Contents

Definition 1.1. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field $\mathbb{K}$, and consider polynomials $f_1, \dots, f_m \in R$. Their zero set

$$V = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n \mid f_1(\alpha_1, \dots, \alpha_n) = 0, \dots, f_m(\alpha_1, \dots, \alpha_n) = 0\}$$

is an *algebraic set* in $\mathbb{K}^n$. These are our basic objects of study, and include many familiar examples such as those listed below.

Example 1.2. If $f_1, \dots, f_m \in \mathbb{K}[x_1, \dots, x_n]$ are homogeneous linear polynomials, their zero set is a vector subspace of $\mathbb{K}^n$. If V, W are vector subspaces of $\mathbb{K}^n$, then we have the following inequality of vector space dimensions:

$$\operatorname{rank}_{\mathbb{K}}(V \cap W) \geqslant \operatorname{rank}_{\mathbb{K}} V + \operatorname{rank}_{\mathbb{K}} W - n.$$

An easy way to see this inequality is via the exact sequence

$$0 \longrightarrow V \cap W \xrightarrow{\ \alpha\ } V \oplus W \xrightarrow{\ \beta\ } V + W \longrightarrow 0,$$

where $\alpha(u) = (u, u)$ and $\beta(v, w) = v - w$. Then

$$\begin{aligned} \operatorname{rank}_{\mathbb{K}}(V \cap W) &= \operatorname{rank}_{\mathbb{K}} V \oplus W - \operatorname{rank}_{\mathbb{K}}(V + W) \\ &\geqslant \operatorname{rank}_{\mathbb{K}} V + \operatorname{rank}_{\mathbb{K}} W - n. \end{aligned}$$

Example 1.3. A *hypersurface* is a zero set of one equation. The circle of unit radius in $\mathbb{R}^2$ is a hypersurface—it is the zero set of the polynomial $x^2 + y^2 - 1$.

Example 1.4. If $f \in \mathbb{K}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree d, then $f(\alpha_1, \dots, \alpha_n) = 0$ implies that

$$f(c\alpha_1, \dots, c\alpha_n) = c^d f(\alpha_1, \dots, \alpha_n) = 0$$

for all $c \in \mathbb{K}$. Hence if an algebraic set $V \subset \mathbb{K}^n$ is the zero set of homogeneous polynomials, then, for all $(\alpha_1, \dots, \alpha_n) \in V$ and $c \in \mathbb{K}$, we have $(c\alpha_1, \dots, c\alpha_n) \in V$. In this case, the algebraic set V is said to be a *cone*.

Example 1.5. For integers $m, n \geqslant 2$, consider the set $V \subset \mathbb{K}^{mn}$ of all $m \times n$ matrices over $\mathbb{K}$ which have rank less than a fixed integer t. A matrix has rank less than t if and only if its size t minors (i.e., the determinants of $t \times t$ submatrices) all equal zero. Take

$$R = \mathbb{K}[x_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n],$$

which is a polynomial ring in mn variables arranged as an $m \times n$ matrix. Then V is the solution set of the $\binom{m}{t}\binom{n}{t}$ polynomials which arise as the size t minors of the matrix (x_{ij}). Hence V is an algebraic set (in fact, a cone) in $\mathbb{K}^{mn}$.

Exercise 1.6 ([94])**.** Let $\mathbb{K}$ be a finite field.

(1) For every point $p \in \mathbb{K}^n$, construct a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ such that $f(p) = 1$ and $f(q) = 0$ for all points $q \in \mathbb{K}^n \setminus \{p\}$.
(2) Given a function $g : \mathbb{K}^n \longrightarrow \mathbb{K}$, show that there is a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ with $f(p) = g(p)$ for all $p \in \mathbb{K}^n$.
(3) Prove that any subset of $\mathbb{K}^n$ is the zero set of a single polynomial.

Remark 1.7. One may ask: is the zero set of an infinite family of polynomials also the zero set of a finite family? To answer this, recall that a ring is Noetherian if all its ideals are finitely generated, and that the polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ is Noetherian by the Hilbert basis theorem, [4, Theorem 7.5]. Let $\mathfrak{a} \subset R$ be the ideal generated by a possibly infinite family of polynomials $\{g_\lambda\}$. The zero set of $\{g_\lambda\}$ is the same as zero set of *all* polynomials in the ideal $\mathfrak{a}$. But $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (f_1, \dots, f_m)$, so the zero set of $\{g_\lambda\}$ is precisely the zero set of the finitely many polynomials $f_1, \dots, f_m$.

Given a set of polynomials $f_1, \dots, f_m$ generating an ideal $\mathfrak{a} \subset R$, we denote their zero set in $\mathbb{K}^n$ by $\operatorname{Var}(f_1, \dots, f_m)$ or by $\operatorname{Var}(\mathfrak{a})$. Note that $\operatorname{Var}(f) = \operatorname{Var}(f^k)$ for any integer $k \geqslant 1$, hence if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals with the same radical, then $\operatorname{Var}(\mathfrak{a}) = \operatorname{Var}(\mathfrak{b})$. A theorem of Hilbert states that over an algebraically closed field, the converse is true as well:

Theorem 1.8 (Hilbert's Nullstellensatz)**.** *Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbb{K}$. If $\operatorname{Var}(\mathfrak{a}) = \operatorname{Var}(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b} \subset R$, then $\operatorname{rad} \mathfrak{a} = \operatorname{rad} \mathfrak{b}$. Consequently the map $\mathfrak{a} \mapsto \operatorname{Var}(\mathfrak{a})$ is a containment reversing bijection between the set of radical ideals of $\mathbb{K}[x_1, \dots, x_n]$ and algebraic sets in $\mathbb{K}^n$.*

For a proof, solve [4, Problem 7.14]. In particular, the theorem above tells us when polynomial equations have a common solution, and the following corollary is also (and perhaps more appropriately) referred to as the Nullstellensatz:

Corollary 1.9. *Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbb{K}$. Then polynomials $f_1, \dots, f_m \in R$ have a common zero if and only if $(f_1, \dots, f_m) \neq R$.*

Proof. $\operatorname{Var}(R) = \varnothing$, so $\operatorname{Var}(\mathfrak{a}) = \varnothing$ for an ideal $\mathfrak{a}$ if and only if $\operatorname{rad} \mathfrak{a} = R$, which occurs if and only $\mathfrak{a} = R$. □

Corollary 1.10. *Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbb{K}$. Then the maximal ideals of R are precisely the ideals*

$$(x_1 - \alpha_1, \dots, x_n - \alpha_n) \qquad \text{where } \alpha_i \in \mathbb{K}.$$

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. Then $\mathfrak{m} \neq R$ so there exists a point $(\alpha_1, \dots, \alpha_n) \in \operatorname{Var}(\mathfrak{m})$. But then

$$\operatorname{Var}(x_1 - \alpha_1, \dots, x_n - \alpha_n) \subseteq \operatorname{Var}(\mathfrak{m}),$$

so $\mathfrak{m} \subseteq (x_1 - \alpha_1, \dots, x_n - \alpha_n)$. Since each is a maximal ideal of R, they must be equal. □

Exercise 1.11 ([94])**.** Let $\mathbb{K}$ be a field which is *not* algebraically closed. Prove that any algebraic set in $\mathbb{K}^n$ is the zero set of a single polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$.

Krull dimension of a ring

We would like a notion of dimension for algebraic sets which agrees with vector space dimension if the algebraic set is a vector space, and gives a suitable generalization of the inequality in Example 1.2. The situation is certainly more complicated than with vector spaces; for example, not all points of an algebraic set are "similar"—the algebraic set defined by $xy = 0$ and $xz = 0$ is the union of a line and a plane. To obtain a good theory of dimension, we recall some notions from commutative algebra.

Definition 1.12. Let R be a ring. The *spectrum* of R, denoted $\operatorname{Spec} R$, is the set of prime ideals of R with the *Zariski topology*, which is the topology where the closed sets are

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\} \qquad \text{for ideals } \mathfrak{a} \subseteq R.$$

It is easily verified that this is indeed a topology: the empty set is both open and closed, an intersection of closed sets is closed since $\bigcap_\lambda V(\mathfrak{a}_\lambda) = V(\bigcup_\lambda \mathfrak{a}_\lambda)$, and the union of two closed sets is closed since

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}).$$

The *height* of a prime ideal $\mathfrak{p}$, denoted $\operatorname{height} \mathfrak{p}$, is the supremum of integers t such that there exists a chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \cdots \supsetneq \mathfrak{p}_t, \qquad \text{where } \mathfrak{p}_i \in \operatorname{Spec} R.$$

The height of an arbitrary ideal $\mathfrak{a} \subset R$ is

$$\operatorname{height} \mathfrak{a} = \inf\{\operatorname{height} \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{a} \subseteq \mathfrak{p}\}.$$

The *Krull dimension* of R is

$$\dim R = \sup\{\operatorname{height} \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Note that for every prime ideal $\mathfrak{p}$ of R, we have $\dim R_\mathfrak{p} = \operatorname{height} \mathfrak{p}$.

Example 1.13. The prime ideals of $\mathbb{Z}$ are (0) and (p) for prime integers p. Consequently the longest chains of prime ideals in $\operatorname{Spec} \mathbb{Z}$ are those of the form $(0) \subsetneq (p)$, and so $\dim \mathbb{Z} = 1$. More generally, if R is a principal ideal domain which is not a field, then $\dim R = 1$.

Theorem 1.14 (Krull)**.** *Let R be a Noetherian ring. If an ideal $\mathfrak{a} \subsetneq R$ is generated by n elements, then each minimal prime $\mathfrak{p}$ of $\mathfrak{a}$ has* $\operatorname{height} \mathfrak{p} \leqslant n$*. In particular, every ideal $\mathfrak{a} \subsetneq R$ has finite height.*

The above theorem implies that every proper principal ideal of a Noetherian ring has height at most one, which is *Krull's principal ideal theorem.* For a proof of Theorem 1.14, see [4, Corollary 11.16]. While it is true that every prime ideal in a Noetherian ring has finite height, the Krull dimension is the *supremum* of the heights of prime ideals, and this supremum may be infinite, see [4, Problem 11.4] for an example due to Nagata. The following theorem implies, in particular, that local rings have finite Krull dimension; see [4, Theorem 11.14] for a proof.

Theorem 1.15 (Main theorem of dimension theory)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and d a nonnegative integer. The following conditions are equivalent:*

(1) $\dim R = d$;
(2) $\operatorname{height} \mathfrak{m} = d$;
(3) *d is the least number of generators of an $\mathfrak{m}$-primary ideal;*
(4) *d is the least integer such that there exist elements $x_1, \ldots, x_d \in \mathfrak{m}$ for which $R/(x_1, \ldots, x_d)$ is an Artinian ring;*
(5) *For $n \gg 0$, the function $\ell(R/\mathfrak{m}^n)$ is a polynomial in n of degree d.*

Definition 1.16. Let $(R, \mathfrak{m})$ be a local ring of dimension d. Elements $x_1, \ldots, x_d$ are a *system of parameters* for R if $\operatorname{rad}(x_1, \ldots, x_d) = \mathfrak{m}$.

Theorem 1.15 guarantees that every local ring has a system of parameters.

Example 1.17. Let $\mathbb{K}$ be a field, and take

$$R = \mathbb{K}[x, y, z]_{(x,y,z)}/(xy, xz).$$

Then R has a chain of prime ideals $(x) \subsetneq (x, y) \subsetneq (x, y, z)$, so $\dim R \geqslant 2$. On the other hand, the maximal ideal (x, y, z) is the radical of the 2-generated ideal $(y, x - z)$, implying that $\dim R \leqslant 2$. It follows that $\dim R = 2$ and that $y, x - z$ is a system of parameters for R.

Exercise 1.18. Let $\mathbb{K}$ be an arbitrary field. For the following local rings $(R, \mathfrak{m})$, compute $\dim R$ by examining $\ell(R/\mathfrak{m}^n)$ for $n \gg 0$. In each case, find a system of parameters for R and a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}, \qquad \text{where } d = \dim R.$$

(1) $R = \mathbb{K}[x^2, x^3]_{(x^2,x^3)}$.
(2) $R = \mathbb{K}[x^2, xy, y^2]_{(x^2,xy,y^2)}$.
(3) $R = \mathbb{K}[w, x, y, z]_{(w,x,y,z)}/(wx - yz)$.
(4) $R = \mathbb{Z}_{(p)}$ where p is a prime integer.

For a finitely generated domain over a field, the dimension may also be computed as the transcendence degree of a field extension:

Theorem 1.19. *If R is a finitely generated domain over a field $\mathbb{K}$, then*

$$\dim R = \operatorname{tr.deg}_{\mathbb{K}} \operatorname{Frac}(R),$$

where $\operatorname{Frac}(R)$ *is the fraction field of R. Moreover, any chain of primes in* $\operatorname{Spec} R$ *can be extended to a chain of length* $\dim R$. *Hence* $\dim R_{\mathfrak{m}} = \dim R$ *for every maximal ideal* $\mathfrak{m}$ *of R, and*

$$\operatorname{height} \mathfrak{p} + \dim R/\mathfrak{p} = \dim R \qquad \textit{for all } \mathfrak{p} \in \operatorname{Spec} R.$$

When $\mathbb{K}$ is algebraically closed, this is [4, Corollary 11.27]; for the general case see [114, Theorem 5.6, Exercise 5.1].

Example 1.20. If $\mathbb{K}$ is a field, then the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_d]$ has dimension d since $\operatorname{tr.deg}_{\mathbb{K}} \mathbb{K}(x_1, \ldots, x_d) = d$.

If $f \in R$ is a nonzero polynomial, then the minimal primes of the ideal (f) are the principal ideals generated by irreducible factors of f and these have height 1 by Theorem 1.14. It follows that $\dim R/(f) = n - 1$.

Remark 1.21. We say that a ring R is $\mathbb{N}$-*graded* if $R = \oplus_{n \geqslant 0} R_n$ as an Abelian group, and $R_m R_n \subseteq R_{m+n}$ for integers $m, n \geqslant 0$. Assume that R is finitely generated over a field $R_0 = \mathbb{K}$. Then $\mathfrak{m} = \oplus_{n>0} R_n$ is the (unique) *homogeneous maximal ideal* of R.

Let M be a finitely generated $\mathbb{Z}$-graded R-module, i.e., $M = \oplus_{n \in \mathbb{Z}} M_n$ as an Abelian group, and $R_m M_n \subseteq M_{m+n}$ for all $m \geqslant 0$ and $n \in \mathbb{Z}$. The *Hilbert-Poincaré series* of M is the generating function for $\dim_{\mathbb{K}} M_n$, i.e., the series

$$P(M, t) = \sum_{n \in \mathbb{Z}} (\dim_{\mathbb{K}} M_n) t^n \in \mathbb{Z}[[t]][t^{-1}].$$

It turns out that $P(M, t)$ is a rational function of t of the form

$$\frac{f(t)}{\prod_i (1 - t^{k_i})} \qquad \text{where } f(t) \in \mathbb{Z}[t],$$

[4, Theorem 11.1], and that the dimension of R is precisely the order of the pole of $P(R,t)$ at $t=1$.

Example 1.22. Let R be the polynomial ring $\mathbb{K}[x_1,\dots,x_d]$ where $\mathbb{K}$ is a field. The vector space dimension of R_n is the number of monomials of degree n, which is the binomial coefficient $\binom{n+d-1}{n}$. Hence

$$P(R,t) = \sum_{n\geqslant 0} \binom{n+d-1}{n} t^n = \frac{1}{(1-t)^d}.$$

Exercise 1.23. Compute $P(R,t)$ in the following cases:

(1) $R=\mathbb{K}[wx,wy,zx,zy]$ where each of wx,wy,zx,zy have degree 1.
(2) $R=\mathbb{K}[x^2,x^3]$ where the grading is induced by $\deg x = 1$.
(3) $R=\mathbb{K}[x^4,x^3y,xy^3,y^4]$ where $\deg x^4 = \deg y^4 = 1$.

Dimension of an algebraic set

For the sake of simplicity we work over an algebraically closed field $\mathbb{K}$.

Definition 1.24. An algebraic set V is *irreducible* if it is not the union of two algebraic sets which are proper subsets of V.

Exercise 1.25. Prove that an algebraic set $V\subset\mathbb{K}^n$ is irreducible if and only if $V=\mathrm{Var}(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of $\mathbb{K}[x_1,\dots,x_n]$.

Remark 1.26. Every algebraic set can be uniquely written as a finite union of irreducible algebraic sets where there are no redundant terms in the union. Let $V=\mathrm{Var}(\mathfrak{a})$ where $\mathfrak{a}$ is a radical ideal. Then

$$\mathfrak{a} = \mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_n \quad \text{for } \mathfrak{p}_i\in\operatorname{Spec} R,$$

and assume this intersection is irredundant. Then

$$V = \mathrm{Var}(\mathfrak{p}_1)\cup\cdots\cup\mathrm{Var}(\mathfrak{p}_n),$$

and $\mathrm{Var}(\mathfrak{p}_i)$, are precisely the irreducible components of V. Note that the map $\mathfrak{a}\mapsto\mathrm{Var}(\mathfrak{a})$ gives us the following bijections:

$$\begin{array}{lcl}
\text{radical ideals of } \mathbb{K}[x_1,\dots,x_n] & \longleftrightarrow & \text{algebraic sets in } \mathbb{K}^n, \\
\text{prime ideals of } \mathbb{K}[x_1,\dots,x_n] & \longleftrightarrow & \text{irreducible algebraic sets in } \mathbb{K}^n, \\
\text{maximal ideals of } \mathbb{K}[x_1,\dots,x_n] & \longleftrightarrow & \text{points of } \mathbb{K}^n.
\end{array}$$

Definition 1.27. Let $V=\mathrm{Var}(\mathfrak{a})$ be the algebraic set defined by an ideal $\mathfrak{a}\subset\mathbb{K}[x_1,\dots,x_n]$. The *coordinate ring* of V, denoted $\mathbb{K}[V]$, is the ring $R/\mathfrak{a}$.

The points of V correspond to maximal ideals of $\mathbb{K}[x_1,\dots,x_n]$ containing $\mathfrak{a}$, and hence to the maximal ideals of $\mathbb{K}[V]$. Let $p\in V$ be a point corresponding to a maximal ideal $\mathfrak{m}\subset\mathbb{K}[V]$. The *local ring of V at p* is the ring $\mathbb{K}[V]_{\mathfrak{m}}$.

Definition 1.28. The *dimension* of an irreducible algebraic set V is the Krull dimension of its coordinate ring $\mathbb{K}[V]$. For a (possibly reducible) algebraic set V, we define

$$\dim V = \sup\{\dim V_i \mid V_i \text{ is an irreducible component of } V\}.$$

Example 1.29. The irreducible components of the algebraic set

$$V = \mathrm{Var}(xy, xz) = \mathrm{Var}(x) \cup \mathrm{Var}(y, z) \quad \text{ in } \mathbb{K}^3$$

are the plane $x = 0$ and the line $y = z = 0$. The dimension of the plane is $\dim \mathbb{K}[x, y, z]/\,(x) = 2$, and of the line is $\dim \mathbb{K}[x, y, z]/\,(y, z) = 1$, so $\dim V = 2$.

Example 1.30. Let V be a d-dimensional vector subspace of $\mathbb{K}^n$. After a linear change of variables, we may assume that the $n-d$ homogeneous linear polynomials defining the algebraic set V are a subset of the variables of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$, i.e., $V = \mathrm{Var}(x_1, \dots, x_{n-d})$. Hence

$$\dim V = \dim \mathbb{K}[x_1, \dots, x_n]/\,(x_1, \dots, x_{n-d}) = d.$$

The dimension of an algebraic set, as we have defined it here, has several desirable properties:

Theorem 1.31. *Let $\mathbb{K}$ be an algebraically closed field, and let V and W be algebraic sets in $\mathbb{K}^n$.*

(1) *If V is a vector space, then $\dim V$ equals the vector space dimension $\mathrm{rank}_{\mathbb{K}} V$.*
(2) *Let W be an irreducible algebraic set of dimension d, and V an algebraic set defined by m polynomials. Then every nonempty irreducible component of $V \cap W$ has dimension at least $d - m$.*
(3) *Every nonempty irreducible component of $V \cap W$ has dimension greater than or equal to $\dim V + \dim W - n$.*
(4) *If $\mathbb{K} = \mathbb{C}$, then $\dim V$ is half of the dimension of V as a real topological space.*

Note that Theorem 1.31(3) generalizes the inequality of vector space dimensions we observed in Example 1.2.

Sketch of proof. (1) was observed in Example 1.30.

(2) Let $\mathfrak{p}$ be the prime ideal of $R = \mathbb{K}[x_1, \dots, x_n]$ such that $W = \mathrm{Var}(\mathfrak{p})$, and let $V = \mathrm{Var}(f_1, \dots, f_m)$. An irreducible component of $V \cap W$ corresponds to a minimal prime $\mathfrak{q}$ of $\mathfrak{p} + (f_1, \dots, f_m)$. But then $\mathfrak{q}/\mathfrak{p}$ is a minimal prime of the ideal $(f_1, \dots, f_m)\, R/\mathfrak{p}$, so $\mathrm{height}\, \mathfrak{q}/\mathfrak{p} \leqslant m$ by Theorem 1.14. Hence

$$\dim R/\mathfrak{q} = \dim R/\mathfrak{p} - \mathrm{height}\, \mathfrak{q}/\mathfrak{p} \geqslant d - m.$$

(3) Replacing V and W by irreducible components, we may assume that $V = \mathrm{Var}(\mathfrak{p})$ and $W = \mathrm{Var}(\mathfrak{q})$ for prime ideals $\mathfrak{p}, \mathfrak{q} \subset \mathbb{K}[x_1, \dots, x_n]$. Let $\mathfrak{q}' \subset \mathbb{K}[x_1', \dots, x_n']$ be the ideal obtained from $\mathfrak{q}$ by replacing each x_i by a new variable x_i'. We may regard $V \times W$ as an algebraic set in $\mathbb{K}^n \times \mathbb{K}^n = \mathbb{K}^{2n}$, i.e., as the zero set of the ideal

$$\mathfrak{p} + \mathfrak{q}' \subset \mathbb{K}[x_1, \dots, x_n, x_1', \dots, x_n'] = S.$$

Using the fact that $\mathbb{K}$ is algebraically closed, it is not hard to see that $\mathfrak{p} + \mathfrak{q}'$ is a prime ideal of S, so $V \times W$ is irreducible and has dimension $\dim V + \dim W$. Let $\mathfrak{d} = (x_1 - x_1', \dots, x_n - x_n')$, in which case $\Delta = \mathrm{Var}(\mathfrak{d})$ is the 'diagonal' in $\mathbb{K}^n \times \mathbb{K}^n$. Then

$$\mathbb{K}[V \cap W] = \frac{\mathbb{K}[x_1, \dots, x_n]}{\mathfrak{p} + \mathfrak{q}} \cong \frac{\mathbb{K}[x_1, \dots, x_n, x_1', \dots, x_n']}{\mathfrak{p} + \mathfrak{q}' + \mathfrak{d}} = \mathbb{K}[(V \times W) \cap \Delta].$$

The ideal $\mathfrak{d}$ is generated by n elements, so Krull's Theorem 1.14 implies that

$$\begin{aligned} \dim(V \cap W) &= \dim S/(\mathfrak{p} + \mathfrak{q}' + \mathfrak{d}) \\ &\geqslant \dim S/(\mathfrak{p} + \mathfrak{q}') - n = \dim V + \dim W - n. \end{aligned}$$

(4) We skip the proof, but point out that an irreducible complex algebraic set of dimension d is the union of a $\mathbb{C}$-manifold of dimension d and an algebraic set of lower dimension. $\square$

An extended example

Example 1.32. Consider the algebraic set V of 2×3 complex matrices of rank less than 2. Take the polynomial ring $R = \mathbb{C}[u, v, w, x, y, z]$. Then $V = \operatorname{Var}(\mathfrak{a})$, where $\mathfrak{a}$ is the ideal generated by the polynomials

$$\Delta_1 = vz - wy, \quad \Delta_2 = wx - uz, \quad \Delta_3 = uy - vx.$$

Exercise 1.34 shows that $\mathfrak{a}$ is a prime ideal. We compute $\dim V$ from four different points of view.

