisfies the descending chain condition.

Proof. The implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) follow from earlier results, so we focus on (4) $\implies$ (1). Let $x \in M$. The descending chain

$$Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2 x \supseteq \dots$$

stabilizes, so $\mathfrak{m}^{t+1}x = \mathfrak{m}^t x$ for some t. But then Nakayama's lemma implies $\mathfrak{m}^t x = 0$, and it follows that M is $\mathfrak{m}$-torsion. Since $\operatorname{soc}(M)$ is a vector space with DCC, it must be finite-dimensional. □

Example 4.4. Let $(R, \mathfrak{m}, K)$ be a discrete valuation ring with maximal ideal $\mathfrak{m} = Rx$. (For example, R may be a power series ring $K[[x]]$ or the ring of p-adic integers $\widehat{\mathbb{Z}}_p$, in which case $x = p$.) We claim that $E_R(K) \cong R_x/R$. To see this, note that $\operatorname{soc}(R_x/R)$ is a 1-dimensional K-vector space generated by the image of $1/x \in R_x$, and that every element of R_x/R is killed by a power of x.

The next result explains the notion of duality in the current context.

Theorem 4.5. *Let $(R, \mathfrak{m}, K)$ be a complete Noetherian local ring, and use $(-)^\vee$ to denote the functor $\operatorname{Hom}_R(-, E_R(K))$.*

(1) If M has ACC then $M^\vee$ has DCC, and if M has DCC then $M^\vee$ has ACC. Hence the category of R-modules with DCC is anti-equivalent to the category of R-modules with ACC.
(2) If M has ACC or DCC, then $M^{\vee\vee} \cong M$.

Proof. Let $E = E_R(K)$. If M with ACC, consider a presentation

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

Applying $(-)^\vee$, we get an exact sequence $0 \longrightarrow M \longrightarrow (R^n)^\vee \longrightarrow (R^m)^\vee$. Since $(R^n)^\vee \cong E^n$ has DCC, so does its submodule M. Applying $(-)^\vee$ again, we get the commutative diagram with exact rows

$$\begin{array}{ccccccc}
(R^m)^{\vee\vee} & \longrightarrow & (R^n)^{\vee\vee} & \longrightarrow & M^{\vee\vee} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R^m & \longrightarrow & R^n & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Since $R \longrightarrow R^{\vee\vee}$ is an isomorphism, it follows that $M \longrightarrow M^{\vee\vee}$ is an isomorphism as well.

If M has DCC, we embed it in E^m and obtain an exact sequence

$$0 \longrightarrow M \longrightarrow E^m \longrightarrow E^n.$$

Applying $(-)^\vee$ gives an exact sequence $(E^n)^\vee \longrightarrow (E^m)^\vee \longrightarrow M^\vee \longrightarrow 0$. The surjection $R^n \cong (E^m)^\vee \twoheadrightarrow M^\vee$ shows that M has ACC, while a similar commutative diagram gives the isomorphism $M^{\vee\vee} \cong M$. □

Remark 4.6. Let M be a finitely generated module over a complete local ring $(R, \mathfrak{m}, K)$. Then

$$\begin{aligned}\operatorname{Hom}_R(K, M^\vee) \cong \operatorname{Hom}_R(K \otimes_R M, E_R(K)) \cong \operatorname{Hom}_R(M/\mathfrak{m}M, E_R(K))\\ \cong \operatorname{Hom}_K(M/\mathfrak{m}M, K),\end{aligned}$$

so the number of generators of M as an R-module equals the vector space dimension of $\operatorname{soc}(M^\vee)$.

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.
[3] H. Matsumura, *Commutative algebra*, W. A. Benjamin, Inc., New York, 1970.
[4] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958) 511–528.
[5] R. Hartshorne, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 5–99.