As a topological manifold: The set of rank one matrices is the union of the sets

$$\left\{ \begin{pmatrix} a & b & c \\ ad & bd & cd \end{pmatrix} \mid (a, b, c) \in \mathbb{C}^3 \setminus \{\mathbf{0}\}, d \in \mathbb{C} \right\}$$

and

$$\left\{ \begin{pmatrix} ad & bd & cd \\ a & b & c \end{pmatrix} \mid (a, b, c) \in \mathbb{C}^3 \setminus \{\mathbf{0}\}, d \in \mathbb{C} \right\},$$

each of which is a copy of $\mathbb{C}^3 \setminus \{\mathbf{0}\} \times \mathbb{C}$ and hence has dimension 8 as a topological space. The set V is the union of these along with one more point corresponding to the zero matrix. Hence V has topological dimension 8 and so $\dim V = 8/2 = 4$.

Using transcendental degree: The ideal $\mathfrak{a}$ is prime so $\dim R/\mathfrak{a}$ can be computed as $\operatorname{tr.deg}_{\mathbb{C}} \mathbb{L}$, where $\mathbb{L}$ is the fraction field of $R/\mathfrak{a}$. In the field $\mathbb{L}$ we have $v = uy/x$ and $w = uz/x$, so

$$\mathbb{L} = \mathbb{C}(u, x, y, z)$$

where u, x, y, z are algebraically independent over $\mathbb{C}$. Hence Theorem 1.19 implies that $\dim R/\mathfrak{a} = 4$.

By finding a system of parameters: In the polynomial ring R we have a chain of prime ideals

$$\mathfrak{a} \subsetneq (u, x, vz - wy) \subsetneq (u, v, x, y) \subsetneq (u, v, w, x, y) \subsetneq (u, v, w, x, y, z)$$

which gives a chain of prime ideals in $R/\mathfrak{a}$ showing that $\dim R/\mathfrak{a} \geqslant 4$. Consider the four elements $u, v - x, w - y, z \in R$. Then the ideal

$$\mathfrak{a} + (u, v - x, w - y, z) = \left(u, v - x, w - y, z, x^2, xy, y^2\right)$$

contains $\mathfrak{m}^2$ and hence it is $\mathfrak{m}$-primary. This means that the image of $\mathfrak{m}$ in $R/\mathfrak{a}$ is the radical of a 4-generated ideal, so $\dim R/\mathfrak{a} \leqslant 4$. It follows that $\dim R/\mathfrak{a} = 4$ and that the images of $u, v - x, w - y, z$ in $R/\mathfrak{a}$ are a system of parameters for $R/\mathfrak{a}$.

From the Hilbert-Poincaré series: Exercise 1.34 shows that $R/\mathfrak{a}$ is isomorphic to the ring $S = \mathbb{C}[as, bs, cs, at, bt, ct]$, under a degree preserving isomorphism, where each of the monomials as, bs, cs, at, bt, ct is assigned degree 1. The vector space dimension of S_n is the product of the number of monomials of degree n in a, b, c with the number of monomials of degree n in s, t, i.e., $\binom{n+2}{2}\binom{n+1}{1}$. It follows that

$$P(S,t) = \sum_n \binom{n+2}{2}\binom{n+1}{1} t^n = \frac{1+2t}{(1-t)^4},$$

which has a pole of order 4 at $t = 1$. Hence $\dim S = 4$.

Exercise 1.33 (Hochster)**.** Let R and S be $\mathbb{K}$-algebras and $\varphi : R \longrightarrow S$ a surjective $\mathbb{K}$-algebra homomorphism. Let $\{s_i\}$ be a $\mathbb{K}$-vector space basis for S, and $r_i \in R$ elements with $\varphi(r_i) = s_i$. Let $\mathfrak{a}$ be an ideal contained in $\ker(\varphi)$. If every element of R is congruent to an element in the $\mathbb{K}$-span of $\{r_i\}$ modulo $\mathfrak{a}$, prove that $\mathfrak{a} = \ker(\varphi)$.

Exercise 1.34 (Hochster)**.** Let $R = \mathbb{K}[u, v, w, x, y, z]$ and S the subring of the polynomial ring $\mathbb{K}[a, b, c, s, t]$ generated over $\mathbb{K}$ by the monomials as, bs, cs, at, bt, ct, i.e., $S = \mathbb{K}[as, bs, cs, at, bt, ct]$. Consider the $\mathbb{K}$-algebra homomorphism $\varphi : R \longrightarrow S$ where

$$\begin{aligned} \varphi(u) &= as, \qquad \varphi(v) = bs, \qquad \varphi(w) = cs,\\ \varphi(x) &= at, \qquad \varphi(y) = bt, \qquad \varphi(z) = ct. \end{aligned}$$

Prove that $\ker(\varphi) = (vz - wy,\ wx - uz,\ uy - vx)$, and conclude that this ideal is prime. This shows that

$$\mathbb{K}[as, bs, cs, at, bt, ct] \cong \mathbb{K}[u, v, w, x, y, z]/(vz - wy, wx - uz, uy - vx).$$

Tangent spaces and regular rings

Working over an arbitrary field, one can consider partial derivatives of polynomial functions with respect to the variables, e.g.,

$$\frac{\partial}{\partial x}(x^3 + y^3 + z^3 + xyz) = 3x^2 + yz.$$

Definition 1.35. Let $V = \mathrm{Var}(\mathfrak{a}) \subset \mathbb{K}^n$ be an algebraic set. The *tangent space* to V at a point $p = (\alpha_1, \dots, \alpha_n)$ is the algebraic set $T_p(V) \subset \mathbb{K}^n$ which is the solution set of the linear equations

$$\sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_p (x_i - \alpha_i) = 0 \quad \text{for} \quad f \in \mathfrak{a}.$$

An easy application of the product rule shows that to obtain the defining equations for $T_p(V)$, it is sufficient to consider the linear equations arising from a generating set for $\mathfrak{a}$.

For the rest of this lecture, we work over an algebraically closed field. Let V be an irreducible algebraic set. Then V is *nonsingular* or *smooth* at a point $p \in V$ if $\dim T_p(V) = \dim V$, and V is *singular* at p otherwise.

Example 1.36. The circle $\mathrm{Var}(x^2 + y^2 - 1)$ is smooth at all points as long as the field does not have characteristic 2.

Take a point $p = (\alpha, \beta)$ on the cusp $\mathrm{Var}(x^2 - y^3)$. Then the tangent space at p is the space defined by the polynomial equation

$$2\alpha(x-\alpha) - 3\beta^2(y-\beta) = 0.$$

This is a line in $\mathbb{K}^2$ if $p \neq (0,0)$, whereas the tangent space to the cusp at $p = (0,0)$ is all of $\mathbb{K}^2$. Hence the cusp has a unique singular point at the origin.

Definition 1.37. A local ring $(R, \mathfrak{m})$ of dimension d is a *regular local ring* if its maximal ideal $\mathfrak{m}$ can be generated by d elements.

Theorem 1.38. *Let p be a point of an irreducible algebraic set V. Then the dimension of $T_p(V)$ is the least number of generators of the maximal ideal of the local ring of V at p.*

Hence $\dim T_p(V) \geqslant \dim V$, *and p is a nonsingular point of V if and only if the local ring of V at p is a regular local ring.*

Proof if V is a hypersurface. Let $V = \mathrm{Var}(f)$ for $f \in \mathbb{K}[x_1, \ldots, x_n]$. After a linear change of coordinates, we may assume $p = (0, \ldots, 0)$ is the origin in $\mathbb{K}^n$. Then $T_p(V)$ is the algebraic set in $\mathbb{K}^n$ defined by the linear equation

$$\sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_p x_i = 0,$$

so $\dim T_p(V) \geqslant n-1$ and V is smooth at p if and only if some partial derivative $\partial f / \partial x_i$ does not vanish at p, i.e., if and only if $f \notin (x_1, \ldots, x_n)^2$. The local ring of V at p is $\mathbb{K}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)} / (f)$ and the minimal number of generators of its maximal ideal $\mathfrak{m}$ is the vector space dimension of

$$\mathfrak{m}/\mathfrak{m}^2 = (x_1, \ldots, x_n) / ((x_1, \ldots, x_n)^2 + (f)),$$

which equals $n-1$ precisely if $f \notin (x_1, \ldots, x_n)^2$. $\square$

Lecture 2. Sheaves: a potpourri of algebra, analysis and topology (UW)

In this lecture we are going to get a first glimpse at the interplay of algebra, geometry and topology that is commonly known as "sheaf theory". The goal of this lecture is to introduce sheaves by example and to motivate their further study by pointing out some of their connections to algebra, analysis and topology. In consequence, there is little proof and much hand-waving. Some of this will be fixed by later lectures, but substantial parts will not. Because of its motivational character the reader may want to revisit this lecture later, when equipped with more of the algebraic tools that are necessary to fully appreciate some of the principles outlined here.

There are many sources for further reading on sheaves and their applications. For the algebro-geometric approach to sheaves on varieties and schemes we refer to Hartshorne's excellent book [63]. For more of the differential aspects of the theory one can consult the monumental book by Griffiths and Harris [53], the very nice book by Björk [9], the classic by Godement [47], or the work by Morita [121]. Much of the required homological background can be found in Weibel's excellent reference book [158], while the books [84, 46] by Iversen, and Gelfand and Manin shows the workings of homological algebra in action in the context of sheaf theory. As perhaps the best expositions of how to link calculus and cohomology we recommend [11, 111], and [29, 88] for connections with D-modules and singularity theory. This is not by any account a complete list, but these books are excellent starting points.

The point of view of this lecture is to consider sheaves as spaces of functions. They are typically defined by local conditions and it is of interest to determine the global functions with the required local properties. For example, the Mittag-Leffler problem specifies the principal parts of a holomorphic function on a Riemann surface at finitely many points and asks for the existence of a global holomorphic function with the appropriate principal parts [53]. Such local-to-global problems are often nontrivial to solve and give rise to the notion of sheaf cohomology (which is also discussed in Lectures 12,13,**??** and 19), a special type of derived functor (to be discussed in Lecture 3). These derived functors can be viewed as a kind of shadow of a chosen functor when evaluated at some specific argument. Particularly in topology such derived functors have a highly hands-on nature; sheaf cohomology (when applied to the right kind of sheaf) "is" singular cohomology.

The nature of sheaves is truly multidisciplinary and so there are several algebraic constructions that sheaf theory gave rise to. One of these is the Čech complex. It may be viewed as a notion born out of the familiar Seifert–van Kampen theorem that expresses the fundamental group of the union of two topological spaces in terms of the fundamental groups of the two spaces, their intersection, and information how these three groups interact. This Čech complex idea is a leading theme in the study of algebraic varieties that we undertake under the umbrella of "local cohomology" in this book.

To close these introductory remarks let us comment that our point of view that sheaves be actual functions reflects the way that sheaves were initially conceived by the French school around Leray and Cartan, but these days one typically describes them in more abstract ways. Namely, Definition 2.3 below distills from the idea of a function space the crucial abstract properties; this leads to a very flexible theory with many areas of application. Despite this abstraction, one may always view any sheaf as a collection of functions although the target of the functions may be hard to understand—see Remark 12.31.

2.1. **The basics of sheaves.** We assume that the reader is familiar with the basic concepts of point set topology. Let us fix a space X with topology $\mathcal{T}_X$,

Example 2.1. a) While in a space like $\mathbb{R}^n$ open and closed sets abound, a topology can quite sparse. In the extreme case, the only sets in $\mathcal{T}_X$ are X and the empty set $\emptyset$. This scenario is known as the *trivial topology.*

b) In the other extreme, all subsets of X are open, and hence all of them are closed as well. In that case, X is said to be given the *discrete topology.*

c) Most situations are of course somewhere between these two extreme cases. The cases that interest us most will typically have distinctly fewer open sets than one is accustomed to in $\mathbb{R}^n$. For example, let us consider the *spectrum* of the ring $R = \mathbb{C}[x]$. By definition, points in this set correspond to prime ideals in R, which are the ideals $\{(x-c)\}_{c\in\mathbb{C}}$ together with the ideal (0). One declares to be closed any collection $\mathrm{Var}(I)$ of prime ideals that happens to be the set of all prime ideals containing an ideal I of R, this is the *variety of* I. This notion is set up in such a way that the variety of the ideal I is the same as the variety of the radical of I. So the closed sets are in this case the empty set, any finite (!) collection of ideals of the type $(x-c)$, and the entire space. The topology on $\mathrm{Spec}(R)$ having as closed sets precisely all varieties of ideals is the *Zariski topology* from Definition 1.12.

In this last example, the topology is fairly coarse. This is best illustrated by looking at the Hausdorff property: the topology $\mathcal{T}_X$ is *Hausdorff* if all pairs $x \neq y$ in X can be enclosed in disjoint open sets—there are $U \ni x, V \ni y$ in $\mathcal{T}_X$ with $U \cap V = \emptyset$. The spectrum of a ring, with its Zariski topology, is almost never Hausdorff.

Choose now a second topological space $(F, \mathcal{T}_F)$. Recall that a *continuous* map from X to F is a function $f : X \longrightarrow F$ such that whenever V is an open set in F then $f^{-1}(V) = \{x \in X | f(x) \in V\}$ is an open set in X. We attach to each open set $U \in \mathcal{T}_X$ the space of all continuous functions $C(U, F)$ from U to F. The following is our running example for this lecture.

Example 2.2. To be concrete, let us take as $X = \mathbb{S}^1$, the unit circle with its usual collection of open sets inherited through the embedding of X into $\mathbb{R}^2$. So any proper open set is the union of open connected arcs. On the other side we let F be the set of integers $\mathbb{Z}$ with the discrete topology (every subset is open). For an open set U in X let $f : X \longrightarrow F$ be in $C(X, F)$. This has the effect that the preimage of any number $z \in \mathbb{Z} \cap f(U)$ is an open set in U. But since the collection of all points in U that do *not* map to z is also open (it's the union of the preimages of all $z' \in \mathbb{Z}$, $z' \neq z$), every $z \in \mathbb{Z}$ decomposes U into two open sets: $f^{-1}(z)$ and $U \setminus f^{-1}(z)$. However, if U is a connected open set in X (an open arc along the unit circle, or the entire circle), then U cannot be written as the union of two disjoint nonempty open subsets. It follows that if U is connected then $f(U)$ is a singleton $z \in \mathbb{Z}$ and so $C(U, F) = \mathbb{Z}$.

We write from now on $\mathcal{Z}(U)$ for $C(U, \mathbb{Z})$. Suppose we have a containment of two open sets $U' \subseteq U$ in the topological space X, and let $f : U \longrightarrow F$ be continuous.

Then one can define a new map $f' : U' \longrightarrow F$ by combining f with the embedding $\iota_{U',U} : U' \hookrightarrow U$. This affords a *restriction* map $\rho_{U,U';\mathcal{F}} : C(U,F) \longrightarrow C(U',F)$. Of course, compositions of restrictions are restrictions again.

In order to prepare for the definition to come, we slightly change the point of view. Namely, we consider the space $\mathbb{Z} \times X$ with its natural projection $\pi_{\mathcal{Z}} : \mathbb{Z} \times X \longrightarrow X$. Then the elements of $C(U,F)$ can for any open set U in X be identified with the continuous maps $f : U \longrightarrow \mathbb{Z} \times X$ such that $\pi_{\mathcal{Z}} \circ f$ is the identity on U. In this way, the elements of $\mathcal{Z}(U)$ become *sections* (that is, continuous lifts) for the projection $\pi_{\mathcal{Z}}$.

Definition 2.3. Let $(X, \mathcal{T}_X)$ be a topological space. A *sheaf* (of sets) on X is a choice $\mathcal{F}$ of a topological *sheaf space* (or *espace étalé*) $(F, \mathcal{T}_F)$ together with a surjective map $\pi_{\mathcal{F}} : F \twoheadrightarrow X$. The sheaf $\mathcal{F}$ associates to each open set U the set $\mathcal{F}(U) = C(U,F)$ of continuous functions $f : U \longrightarrow F$ for which $\pi_{\mathcal{F}} \circ f$ is the identity on U.

Then for each open containment $\iota_{U',U} : U' \hookrightarrow U$ there a restriction map

♣ check other notations of restriction maps

$$\begin{array}{rcl} \rho_{U',U} = \rho_{U',U;\mathcal{F}} & : & \mathcal{F}(U) \longrightarrow \mathcal{F}(U') \\ f & \mapsto & f \circ \iota_{U',U} \end{array}$$

satisfying

$$\rho_{U'',U'} \circ \rho_{U',U} = \rho_{U'',U}$$

for any three open sets $U'' \subseteq U' \subseteq U$ in $\mathcal{T}_X$.

The elements of $\mathcal{F}(U)$ are the *sections* of $\mathcal{F}$ over U; if $U = X$ one calls them *the global sections*.

The notion of a sheaf arose from the idea to consider the sections (i.e., the continuous lifts) of a bundle map $\pi : E \twoheadrightarrow X$ over the base space X with fiber F_0 and total space E. To allow for greater flexibility, the bundle E was later replaced by an arbitrary space F surjecting onto X. An important class of sheaves, including our running example, are the *constant sheaves*. These arise when the sheaf space F is the product $F_0 \times X$ of X with a space F_0 which is equipped with the discrete topology. In that case, the sections of $\mathcal{F}$ over the open set U are identified with the continuous maps from U to F_0. In our example, $F_0 = \mathbb{Z}$.

Remark 2.4. We will see in Lecture 12 that one may specify a sheaf without knowing the sheaf space F. Namely, one may select the collections of sections $\mathcal{F}(U)$ as long as they fit the crucial properties from our definition. This is useful since for many important sheaves, particularly those sheaves that arise in algebraic geometry, the sheaf space F is very obscure and its topology $\mathcal{T}_F$ highly complicated.

In general, the sections $\mathcal{F}(U)$ of a sheaf form a set without further structure. There are, however, more special kinds of sheaves: sheaves of groups (specifically, Abelian groups), sheaves of rings (specifically, commutative rings), or sheaves of R-algebras where R is a fixed ring. In these cases, $\mathcal{F}(U)$ is for all U of the appropriate algebraic structure, and the restriction maps are morphisms in the corresponding category.

We continue our Example 2.2 from above.

Example 2.5 (The constant sheaf $\mathcal{Z}$)**.** With $X = \mathbb{S}^1$, let $\mathcal{Z}(U) = C(U, \mathbb{Z})$ where $\mathbb{Z}$ has the discrete topology. Since $\mathbb{Z}$ is an Abelian group, $\mathcal{Z}(U)$ is an Abelian group as well, with pointwise addition of maps, and the restrictions $\mathcal{Z}(U) \longrightarrow \mathcal{Z}(U')$ for $U' \subseteq U$ are group homomorphisms.

It turns out that the sheaf $\mathcal{Z}$ can be used to show that the circle is not contractible. The remainder of this lecture is devoted to an outline of two mechanisms that connect $\mathcal{Z}$ to the topology of the circle: Čech complex, and derived functors.

2.2. **Čech cohomology.** Cover $\mathbb{S}^1$ with two sets U_1, U_2 as follows. Picturing $\mathbb{S}^1$ as unit circle embedded into the complex line $\mathbb{C}^1$ let $U_1 = \mathbb{S}^1 \setminus \{-1\}$ and $U_2 = \mathbb{S}^1 \setminus \{1\}$, and write $U_{1,2} = U_1 \cap U_2$.[1] Since both U_1 and U_2 are connected, $\mathcal{Z}(U_i) = C(U_i, \mathbb{Z}) = \mathbb{Z}$ for $i = 1, 2$. The restriction maps $\mathcal{Z}(\mathbb{S}^1) \longrightarrow \mathcal{Z}(U_i)$ are isomorphisms since the number of connected components of all three sets involved is the same, equal to one. With $U_{1,2}$, however, the story is different. The two different connected components of $U_{1,2}$ may be mapped to distinct numbers of $\mathbb{Z}$ and hence $\mathcal{Z}(U_{1,2}) \cong \mathbb{Z} \times \mathbb{Z}$. In fact, in general, the assignment $U \longrightarrow \mathcal{Z}(U)$ satisfies clearly

Lemma 2.6. *Let $\mathcal{Z}(-)$ be the sheaf of Abelian groups on X sending U to $\mathcal{Z}(U) = C(U, \mathbb{Z})$. Then $\mathcal{Z}(U) = \prod \mathbb{Z}$ where the product ranges over all connected components of U.* □

The four open sets X, U_1, U_2 and $U_{1,2}$ give a small commutative diagram of embeddings

$$\begin{tikzcd} & U_1 \arrow[dl] & \\ \mathbb{S}^1 & & U_{1,2} \arrow[ul] \arrow[dl] \\ & U_2 \arrow[ul] & \end{tikzcd}$$

The restriction maps furnish a commutative diagram on the level of sections:

$$\begin{tikzcd} & \mathcal{Z}(U_1) \arrow[dr] & \\ \mathcal{Z}(\mathbb{S}^1) \arrow[ur] \arrow[dr] & & \mathcal{Z}(U_{1,2}) \\ & \mathcal{Z}(U_2) \arrow[ur] & \end{tikzcd}$$

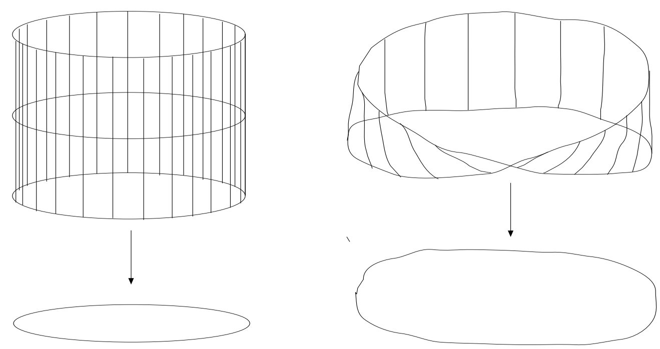
Let us discuss the maps $\mathcal{Z}(U_i) \longrightarrow \mathcal{Z}(U_{1,2})$ for $i = 1, 2$. A section on $U_{1,2}$ is given by a pair of integer numbers (a, b). If this element were to come from a section on U_i, one would obviously have to have $a = b$ since each U_i is connected. It follows that the image of $\mathcal{Z}(U_i) \longrightarrow \mathcal{Z}(U_{1,2})$ is $\mathbb{Z} \cdot (1, 1)$ for both $i = 1, 2$. In a sense, the quotient of $\mathcal{Z}(U_{1,2})$ by the images of $\mathcal{Z}(U_i)$ measures the insufficiency of knowing the value of a section in one point in order to determine the entire section. In more fancy terms, it expresses the variety[2] of possible $\mathbb{Z}$-bundles over $\mathbb{S}^1$. Since U_1, U_2 are contractible, any bundle on them is constant (given by the product of U_i with the fiber of the bundle). The question then arises how the sections on the two open sets are identified along the two parts of their intersection $U_{1,2}$. Choose a generator for $\mathbb{Z}$ over U_1 and U_2, and suppose the two generators agree over one connected component of $U_{1,2}$. On the other connected component, these generators may be

[1] While analysts usually refer to $\mathbb{C}^1$ as the complex "plane", it is only of complex dimension 1, and hence a "line".

[2] Of course, "variety" is here used in the colloquial and not the technical sense of the word as in Example 2.1, c.

either identified the nose, or each is identified with the (group law) opposite of the other. In the former case the trivial bundle arises on $\mathbb{S}^1$, in the latter case the total space of the bundle is a "discrete Möbius band". The trivial bundle corresponds to the section $(1,1)$ over $U_{1,2}$ while the Möbius strip is represented by $(1,-1)$.

FIGURE 1. The trivial bundle and the Möbius bundle over the circle



In terms of sections of $\mathcal{Z}$ on X, a pair $(a,b) \in \mathcal{Z}(U_1) \times \mathcal{Z}(U_2)$ gives rise to an element of $\mathcal{Z}(X)$ if and only if $\rho_{U_{1,2},U_1}(a) = \rho_{U_{1,2},U_2}(b)$.

Algebraically this can be read as follows. Consider the following complex:

$$0 \longrightarrow \mathcal{Z}(\mathbb{S}^1) \longrightarrow \underbrace{\mathcal{Z}(U_1) \times \mathcal{Z}(U_2)}_{\check{C}^0} \xrightarrow{d^0} \underbrace{\mathcal{Z}(U_{1,2})}_{\check{C}^1} \longrightarrow 0.$$

We introduce signs in the maps that are of combinatorial nature: the map $1 \times \mathcal{Z}(U_2) \longrightarrow \mathcal{Z}(U_{1,2})$ in the complex above is *negative* the restriction map. The sign occurs since the permutation $(2,1)$ that arises by juxtaposing the index set $\{2\}$ of U_2 with the new index $\{1\}$ that occurs in the index set of $U_{1,2}$ is odd. The benefit is that now we have an actual complex: the composition of two consecutive maps is zero.

The discussion above reveals that the complex is exact on the left and in the middle, and has a free group of dimension one as cohomology on the right, representing the existence of interesting $\mathbb{Z}$-bundles on $\mathbb{S}^1$.

It is entirely reasonable to ask what would happen if we covered $\mathbb{S}^1$ with other open sets than the ones we chose. In particular, what happens if the number of sets is changed?

Example 2.7. Let us cover the circle with three open sets, the complements of the three third roots of unity; call them U_i, $1 \leqslant i \leqslant 3$. Their intersections $U_{i,j}$ are homeomorphic to pairs of intervals, and their triple intersection $U_{1,2,3}$ is the complement of all three roots, hence homeomorphic to three disjoint intervals.

It follows that there is a diagram of restriction maps that involves eight open sets, including the full circle:

$$
\begin{tikzcd}
 & \mathcal{Z}(U_1) \arrow[r] \arrow[dr] & \mathcal{Z}(U_{1,2}) \arrow[dr] & \\
\mathcal{Z}(\mathbb{S}^1) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{Z}(U_2) \arrow[ur] \arrow[dr] & \mathcal{Z}(U_{1,3}) \arrow[r] & \mathcal{Z}(U_{1,2,3}) \\
 & \mathcal{Z}(U_3) \arrow[ur] \arrow[r] & \mathcal{Z}(U_{2,3}) \arrow[ur] &
\end{tikzcd}
$$

which algebraically corresponds to the following complex $C^\bullet$

$$
0 \longrightarrow \mathcal{Z}(\mathbb{S}^1) \longrightarrow \underbrace{\prod_{i=1}^{3} \mathcal{Z}(U_i)}_{\check{C}^0} \xrightarrow{d^0} \underbrace{\prod_{1\leqslant i<j\leqslant 3} \mathcal{Z}(U_{i,j})}_{\check{C}^1} \xrightarrow{d^1} \underbrace{\mathcal{Z}(U_{1,2,3})}_{\check{C}^2} \longrightarrow 0
$$

In this complex all groups are free and of ranks 1, 3, 6 and 3 respectively. The first map is given by the matrix $(1,1,1)^T$. The differentials d^i are given by

$$
d^0 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} : \check{C}^1 \longrightarrow \check{C}^2
$$

and

$$
d^1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} : \check{C}^2 \longrightarrow \check{C}^3.
$$

As in the case of two open sets, in these matrices the sign of the entry that corresponds to the restriction from U_I to $U_{I\cup\{j\}}$ corresponds to the sign of the permutation (I,j).

Again there is just one cohomology group, in degree one, and this cohomology is a free group of rank one generated by the element $(1,-2,-2,1,1,-2) \in C^1$.

The preceding examples are supposed to suggest that there is a certain invariance to the computations induced by open covers; this is indeed the case. The following procedure attaches a complex to an open cover of a space equipped with a sheaf, and we shall investigate the independence of its cohomology groups from the chosen cover.

Definition 2.8. Fix a topological space X and a sheaf $\mathcal{F}$ on X, and let I be a totally ordered index set. Given any open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X define for a finite $J \subseteq I$ the open set $U_J = \bigcap_{i\in J} U_i$. We define a complex $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$ whose t-th term (where $t \geq 0$) is $\prod_{|J|=t+1} \mathcal{F}(U_J)$.

To define the maps of the complex, fix $t \in \mathbb{N}$. Let $J \subseteq I$ with $|J| = t$ and pick $j \in I \setminus J$. Put $\mathrm{sgn}(J,j)$ to be -1 raised to the number of elements of J that are bigger than j. In other words, $\mathrm{sgn}(J,j)$ is the sign of the permutation (J,j). Then define

$$
d^t : \check{C}^t(\mathfrak{U};\mathcal{F}) \longrightarrow \check{C}^{t+1}(\mathfrak{U};\mathcal{F})
$$

as the sum of the maps

$$\mathrm{sgn}(J,j)\cdot\rho_{U_{J\cup\{j\}},U_J;\mathcal{F}} \quad : \quad \mathcal{F}(U_J)\longrightarrow\mathcal{F}(U_{J\cup\{j\}}).$$

Because of the sign choices, $d^{t+1}\circ d^t=0$, so $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$ is a cohomological complex

(2.8.1)

$$0\longrightarrow\underbrace{\prod_{i\in I}\mathcal{F}(U_i)}_{\check{C}^0(\mathfrak{U};\mathcal{F})}\xrightarrow{d^0}\underbrace{\prod_{i,j\in I}\mathcal{F}(U_{\{i,j\}})}_{\check{C}^1(\mathfrak{U};\mathcal{F})}\xrightarrow{d^1}\cdots\xrightarrow{d^{t-1}}\underbrace{\prod_{J\subseteq I,|J|=t+1}\mathcal{F}(U_J)}_{\check{C}^t(\mathfrak{U};\mathcal{F})}\xrightarrow{d^t}\cdots$$

called the *Čech complex* associated to $\mathcal{F}$ and the open cover $\mathfrak{U}$. Its cohomology is denoted $\check{H}(\mathfrak{U};\mathcal{F})$.

We note that $\mathcal{F}(X)$ is not part of the Čech complex. However, the following exercise justifies the indexing in the Čech complex—the global sections are determined by the sections on the cover and $\mathcal{F}(X)$ can be read off $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$.

Exercise 2.9. Let $\mathcal{F}$ be a sheaf on the space X and assume that $\mathfrak{U}$ is an open cover for X. Prove that the kernel of the initial differential d^0 in the Čech complex $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$ is identified with $\mathcal{F}(X)$.

If you found this easy, look up the idea of a refinement of an open cover by another cover in Definition 2.12. Then prove that the identification of $\ker(d^0)$ with $\mathcal{F}(X)$ that you found is independent of the refinement.

Remark 2.10. Suppose that

$$0\longrightarrow A'\longrightarrow A\longrightarrow A''\longrightarrow 0$$

is a short exact sequence of groups. We consider the three induced constant sheaves $\mathcal{A}',\mathcal{A},\mathcal{A}''$ on the space X. Suppose there is a cover $\mathfrak{U}$ such that evaluating the above sequence at any intersection U_I of open sets in the cover gives a short exact sequence

$$0\longrightarrow\check{C}(U_I;A')\longrightarrow\check{C}(U_I;A)\longrightarrow\check{C}(U_I;A'')\longrightarrow 0.$$

Then one obtains a short exact sequence of Čech complexes which in turn yields a long exact sequence of cohomology groups

$$\cdots\longrightarrow\check{H}^i(\mathfrak{U};\mathcal{A}')\cdots\longrightarrow\check{H}^i(\mathfrak{U};\mathcal{A})\cdots\longrightarrow\check{H}^i(\mathfrak{U};\mathcal{A}'')\cdots\longrightarrow\check{H}^{i+1}(\mathfrak{U};\mathcal{A}')\longrightarrow\cdots$$

We now come to the fundamental theorem about the Čech complex, it says that there is a stable "limit" version that can be obtained by choosing finer and finer covers. The cohomology groups of a Čech complex attached to a very fine cover are hence attached to the underlying space rather than the cover.

Theorem 2.11. *For reasonable spaces X, for a suitable fixed sheaf $\mathcal{F}$ on X, and for all sufficiently fine covers $\mathfrak{U}$, the cohomology groups of the associated Čech complex are well defined (i.e., are naturally isomorphic for all sufficiently fine covers).*

It is adequate to include some discussion of this theorem: what kinds of spaces are "reasonable", which sheaves are "suitable", and what does "fine" mean in the corresponding context? This is a complicated question since quite different sorts of spaces fit the bill. The simplest class is formed by the n-dimensional real manifolds. In that case, if all sets in the cover $\mathfrak{U}$ as well as all their finite intersections are homeomorphic to $\mathbb{R}^n$, then $\check{H}^\bullet(\mathfrak{U};\mathcal{F})$ is independent of the cover $\mathfrak{U}$. More generally, if X is a paracompact space then $\check{H}^\bullet(\mathcal{F},\mathfrak{U})$ is independent of the cover provided that any finite intersection of elements in the open cover $\mathfrak{U}$ is contractible. In both situations the sheaf $\mathcal{F}$ can be arbitrary. On the other hand, if X is a scheme and $\mathcal{F}$ a quasi-coherent sheaf[3] then any cover $\mathfrak{U}$ consisting of affine schemes (spectra of rings) satisfies Theorem 2.11. It is often interesting to consider certain non-quasi-coherent sheaves on schemes, such as the constant sheaf. In Lecture 19 we will consider a method that replaces the constant sheaf by a complex of quasi-coherent sheaves. This, in conjunction with an affine cover, can be used to compute the cohomology of the constant sheaf, which in turn yields interesting topological information.

To understand in what sense the cohomology of the Čech complex can be independent of the cover, we need to introduce some notation.

Definition 2.12. Let X be a topological space and suppose that $\mathfrak{U}, \mathfrak{V}$ are two covers with index sets I, I'. We call $\mathfrak{U}$ a *refinement* of $\mathfrak{V}$, $\mathfrak{U} < \mathfrak{V}$, if there is a map $\tau : I \longrightarrow I'$ with $U_i \subseteq V_{\tau(i)}$. Note that there is no need for τ to be injective.

Suppose now that $\mathfrak{U} < \mathfrak{V}$ with index map $\tau : I \longrightarrow I'$. This induces $\mathcal{F}(V_{\tau(i)}) \longrightarrow \mathcal{F}(U_i)$ by restriction, and we use this to construct a morphism $\check{C}^t(\mathfrak{V}; \mathcal{F}) \longrightarrow \check{C}^t(\mathfrak{U}; \mathcal{F})$ and a map of complexes

$$\check{\tau} : \check{C}^\bullet(\mathfrak{V}; \mathcal{F}) \longrightarrow \check{C}^\bullet(\mathfrak{U}; \mathcal{F}).$$

as follows. Let c be an element of $\check{C}^t(\mathfrak{V}; \mathcal{F})$ and let J' be a subset of I' of cardinality $t+1$. Write c_J for the component of c over V_J so that the canonical projection $\prod_{J \subseteq I, |J| = t+1} C(V_J, F) \longrightarrow C(V_J, F)$ sends c to c_J. Now define $\check{\tau}(c)$ to be the element $\widetilde{c} \in \check{C}^t(\mathfrak{U}; \mathcal{F})$ for which

$$\widetilde{c}_J = \rho_{U_J, V_{\tau(J)}}(c_{\tau(J)}).$$

This map clearly depends on the assignment $i \mapsto \tau(i)$, but by good fortune it always maps cocycles to cocycles. Amazingly, the induced map on cohomology is *independent* of τ. Indeed, as one can show, any two index maps give rise to homotopic maps of complexes.

Considering all possible covers of the space and their associated Čech complexes, one can form the direct limit [4] $\check{C}^\bullet(X; \mathcal{F})$ of the complexes, where the limit goes over all open covers of X. Since taking cohomology of complexes commutes with direct limits (see Theorem 4.30 and also Remark **??**),

$$\check{H}^t(X; \mathcal{F}) := \varinjlim_{\mathfrak{U}} H^t(\mathfrak{U}; \mathcal{F})$$

is the cohomology of the limit complex, called the *Čech cohomology groups of $\mathcal{F}$ on X*. The term "well-defined" in Theorem 2.11 refers to the fact that if $\mathfrak{U}$ and $\mathfrak{V}$ are both "fine enough" in the appropriate context and one refines the other, then the induced maps on t-th cohomology are isomorphisms, and $\check{H}^t(\mathfrak{U}; \mathcal{F}) \cong \check{H}^t(X; \mathcal{F})$ for all t.

One can prove the following fundamental principle which may be viewed as the non plus ultra of the Mayer–Vietoris principle, and provides a quantitative criterion for the applicability of Theorem 2.11, see [47, Théorème 5.4.1 + Corollaire]:

[3] Roughly speaking, a quasi-coherent sheaf is a sheaf whose sections look (locally) like a module over the ring that gave rise to the (local patch of the) scheme. Detailed accounts on this will come in Lectures 12 and 13

[4] The notion of a direct limit is discussed in detail in Lecture 4. Readers unacquainted with this notion may on first reading want to disregard the limit issue and take the existence of the groups $\check{H}^t(X; \mathcal{F})$ on faith.

Theorem 2.13. *Suppose $\mathfrak{U}$ is an open cover for X. Let $\mathcal{F}$ be a sheaf on X. If $\check{H}^i(U;\mathcal{F}) = 0$ for all $i > 0$ and all finite intersection U of elements of $\mathfrak{U}$ then*

$$H^i(\mathfrak{U};\mathcal{F}) = \check{H}^i(X;\mathcal{F})$$

for all i.

This seems to be a ridiculous test for "fineness" of $\mathfrak{U}$, since it requires information about the behavior of the Čech mechanism on all sets of the cover and yet only yields information about the one space X. However, it is quite handy as we will see later. In fact, it comes with its own definition:

Definition 2.14. A sheaf $\mathcal{F}$ on the topological space X that has no higher Čech cohomology on X is called *acyclic on X*.

Note that if $U \subseteq X$ is open then acyclicity of $\mathcal{F}$ on U and on X do not imply one another.

The prototype of an acyclic sheaf is the *flasque* sheaf where every restriction map is surjective. A more specialized sort are the *injective sheaves*[5], and we will see in Remark 12.25 and the surrounding discussion that there are "enough" of these. Lecture 12 discusses both of these types. Typically, flasque and injective sheaves have tremendous numbers of sections. (To illustrate: let $\mathcal{I}(U)$ be the sheaf of *all* maps from U to a fixed target F_0; this produces a flasque and injective sheaf with sheaf space $F_0 \times X$. Injective sheaves are generally not far from this type.) On differentiable manifolds one has access to nicer (and weaker) versions of acyclic sheaves such as "soft" and "fine" sheaves whose sections are smooth functions.

In some ways the main theme of the summer school is the study of quasi-coherent sheaves (see Definition 12.22) on varieties and their Čech cohomology behavior. Typically, geometrically interesting sheaves $\mathcal{F}$ allow on any space X for a scenario that satisfies the hypotheses of the Acyclicity Theorem, see [53, pages 40-41]. We shall discuss in later lectures acyclicity in the theory of quasi-coherent sheaves. For example, it will turn out (Remark 12.40) that if X is an affine variety equipped with its Zariski topology, and $\mathcal{F}$ is a quasi-coherent sheaf, then all Čech cohomology groups of $\mathcal{F}$ on X with positive index are zero. Affine sets relate to quasi-coherent sheaf cohomology the way contractible sets relate to singular cohomology: they are trivial in that particular sense.

♣ `what to do about GL-affine-coho?`

We close this train of thoughts with the remark that if we computed $\check{H}^\bullet(\mathbb{S}^1;\mathcal{F})$ where $\mathcal{F}$ is the sheaf that assigns to each open U the set of continuous functions from U into a field of characteristic zero (such as $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ — each endowed with the discrete topology) then we would get the tensor product of $\check{H}^\bullet(\mathbb{S}^1;\mathcal{Z})$ with the corresponding field.

On the other hand, replacing $\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$ where $n \in \mathbb{Z}$ has more interesting effects since a tensor product with $\mathbb{Z}/n\mathbb{Z}$ does not necessarily preserve exact sequences (in contrast, tensoring with a field of characteristic zero is an exact functor).

Exercise 2.15. Recall that the real projective plane $\mathbb{RP}^2$ arises as the quotient of the 2-sphere $\mathbb{S}^2$ by identifying antipodal points. Cover $\mathbb{RP}^2$ with three open hemispheres whose pairwise and triple intersections are unions of disjoint contractible sets.

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[5] defined by a universality condition that mirrors that of injective modules over a ring, Definition A.1

Use this cover to prove that $\check{H}^1(\mathbb{RP}^2;\mathcal{Z})$ is zero. In contrast, prove that the sheaf $\mathcal{Z}_2$ given by $\mathcal{Z}_2(U) = C(U;\mathbb{Z}/2\mathbb{Z})$ satisfies $\check{H}^1(\mathbb{S}^2;\mathcal{Z}_2) \cong \mathbb{Z}/2\mathbb{Z}$. In particular, the map $\check{H}^1(\mathbb{RP}^2;\mathcal{Z}) \longrightarrow \check{H}^1(\mathbb{RP}^2;\mathcal{Z}_2)$ induced by the natural projection $\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is not surjective. (In geometric terms: the real projective plane is a non-orientable surface and as a result the orientation module $H^2(\mathbb{RP}^2;\mathcal{Z})$ is not free but isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This is reflected in the non-vanishing of the cokernel we consider here.)

2.3. **Calculus versus topology: the use of resolutions.** We shall now switch gears and investigate what happens if instead of making the cover increasingly fine, we replace $\mathcal{F}$ by a resolution of more "flexible" sheaves. Specifically, let $\mathcal{D}$ be the sheaf on $\mathbb{S}^1$ that attaches to each open set U the ring of real-valued infinitely many times differentiable functions.[6] Note that in this case, since $\mathbb{R}$ is a vector space, $\mathcal{D}(U)$ is always a vector space as well.

Example 2.16 (Poincaré Lemma, Version 1)**.** Consider the covering of $\mathbb{S}^1 = U_1 \cup U_2$ from Example 2.5: $U_1 = \mathbb{S}^1 \setminus \{-1\}$, $U_2 = \mathbb{S}^1 \setminus \{1\}$. In the same way as before we get a complex of the form

$$0 \longrightarrow \mathcal{D}(\mathbb{S}^1) \longrightarrow \underbrace{\mathcal{D}(U_1) \times \mathcal{D}(U_2)}_{\check{C}^0(\mathfrak{U};\mathcal{D})} \longrightarrow \underbrace{\mathcal{D}(U_{1,2})}_{\check{C}^1(\mathfrak{U};\mathcal{D})} \longrightarrow 0,$$

the sheaf $\mathcal{D}$ replacing the sheaf $\mathcal{Z}$.

Suppose we have a pair of functions (f_1, f_2) in $\mathcal{D}(U_1) \times \mathcal{D}(U_2)$ that is in the kernel of the restriction to $U_{1,2}$. Then on $U_{1,2}$, these two functions agree. It follows that neither of the two functions has a non-removable singularity at 1 or -1, since f_1 has no singularity in 1, f_2 has no singularity in -1, and they agree on the overlap. In particular, there is a function $f \in \mathcal{D}(\mathbb{S}^1)$ such that each f_i is the restriction of f to U_i, as predicted by Exercise 2.9.

Now consider the cohomology at the last spot in this complex. Every section on $U_{1,2}$ is in the kernel of the zero map, so we are interested in the failure of $\mathcal{D}(U_1) \times \mathcal{D}(U_2) \longrightarrow \mathcal{D}(U_{1,2})$ to be surjective. Let f be a section on $U_{1,2}$; it has singularities in no places but 1 and -1. Let u be a C^∞-function on $U_{1,2}$ that is zero near -1 and identically equal to 1 near 1; of course $f = uf + (1-u)f$. Clearly uf can be extended to a C^∞-function on U_1 and $(1-u)f$ can be extended to a C^∞-function on U_2. It follows that f is in the image of $\mathcal{D}(U_1) \times \mathcal{D}(U_2) \longrightarrow \mathcal{D}(U_{1,2})$ and hence the sheaf $\mathcal{D}$ gives rise to a Čech complex on $\mathbb{S}^1$ with unique cohomology group $\check{H}^0(\mathbb{S}^1;\mathcal{D}) = \mathcal{D}(\mathbb{S}^1)$.

We remark that this behavior of the sheaf $\mathcal{D}$ generalizes in two ways. Firstly, if we had taken any other open cover of $\mathbb{S}^1$ we would again have ended up with a Čech complex with unique cohomology in degree zero. In terms of the limit over all open covers, this is saying that $\mathcal{D}$ has no higher Čech cohomology on the circle.

[6] Strictly speaking, this construction does not fall under the umbrella of our definition of a sheaf: we have defined a sheaf as the *continuous functions*, rather than *differentiable* ones, into a fixed target space. However, in Remark 12.31 it is outlined how to find a projection $\pi_D : D \longrightarrow X$ such that the C^∞-maps $\mathcal{D}(U)$ on U are exactly the continuous lifts from U to D. Unfortunately, this D is not a nice space. On the good side, it is a general paradigm that in the definition of sheaves one can replace "continuous" by "differentiable", or "analytic", or "C^∞" if the circumstances permit—that is, when both X and F are in the appropriate category. Each such choice will yield a sheaf on the base space, even though the sheaf space that gives rise to $\mathcal{F}$ as the continuous functions into it may be fairly obscure.

Secondly, $\check{H}^{>0}(M;\mathcal{D})$ actually vanishes for all smooth manifolds M. The key is the existence of partitions of unity for $\mathcal{D}$:

Definition 2.17. Let M be a C^∞-manifold and $\mathfrak{U} = \{U_i\}_{i\in I}$ a locally finite open cover of M (so that for all $m \in M$ only a finite number of open sets U_i contain m).

A *partition of unity subordinate to* $\mathfrak{U}$ is a collection of smooth functions $\{f_i\}_{i\in I}$, $f_i : M \longrightarrow \mathbb{R}$ such that $f_i|_{M\setminus U_i} = 0$ and $\sum f_i = 1$. (Note that this is a finite sum near every point of M.)

The partition in our case was $1 = u + (1-u)$; its existence allows for the sections of $\mathcal{D}$ on $U_{1,2}$ to be writable as sum of sections over U_1 and U_2. Partitions of unity make the sheaf $\mathcal{D}$ sufficiently fine and flexible so that $\mathcal{D}$ has no cohomology. On the other hand, their absence for the sheaf $\mathcal{Z}$ leads to nonzero cohomology.

In order to play off the sheaf $\mathcal{D}$ against the constant sheaf $\mathcal{Z}$ we shall replace $\mathbb{Z}$ by $\mathbb{R}$: let $\mathcal{R}$ be the sheaf that sends the open set $U \subseteq X$ to the continuous functions $C(U,\mathbb{R})$ where $\mathbb{R}$ is endowed with the discrete topology. The sections of this sheaf are the locally constant maps from U to $\mathbb{R}$. (Note that $C(U,\mathbb{R})$ is, when $\mathbb{R}$ has the discrete topology, identified with the maps $f : U \longrightarrow U \times \mathbb{R}$ for which $U \longrightarrow U \times \mathbb{R} \xrightarrow{nat} U$ is the identity. In particular, $\mathcal{R}$ fits definition 2.3 with sheaf space $F = \mathbb{S}^1 \times \mathbb{R}$ and projection $\pi_{\mathcal{R}} : \mathbb{S}^1 \times \mathbb{R} \xrightarrow{nat} \mathbb{S}^1$.)

The elements of $\mathcal{R}(U)$ are differentiable functions. Hence (see Definition 2.19) $\mathcal{R}$ may be viewed as a subsheaf of $\mathcal{D}$.

Example 2.18 (Poincaré Lemma, Version 2)**.** Let U be any proper open subset of $\mathbb{S}^1$. There is an exact sequence

$$0 \longrightarrow \mathcal{R}(U) \longrightarrow \underbrace{\mathcal{D}(U)}_{\text{degree } 0} \xrightarrow{\frac{d}{dt}} \underbrace{\mathcal{D}(U)}_{\text{degree } 1} \longrightarrow 0 \tag{2.18.1}$$

where the first map is the inclusion of the locally constant maps from U to $\mathbb{R}$ into the smooth maps from U to $\mathbb{R}$, while the second map is differentiation by arclength. To see that this sequence is indeed exact, note that each U is the union of disjoint open subsets that are diffeomorphic to the real line. On such open arcs, however, it is clear that a) the constants are the only functions that are annihilated by differentiation, and b) every smooth function can be integrated to a smooth function. On the other hand, as we will discuss below, the sequence is *not exact* on the right if $U = \mathbb{S}^1$!

The sequences above, with the appropriate restriction maps, splice together to commutative diagrams

$$\begin{array}{ccccc} \mathcal{R}(U) & \longrightarrow & \mathcal{D}(U) & \xrightarrow{\frac{d}{dt}} & \mathcal{D}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}(V) & \longrightarrow & \mathcal{D}(V) & \xrightarrow{\frac{d}{dt}} & \mathcal{D}(V) \end{array}$$

for any inclusion $V \subseteq U$ of open sets in $\mathbb{S}^1$. This prompts

Definition 2.19. A *morphism* φ *between sheaves* $\mathcal{F}, \mathcal{G}$ on X is a collection of maps $\{\varphi_U\}_{U\in\mathcal{T}_X}$ such that for all inclusions $V \subseteq U$ of open sets one has equality of the two maps $\rho_{V,U;\mathcal{G}} \circ \varphi_U$ and $\varphi_V \circ \rho_{V,U;\mathcal{F}}$ from $\mathcal{F}(U)$ to $\mathcal{G}(V)$. In other words, the maps given by φ "commute" with the restriction maps:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho_{V,U;\mathcal{F}}"'] & \mathcal{G}(U) \arrow[d, "\rho_{V,U;\mathcal{G}}"] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V) \end{tikzcd}$$

is a commutative diagram.

If $\mathcal{F}, \mathcal{G}$ are sheaves of Abelian groups or rings or algebras etc., a morphism of such sheaves is a collection as above with the additional property that all maps in question are morphisms in the appropriate category.

The exact sequences (2.18.1) form an *exact sequence of sheaves*. This means that the sequences (2.18.1) are exact for all sufficiently small open sets (i.e., X has a base of open sets with this property). We stress that in order to be an exact sequence of sheaves it is *not required* that the sequences are exact for *all* open sets. This "minor" detail makes all the difference between a useless and a very interesting theory of sheaves and their morphisms.

In fact, we shall now study what happens when we look at the sequence of global sections. Naturally, the only functions on all of $\mathbb{S}^1$ that are in the kernel of the differentiation map are the constants. Hence (2.18.1) is exact "on the left and in the middle" even if $U = \mathbb{S}^1$.

However, there are C^∞-functions on $\mathbb{S}^1$ which are *not a derivative*. This fairly odd seeming statement stems from the fact that one cannot compute in *local* coordinates, where of course each function *is a derivative*. Indeed, the (nonzero) constant functions on the circle are not derivatives: following the value of $g \in \mathcal{D}(\mathbb{S}^1)$ around the circle, the main theorem of integral calculus states that $g(1) = g(1) + \int_{t=0}^{2\pi} g'(\exp(\sqrt{-1}t))\,dt$, and so the average value of any derivative $f(x) = g'(x)$ on the circle must be zero. On the other hand, for any C^∞-function $f : \mathbb{S}^1 \longrightarrow \mathbb{R}$ the function $g = f - \int_{t=0}^{2\pi} f(\exp(\sqrt{-1}t))\,dt/2\pi$ is always a derivative, namely that of $\int_{\tau=0}^{t} g(\exp(\sqrt{-1}\tau))\,d\tau$. It follows that the complex of global sections

$$0 \longrightarrow \underbrace{\mathcal{D}(\mathbb{S}^1)}_{\text{degree } 0} \longrightarrow \underbrace{\mathcal{D}(\mathbb{S}^1)}_{\text{degree } 1} \longrightarrow 0$$

has cohomology in degrees 0 and 1, and both cohomology groups are isomorphic to the space of constant functions on $\mathbb{S}^1$ (although for different reasons).

The mechanism that we discussed here gives rise to a second type of cohomology that one can compute from a sheaf.

Definition 2.20. Suppose $\mathcal{F}$ is a sheaf on the space X. Let $\{\mathcal{G}^t\}_{t\in\mathbb{N}}$ be sheaves on X and suppose that for all $t \in \mathbb{N}$ there are sheaf morphisms $d^t : \mathcal{G}^t \longrightarrow \mathcal{G}^{t+1}$ with $d^{t+1} \circ d^t = 0$. Assume that the topology $\mathcal{T}_X$ has a base consisting of open sets U for which the complex

$$0 \longrightarrow \mathcal{G}^0(U) \longrightarrow \mathcal{G}^1(U) \longrightarrow \cdots \longrightarrow \mathcal{G}^n(U) \longrightarrow \cdots \tag{2.20.1}$$

has a unique cohomology group, in degree 0, isomorphic to $\mathcal{F}(X)$. Then $\mathcal{G}^\bullet$ is a *resolution* of $\mathcal{F}$.

If each $\mathcal{G}^t$ is acyclic on X then the resolution is called *acyclic* and the cohomology of the complex (2.20.1) with $U = X$ is the *derived functor* or *sheaf cohomology of* $\mathcal{F}$, denoted $H^i(X;\mathcal{F})$.

Remark 2.21. The quantity $H^i(X;\mathcal{F})$ does not, in fact, depend on the particular acyclic resolution (see Theorem 2.22).

We have verified that $\mathcal{D}$ is acyclic on $\mathbb{S}^1$ in Example 2.16. This means that $H^\bullet(\mathbb{S}^1;\mathcal{D})$ can be computed from the acyclic resolution $0 \longrightarrow \mathcal{D} \longrightarrow 0$ which implies immediately that $H^i(\mathbb{S}^1;\mathcal{D}) = 0$ for positive i.

We also computed just before Definition 2.20 that the constant sheaf $\mathcal{R}$ has an acyclic resolution of the form $0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{D} \longrightarrow 0$, so $H^i(\mathbb{S}^1;\mathcal{R})$ vanishes for $i \neq 0, 1$. Moreover, $H^0(\mathbb{S}^1;\mathcal{R}) \cong \mathbb{R} \cong H^1(\mathbb{S}^1;\mathcal{R})$. These agree with the singular cohomology groups of $\mathbb{S}^1$.

2.4. **Čech versus derived functors: a comparison.** We now compare the two approaches we have taken. Starting with the constant sheaf $\mathcal{R}$ on $\mathbb{S}^1$ we have discovered that there is the principle of Čech cohomology that associates to this sheaf the groups $\check{H}^i(\mathbb{S}^1;\mathcal{R}) = 0$ if $i > 1$, and $\check{H}^0(\mathbb{S}^1;\mathcal{R}) \cong \check{H}^1(\mathbb{S}^1;\mathcal{R}) \cong \mathbb{R}$. On the other hand, we have the calculus approach which gave us derived functor cohomology that turned out to return (at least in appearance) the same results as Čech cohomology. Let's try to investigate this similarity between Čech complexes and the calculus approach.

We take an open cover $\mathfrak{U}$ of $\mathbb{S}^1$, fine enough so that each finite intersection U_I of the open sets is an open arc. By the discussion following Theorem 2.11 then $\check{H}^\bullet(\mathfrak{U};\mathcal{R}) = \check{H}^\bullet(\mathbb{S}^1;\mathcal{R})$ and $\check{H}^\bullet(\mathfrak{U};\mathcal{D}) = \check{H}^\bullet(\mathbb{S}^1;\mathcal{D})$. Moreover, for each U_I there is a short exact sequence

$$0 \longrightarrow \mathcal{R}(U_I) \longrightarrow \mathcal{D}(U_I) \longrightarrow \mathcal{D}(U_I) \longrightarrow 0$$

and that means that there is a short exact sequence of complexes

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U};\mathcal{R}) \longrightarrow \check{C}^\bullet(\mathfrak{U};\mathcal{D}) \longrightarrow \check{C}^\bullet(\mathfrak{U};\mathcal{D}) \longrightarrow 0.$$

By Remark‘2.10 there results a long exact sequence of cohomology groups

$$0 \longrightarrow \check{H}^0(\mathbb{S}^1;\mathcal{R}) \longrightarrow \underbrace{\check{H}^0(\mathbb{S}^1;\mathcal{D})}_{=\mathcal{D}(\mathbb{S}^1)} \longrightarrow \underbrace{\check{H}^0(\mathbb{S}^1;\mathcal{D})}_{=\mathcal{D}(\mathbb{S}^1)} \longrightarrow \check{H}^1(\mathbb{S}^1;\mathcal{R}) \longrightarrow 0$$

where the zero on the right comes from the fact that by Example 2.16 we have $\check{H}^{\geq 1}(\mathbb{S}^1;\mathcal{D}) = 0$. We deduce that $\check{H}^{\geq 2}(\mathbb{S}^1;\mathcal{R})$ is zero, and that $\check{H}^0(\mathbb{S}^1;\mathcal{R})$ and $\check{H}^1(\mathbb{S}^1;\mathcal{R})$ arise naturally as kernel and cokernel of the differentiation map $\frac{d}{dt} : C^\infty(\mathbb{S}^1) \longrightarrow C^\infty(\mathbb{S}^1)$.

This exhibits an explicit isomorphism of the vector space of real constant functions with $\check{H}^0(\mathbb{S}^1;\mathcal{R}) = \mathcal{R}(\mathbb{S}^1)$ (these are the elements of $\mathcal{D}(\mathbb{S}^1)$ that have zero derivative), and another isomorphism of the vector space of real constant functions with $\check{H}^1(\mathbb{S}^1;\mathcal{R})$ (since they are a set of representatives for the quotient of $\mathcal{D}(\mathbb{S}^1)$ by its submodule of functions that have no integral). In particular, the Čech cohomology of $\mathcal{R}$ on $\mathbb{S}^1$ can be "read off" the global sections of the morphism $\mathcal{D} \longrightarrow \mathcal{D}$ given by differentiation. In Lecture 19 we shall revisit this theme of linking differential calculus with sheaves and topology. The main ideas, sketched here in one example, are the following. One can get topological information from the Čech approach since very fine open covers turn the computation effectively into a triangulation of the underlying space. On the other hand, one can take an algebraic-analytic approach and replace the given sheaf, say $\mathcal{R}$, by a suitable complex of sheaves that have themselves no higher Čech cohomology (such as $\mathcal{D}$) and consider the cohomology in the resulting complex of global sections.

The following statement enunciates what the discussion above is supposed to convey. It also places Theorem 2.13 in the context of derived functor cohomology.

Theorem 2.22 (Acyclicity Principle, [47])**.** *Let X be a topological space and $\mathcal{F}$ a sheaf of Abelian groups on X. Suppose $\mathcal{G}^\bullet$ is a finite complex of sheaves of Abelian groups and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of X. Assume that the following hypotheses hold:*

(1) For all finite index sets $\emptyset \neq I' \subseteq I$, with $U_{I'} = \bigcap_{i\in I'} U_i$,

$$0 \longrightarrow \mathcal{F}(U_{I'}) \longrightarrow \mathcal{G}^\bullet(U_{I'})$$

is an exact sequence of Abelian groups.

(2) Each sheaf $\mathcal{G}^t$ is acyclic on $\mathfrak{U}$ and on X: for all finite index sets I and for all $j > 0$,

$$\check{H}^j(X; \mathcal{G}^t) = \check{H}^j(U_{I'}; \mathcal{G}^t) = 0.$$

Then there are natural isomorphisms

$$\check{H}^j(\mathfrak{U}; \mathcal{F}) \cong H^j(\cdots \longrightarrow \mathcal{G}^{j-1}(X) \longrightarrow \mathcal{G}^j(X) \longrightarrow \mathcal{G}^{j+1}(X) \longrightarrow \cdots) \cong H^j(X; \mathcal{F})$$

between the Čech cohomology of $\mathcal{F}$ relative to the open cover $\mathfrak{U}$, the cohomology of the complex of global sections of $\mathcal{G}^\bullet$, and the derived functor cohomology of $\mathcal{F}$ on X.

The main statement, in Grothendieck language, of this theorem is the following. If Γ is an additive covariant functor between Abelian categories (such as the global section functor on the category of sheaves of Abelian groups on X to the category of Abelian groups), $\mathcal{F}$ an object in the source category and $\mathcal{G}^\bullet$ a Γ-acyclic resolution of $\mathcal{F}$ then the cohomology of $\Gamma(\mathcal{G}^\bullet)$ is the derived functor of Γ evaluated on $\mathcal{F}$. For example, the global section functor associates to $\mathcal{Z}$ the Abelian group $\mathbb{Z}^{\pi_0(X)}$, the product of copies of $\mathbb{Z}$ indexed by the components of X. We know now that its derived functors appear as the cohomology groups of a Čech complex on an open cover with small open sets. This motivates why one should want to know about such derived functors: for paracompact spaces there is a natural isomorphism between the Čech cohomology groups of $\mathcal{Z}$ on X and the singular ("="derived functor) cohomology groups $H^\bullet_{\text{sing}}(\mathbb{S}^1; \mathbb{Z})$ with $\mathbb{Z}$-coefficients. For example, $H^1(\mathbb{S}^1; \mathcal{R}) \cong \mathbb{R}$ "because" $H^1_{\text{sing}}(\mathbb{S}^1; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$

On differentiable manifolds it gets even better (as we will see in Lecture 19) where sheaves provide a link between topology of a manifold and integrals of differential forms. This is indicated by the fact that we could use the sheaf of C^∞-functions to resolve $\mathcal{R}$ on $\mathbb{S}^1$. On algebraic manifolds this will lead in Theorem 19.28 to a connection between local cohomology, differential forms, and singular cohomology.

Lecture 3. Resolutions and derived functors (GL)

This lecture is intended to be a whirlwind introduction to, or review of, resolutions and derived functors – with tunnel vision. That is, we'll give unabashed preference to topics relevant to local cohomology, and do our best to draw a straight line between the topics we cover and our final goals. At a few points along the way, we'll be able to point generally in the direction of other topics of interest, but other than that we will do our best to be single-minded.

Appendix A contains some preparatory material on injective modules and Matlis theory. In this lecture, we will cover roughly the same ground on the projective/flat side of the fence, followed by basics on projective and injective resolutions, and definitions and basic properties of derived functors.

Throughout this lecture, let us work over an unspecified commutative ring R with identity. Nearly everything said will apply equally well to noncommutative rings (and some statements need even less!).

In terms of module theory, fields are the simple objects in commutative algebra, for all their modules are *free*. The point of resolving a module is to measure its complexity against this standard.

Definition 3.1. A module F over a ring R is *free* if it has a *basis*, that is, a subset $B \subseteq F$ such that B generates F as an R-module and is linearly independent over R.

It is easy to prove that a module is free if and only if it is isomorphic to a direct sum of copies of the ring. The cardinality of a basis S is the *rank* of the free module. (To see that the rank is well-defined, we can reduce modulo a maximal ideal of R and use the corresponding result for—what else?—fields.)

In practice and computation, we are usually satisfied with free modules. Theoretically, however, the properties that concern us are *projectivity* and *flatness*. Though the definition of freeness given above is "elementary", we could also have given an equivalent definition in terms of a universal lifting property. (It's a worthwhile exercise to formulate this property, and you'll know when you've got the right one because the proof is trivial.) For projective modules, we reverse the process and work from the categorical definition to the elementary one.

Definition 3.2. An R-module P is *projective* if whenever there exist a surjective homomorphism of R-modules $f : M \longrightarrow N$ and an arbitrary homomorphism of R-modules $g : P \longrightarrow N$, there is a lifting $h : P \longrightarrow M$ so that $fh = g$. Pictorially, we have

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "g"] & \\ M \arrow[r, "f"'] & N \arrow[r] & 0 \end{tikzcd}$$

with the bottom row forming an exact sequence of R-modules.

Here is another way to word the definition which highlights our intended uses for projective modules. Let $\mathcal{F}$ be a *covariant* functor from R-modules to abelian groups. Recall that $\mathcal{F}$ is said to be *left-exact* if for each short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0\,,$$

there is a corresponding induced exact sequence

$$0 \longrightarrow \mathcal{F}(M') \longrightarrow \mathcal{F}(M) \longrightarrow \mathcal{F}(M'') .$$

If in addition the induced map $\mathcal{F}(M) \longrightarrow \mathcal{F}(M'')$ is surjective, then we say that $\mathcal{F}$ is *exact*. It is easy to show that for any R-module N, $\operatorname{Hom}_R(N, -)$ is a covariant left-exact functor.

Exercise 3.3. Prove that P is projective if and only if $\operatorname{Hom}_R(P, -)$ is exact.

Here are the first four things that you should check about projectives, plus one.

(1) Free modules are projective.
(2) A module P is projective if and only if there is a module Q such that $P \oplus Q$ is free.
(3) Arbitrary direct sums of projective modules are projective.
(4) Freeness and projectivity both localize.
(5) Over a Noetherian local ring R, all projectives are free.

Example 3.4. Despite their relatively innocuous definition, projective modules are even now a very active area of research. Here are a couple of highlights.

(1) Let R be a polynomial ring over a field $\mathbb{K}$. Then all finitely generated projective R-modules are free. This is the content of the rightly renowned Quillen-Suslin theorem [129, 152], also known as Serre's Conjecture, pre-1978 (see [6]). It's less well-known that the Quillen-Suslin theorem holds as well when $\mathbb{K}$ is a discrete valuation ring. Closely related is the Bass-Quillen Conjecture, which asserts for any regular ring R that every projective module over $R[T]$ is extended from R. Quillen and Suslin's solutions of Serre's Conjecture proceed by proving this statement when R is a regular ring of dimension at most 1. Popescu's celebrated theorem of "General Néron Desingularization" [128, 153], together with results of Lindel [101], proves Bass-Quillen for regular local rings $(R, \mathfrak{m})$ such that either R contains a field, $\operatorname{char}(R/\mathfrak{m}) \notin \mathfrak{m}^2$, or R is excellent and Henselian.
(2) In the ring $R = \mathbb{Z}[\sqrt{-5}]$, the ideal $\mathfrak{a} = (3, 2 + \sqrt{-5})$ is projective but not free as an R-module. Indeed, $\mathfrak{a}$ is not principal, so cannot be free (prove this!), while the obvious surjection $R^2 \longrightarrow \mathfrak{a}$ has a splitting given by

$$x \mapsto x \cdot \left(\frac{-1 + \sqrt{-5}}{2 + \sqrt{-5}}, \frac{2 - \sqrt{-5}}{3} \right)$$

so that $\mathfrak{a}$ is a direct summand of R^2. (This is of course directly related to the fact that R is not a UFD.)
(3) Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, the coordinate ring of the real 2-sphere. Then the homomorphism $R^3 \longrightarrow R$ defined by the row vector $\nu = [x, y, z]$ is surjective, so the kernel P satisfies $P \oplus R \cong R^3$. However, it can be shown that P is not free. Every element of R^3 gives a vector field in $\mathbb{R}^3$, with ν defining the vector field pointing straight out from the origin. An element of P thus gives a vector field that is tangent to the 2-sphere in $\mathbb{R}^3$. If P were free, a basis would define two linearly independent vector fields on the 2-sphere. But hedgehogs can't be combed!

As we noted above, the definition of projectivity amounts to saying that some usually half-exact functor is exact. You can also check easily that an R-module I

is *injective* if and only if the *contravariant* functor $\operatorname{Hom}_R(-, I)$ is exact. Our next step is to mimic these two statements for the other half-exact functor that we're familiar with.

Recall that for a given R-module M, the functor $- \otimes_R M$ is right-exact. (The proof is "elementary", in that the best way to approach it is by chasing elements.) It's clear that if we take $M = R$, then $A \otimes_R M$ and $B \otimes_R M$ are nothing but A and B again, so that in fact $- \otimes_R R$ is exact. With an eye toward defining the Tor and Ext functors below, we give this property its rightful name.

Definition 3.5. An R-module M is *flat* provided $- \otimes_R M$ is an exact functor.

We have already observed that the free module R is flat, and it is easy to check that the direct sum of a family of flat modules is flat. Thus free modules are trivially flat, and it follows immediately from the distributivity of $\otimes$ over $\oplus$ that projective modules are flat as well. In fact, it is very nearly true that the only flat modules are the projectives. Specifically,

Theorem 3.6 (Govorov and Lazard [49, 96], see [31])**.** *An R-module M is flat if and only if M is a direct limit of a directed system*[7] *of free modules. In particular, a finitely generated flat module is projective.*

Having defined the three classes of modules to which we will compare all others, let us move on to resolutions.

Definition 3.7. Let M be an R-module.

- An *injective resolution* of M is an exact sequence of the form
$$E^\bullet : 0 \longrightarrow M \longrightarrow E^0 \xrightarrow{\varphi^1} E^1 \xrightarrow{\varphi^2} E^2 \longrightarrow \cdots$$
with each E^n injective.
- A *projective resolution* of M is an exact sequence of the form
$$P_\bullet : \cdots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \longrightarrow M \longrightarrow 0$$
with each P_n projective.
- A *flat resolution* of M is an exact sequence of the form
$$F_\bullet : \cdots \longrightarrow F_2 \xrightarrow{\rho_2} F_1 \xrightarrow{\rho_1} F_0 \longrightarrow M \longrightarrow 0$$
with each F_n flat.

Remark 3.8. Each of the resolutions above exist for any R-module M; another way to say this is that the category of R-modules *has enough projectives* and *enough injectives*. (Since projectives are flat, there are of course also enough flats.) In contrast, the category of sheaves over projective space does *not* have enough projectives, as we'll see in Lecture 12!

Slightly more subtle is the question of minimality. Let us deal with injective resolutions first. We say that $E^\bullet$ as above is a *minimal* injective resolution if each E^n is the injective hull of the image of $\varphi^n : E^{n-1} \longrightarrow E^n$. As in the proof of Theorem A.21, we see that E is an injective hull for a submodule M if and only if for all $\mathfrak{p} \in \operatorname{Spec} R$, the map $\operatorname{Hom}_R(R/\mathfrak{p}, M)_\mathfrak{p} \longrightarrow \operatorname{Hom}_R(R/\mathfrak{p}, E)_\mathfrak{p}$ is an isomorphism. Therefore, $E^\bullet$ is a *minimal* injective resolution if and only if the result of applying $\operatorname{Hom}_R(R/\mathfrak{p}, -)_\mathfrak{p}$ to each homomorphism in $E^\bullet$ is the zero map.

[7]For "direct limit of a directed system" in this statement, you can substitute "union of submodules" without too much loss of sense. For more on direct limits, see Lecture **??**.

The *injective dimension* of M, $\mathrm{id}_R M$, is the minimal length of an injective resolution of M. (If no resolution of finite length exists, we say $\mathrm{id}_R M = \infty$.) We have $\mathrm{id}_R M = 0$ if and only if M is injective. Theorem A.25 shows that not only is this concept well-defined, it can be determined in terms of the Bass numbers of M. Observe that all this bounty springs directly from the structure theory of injective modules over Noetherian rings, Theorem A.21.

In contrast, the theory of minimal projective resolutions works best over local rings R, where, not coincidentally, all projective modules are free. See Lecture 8 for more in this direction. In any case, we define the *projective dimension* of M, $\mathrm{pd}_R M$, as the minimal length of a projective resolution of M, or ∞ if no finite resolution exists.

Finally, for completeness, we mention that *flat (or weak) dimension* is the minimal length of a flat resolution. For finitely generated modules over Noetherian rings, this turns out to be exactly the same as projective dimension, so we won't have much need for it.

One main tool for proving existence and uniqueness of derived functors will be the following Comparison Theorem. It comes in two dual flavors, the proof of each being immediate from the definitions.

Theorem 3.9 (Comparison Theorem)**.** *Let $f : M \longrightarrow N$ be a homomorphism of R-modules.*

(1) Assume that we have a diagram

$$\begin{array}{lccccccccccc}
J^\bullet : & 0 & \longrightarrow & M & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \cdots \\
& & & \big\downarrow{\scriptstyle f} & & & & & & & & \\
I^\bullet : & 0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots
\end{array}$$

with $J^\bullet$ exact and $I^\bullet$ a complex of injective modules. Then there is a lifting $\varphi^\bullet : J^\bullet \longrightarrow I^\bullet$ of f, and $\varphi^\bullet$ is unique up to homotopy.

(2) Assume that we have a diagram of homomorphisms of R-modules

$$\begin{array}{lccccccccccc}
P_\bullet : & \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
& & & & & & & & & \big\downarrow{\scriptstyle f} & & \\
Q_\bullet : & \cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

with $P_\bullet$ a complex of projective modules, and $Q_\bullet$ exact. Then there is a lifting $\varphi_\bullet : P_\bullet \longrightarrow Q_\bullet$ of f, and $\varphi_\bullet$ is unique up to homotopy.

Recall that two degree-zero maps of complexes $\varphi_\bullet, \psi_\bullet : (F_\bullet, \partial^F) \longrightarrow (G_\bullet, \partial^G)$ are *homotopic* (or *homotopy-equivalent*) if there is a map of degree -1, $s : F_\bullet \longrightarrow G_\bullet$, so that

$$\varphi_\bullet - \psi_\bullet = \partial^G s - s\partial^F \,.$$

Exercise 3.10. Prove that homotopic maps induce the same homomorphism in homology.

At last we define derived functors. The basic strategy is as follows: for a half-exact additive functor $\mathcal{F}$ and module M, resolve M by modules that are *acyclic* for $\mathcal{F}$, apply $\mathcal{F}$ to the complex obtained by deleting M from the resolution, and take (co)homology. The details vary according to whether $\mathcal{F}$ is left- or right-exact and co- or contravariant. We give here the one most relevant to our purposes, and leave it to the reader to formulate the others.

Definition 3.11. Let $\mathcal{F}$ be an additive, covariant, left-exact functor (for example, $\mathrm{Hom}_R(M, -)$ for some fixed R-module M). Let $M \longrightarrow E^\bullet$ be an injective resolution. Then $\mathcal{F}(E^\bullet)$ is a complex; the i^{th} right derived functor of $\mathcal{F}$ on M is defined by $R^i\mathcal{F}(M) := H^i(\mathcal{F}(E^\bullet))$.

Remark 3.12. Derived functors, both the flavor defined above and the corresponding ones for other variances and exactnesses, satisfy appropriate versions of the following easily-checked properties. Let $\mathcal{F}$ be as in Definition 3.11. Then

(1) $R^i\mathcal{F}$ is well-defined up to isomorphism (use the Comparison Theorem). More generally, any homomorphism $f : M \longrightarrow N$ gives rise to homomorphisms $R^i\mathcal{F}(f) : R^i\mathcal{F}(M) \longrightarrow R^i\mathcal{F}(N)$ for every $i \geq 0$. In particular, if $\mathcal{F}$ is multiplicative (so that $\mathcal{F}$ takes multiplication by $r \in R$ to multiplication by r), then so is $R^i\mathcal{F}$.
(2) $R^0\mathcal{F} = \mathcal{F}$, and $R^i\mathcal{F}(E) = 0$ for all $i > 0$ if E is injective.
(3) For every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules, there are *connecting homomorphisms* δ^i and a long exact sequence

$$\cdots \longrightarrow R^i\mathcal{F}(M) \longrightarrow R^i\mathcal{F}(M'') \xrightarrow{\delta^i} R^{i+1}\mathcal{F}(M') \longrightarrow R^{i+1}\mathcal{F}(M) \longrightarrow \cdots .$$

For our purposes, there are three main examples of derived functors. We define two of them here; the third will make its grand entrance in Lecture 7.

Definition 3.13. Let M and N be R-modules.

(1) The Ext *functors* $\mathrm{Ext}^i_R(M, N)$, $i \geq 0$, are the right derived functors of $\mathrm{Hom}_R(M, -)$.
(2) The Tor *functors* $\mathrm{Tor}^R_i(M, N)$, $i \geq 0$, are the left derived functors of $- \otimes_R N$.

A sharp eye might see that we've smuggled a few theorems in with this definition. There are two potential descriptions of Ext: while we chose to use the right derived functors of the left-exact covariant functor $\mathrm{Hom}_R(M, -)$, we could also have used the right derived functors of the left-exact *contravariant* functor $\mathrm{Hom}_R(-, N)$. More concretely, our definition gives the following recipe for computing Ext: Given M and N, let $I^\bullet$ be an injective resolution of N, and compute $\mathrm{Ext}^i_R(M, N) = H^i(\mathrm{Hom}_R(M, I^\bullet))$. An alternative definition would proceed by letting $P_\bullet$ be a projective resolution of M, and computing $\mathrm{Ext}^i_R(M, N) = H^i(\mathrm{Hom}_R(P_\bullet, N))$. It is a theorem (which we will not prove) that the two approaches agree. Similarly, $\mathrm{Tor}^R_i(M, N)$ can be computed either by applying $M \otimes_R -$ to a flat resolution of N, or by applying $- \otimes_R N$ to a flat resolution of M.

Here are two examples of computing Tor and Ext.[8]

[8]It's possible that these examples are too namby-pamby. Another possibility would be to replace them by the 0134 and 2-by-3 examples. I'm open to suggestions.

Example 3.14. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x, y, z]$. Denote the residue field $R/(x, y, z)R$ again by $\mathbb{K}$. We assert that

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]} R^3 \xrightarrow{\left[\begin{smallmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{smallmatrix}\right]} R^3 \xrightarrow{[\,x\ y\ z\,]} R \longrightarrow 0$$

is a (truncated) free resolution of $\mathbb{K}$. (You can check this directly and laboriously, or wait until Lectures 6 and 8.) From it we can calculate $\mathrm{Tor}_i^R(\mathbb{K}, \mathbb{K})$ and $\mathrm{Ext}_R^i(\mathbb{K}, R)$ for all $i \geq 0$. For the Tor_i, we apply $- \otimes_R \mathbb{K}$ to the resolution. Each free module R^b becomes $R^b \otimes_R \mathbb{K} \cong \mathbb{K}^b$, and each matrix is reduced modulo the ideal (x, y, z). The result is the complex

$$0 \longrightarrow \mathbb{K} \xrightarrow{0} \mathbb{K}^3 \xrightarrow{0} \mathbb{K}^3 \xrightarrow{0} \mathbb{K} \longrightarrow 0$$

with zero differentials at every step. Thus

$$\mathrm{Tor}_i^R(\mathbb{K}, \mathbb{K}) \cong \begin{cases} \mathbb{K} & \text{for } i = 0; \\ \mathbb{K}^3 & \text{for } i = 1; \\ \mathbb{K}^3 & \text{for } i = 2; \\ \mathbb{K} & \text{for } i = 3; \\ 0 & \text{for } i \geq 4. \end{cases}$$

Applying $\mathrm{Hom}_R(-, R)$ has the effect of replacing each matrix in our resolution of $\mathbb{K}$ by its transpose, which yields

$$0 \longleftarrow R \xleftarrow{[\,x\ y\ z\,]} R^3 \xleftarrow{\left[\begin{smallmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{smallmatrix}\right]} R^3 \xleftarrow{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]} R \longleftarrow 0\,.$$

Noting the striking similarity of this complex to the one we started with, we conclude that

$$\mathrm{Ext}_R^i(\mathbb{K}, R) \cong \begin{cases} \mathbb{K} & \text{if } i = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.15. Let $\mathbb{K}$ again be a field and put $R = \mathbb{K}[x, y]/(xy)$. Set $M = R/(x)$ and $N = R/(y)$. To compute Tor_i and Ext^i, let us start with a projective resolution of M. As the kernel of multiplication by x is the ideal y, and vice versa, we obtain the free resolution

$$F_\bullet : \ \cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow M \longrightarrow 0\,.$$

Computing $\mathrm{Tor}_i^R(M, N)$ requires that we truncate $F_\bullet$ and apply $- \otimes_R N$. In effect, this replaces each copy of R by $N = R/(y) \cong \mathbb{K}[x]$:

$$F_\bullet : \ \cdots \xrightarrow{x} R/(y) \xrightarrow{y} R/(y) \xrightarrow{x} R/(y)\,.$$

Since y kills $R/(y)$ while x is a nonzerodivisor on $R/(y)$, computing kernels and images quickly reveals that

$$\mathrm{Tor}_i^R(M, N) \cong \begin{cases} \mathbb{K} & \text{for } i \geq 0 \text{ even, and} \\ 0 & \text{for } i \geq 0 \text{ odd.} \end{cases}$$

Similarly, applying $\operatorname{Hom}_R(-,N)$ replaces each R by $N = R/(y)$, but this time reverses all the arrows:

$$\operatorname{Hom}_R(F_\bullet, N):\ R/(y) \xrightarrow{x} R/(y) \xrightarrow{y} R/(y) \xrightarrow{x} \cdots.$$

We see that

$$\operatorname{Ext}^i_R(M,N) \cong \begin{cases} 0 & \text{for } i \geq 0 \text{ even, and} \\ \mathbb{K} & \text{for } i \geq 0 \text{ odd.} \end{cases}$$

Finally, apply $\operatorname{Hom}_R(-,R)$ to find that

$$\operatorname{Ext}^i_R(M,R) \cong \begin{cases} N & \text{for } i = 0 \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

We finish this section with the main properties of Ext and Tor that we'll use repeatedly in the lectures to follow. They follow directly from the properties of derived functors listed above.

Theorem 3.16. *Let R be a ring and*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

a short exact sequence of R-modules. Then for any R-module N, there are three long exact sequences

$$\cdots \longrightarrow \operatorname{Ext}^i_R(N,M) \longrightarrow \operatorname{Ext}^i_R(N,M'') \longrightarrow \operatorname{Ext}^{i+1}_R(N,M') \longrightarrow \operatorname{Ext}^{i+1}_R(N,M) \longrightarrow \cdots$$

$$\cdots \longrightarrow \operatorname{Ext}^{i+1}_R(M,N) \longrightarrow \operatorname{Ext}^{i+1}_R(M',N) \longrightarrow \operatorname{Ext}^i_R(M'',N) \longrightarrow \operatorname{Ext}^i_R(N,M) \longrightarrow \cdots$$

$$\cdots \longrightarrow \operatorname{Tor}^{i+1}_R(M,N) \longrightarrow \operatorname{Tor}^{i+1}_R(M'',N) \longrightarrow \operatorname{Tor}^i_R(M',N) \longrightarrow \operatorname{Tor}^i_R(M,N) \longrightarrow \cdots$$

Theorem 3.17. *Let M be an R-module. Each of the following three sets of conditions are equivalent:*

(1a) *M is injective;*
(1b) $\operatorname{Ext}^i_R(-,M) = 0$ *for all $i \geq 1$;*
(1c) $\operatorname{Ext}^1_R(-,M) = 0$.

(2a) *M is projective;*
(2b) $\operatorname{Ext}^i_R(M,-) = 0$ *for all $i \geq 1$;*
(2c) $\operatorname{Ext}^1_R(M,-) = 0$.

(3a) *M is flat;*
(3b) $\operatorname{Tor}^i_R(-,M) = 0$ *for all $i \geq 1$;*
(3c) $\operatorname{Tor}^1_R(-,M) = 0$.

4.1. **Motivation.** One of the most fundamental results in algebraic topology, named after Herbert Seifert and Egbert Rudolf van Kampen, expresses the fundamental group of the space $X = U_a \cup U_b$ in terms of the fundamental groups of the subsets U_a, U_b, $U_c = U_a \cap U_b$ of X, and information how these three groups interact. To be precise, the three groups fit into a diagram of the following sort:

(4.0.1)
$$\begin{tikzcd} & \pi_1(U_a) \\ \pi_1(U_c) \arrow[ur, "\varphi_{c,a}"] \arrow[dr, "\varphi_{c,b}"'] & \\ & \pi_1(U_b) \end{tikzcd}$$

with $\varphi_{c,a} = \pi_1(U_c \hookrightarrow U_a)$, $\varphi_{c,b} = \pi_1(U_c \hookrightarrow U_b)$. Assume now that U_a, U_b and U_c are path-connected. According to the Seifert–van Kampen Theorem, the fundamental group of X is a group G with morphisms from $\pi_1(U_a)$, $\pi_1(U_b)$ to G such that the following diagram commutes:

(4.0.2)
$$\begin{tikzcd} & \pi_1(U_a) \arrow[dr, dashed, "\varphi_a"] & \\ \pi_1(U_c) \arrow[ur] \arrow[dr] & & G = \pi_1(X) \\ & \pi_1(U_b) \arrow[ur, dashed, "\varphi_b"'] & \end{tikzcd}$$

with $\varphi_a = \pi_1(U_a \hookrightarrow X)$, $\varphi_b = \pi_1(U_b \hookrightarrow X)$. This may be phrased as saying that every one of the three given fundamental groups maps to G in a way that is compatible with the "internal" maps (4.0.1), the commutativity of (4.0.2) being the necessary condition for the map $\pi_1(U_c) \longrightarrow G$ to be well-defined.

Of course, this property alone does not determine G in any way. The trivial group fits into (4.0.2) and so do many others. Suppose a second space Y were given, together with a continuous map $f : X \longrightarrow Y$. This would by restriction induce maps from U_a, U_b and U_c to Y, so that with $\psi_a = \pi_1(U_a \hookrightarrow Y)$ and $\psi_b = \pi_1(U_b \hookrightarrow Y)$ we have a diagram of the form

(4.0.3)
$$\begin{tikzcd} & \pi_1(U_a) \arrow[dr, dashed, "\varphi_a"'] \arrow[drr, "\psi_a"] & & \\ \pi_1(U_c) \arrow[ur] \arrow[dr] & & G \arrow[r, dotted] & H \\ & \pi_1(U_b) \arrow[ur, dashed, "\varphi_b"] \arrow[urr, "\psi_b"'] & & \end{tikzcd}$$

where $H = \pi_1(Y)$. Since π_1 is a functor, the dotted arrow is just $\pi_1(f)$) and then every triangle commutes

It turns out that the dotted morphisms can be filled in in a *unique* manner, once $\pi_1(Y)$ and the maps $\varphi_\bullet$, $\psi_\bullet$ have been chosen. Thus, the information contained in all diagrams of type (4.0.3) pins down completely both $\pi_1(X)$ as well as its behavior under morphisms.

Exercise 4.1. Suppose G' is a second group that fits into a commutative diagram

(4.1.1)
$$\begin{tikzcd} & \pi_1(U_a) \arrow[dr, dashed, "\varphi'_a"] & \\ \pi_1(U_c) \arrow[ur, "\varphi_{c,a}"] \arrow[dr, "\varphi_{c,b}"'] & & G' \\ & \pi_1(U_b) \arrow[ur, dashed, "\varphi'_b"'] & \end{tikzcd}$$

and assume that one can find a unique dotted arrow $\psi : G' \dashrightarrow H$ for every choice of a group H and all maps ψ_i' such that the following diagram commutes:

(4.1.2)
$$\begin{tikzcd}
 & \pi_1(U_a) \arrow[dr, dashed, "\varphi_a'"'] \arrow[drr, "\psi_a'"] & & \\
\pi_1(U_c) \arrow[ur, "\varphi_{c,a}"] \arrow[dr, "\varphi_{c,b}"'] & & G' \arrow[r, dotted] & H \\
 & \pi_1(U_b) \arrow[ur, dashed, "\varphi_b'"] \arrow[urr, "\psi_b'"'] & &
\end{tikzcd}$$

Then prove that there is a unique isomorphism $G \cong G'$, completing an isomorphism between (4.0.3) and (4.1.2) that is the identity for every other group in the diagrams.

This type of construction (namely, defining G to be the object that fits into a certain type of diagram with maps that are unique) is called a *universal property* with respect to the diagram type because it matters not how the corner and morphism variables are chosen. The object so defined is unique up to unique diagram preserving isomorphism. Our particular diagram is known as *the pushout*, and the group G is the *amalgamated sum* of $\pi_1(U_a)$ and $\pi_1(U_b)$, with respect to $\pi_1(U_a \cap U_b)$.

Universal properties abound in algebra and elsewhere, but often it is hard to decipher which explicit construction is hidden behind the diagram. The pushout case is at least somewhat translucent. The amalgamated sum $G_a *_{G_c} G_b$ of the groups G_a, G_b relative to the pushout $\begin{tikzcd}[row sep=tiny] & G_c \\ G_b \arrow[ur] \arrow[dr] & \\ & G_a \end{tikzcd}$ is the following group. Let G be the collection of all words w in the alphabet $G_a \sqcup G_b$. Introduce an equivalence relation on this set induced by $w \equiv w'$ if w arises from w' either

- by insertion of an identity element from some G_a or G_b, or
- by replacing two consecutive letters from the same G_i by their product in that G_i, or
- by replacing a letter $g_i \in G_i \cap \operatorname{image}(\varphi_{c,i})$ by $\varphi_{c,j}(g_c)$ where $\varphi_{c,i}(g_c) = g_i$ for $\{i, j\} = \{a, b\}$.

The quotient set is a group under composition of words, the inverse of $w = g_1 g_2 \cdots g_k$ being given by the word $g_k^{-1} \cdots g_2^{-1} g_1^{-1}$. In particular, in favorable cases one may find generators and relations for G if such are known for G_a, G_b and G_c and if the maps $\varphi_{c,i} : G_c \longrightarrow G_a, G_b$ are given explicitly.

Exercise 4.2. Define $F_1 = \mathbb{Z}$ and then inductively $F_{t+1} = F_t *_1 \mathbb{Z}$, the amalgamated sum of F_t and the integers over the one-element subgroup consisting of the identity. The group F_t is the *free group on t letters*; it consists of all words that can be formed from t distinct letters and their formal inverses.

Show that the notion of rank that one is accustomed to from free *commutative* groups makes no sense here by proving that F_2 contains F_3 as subgroup.

Exercise 4.3. Let $X = \mathbb{S}^1$ be the 1-sphere. As in Lecture 2, cover X with open sets $U_a = X \setminus \{-1\}$, $U_b = X \setminus \{1\}$. In this context, what does the Seifert–van Kampen theorem say about $\pi_1(X)$?

The *Hurewicz map* from homotopy to homology induces in degree one an isomorphism between the quotient $(\pi_1(X))^{\mathrm{Ab}}$ of $\pi_1(X)$ modulo its commutator with the singular homology $H_1(X; \mathbb{Z})$. This map is functorial and in particular commutes with the maps in the Seifert–van Kampen pushout diagram. It follows that on the level of homology there is a diagram

$$
\begin{tikzcd}
 & H_1(U_a;\mathbb{Z}) \arrow[dr, dashed] & \\
H_1(U_c;\mathbb{Z}) \arrow[ur] \arrow[dr] & & H_1(X;\mathbb{Z}) \\
 & H_1(U_b,\mathbb{Z}) \arrow[ur, dashed] &
\end{tikzcd}
\tag{4.3.1}
$$

that arises as *Abelianized* version of diagram (4.0.2). Let us investigate whether this is the pushout of the homology groups.

Suppose G is a *commutative* group and assume that there is a diagram

$$
\begin{tikzcd}
 & H_1(U_a;\mathbb{Z}) \arrow[dr, dashed] \arrow[drr] & & \\
H_1(U_c;\mathbb{Z}) \arrow[ur] \arrow[dr] & & H_1(X;\mathbb{Z}) & G = G^{\mathrm{Ab}} \\
 & H_1(U_b;\mathbb{Z}) \arrow[ur, dashed] \arrow[urr] & &
\end{tikzcd}
\tag{4.3.2}
$$

The Hurewicz functor $h : \pi_1(-) \longrightarrow H_1(-;\mathbb{Z})$ can be used to lift this to the diagram

$$
\begin{tikzcd}
 & \pi_1(U_a) \arrow[dr, dashed] \arrow[drr] & & \\
\pi_1(U_c) \arrow[ur] \arrow[dr] & & \pi_1(X) \arrow[r, dotted] & G \\
 & \pi_1(U_b) \arrow[ur, dashed] \arrow[urr] & &
\end{tikzcd}
\tag{4.3.3}
$$

Pushout properties imply that there is a unique dotted arrow. Since G is commutative, $\pi_1(X) \dashrightarrow G$ factors as $\pi_1(X) \xrightarrow{h_X} H_1(X;\mathbb{Z}) \dashrightarrow G$, and this factorization is unique (prove that!). This shows that there is a unique natural dotted arrow $H_1(X;\mathbb{Z}) \dashrightarrow G$ in diagram (4.3.2). So $H_1(X;\mathbb{Z})$ is the pushout of $H_1(U_a;\mathbb{Z})$ and $H_1(U_b;\mathbb{Z})$ relative to $H_1(U_c;\mathbb{Z})$ *in the category* Ab *of Abelian groups.*

Exercise 4.4. Let X be the figure of 8, U_a the complement of the lowest point, and U_b the complement of the highest point. Determine $\pi_1(X)$. Find $H_1(X;\mathbb{Z})$ in two ways: as Abelianization of $\pi_1(X)$, and via the Mayer–Vietoris sequence.

Exercise 4.5. Use the Exercise 4.4 to show that $H_1(X;\mathbb{Z})$ is typically *not* the pushout of the relevant homology groups in the category Groups of all groups.

Let G_a, G_b and G_c be the groups in a pushout diagram. Let H_a, H_b and H_c be their Abelianizations. Prove that the H-pushout in the category of Abelian groups is the Abelianization of the G-pushout in the category of all groups, which is also the Abelianization of the H-pushout in the category of all groups.

4.2. **Axiomatization.** In this subsection we distill the essential properties of a pushout in order to form a more general concept, that of a *direct limit.* To begin with, we shall call the underlying structure of a pushout diagram (namely, the indices a, b and c together with the information "object at c maps to both the object at a and the object at b") by its proper name: a partially ordered set, or *poset* for short. Posets will (usually) have names like I or J, and their elements are (usually) $i, i', j, \ldots$ with order relation $i \leqslant i'$, or $i < i'$ if $i = i'$ is impossible. We denote by $\left\{ \begin{tikzcd}[row sep=tiny] & a \\ c \arrow[ur] \arrow[dr] & \\ & b \end{tikzcd} \right\}$ the pushout poset where $c \leqslant a, b$.

The process of decorating the vertices of a poset I with groups (or other algebraic structures) comes with a collection of maps in the appropriate category that are compatible with the relations in the poset. That means that any resulting subdiagram that one can ask to be commutative *must be commutative*.

The following describes in fancy language what is happening. Read the poset I as a *category* whose objects are the vertices of the poset. Whenever $i \leqslant j$ then there is precisely one morphism from i to j, which, abusing notation, we write as $i \leqslant j$. The convention $(j \leqslant k) \circ (i \leqslant j) := (i \leqslant k)$ then makes a category out of the poset. Obviously for $i \leqslant i' \leqslant j$ and $i \leqslant i'' \leqslant j$ the compositions $(i' \leqslant j) \circ (i \leqslant i')$ and $(i'' \leqslant j) \circ (i \leqslant i'')$ are the same, namely $i \leqslant j$, so every subdiagram of the category I that can be commutative is in fact commutative.

Now pick a second category, $\mathcal{A}$, and a covariant functor Φ from I to $\mathcal{A}$. The effect of Φ on I is the decoration of the vertices of I with objects from $\mathcal{A}$, and the decoration of the relations in I with morphisms in $\mathcal{A}$. Functors being functors, this leads to commutative subdiagrams within $\Phi(I)$.

Definition 4.6. A *diagram* or *direct system* $(I, \mathcal{A}, \Phi)$ *over* I *in* $\mathcal{A}$ is a category $\mathcal{A}$, a poset I, and a covariant functor $\Phi : I \longrightarrow \mathcal{A}$. We often abuse language and call $\Phi(I)$, or even Φ, the direct system. Moreover, we typically write $\varphi_{i,j}$ for $\Phi(i \leqslant j)$.

The collection of all diagrams over I in $\mathcal{A}$ forms the object set of a category $\mathfrak{Dir}_I^{\mathcal{A}}$ whose morphisms are the natural transformations between the objects (which are functors).

Now that we have generalized the pushout diagram to arbitrary posets we can put down the definition of a direct limit:

Definition 4.7. Let $(I, \mathcal{A}, \Phi)$ be a diagram in $\mathcal{A}$ over I. For simplicity we shall write A_i for $\Phi(i)$ for any $i \in I$. The *direct limit* $\varinjlim_{i \in I} A_i$ of $\Phi(I)$, so it exists, is

- an object A of $\mathcal{A}$,
- together with a morphism $\varphi_i : A_i \dashrightarrow A$ for every i such that the diagram

(4.7.1)
$$\begin{tikzcd} A_i \arrow[dr, dashed, "\varphi_i"] \arrow[dd, "\varphi_{i,j}"'] & \\ & A \\ A_j \arrow[ur, dashed, "\varphi_j"'] & \end{tikzcd}$$

  commutes for all $i \leqslant j$,
- so that whenever another object A' of A is given, together with morphisms $\psi_i : A_i \longrightarrow A'$ such that every diagram

(4.7.2)
$$\begin{tikzcd} A_i \arrow[dr, "\psi_i"] \arrow[dd, "\varphi_{i,j}"'] & \\ & A' \\ A_j \arrow[ur, "\psi_j"'] & \end{tikzcd}$$

  commutes for $i \leqslant j \in I$, then there is a *unique* $\mathcal{A}$-morphism $\psi : A \cdots\to A'$ such that

(4.7.3)
$$\begin{tikzcd} A_i \arrow[dr, dashed, "\varphi_i"] \arrow[drr, "\psi_i"] \arrow[dd, "\varphi_{i,j}"'] & & \\ & A \arrow[r, dotted, "\psi"] & A' \\ A_j \arrow[ur, dashed, "\varphi_j"] \arrow[urr, "\psi_j"'] & & \end{tikzcd}$$

  commutes for every $i \leqslant j$.

The object A' in conjunction with the maps $\{\psi_i\}$ is called a *test object* for $(A, \{\varphi_i\})$. If we need to stress the underlying category in which the limit is taken, we shall write $\varinjlim_I^{\mathcal{A}} A_i$.

Exercise 4.8. Prove that if $(I, \mathcal{A}, \Phi)$ allows a direct limit then the limit is unique up to unique isomorphism. That is, if A and A' both satisfy the stipulating conditions then there is a unique $\mathcal{A}$-morphism ψ such that

(4.8.1)
$$\begin{tikzcd} A_i \arrow[rr, dashed, "\varphi_i"] \arrow[d, "\varphi_{i,j}"'] \arrow[drr, dashed, "\varphi_i'" near start] & & A \arrow[d, dotted, "\psi"] \\ A_j \arrow[rr, dashed, "\varphi_j'"'] \arrow[urr, dashed, "\varphi_j"' near start] & & A' \end{tikzcd}$$

commutes for all $i \leqslant j \in I$. Moreover, this unique morphism is an isomorphism.

Morally speaking, the direct limit $\varinjlim_I A_i$ is an object of $\mathcal{A}$, positioned "after" all A_i (so that the maps $\varphi_i : A_i \longrightarrow \varinjlim_I A_i$ exist) that is "as close" to all A_i as possible (so that the factorizations $A_i \xrightarrow{\psi_i} A'$ into $A_i \xrightarrow{\varphi_i} \varinjlim_I A_i \xrightarrow{\psi} A'$ become possible).

4.3. **Existence.** The "universal" definition of a direct limit as given above is useful in proving uniqueness. In general, universal definitions keep the actual value and structure of the object in question well hidden. To be practical we need to have a formula that expresses $\varinjlim_I A_i$ in terms of the A_i and the maps between them. As we have seen in Exercise 4.5, the construction will depend on the category $\mathcal{A}$ as well, not just the obvious variables I, Φ. We shall specifically be interested in the categories Groups, Ab, and R-mods (of all groups, Abelian groups, and R-modules for a fixed ring R, respectively). So until further notice is given $\mathcal{A}$ is one of these three categories.

If I is a poset without any comparable elements, then $\varinjlim_I A_i$ must satisfy precisely the conditions that one expects the *categorical sum* (or coproduct) $\coprod_I A_i$ of all the A_i to satisfy. Namely, $A = \varinjlim_I A_i$ permits an arrow $\varphi_i : A_i \dashrightarrow A$, $i \in I$, without commutativity conditions since I has no comparable elements with an implied arrow $\psi : A \cdots\rightarrow A'$ for *any* collection of morphisms $\psi_i : A_i \longrightarrow A'$. In particular, the existence of direct limits in $\mathcal{A}$ over every index set implies the existence of arbitrary categorical sums.

Exercise 4.9. Show that in the category of finite Abelian groups direct limits do not necessarily exist.

Name three other, fairly popular, categories with the same defect.

Consider the category Groups; we investigate the existence of direct limits. The following generalizes the idea of free groups in Exercise 4.2 to free sums. For objects $\{A_i\}_{i \in I}$ of Groups, where I has no relations, let $e_i : A_i \dashrightarrow \coprod_I A_i$ be the canonical map arising from the coproduct construction as a direct limit. Explicitly, the coproduct is the group of all words in the (disjoint union of the) elements of the participating groups subject to the rules that identify the words w and w' if w arises from w'

- by insertion of the identity element of any A_i in any position, or
- by substituting two adjacent elements of the same A_i by their product in that group.

The map e_i sends $a_i \in A_i$ to the one-letter word $a_i \in \coprod_I A_i$. This construction may be viewed as the amalgamated sum of the A_i relative to the trivial group.

Note that our category $\mathcal{A}$ has arbitrary coproducts, namely the familiar direct sums in Ab and R-mods, and the free sum in Groups.

Remark 4.10. Even for two-element index sets I that are incomparable, the categorical coproduct and product may not be the same. One is used to such equality from Ab and its reasonable subcategories, but Exercise 4.5 shows that coproduct and product of two copies of $\mathbb{Z}$ are not the same when taken in Groups (F_2 is not Abelian).

If now I is any poset and $(I, \mathcal{A} = \text{Groups}, \Phi)$ is a direct system then the direct limit $\varinjlim_I A_i$ is given as the quotient

$$A = \frac{\coprod_{i\in I} A_i}{(\{e_i(a_i) * (\varphi_{i,j} \circ e_i(a_i))^{-1}\}_{i \leqslant j \in I, a_i \in A_i})}$$

where the denominator denotes the smallest normal subgroup containing the specified elements. Explicitly, we identify words if they are already the same word in the coproduct, or if w contains two adjacent letters a_i and a_j such that $i \leqslant j$, $a_j * \varphi_{i,j}(a_i) = 1_{A_j}$, and w' arises from w by cancellation of a_i and a_j.

Exercise 4.11. Prove that A is $\varinjlim_I A_i$ by verifying the correctness of the universal property.

Let us now consider direct limits in Ab. We switch from multiplicative notation in Groups to additive notation in Ab.

Exercise 4.12. Show that the direct limit of Abelian groups within the category of Abelian groups is the Abelianization of the direct limit when taken in Groups. (Compare the discussion in Exercise 4.5.)

It follows by simplifying the formula for Groups that if $\{A_i\}_{i\in I}$ are in Ab then in Ab,

$$\varinjlim_I A_i = \frac{\bigoplus_{i\in I} A_i}{\{(\ldots, 0, \underbrace{a_i}_{\in A_i}, 0, \ldots, 0, \underbrace{-\varphi_{i,j}(a_i)}_{\in A_j}, 0, \ldots)\}} \tag{4.12.1}$$

where $\bigoplus$ denotes the coproduct (direct sum) of Abelian groups.

Finally, suppose that $\mathcal{A} = R$-mods. The direct sum of the A_i permits a (well-known) componentwise R-action and with this action the coproduct in the category Ab becomes not only an R-module but indeed the coproduct in R-mods.

Consider now the forgetful functor

$$F_{\text{Ab},R\text{-mods}} : R\text{-mods} \longrightarrow \text{Ab},$$

and let $(I, R\text{-mods}, \Phi)$ be a direct system in R-mods with $\Phi(i) = A_i$. Since for any R-mods-morphism φ the Ab-morphism $F_{\text{Ab},R\text{-mods}}(\varphi)$ has the same effect on elements as φ itself, $\varinjlim_I^{\text{Ab}} A_i$ is an R-module and the natural maps $\varphi_i : A_i \longrightarrow \varinjlim_I^{\text{Ab}} A_i$ are R-linear. Next, if $\{\psi_i : A_i \longrightarrow A'\}$ is a test object for A in R-mods then there is a unique map of Abelian groups $\psi : \varinjlim_I^{\mathrm{Ab}} A_i \dashrightarrow A'$ such that $\psi \circ \varphi_i = \psi_i$. Each element $a \in \varinjlim_I^{\mathrm{Ab}} A_i$ is a finite sum $\sum \varphi_i(a_i)$ and hence

$$\begin{aligned} \psi(ra) = \psi(r \sum \varphi_i(a_i)) &= \sum \psi(\varphi_i(ra_i)) \\ &= \sum \psi_i(ra_i) \\ &= r \sum \psi_i(a_i) \\ &= r \sum \psi \circ \varphi_i(a_i) = r\psi(a). \end{aligned}$$

Therefore ψ is R-linear. It follows that

$$\varinjlim_I^{\mathrm{Ab}} F_{\mathrm{Ab},R\text{-mods}}(A_i) = F_{\mathrm{Ab},R\text{-mods}} \varinjlim_I^{R\text{-mods}} (A_i).$$

4.4. **Limits of diagrams.** We now consider the question of existence of direct limits in categories of diagrams over $\mathcal{A}$. To start, assume that $\mathcal{A}$ is a category that has direct limits for every I-diagram in $\mathcal{A}$, for *one particular fixed* I.

Suppose that $\eta'' : \Phi \longrightarrow \Phi''$ is a morphism of I-diagrams. Hence for $i \leqslant j \in I$ we get a commutative diagram

$$\begin{tikzcd} A_i \arrow[rr, "\eta''_i"] \arrow[d, "\varphi_{i,j}"] \arrow[drr, dashed, "\varphi_i"] & & A''_i \arrow[d, "\varphi''_{i,j}"] \arrow[drr, dashed, "\varphi''_i"] & & \\ A_j \arrow[rr, "\eta''_j"] \arrow[r, dashed, "\varphi_j"] & \varinjlim_I A_i \arrow[rrr, dotted, "\underrightarrow{\eta}''"'] & A''_j \arrow[rr, dashed, "\varphi''_j"] & & \varinjlim_I A''_i \end{tikzcd}$$

Here as usual the dashed arrows are the universal morphisms from the direct limit definition for Φ, Φ''. The commutativity of the diagrams gives compatible maps $A_i \longrightarrow A''_i \longrightarrow \varinjlim_I A''_i$ which induce through the universal property of $\varinjlim_I A_i$ the dotted arrow. Given two morphisms of three I-diagrams such as in $A' \xrightarrow{\eta'} A \xrightarrow{\eta''} A''$ then the uniqueness part of the universal property ensures that the induced diagram

(4.12.2)
$$\begin{tikzcd} \varinjlim_I A'_i \arrow[d, dotted, "\underrightarrow{\eta'}"'] \arrow[dr, dotted, "\underrightarrow{(\eta'' \circ \eta')}"] & \\ \varinjlim_I A_i \arrow[r, dotted, "\underrightarrow{\eta''}"'] & \varinjlim_i A''_i \end{tikzcd}$$

commutes. We conclude that

Proposition 4.13. *The process of taking direct limits is a functor*

$$\varinjlim_I(-) \ : \ \mathfrak{Dir}_I^{\mathcal{A}} \longrightarrow \mathcal{A}.$$

Let J be a second poset and consider the category $\mathfrak{Dir}_J^{\mathcal{A}}$ of J-diagrams in $\mathcal{A}$. If we consider objects labeled by $I \times J$, we use upper indices for I and lower indices for J. Let Φ be an I-diagram in $\mathfrak{Dir}_J^{\mathcal{A}}$. So Φ is a collection of J-diagrams $\Phi^i_\bullet$, one for every $i \in I$, with inner maps $\varphi^i_{j,j'} : A^i_j \longrightarrow A^i_{j'}$ specific to the J-diagram $\Phi^i_\bullet$, and a collection of outer maps (morphisms of J-diagrams) $\varphi_j^{i,i'} : A^i_j \longrightarrow A^{i'}_j$ that commute with the inner maps. In other words, $\Phi = \{A^i_j\}_{i\in I, j\in J}$ is a diagram in $\mathcal{A}$ over $I \times J$ with maps $\varphi_j^{i,i'} : A^i_j \longrightarrow A^{i'}_j$ and $\varphi^i_{j,j'} : A^i_j \longrightarrow A^i_{j'}$, where the latter type of map is considered "inner" (specific to a particular J-diagram) and the former maps are "outer", giving a morphism of J-diagrams in $\mathcal{A}$.

Proposition 4.14. *If $\mathcal{A}$ has direct limits for all I-diagrams then $\mathfrak{Dir}_J^{\mathcal{A}}$ has direct limits for all I-diagrams.*

Proof. Let $\Phi = \{\Phi^i_\bullet\}_I = \{A^i_j\}_{i\in I, j\in J}$ be an I-diagram in $\mathfrak{Dir}_J^{\mathcal{A}}$. We picture this with the J-morphisms in vertical and the I-morphisms in horizontal direction. What we are looking for is another J-diagram $\{A_j\}$ (a "column") with universal morphisms $\varphi^i_\bullet : A^i_\bullet \dashrightarrow A_\bullet$ of J-diagrams that fit into

(4.14.1)
$$\begin{tikzcd}
 & & & & A'_j \arrow[dd] \\
A^i_j \arrow[r] \arrow[urrrr] \arrow[drr, dashed] \arrow[dd] & A^{i'}_j \arrow[urrr] \arrow[dr, dashed] \arrow[dd] & & & \\
 & & A_j \arrow[uurr, dotted] \arrow[dd] & & A'_{j'} \\
A^i_{j'} \arrow[r] \arrow[urrrr] \arrow[drr, dashed] & A^{i'}_{j'} \arrow[urrr] \arrow[dr, dashed] & & & \\
 & & A_{j'} \arrow[uurr, dotted] & &
\end{tikzcd}$$

for every test object $\{A'_j\}$ in $\mathfrak{Dir}_J^{\mathcal{A}}$.

By considering test J-diagrams $\Psi = \{A'_j\}$ with $A'_j = 0$ unless $j = j_0$, one sees that the only possibility for A_j is $\varinjlim_I A^i_j$ and $\varphi^i_j : A^i_j \longrightarrow A_j$ must for every fixed j be the universal morphism that belongs to this direct limit.

It follows that for every test object $\Psi' = \{A'_j\}$, the row with index j in (4.14.1) is always the direct limit diagram for the system $\{A^i_j\}_{i\in I}$ with test object A'_j. In particular, there is no choice regarding either the object $\varinjlim_I \Phi^i$, nor the universal morphism $\Phi \longrightarrow \varinjlim_I \Phi^i$. Moreover, the inner (vertical) map $A_j \longrightarrow A_{j'}$ of $\varinjlim_I \Phi^i$ must arise as the universal map from the test object $\{A_{j'}, A^i_j \longrightarrow A^i_{j'} \longrightarrow A_{j'}\}$ for the direct limit $\varinjlim_I A^i_j$. The uniqueness of the universal map implies that

(4.14.2)
$$\begin{tikzcd}
A_j \arrow[d] \arrow[dr] & \\
A_{j'} \arrow[r] & A_{j''}
\end{tikzcd}$$

commutes, so by Proposition 4.13 we obtain actually a J-diagram in $\mathcal{A}$.

The only remaining question is whether the various universal maps of horizontal I-diagrams in $\mathcal{A}$ combine to a universal map of I-diagrams in $\mathfrak{Dir}_J^{\mathcal{A}}$. In other words, we need to know whether the diagram

$$
\begin{tikzcd}
A_j \arrow[r, dotted] \arrow[d] & A'_j \arrow[d] \\
A_{j'} \arrow[r, dotted] & A'_{j'}
\end{tikzcd}
\tag{4.14.3}
$$

commutes for all $j \leqslant j' \in J$. Note that

$$
\begin{tikzcd}
A^i_j \arrow[r] \arrow[d] & A'_j \arrow[d] \\
A^i_{j'} \arrow[r] & A'_{j'}
\end{tikzcd}
$$

commutes because the horizontal maps are maps of J-diagrams. This gives two test objects for $A_j = \varinjlim_I A^i_j$, namely via the maps $A^i_j \longrightarrow A'_j \longrightarrow A'_{j'}$ (which factors as $A^i_j \longrightarrow A_j \longrightarrow A'_j \longrightarrow A'_{j'}$ by universality of A_j), and $A^i_j \longrightarrow A^i_{j'} \longrightarrow A'_{j'}$ (which factors as $A^i_j \longrightarrow A^i_{j'} \longrightarrow A_{j'} \longrightarrow A'_{j'}$ by universality of $A_{j'}$). Since the morphism $A_j \longrightarrow A_{j'}$ is the universal map to $\{A^i_j \longrightarrow A^i_{j'} \longrightarrow A_{j'}\}_{i\in I}$, the maps $A^i_j \longrightarrow A^i_{j'} \longrightarrow A_{j'} \longrightarrow A'_{j'}$ and $A^i_j \longrightarrow A_j \longrightarrow A_{j'} \longrightarrow A'_{j'}$ are identical. Hence the two test objects are the same. The uniqueness of factorizations through a direct limit now implies that diagram 4.14.3 commutes. □

Remark 4.15. Suppose that the notion of a complex makes sense in $\mathcal{A}$. Suppose also that $\mathcal{A}$ has direct limits for all I-diagrams. Note that a complex in $\mathcal{A}$ may be viewed as a diagram in $\mathcal{A}$ over $\mathbb{Z}$. It follows that if Φ is a direct system of complexes in $\mathcal{A}$, indexed by I, then there is a limit complex.

4.5. **Exactness.** From now on we shall assume that the category $\mathcal{A}$ is Abelian, and that it has direct limits over any poset. In particular, of the three standard categories we look at, this eliminates Groups (since, amongst other things, morphisms must form Abelian groups now), but it allows for $\mathcal{A} = \mathrm{Ab}$ and $\mathcal{A} = R$-mods. The reason is that we wish to talk about homology in $\mathcal{A}$.

Fix a poset I and an Abelian category $\mathcal{A}$. Recall (Definition 4.6) that the objects of $\mathfrak{Dir}^{\mathcal{A}}_I$ are the direct systems $(I, \mathcal{A}, \Phi)$, and the morphisms $\mathrm{Hom}_{\mathfrak{Dir}^{\mathcal{A}}_I}((I, \mathcal{A}, \Phi), (I, \mathcal{A}, \Phi'))$ are exactly the natural transformations from Φ to Φ'. Such a transformation η "is" a prism with base I where all corners are decorated with objects of $\mathcal{A}$, all edges are decorated with morphisms of $\mathcal{A}$, and all diagrams commute.

One may now use Abelianness of $\mathcal{A}$ and make the following

Definition 4.16. Let Φ', Φ, Φ'' be three objects of $\mathfrak{Dir}^{\mathcal{A}}_I$, and suppose $\eta' : \Phi' \longrightarrow \Phi$ and $\eta'' : \Phi \longrightarrow \Phi''$ are morphisms of diagrams.

Then η' gives rise to a new I-diagram, the *kernel* of η', obtained by $(\ker(\eta'))_i = \ker(\eta'_i) \subseteq A'_i$ and whose interior maps are simply the restrictions of those of A'. We say that η' is a *monomorphism* if for every $i \in I$ the induced $\mathcal{A}$-morphism $\eta'_i : A'_i \longrightarrow A_i$ is a monomorphism, so that $\ker(\eta')$ is the zero diagram.

Moreover, η'' gives rise to a new I-diagram, the *cokernel* of η'', obtained by $(\mathrm{coker}(\eta''))_i = \mathrm{coker}(\eta''_i)$ and whose interior maps are simply those induced by η''

via the universal cokernel property. Note that $\operatorname{coker}(\eta'')$ is well-defined since η'' is a natural transformation. We say that η'' is an *epimorphism* if for every $i \in I$ the induced $\mathcal{A}$-morphism $\eta''_i : A_i \longrightarrow A''_i$ is an epimorphism, so that $\operatorname{coker}(\eta'')$ is the zero diagram.

We call $\Phi' \longrightarrow \Phi \longrightarrow \Phi''$ is a *short exact sequence of I-diagrams*, if η' is a monomorphism, η'' is an epimorphism, $\eta'' \circ \eta' = 0$, η' induces an isomorphism between Φ' and $\ker(\eta'')$ and η'' induces an isomorphism between $\operatorname{coker}(\eta')$ and Φ''. In down to earth language, every $0 \longrightarrow A'_i \longrightarrow A_i \longrightarrow A''_i \longrightarrow 0$ is an exact sequence in $\mathcal{A}$.

With this terminology, $\mathfrak{Dir}_I^{\mathcal{A}}$ becomes an Abelian category: a morphism with zero kernel and zero cokernel is an isomorphism.

Remark 4.17. Let Fil be the category of filtered Abelian groups. Objects of Fil look exactly like certain diagrams, and their morphisms are defined in the same way. Nonetheless, Fil is not Abelian, since any refinement of a filtration of a fixed group G gives a monomorphism that is also an epimorphism but not an isomorphism. The trouble is that the cokernel is defined “incorrectly” in Fil. Namely, if $F_i \subseteq F'_i$ are refining filtrations exhausting G then the cokernel filtration to the inclusion is $\frac{F'_i+G}{F_i+G} = 0$. If F, F' are considered as direct systems, however, the cokernel is zero if and only if $F_i = F'_i$ throughout.

Exercise 4.18. Verify that $\varinjlim_I(-)$ is an *additive* functor. That is, the map $\operatorname{Hom}_{\mathfrak{Dir}_I^{\mathcal{A}}}(\Phi', \Phi) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(\varinjlim_I \Phi', \varinjlim_I \Phi)$ sending η to $\underrightarrow{\eta}$ is a group homomorphism that is natural with respect to composition of morphisms.

We now come to the central question of this lecture, the exactness of the direct limit functor. Let $0 \longrightarrow \Phi' \longrightarrow \Phi \longrightarrow \Phi'' \longrightarrow 0$ be an exact sequence of I-diagrams. (I.e., for any $i \in I$ the sequence $0 \longrightarrow A'_i \longrightarrow A_i \longrightarrow A''_i \longrightarrow 0$ is exact.)

When is $0 \longrightarrow \varinjlim_I \Phi' \longrightarrow \varinjlim_I \Phi \longrightarrow \varinjlim_I \Phi'' \longrightarrow 0$ exact?

Let us record an example showing that this is not a trivial issue.

Example 4.19. Let $I = \left\{\begin{tikzcd}[row sep=tiny] & a \\ c \arrow[ur] \arrow[dr] & \\ & b\end{tikzcd}\right\}$ be the pushout poset and consider the following sequence of I-diagrams in Ab:

$$0 \longrightarrow \underbrace{\begin{pmatrix}\begin{tikzcd}[row sep=tiny] & \mathbb{Z} \\ 0 \arrow[ur] \arrow[dr] & \\ & \mathbb{Z}\end{tikzcd}\end{pmatrix}}_{\Phi'} \xrightarrow{\eta'} \underbrace{\begin{pmatrix}\begin{tikzcd}[row sep=tiny] & \mathbb{Z} \\ \mathbb{Z} \arrow[ur] \arrow[dr] & \\ & \mathbb{Z}\end{tikzcd}\end{pmatrix}}_{\Phi} \xrightarrow{\eta''} \underbrace{\begin{pmatrix}\begin{tikzcd}[row sep=tiny] & 0 \\ \mathbb{Z} \arrow[ur] \arrow[dr] & \\ & 0\end{tikzcd}\end{pmatrix}}_{\Phi''} \longrightarrow 0$$

Here, the outermost two systems have zero objects at every vertex, and the maps η'_i, η''_i are isomorphisms whenever that is conceivable, and the zero morphism otherwise.

From Exercise 4.12 and the subsequent remarks it is clear that

$$\begin{aligned}
\varinjlim_I A'_i &\cong \mathbb{Z} \oplus \mathbb{Z}, \\
\varinjlim_I A_i &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/\left((1,-1,0),(1,0,-1)\right) \cong \mathbb{Z}, \text{and} \\
\varinjlim_I A''_i &\cong \mathbb{Z}/\mathbb{Z} \cong 0.
\end{aligned}$$

So on the level of limits, we have a complex

$$0 \longrightarrow \underbrace{\varinjlim_I A_i'}_{\cong \mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\eta'} \underbrace{\varinjlim_I A_i}_{\cong \mathbb{Z}} \xrightarrow{\eta''} \underbrace{\varinjlim_I A_i''}_{\cong 0} \longrightarrow 0.$$

Clearly, $\underrightarrow{\eta'}$ is not injective and hence $\varinjlim_I(-)$ is not an exact functor.

Definition 4.20. For a poset I and an object A of $\mathcal{A}$, let A_I be the *constant diagram* along I with coefficient A. That is, $(A_I)_i = A$ for all $i \in I$, and for all $i \leqslant j \in I$ the morphism $A_I(i \leqslant j)$ is the identity on A.

Note that $(-)_I$ is a functor from $\mathcal{A}$ to $\mathfrak{Dir}_I^{\mathcal{A}}$, which is obviously exact.

Exercise 4.21. Let A be an object of $\mathcal{A}$ and let Φ be an I-diagram in $\mathcal{A}$. Prove that a morphism of diagrams $\Phi \longrightarrow A_I$ determines a unique morphism $\varinjlim_I \Phi \longrightarrow A$. Prove conversely that a morphism $\varinjlim_I \Phi \longrightarrow A$ determines a unique morphism $\Phi \longrightarrow A_I$, and that these two procedures are inverses of each other.

The exercise establishes that there is an identification

$$\mathrm{Hom}_{\mathfrak{Dir}_I^{\mathcal{A}}}(\Phi, A_I) = \mathrm{Hom}_{\mathcal{A}}(\varinjlim_I \Phi, A).$$

Definition 4.22. Suppose one has two categories $\mathcal{B}, \mathcal{C}$ and two functors $F : \mathcal{B} \longrightarrow \mathcal{C}$, $G : \mathcal{C} \longrightarrow \mathcal{B}$. Assume that for any objects B, C one has an identification

$$\mathrm{Hom}_{\mathcal{B}}(B, G(C)) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{\mathcal{C}}(F(B), C) \tag{4.22.1}$$

as functors from $\mathcal{B} \times \mathcal{C}$ to the category Sets.

Then (F, G) form an *adjoint pair*, F is the *left adjoint* of G and G is the *right adjoint* of F.

The condition of the definition that the Hom-sets be identified as functors is stronger than what Exercise 4.21 asserts. Namely, the functor property requires that the identification is functorial under morphisms in both arguments.

Example 4.23. The functor $(-) \otimes_R (B)$ is left adjoint to $\mathrm{Hom}_R(B, -)$ for any R-module B.

Exercise 4.24. Extending Exercise 4.21, verify that $\varinjlim_I(-)$ and $(-)_I$ are adjoints.

The benefit of adjoint pairs is the following.

Proposition 4.25. *Let (F, G) be an adjoint pair of the categories $\mathcal{B}$ and $\mathcal{C}$.*

- *Then there are* adjunction morphisms *(i.e., natural transformations) of functors*
$$\sigma : F \circ G \longrightarrow \mathrm{Id}_{\mathcal{C}}, \qquad \tau : \mathrm{Id}_{\mathcal{B}} \longrightarrow G \circ F$$
such that $F \xrightarrow{F\circ\tau} FGF \xrightarrow{\sigma\circ F} F$ and $G \xrightarrow{\tau\circ G} GFG \xrightarrow{G\circ\sigma} F$ are the identity transformations on F and G respectively.
- *The adjunction morphisms determine the* Hom*-set equivalence (4.22.1).*
- *If in addition $\mathcal{B}, \mathcal{C}$ are Abelian, and F, G are additive then F is right exact and G is left exact.*

Proof. The first part follows by substituting $C = F(B)$ and $B = G(C)$ into the Equation (4.22.1); the equational properties follow from plugging in.

The second part comes about as follows. Let $u : B \longrightarrow G(C)$ be a morphism in $\mathcal{B}$. Its corresponding morphism $\alpha(u) : F(B) \longrightarrow C$ is obtained as $F(B) \xrightarrow{F(u)} F(G(C)) \xrightarrow{\sigma(C)} C$; the converse direction is similar.

For the last part note first that the second part says that $\alpha(u : B \longrightarrow G(Y)) = \sigma_C \circ F(u)$ and $\beta(v : F(B) \longrightarrow C) = G(v) \circ \tau_B$. Then let $v : C \longrightarrow C'$ be a monomorphism and let B be the kernel of $G(v)$, so $0 \longrightarrow B \xrightarrow{u} G(C) \xrightarrow{G(v)} G(C')$ is exact. Apply F and consider the resulting diagram:

$$\begin{tikzcd}
F(B) \arrow[r, "F(u)"] & FG(C) \arrow[r, "FG(v)"] \arrow[d, "\sigma_C"'] & FG(C') \arrow[d, "\sigma_{C'}"'] \\
& C \arrow[r, "v"'] & C'
\end{tikzcd}$$

Since $G(v) \circ u = 0$, the top row is zero. By naturality of σ, the diagram commutes, so $F(B) \longrightarrow FG(C) \longrightarrow C \longrightarrow C'$ is zero. Since v is a monomorphism, $\sigma_C \circ F(u) = 0$, but by the first sentence of the paragraph above the diagram that means that $u = 0$. In the light of u being a monomorphism this implies that $B = 0$, so $G(v)$ is injective and hence G is left exact. $\square$

Exercise 4.26. Complete the proof of the third part of the proposition by showing that F is right exact.

We have established that the direct limit functor from $\mathfrak{Dir}_I^{\mathcal{A}}$ to $\mathcal{A}$ is always right exact, but not exact on the pushout diagram. The right exactness (and general derived functor patterns) indicates that if a short exact sequence in $\mathfrak{Dir}_I^{\mathcal{A}}$ fails to provide a short exact sequence in $\mathcal{A}$ then in some way the fault is with the rightmost I-diagram, since it is its first left derived functor that keeps the limit sequence inexact. Instead of discussing acyclic object over arbitrary I we shall identify a property of I that forces all I-diagrams to be $\varinjlim(-)$-acyclic. Justification of this approach comes from the fact that, apart from the pushout, the most important posets appear to fall into that category.

Definition 4.27. The poset I is said to be *confluent* if for all $i, j \in I$ there is $k \in I$ such that $i \leqslant k$ and $j \leqslant k$.

The traditional word is "directed". However, as a graph *every* poset is directed, and in the presence of "direct" limits, "directed" occasionally causes confusion.

Example 4.28.
(1) The most important poset in topology is the pushout diagram and that is clearly not confluent.
(2) The most important poset in algebraic geometry is the poset of open sets $\{U_i\}$ in a scheme X that contain a chosen point P. The order relation is reverse inclusion. Since the intersection of two open sets containing P is another such set, U is confluent.
(3) The two most important direct systems in algebra are the natural numbers $\mathbb{N}$ (obviously confluent), and the collection of all finitely generated

R-modules $\{M_i\}$ of some fixed R-module M. The order relation is containment, and since the sum of two finitely generated submodules is another such module, $\{M_i\}$ is confluent.

At this point we restrict ourselves to categories whose objects have *elements*. For example, subcategories of Ab are permitted.

Recall that $\varphi : A_i \dashrightarrow \varinjlim_I A_i$ are the universal limit maps. In general, the elements of $\varinjlim_I A_i$ are sums $\sum_{i \in I} \varphi_i(a_i)$ where all but finitely many a_i are zero. As it turns out, the crucial property of confluent posets I is that every element $\{a_i\}_{a_i \in A_i}$ of $\varinjlim_I A_i$ can be represented as the coset of (φ_i of) a *single* element a_i. This follows immediately from the formula (4.12.1), and confluence. This allows for the following vanishing test in confluent diagrams.

Lemma 4.29. *Let I be a confluent poset, suppose that $\mathcal{A}$ has direct limits for all I-diagrams, and let Φ be an I-diagram in $\mathcal{A}$. Then $a = \{a_i\} \in \bigoplus_I \Phi_i$ maps to $0 = \overline{a} =\in \varinjlim_I(\Phi)$ provided that there exists $j \in I$ such that*

(1) $i \leqslant j$ for every i with $a_i \neq 0$;
(2) $\sum_{a_i \neq 0} \varphi_{i,j}(a_i) = 0$.

In other words, in direct limits over confluent diagrams, "vanishing elements vanish at finite time".

Proof. Let $\overline{a}$ be as stipulated. Since only a finite number of the a_i is nonzero, confluence assures the existence of j as stipulated by (1). We are therefore reduced to proving the lemma in the case of one non-vanishing component a_j.

If $\overline{a} = 0$ then a_j is a finite sum of expressions of the type $a_{i'} - \varphi_{i',i''}(a_{i'})$:

$$a_j = \sum_{i'} (a_{i'} - \varphi_{i',i''}(a_{i'})) \in \bigoplus_{i \in I} A_i.$$

Let j' be an index dominating j and all i' with $a_{i'} \neq 0$. Then

$$\begin{aligned} \varphi_{j,j'}(a_j) &= (-a_j) - \varphi_{j,j'}(-a_j) + \sum_{i'} (a_{i'} - \varphi_{i',j'}(a_{i'})) \\ &\qquad + \sum_{i'} [\varphi_{i'',j'}(\varphi_{i',i''}(a_{i'})) - \varphi_{i',i''}(a_{i'})] \in \bigoplus_{i \in I} A_i \end{aligned}$$

(since $\varphi_{i',j'} = \varphi_{i'',j'}\varphi_{i',i''}$), and $\overline{\varphi_{j,j'}(a_j)} = \overline{a}$. We rewrite this as

$$\varphi_{j,j'}(a_j) = \sum_{i'' \leqslant j'} (a_{i''} - \varphi_{i'',j'}(a_{i''})) \in \bigoplus_{i \in I} A_i.$$

We may assume that each i'' appears at most once in the sum. This being a statement about a *direct* sum, components left and right must be equal. In particular, $a_{i''} = 0$ for every $i'' \neq j'$. The equation then simplifies to $\varphi_{j,j'}(a_j) = a_{j'} - \varphi_{j',j'}(a_{j'}) = 0$. $\square$

Note that the converse of the lemma is obvious.

Theorem 4.30. *If I is confluent then $\varinjlim_I(-) : \mathfrak{Dir}_I^{\mathcal{A}} \longrightarrow \mathcal{A}$ is an exact functor.*

Proof. Let $0 \longrightarrow \Phi' \xrightarrow{\eta'} \Phi \xrightarrow{\eta''} \Phi'' \longrightarrow 0$ be a short exact sequence of I-diagrams. In view of Proposition 4.25 we only need to show exactness of $0 \longrightarrow \varinjlim_I A_i' \longrightarrow \varinjlim_I A_i \longrightarrow \varinjlim_I A_i'' \longrightarrow 0$ on the left.

Let $\overline{a'} = \overline{\{a'_i\}} \in \varinjlim_I A'_i$ be in the kernel of $\underrightarrow{\eta'}$. By Lemma 4.29 applied to $\underrightarrow{\eta'}(\overline{a'})$ there is $j \in I$ with $j \geq i$ for all $a'_i \neq 0$ such that

$$\eta'_j(\sum_i \varphi'_{i,j}(a'_i)) = \sum_i \varphi_{i,j}(\eta'_i(a'_i)) = 0.$$

But η'_j is a monomorphism according to the hypothesis, so $\sum_i \varphi_{i,j}(a'_i) = 0$. Since this sum represents $\overline{a'}$, the theorem follows. $\square$

Exercise 4.31. Let $0 \longrightarrow \Phi^{(0)} \longrightarrow \Phi^{(1)} \longrightarrow \ldots \longrightarrow \Phi^{(n)} \longrightarrow 0$ be a complex of diagrams in $\mathcal{A}$ over the confluent poset I.

- If the complex $\Phi^{(\bullet)}$ is exact in $\mathfrak{Dir}_I^{\mathcal{A}}$ then show that the resulting sequence of direct limits is exact too.
- In general, let $\Psi^{(k)} = \{H_i^{(k)}\}$ be the k-th homology I-diagram

$$H_i^{(k)} = \frac{\ker\left(A_i^{(k)} \longrightarrow A_i^{(k+1)}\right)}{\text{image}\left(A_i^{(k-1)} \longrightarrow A_i^{(k)}\right)}.$$

  Show that $\varinjlim_I \Psi^{(k)}$ is naturally isomorphic to the homology of

$$\varinjlim_I A_i^{(k-1)} \longrightarrow \varinjlim_I A_i^{(k)} \longrightarrow \varinjlim_I A_i^{(k+1)}.$$

One says that *direct limits on confluent index sets commute with homology.* We reiterate the point made in Example 4.19 that arbitrary diagrams will fail Theorem 4.30.

4.6. **Direct limits and left adjoints.** It is somewhat remarkable that the following theorem uses no such condition as additivity or Abelianness. It is a pure example of what is called "abstract nonsense".

Theorem 4.32. *Let $F : \mathcal{B} \longrightarrow \mathcal{C}$ be a covariant functor that is left adjoint to $G : \mathcal{C} \longrightarrow \mathcal{B}$. Let I be a poset and assume that $\mathcal{B}$ has direct limits for all I-diagrams.*

Let $\{B_i\}_{i\in I}$ be an I-diagram in $\mathcal{B}$ and $\varinjlim_I B_i$ its direct limit. The I-diagram $\{F(B_i)\}_{i\in I}$ in $\mathcal{C}$ has direct limit $F(\varinjlim_I B_i)$.

Proof. Note first that, since F is a functor, $F(B)$ is indeed an I-diagram over I in $\mathcal{C}$.

We shall show that $F(\varinjlim_I B_i)$ satisfies the appropriate universal condition. Certainly there are compatible maps from $F(B_i)$ to $F(\varinjlim_I B_i)$; they arise by applying F to the universal maps $\varphi_i : B_i \dashrightarrow \varinjlim_I B_i$.

Let C be an object of C and suppose we are given compatible morphisms $\psi_i : F(B_i) \longrightarrow C$:

(4.32.1)
$$\begin{tikzcd} F(B_i) \arrow[d, "F(\varphi_{i,j})"'] \arrow[dr, "\psi_i"] & \\ F(B_j) \arrow[r, "\psi_j"'] & C \end{tikzcd}$$

We need to show that there is a unique morphism $\psi : F(\varinjlim_I B_i) \dashrightarrow C$ through which all ψ_i factor.

From the given data, by applying the functor G, we get compatible maps

$$\begin{tikzcd} G(F(B_i)) \arrow[d, "G(F(\varphi_{i,j}))"'] \arrow[dr, "G(\psi_i)"] & \\ G(F(B_j)) \arrow[r, "G(\psi_j)"'] & G(C) \end{tikzcd}$$

and the adjunction morphisms $\tau_{B_i} : B_i \longrightarrow G(F(B_i))$ produce a commutative diagram

$$\begin{tikzcd} B_i \arrow[d, "\varphi_{i,j}"'] \arrow[dr, "G(\psi_i)\circ\tau_{B_i}"] & \\ B_j \arrow[r, "G(\psi_j)\circ\tau_{B_j}"'] & G(C) \end{tikzcd}$$

for all $i \leqslant j \in I$.

The universal property of $\varinjlim_I B_i$ yields a unique morphism $\varinjlim_I B_i \dashrightarrow G(C)$ compatible with these triangles. In view of the adjoint correspondence of morphism sets (4.22.1) there is a unique morphism $F(\varinjlim_I B_i) \dashrightarrow C$ compatible with the maps in (4.32.1). This means that $F(\varinjlim_I B_i)$ is the direct limit of $\{F(B_i)\}$. $\square$

Let us mention some instances where this theorem can be applied.

Example 4.33. (1) Let $\mathcal{A}$ be Ab or R-mods and take F to be the tensor product (over $\mathbb{Z}$, resp. R) with the object A' of $\mathcal{A}$. As is well-known (and easy to check), $(-)_{\mathcal{A}} \otimes A'$ is left adjoint to $\mathrm{Hom}_{\mathcal{A}}(A', -)$. It follows from the theorem that if $\{A_i\}$ is an I-diagram in $\mathcal{A}$ then there is a natural identification

$$\varinjlim_I (A_i \otimes_{\mathcal{A}} A') = \varinjlim_I (A_i) \otimes_{\mathcal{A}} A',$$

i.e., "direct limits commute with tensor products".

(2) Let I, J be posets and let $\mathcal{A}$ be a category such that $\mathcal{A}$ has direct limits for both all I-diagrams and all J-diagrams.

By Proposition 4.14, $\mathfrak{Dir}_I^{\mathcal{A}}$ has direct limits over J-diagrams, and $\mathfrak{Dir}_J^{\mathcal{A}}$ has direct limits over I-diagrams. Recall from the discussion before Proposition 4.14 that there is an identification of $\mathfrak{Dir}_J^{\mathfrak{Dir}_I^{\mathcal{A}}}$ and $\mathfrak{Dir}_I^{\mathfrak{Dir}_J^{\mathcal{A}}}$ with the diagrams over the poset $I \times J$ (with $(i,j) \leqslant (i',j')$ if and only if $i \leqslant i'$ and $j \leqslant j'$) in $\mathcal{A}$. We continue the custom to decorate objects in this category with upper indices in I and lower indices in J. Let $\{A_j^i\}_{i\in I, j\in j}$ be one such diagram. Then since $\varinjlim_\bullet(-)$ is left adjoint to the functor $(-)_\bullet$ (from Definition 4.20) of constant $\bullet$-diagrams for both $\bullet = I, J$, we have

$$\varinjlim_J \left(\varinjlim_I (A_j^i) \right) = \varinjlim_I \left(\varinjlim_J (A_j^i) \right).$$

Here, $\{A_j^i\}$ is on the left interpreted as a collection of I-diagrams in $\mathcal{A}$ which happen to form a J-diagram in $\mathfrak{Dir}_I^{\mathcal{A}}$. The direct limits of these I-diagrams then forms a J-diagram in $\mathcal{A}$, and it is the direct limit of this system that forms the left hand side of the equation. On the right, the positions of I and J are inverted. One says that "direct limits commute with each other".

Indeed, both constructions can be seen to satisfy the universal property of $\varinjlim_{I\times J} A_i^j$ by virtue of the right adjoints $(-)_I$ and $(-)_J$.

(3) As a particular case of the previous property, direct limits commute with arbitrary direct sums because a direct sum is a direct limit over its index set with no nontrivial relations.

Exercise 4.34. Despite the fact that "direct limits commute with tensor products", show by example that if $\Phi = \{A_i\}$ and $\Psi = \{A'_i\}$ are two I-diagrams in $\mathcal{A} = \mathrm{Ab}$ or R-mods then

$$\varinjlim_I (A_i \otimes_{\mathcal{A}} A'_i) = \varinjlim_I (A_i) \otimes_{\mathcal{A}} \varinjlim_I (A'_i)$$

may be false. (Hint: look at the pushout.)

Note that there is always a natural map from the left to the right hand side of the equation, and the right hand term always agrees with $\varinjlim_I \varinjlim_J (A_i \otimes_{\mathcal{A}} A'_j)$. Prove that if I is confluent then this natural map is an isomorphism.

4.7. **Assorted remarks and exercises on the pushout.** We identify here the derived functors of $\varinjlim_I(-)$ on the pushout poset.

Exercise 4.35. We consider I-diagrams in $\mathcal{A} = \mathrm{Ab}$ where I is arbitrary. For $i \in I$ and an Abelian group M, let $M_{[i]}$ be the I-diagram given by $(M_{[i]})_j = M$ whenever $i \leqslant j$ and $(M_{[i]})_j = 0$ otherwise. Let $\varphi_{j,j'}$ be the identity on M for all $i \leqslant j \leqslant j'$ and the zero map otherwise. For example, on the pushout poset we have

$$M_{[c]} = \left\{ \begin{array}{ccc} & \nearrow & M \\ M & & \\ & \searrow & M \end{array} \right\}, \quad M_{\overline{a}} = \left\{ \begin{array}{ccc} & \nearrow & M \\ 0 & & \\ & \searrow & 0 \end{array} \right\}, \quad M_{\overline{b}} = \left\{ \begin{array}{ccc} & \nearrow & 0 \\ 0 & & \\ & \searrow & M \end{array} \right\}$$

where every morphism is the obvious embedding and M is any object of $\mathcal{A}$. One might call $M_{[i]}$ the constant system on the closure of i, i.e., on the sub-poset given by all j with $i \leqslant j$.

Show that each $M_{[i]}$ is $\varinjlim_I(-)$-acyclic. In other words, show that a short exact sequence of I-diagrams with $M_{[i]}$ as the rightmost system gives a short exact sequence of direct limits. Consider first the case when M is free. Then note that $\varinjlim_I M_{[i]}$ is canonically identified with $(M_{[i]})_i \cong M$. Deal with arbitrary M by resolving M in Ab and noting that this gives you a resolution of $M_{[i]}$ in $\mathfrak{Dir}_I^{\mathrm{Ab}}$. Now use the above identification to show that on the level of limits the sequence stays a resolution, implying acyclicity.

From now on, let I be the pushout poset $\left\{ \begin{array}{ccc} & \nearrow & a \\ c & & \\ & \searrow & b \end{array} \right\}$.

Exercise 4.36. Show that every I-diagram in Ab has a two-step left resolution by I-diagrams of the type $M'_{[a]} \oplus M''_{[b]} \oplus M'''_{[c]}$ where all occurring groups are *free*. Show that for any two such resolutions there is a third such resolution dominating it. In consequence, show that every short exact sequence $0 \longrightarrow \{A_i\} \longrightarrow \{B_i\} \longrightarrow \{C_i\} \longrightarrow 0$ of I-diagrams in Ab gives rise to an exact sequence

$$0 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow \varinjlim_I A_i \longrightarrow \varinjlim_I B_i \longrightarrow \varinjlim_I C_i \longrightarrow 0$$

where A_1, B_1, C_1 are objects of Ab that do *not depend on the resolution*, up to isomorphism. Prove that A_1, B_1, C_1 are natural with respect to maps of I-diagrams: a map of short exact sequences of I-diagrams in Ab results in a morphism of the corresponding six-term exact sequences.

We may hence consider these groups as left derived functors

$$\varinjlim_{I,1} : \mathfrak{Dir}_I^{\mathrm{Ab}} \longrightarrow \mathrm{Ab}$$

of $\varinjlim_I(-)$, and we know that the higher left derived functors $\varinjlim_{k,I}(-)$ are zero for $k > 1$.

Compute $\varinjlim_{I,1}\left(\begin{tikzcd}[row sep=small] & 0 \\ \mathbb{Z} \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}\right)$.

Exercise 4.37. Contemplate the case of $\mathcal{A} = R$-mods in the previous exercise.

Exercise 4.38. Find a poset such that its second left derived functors $\varinjlim_{I,2}^{\mathrm{Ab}}(-)$ is not the zero functor. (Hint: think of the pushout as a baby case of a dualized Čech complex for a 2-set cover of $\mathbb{S}^1$. Find a space with nonzero second homology, cover it with open sets, and "dualize" the corresponding Čech complex.)

Generalize to show that for arbitrary n there are posets for which $\varinjlim_{I,n}^{\mathrm{Ab}}(-)$ is nonzero.

For the last three exercises in this lecture, if M is a pushout diagram in Ab, let K_M be the group of elements in M_c that are sent to zero under both $\varphi_{c,a}$ and $\varphi_{c,b}$.

Exercise 4.39. Show that if K_M is zero then M is $\varinjlim_I$-acyclic. (Hint: Consider the natural inclusion of the constant diagram $(M_c)_{[c]}$ into M and let Q be the cokernel. Prove that one may assume $M = Q$.)

Exercise 4.40. Using result and strategy of the preceding exercise, show that $K_M \cong \varinjlim_{I,1}(M)$.

Exercise 4.41. If $0 \longrightarrow C'_\bullet \longrightarrow C_\bullet \longrightarrow C''_\bullet \longrightarrow 0$ is a short exact sequence of complexes (i.e., of diagrams over the poset $\mathbb{Z}$ with its natural order) then there is long exact homology sequence. Use this to give a quick and dirty proof for $K_M \cong \varinjlim_{I,1}(M)$.

Remark 4.42. The discussion of direct limits can be "turned upside down" by considering universal properties in the opposite category $\mathcal{A}^{opp}$. This leads to *inverse limits*, which have been slighted here. This is mainly because direct limits appear in the more basic theory of local cohomology. It should be noted that inverse limits (and their derived functors on R-mods) also play an important role in local cohomology theory. This role can be summed up in "Greenlees–May duality", which interweaves the completion functor along the variety of an ideal with the local cohomology functor with supports in the ideal. Greenlees–May duality is an upscale version of local duality which is discussed for example in Theorem 11.32. The reader is referred to [102] for details.

LECTURE 5. DIMENSION THEORY, GRÖBNER BASES (AL)

This lecture develops the dimension theory for algebras and modules that was already brought to light in the first lecture in the series. Starting with a filtration on an algebra (ring, module) and working with the graded associated algebra (ring, module), we define dimension and multiplicity by introducing the Hilbert polynomial. For a classical treatment of the dimension theory see [4].

Later we show how to compute this polynomial using the apparatus of Gröbner bases, which is a cornerstone of the computational commutative algebra. A crash-course on Gröbner bases begins with the introduction of orders on the monomials of the polynomial ring $R = \mathbb{K}[x_1, ..., x_n]$. A special attention is paid to the weight monomial orders: a weight defines a filtration on R.

The concept of the initial ideal leads to flat deformations that enable computation of such things as Hilbert polynomials. Besides, initial ideals carry a certain artistic value: these are monomial ideals and, therefore, correspond to staircases in the integer lattice $\mathbb{Z}^n_{\geq 0}$ that can be drawn in $\mathbb{R}^n_{\geq 0}$ for $n \leq 3$.

Several equivalent definitions of a Gröbner basis follow; we put special emphasis on Buchberger's algorithm for computing Gröbner bases. For an introduction to the very basics of computational commutative algebra we recommend [26].

This lecture will be accompanied by examples of computations in the computer algebra system Macaulay 2 [**?**], several applications of which are described in the recent book [32]. Besides, there are several other computer systems specialized on the computational commutative algebra, in particular, CoCoA [22] and Singular [51]. The developers of the latter published a good textbook on commutative algebra [52] showing an abundance of applications of their software.

5.1. **Graded algebras, filtrations, associated graded algebra.** The first mentioning of graded rings and modules was made in Lecture 1. Let us recall what has been said replacing the word *ring* with the word *algebra*.

Definition 5.1. An algebra R over a field $\mathbb{K}$ is called $\mathbb{N}$-graded if $R = \oplus_{i\in\mathbb{N}} R_i$ as a $\mathbb{K}$-vector space, and $R_iR_j \subset R_{i+j}$ for all $i, j \in \mathbb{N}$.

Usually, *graded* means $\mathbb{N}$-*graded* when talking about algebras and their ideals.

Example 5.2. The ring of polynomials $R = \mathbb{K}[x_1, ..., x_n]$ can be graded by the degree, i.e. $R_i = \{f \in R \mid \deg(f) = i\}$.

Exercise 5.3. A two-sided ideal I of a graded algebra $R = \oplus_{i\in\mathbb{N}} R_i$ is called *graded* if $I = \oplus_{i\in\mathbb{N}} I_i$, where $I_i = R_i \cap I$.

Show that if I is a graded ideal of a graded $\mathbb{K}$-algebra R then the quotient $\mathbb{K}$-algebra R/I is graded.

Exercise 5.4. Let $R = \oplus_{i\in\mathbb{N}} R_i$ and $S = \oplus_{i\in\mathbb{N}} S_i$ be graded $\mathbb{K}$-algebras. A $\mathbb{K}$-algebra homomorphism $\varphi : R \to S$ is called graded if $\varphi(R_i) \subset S_i$ for all $i \in \mathbb{N}$.

Show that $\ker\varphi$ is a *graded* (two-sided) ideal of R.

A graded algebra admits a special kind of modules — the graded ones. In case of modules, we allow negative graded parts: a left R-module M is called $\mathbb{Z}$-*graded* if $M = \oplus_{i\in\mathbb{Z}} M_i$ as a $\mathbb{K}$-vector space , where M_i are called the *homogeneous components of degree* i, and $R_i \cdot M_j \subset M_{i+j}$ for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. A submodule N of M is a graded submodule if $N = \oplus_{i\in\mathbb{Z}} N_i$, where $N_i = M_i \cap N$. A grading on M also induces the grading on M/N in this case.

Example 5.5. For a graded $\mathbb{K}$-algebra $R = \oplus_{i\in\mathbb{N}} R_i$, let $R^{\otimes m}$ denote the m-th tensor power of R over $\mathbb{K}$. This module can be graded naturally the following way:

$$(R^m)_i = \sum_{j_1+\ldots+j_m=i} (R_{j_1} \otimes_{\mathbb{K}} \ldots \otimes_{\mathbb{K}} R_{j_m}).$$

Another approach to constructing a graded algebra is via filtrations: a family $\mathcal{F} = \{F_i\}_{i\in\mathbb{N}}$ of $\mathbb{K}$-vector spaces is called a *filtration* on a $\mathbb{K}$-algebra R if

(1) $F_0 \subset F_1 \subset F_2 \subset \ldots \subset R$,
(2) $R = \bigcup_{i\in\mathbb{N}} F_i$,
(3) $F_i \cdot F_j \subset F_{i+j}$.

We also set a convention that $F_i = 0$ for $i < 0$.

If an algebra comes with a filtration then we say that it is *filtered*.

Example 5.6. Every graded algebra is filtered. Take $R = \oplus_{i\in\mathbb{N}} R_i$, then $F_i = \sum_{0\le j\le i} R_i$ for $i \in \mathbb{N}$ form a filtration of R.

For an example of a filtered algebra that is not graded, we refer you to Lecture 17 where the algebra of differential operators is introduced; it possesses only a trivial grading, however, it can be nontrivially filtered. That is why the next concept, the associated graded algebra, is useful: although it may not coincide with the original filtered algebra, many of its properties pass on to its parent.

Definition 5.7. Let R be a $\mathbb{K}$-algebra equipped with filtration $\mathcal{F} = \{F_i\}_{i\in\mathbb{N}}$, then the vector space $\mathrm{gr}_{\mathcal{G}}\, R = \oplus_{i\in\mathbb{N}} (F_i/F_{i-1})$ with the naturally defined multiplication is called the *graded algebra associated to filtration* $\mathcal{F}$ of R, or simply the *associated graded algebra* in case the filtration is implied.

Example 5.8. The associated graded algebra (with respect to the natural filtration as in Example 5.6) of a graded algebra is isomorphic to its parent.

For an algebra R and its filtration $\mathcal{F}$, we define a filtration on a left R-module M to be a collection of $\mathbb{K}$-vector spaces $\mathcal{G} = \{G_i\}_{i\in\mathbb{Z}}$ that satisfies

(1) $\ldots \subset G_{-1} \subset G_0 \subset G_1 \subset \ldots \subset M$,
(2) $M = \bigcup_{i\in\mathbb{Z}} G_i$,
(3) $F_i \cdot G_j \subset F_{i+j}$.

Similarly, the graded module of M associated to the filtration $\mathcal{G}$ is set to be

$$\mathrm{gr}_{\mathcal{G}}\, M = \oplus_{i\in\mathbb{Z}} (G_i/G_{i-1}).$$

Example 5.9. Let $R = \mathbb{K}[x]$ be the ring of univariate polynomials filtered by the degree. For a polynomial $f \in R$ of degree d, the localized ring $R_f = \mathbb{K}[x, f^{-1}]$ is a module over R.

Consider the filtration of R_f by the degree:

$$G_i = \{\frac{g}{f^m} \in R_f \mid g \in R,\ \deg g - dm \le i\}.$$

The i-th component of the associated graded module $\mathrm{gr}_{\mathcal{G}}\, R_f$ consists of rational functions of the form g/f^m, where g is homogeneous and $\deg g - dm = i$. Note that these are not finitely generated.

If a filtration on a module is such that the components of the associated graded module are finitely generated then we call it a *good filtration*.

Exercise 5.10. Show that for every finitely generated module over a polynomial ring there exists a good filtration.

Theorem 5.11. *Let $M = \oplus_i M_i$ be a finitely generated module over the polynomial ring $R = \mathbb{K}[x_1, ..., x_n]$. Then there exist a polynomial $\chi(t) \in \mathbb{Q}[t]$ such that*

$$\sum_{i=0}^{s} \dim_{\mathbb{K}}(M_i) = \chi(s), \text{ for } s >> 0.$$

The polynomial $\chi(t)$ is called the *Hilbert polynomial* of M and is denoted by $\chi(M,t)$.

Exercise 5.12. Show that the Hilbert polynomial of $R = \mathbb{K}[x_1, ..., x_n]$ as a graded module over itself is $\chi(R,t) = \binom{t+n}{n} = (t+n)(t+n-1)...(t+1)$.

Prove that $\chi(M,t)$ is additive in the second argument. What is $\chi(R^m, t)$?

For the polynomial ring $R = \mathbb{K}[x_1, ..., x_n]$ and a module M equipped with a good filtration $\mathcal{G}$, let us define the Hilbert polynomial by $\chi(M, \mathcal{G}, t) = \chi(\operatorname{gr}_{\mathcal{G}} M, t)$.

Definition 5.13. Let $\chi(M, \mathcal{G}, t) = a_d t^d +$ lower degree terms. Then d is called the *dimension* of M and the *multiplicity* of M is defined as $d! a_d$.

Both numbers are nonnegative integers and do not depend on the choice of the good filtration.

5.2. **Hilbert polynomial, function, series.** Let us establish the connection between Hilbert polynomial, Hilbert function and Hilbert-Poincaré series.

Hilbert function is $h : \mathbb{Z} \to \mathbb{Z}$ is defined $h(M,i) = \dim(\operatorname{gr} M)_i$. For $i >> 0$ it may be expressed through the Hilbert polynomial: $h(M,i) = \chi(M,i) - \chi(M, i-1)$.

The Hilbert-Poincaré series, in turn, is $P(M,t) = \sum_{i \in \mathbb{Z}} h(M,i) t^i$. To complete the loop we express the Hilbert polynomial via the Hilbert-Poincaré series (e.g. see [52, Definition 5.1.4]). Let

$$P(M,t) = \frac{G(t)}{(1-t)^s},\ G(t) = \sum_{i=0}^{d} g_i t^i \in \mathbb{Z}[t],\ G(1) \neq 0.$$

Then the Hilbert polynomial is

$$\chi(M,n) = \sum_{i=0}^{d} g_i \begin{pmatrix} s-1+n-i \\ s-1 \end{pmatrix} \in \mathbb{Q}[n].$$

It follows that it is sufficient to find one of the three in order to know the dimension and the multiplicity of a module. Let us see if we can do the computations for a quotient of the ring.

Lemma 5.14. *If f is a homogeneous polynomial of degree d and $I \subset R$ a homogeneous ideal, then*

$$P(R/I, t) = P(R/\langle I, f \rangle, t) + t^d P(R/(I : \langle f \rangle), t).$$

Proof. See the proof of [52, Lemma 5.2.2]. □

Exercise 5.15. Let $I \subset R$ be a monomial ideal with the minimal set of generators $\{x^{\alpha_1}, ..., x^{\alpha_s}\}$. Using Lemma 5.14 construct an algorithm for computing $P(R/I, t)$.

5.3. **Monomial orders.** Let $R = \mathbb{K}[x_1, ..., x_n]$ be the ring of polynomials in n variables with coefficients in the field $\mathbb{K}$.

Definition 5.16. A *monomial order* $\geq$ is an order on the monoid $\{x^\alpha \mid \alpha \in \mathbb{Z}^n_{\geq 0}\}$ that respects multiplication: i.e. $x^\alpha > x^\beta \Rightarrow x^{\alpha+\gamma} > x^{\beta+\gamma}$ for any γ.

If in addition $\geq$

- is a total order, i.e. for any α and β one of the three holds: $x^\alpha > x^\beta$, $x^\beta > x^\alpha$, or $\beta = \alpha$;
- is a well-order: any set of monomials has the minimal element,

then it is called a *term order* .

Example 5.17. One of the standard examples of a term order is the *lexicographic order* a.k.a. dictionary order, since the monomials are ordered as words in a dictionary:

$$x^\alpha \geq x^\beta \Leftrightarrow \text{the first nonzero entry in } \alpha - \beta \text{ is positive.}$$

One way to construct monomial orders is via integer weights; to a *weight vector* $\omega \in \mathbb{Z}^n$ we may associate the *weight order* $\leq_\omega$ by setting

$$x^\alpha \geq_\omega x^\beta \Leftrightarrow \langle \alpha, \omega \rangle \geq \langle \beta, \omega \rangle, \tag{5.17.1}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product. Note that $x^\alpha =_\omega x^\beta$ is possible for $\alpha \neq \beta$; also, if $\omega_i < 0$ then $1 > x_i > x_i^2 > ...$ is an infinite descending sequence. Therefore, such a weight order is a non-term order. However, if a positive weight is used and the order is refined (i.e. the ties are broken) with, for example, the lexicographic order, then such an order becomes a term order.

For a fixed weight vector ω, let $\mathcal{F}_\omega = \{F_{\omega,m}\}_{m\in\mathbb{Z}}$, where

$$F_{\omega,m} = \{ \sum_{\langle \alpha,\omega \rangle \leq m} c_\alpha x^\alpha \}.$$

Exercise 5.18. Prove that $\mathcal{F}_\omega$ is a $\mathbb{Z}$-filtration on R. Show that if $\omega \in \mathbb{Z}^n_{>0}$ then $\mathcal{F}_\omega$ is a good $\mathbb{N}$-filtration.

We shall write $\operatorname{gr}_\omega R$ for the associated graded ring of R with respect to the filtration $\mathcal{F}$ above.

In presence of a term order $\geq$, a polynomial

$$f(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \in R \tag{5.18.1}$$

has the following attributes associated with it:

$$\begin{array}{rcccl} \mathrm{Supp}(f) & = & \text{the support} & = & \{x^\alpha \mid c_\alpha \neq 0\}, \\ \mathrm{le}(f) & = & \text{the leading exponent} & = & \max_{\geq} \mathrm{Supp}(f), \\ \mathrm{lm}(f) & = & \text{the leading monomial} & = & x^{\mathrm{le}(f)}, \\ \mathrm{lc}(f) & = & \text{the leading coefficient} & = & c_{\mathrm{le}(f)}, \\ \mathrm{lt}(f) & = & \text{the leading term} & = & \mathrm{lc}(f)\,\mathrm{lm}(f). \end{array}$$

For an arbitrary monomial order $\geq_\omega$, we use $\mathrm{in}(f) = \mathrm{in}_\omega(f)$ to denote the *initial form* of f, which is the sum of the terms $c_\alpha x^\alpha$ that are maximal. If $\geq_\omega$ happens to be a term order, then $\mathrm{in} = \mathrm{lt}$.

For a weight order $\geq_\omega$, the initial form is viewed as an element of $\operatorname{gr}_\omega R$.

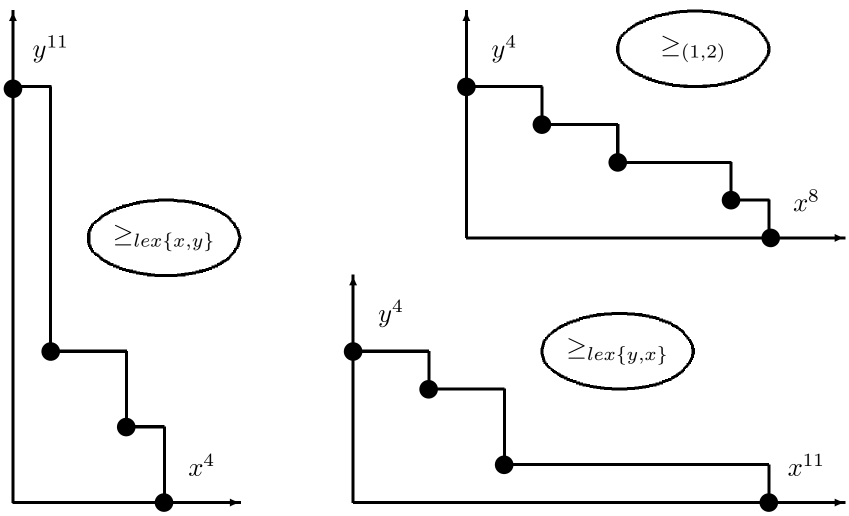


FIGURE 2. Staircases: $\mathrm{in}(I)$ for $I = \langle x^4 + x^2y^3, y^4 - y^2x^3\rangle$ and three different orders.

Definition 5.19. Let I be an ideal of R equipped with an order $\geq$. We define the *initial ideal* of I with respect to $\geq$ as $\mathrm{in}(I) = \mathrm{in}_{\geq}(I) = \langle \mathrm{in}(f) | f \in I\rangle$.

For a weight order ω we write $\mathrm{in}_\omega(I)$ for the initial ideal $\mathrm{in}_{\geq_\omega}(I)$ and think $\mathrm{in}_\omega(I) \subset \mathrm{gr}_\omega R$.

Remark 5.20. The initial $\mathrm{in}_\omega(I)$ is equal to $\mathrm{gr}_\omega I \subset \mathrm{gr}_\omega R$, the associated graded of $I \subset R$ equipped with the filtration $\mathcal{F}_\omega$.

In case of a term order, the initial ideal is guaranteed to be monomial. The monomials that do not belong to the initial ideal $\mathrm{in}(I)$ are called *standard monomials*. A monomial ideal can be represented by a staircase in the nonnegative integer lattice $\mathbb{Z}^n_{\geq 0}$.

Example 5.21. Consider the ideal $I = \langle x^4 + x^2y^3, y^4 - y^2x^3\rangle \subset R = \mathbb{K}[x, y]$. We will consider three monomial orders: $\geq_{lex\{x,y\}}$, $\geq_{lex\{y,x\}}$, and $\geq_{(1,2)}$ — the first two are lexicographic orders with the different order of indeterminates, the last one is the weight order with $\omega = (1, 2)$. Notice that $\geq_{(1,2)}$ is a non-term order, however, the initial ideal $\mathrm{in}_{(1,2)}(I)$ still turns out to be monomial.

Figure 5.21 displays the staircases corresponding to these three orders.

5.4. **Flat deformations.** Fundamental to the computation of the Hilbert-Poincaré series is the following statement [52, Theorem 5.2.6].

Theorem 5.22. *Fix a term order on the polynomial ring R. Then for every homogeneous ideal $I \subset R$,*

$$P(R/I, t) = P(R/\mathrm{in}(I), t).$$

Note that together with the algorithm developed in Exercise 5.15, this theorem enables the computation of $P(R/I,t)$ for any homogeneous ideal I.

Corollary 5.23. *With the notation of the previous theorem,*

$$\begin{aligned} \dim(R/I) &= \dim(R/\operatorname{in}(I)), \\ \dim_{\mathbb{K}}(R/I) &= \dim_{\mathbb{K}}(R/\operatorname{in}(I)). \end{aligned}$$

Proof. All the numbers associated to the Hilbert-Poincaré series are invariant under flat deformations. $\square$

Exercise 5.24. Count the number of monomials under the staircase (standard monomials) in Figure 5.21. Explain why the result does not depend on the order.

5.5. **Gröbner bases.** Is there a recipe for getting initial ideals in Example 5.21? The question is answered positively via Gröbner bases and Buchberger's algorithm.

Definition 5.25. Fix a monomial order on R. A subset G of an ideal $I \subset R$ is a *Gröbner basis* iff $\operatorname{in}(I) = \langle \operatorname{in}(g) \mid g \in G \rangle$.

Example 5.26. For the ideal I and the orders of the Example 5.21, the corresponding Gröbner bases can be computed using `gb` command of *Macaulay2*.

```
i1 : Rxy = QQ[x,y, MonomialOrder=>Lex];

i2 : I = ideal(x^4+x^2*y^3,y^4-y^2*x^3);

o2 : Ideal of Rxy

i3 : gb I

o3 = | y11+y6 xy4-y8 x3y2-y4 x4+x2y3 |

o3 : GroebnerBasis

i4 : Ryx = QQ[y,x, MonomialOrder=>Lex];

i5 : gb substitute(I,Ryx)

o5 = | x11-x6 yx4+x8 y3x2+x4 y4-y2x3 |

o5 : GroebnerBasis

i6 : R12 = QQ[x,y, Weights=>{1,2}];

i7 : gb substitute(I,R12)

o7 = | y4-x3y2 x2y3+x4 x4y2-x7 x8+x4y x7y+x6 |

o7 : GroebnerBasis
```

Now we can read off generators for the corresponding initial ideals, which happen to be the corners of the staircases in Figure 5.21.

5.6. **Buchberger's algorithm.** Let us discuss the basic idea behind the algorithm that makes the `gb` command of Example 5.26 work.

Fix a term order. The following algorithm reduces a polynomial with respect to a finite subset of R:

Algorithm 5.27 (Normal form)**.** $\overline{f}^G = NF(f,G)$

Require: $f \in R$, $G \subset R$.

Ensure: $\overline{f}^G \in R$, such that $\overline{f}^G = f \mod \langle G \rangle$ and $\mathrm{lm}(\overline{f}^G)$ is not divisible by $\mathrm{lm}(g)$ for all $g \in G$.

$f' := f$
while f' is divisible by $\mathrm{lm}(g)$ for some $g \in G$ **do**

$$f' := f' - \frac{\mathrm{lt}(f')}{\mathrm{lt}(g)} \cdot g$$

end while
$\overline{f}^G := f'$

Note that the choice of a reductor g in the **while** loop is not deterministic. However, we shall assume that a strategy for picking reductors is fixed and the outcome of the algorithm is, therefore, uniquely determined.

Remark 5.28. Algorithm 5.27 terminates since the leading monomials of f' at every step form a decreasing sequence, which has to terminate since a term order is, in particular, a well-order.

There is another way to define a Gröbner basis using the concept of the s-polynomial of two polynomials f, g:

$$S(f,g) = \frac{x^\alpha}{\mathrm{lt}(f)} f - \frac{x^\alpha}{\mathrm{lt}(g)} g, \text{ where } x^\alpha = \mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g)).$$

Theorem 5.29 (Buchberger Criterion)**.** *Let G be a finite generating set of an ideal $I \subset R$. Then G is a Gröbner basis if and only if $\overline{S(f,g)}^G = 0$ for all pairs $f, g \in G$.*

Proof. See Chapter 2, §7 of [26]. □

The Buchberger criterion provides an idea for the following algorithm.

Algorithm 5.30 (Buchberger's algorithm)**.** G = Buchberger(F)

Require: F, a finite generating set for an ideal $I \subset R$ equipped with a term order.
Ensure: G, a Gröbner basis of I.

$G := F, Q := \{(f,g) \mid f, g \in F\}$
while $Q \neq \emptyset$ **do**
  Pick a pair $(f,g) \in Q$
  $h := NF(S(f,g))$
  if $h \neq 0$ **then**
    $Q := Q \cup \{(f,h) \mid f \in G\}$
    $G := G \cup \{h\}$
  end if
end while

A proof of termination of Buchberger's algorithm can be found in Chapter 2, §6 of [26].

5.7. **Gröbner bases for modules.** A monomial order $\geq$ on the polynomial ring $R = \mathbb{K}[x_1, ..., x_n]$ can be used to build an order on the free module R^m of rank $m > 0$ in several ways; Let $R^m = Re_1 \oplus ... \oplus Re_m$, we describe two possible orders $\geq_{TOP}$ and $\geq_{POT}$ as follows:

$$\begin{array}{rcl} x^\alpha e_i >_{TOP} x^\beta e_j & \Leftrightarrow & x^\alpha > x^\beta \textbf{ or } (x^\alpha = x^\beta \textbf{ and } i > j) \\ (TOP & = & \text{“term over position”}) \\ x^\alpha e_i >_{POT} x^\beta e_j & \Leftrightarrow & i > j \textbf{ or } (i = j \textbf{ and } x^\alpha > x^\beta) \\ (POT & = & \text{“position over term”}) \end{array}$$

Similarly to how it is done in 5.16, we may introduce the notion of a *term order* for a free module of finite rank. If $\geq$ is a term order, so are both $\geq_{TOP}$ and $\geq_{POT}$.

It is also not very hard to modify the definition of a Gröbner basis and Buchberger's algorithm to make them work in the module setting.

Lecture 6. Complexes from a sequence of ring elements (GL)

In Lecture 3 we postulated or proved the existence of several examples of exact sequences: projective, free, or injective resolutions, as well as long exact sequences in (co)homology. Most of these were quite abstract, and the concrete examples came out of thin air. The problem of actually producing any one of these kinds of resolutions for a given module was essentially ignored. In this lecture, we will give a few, quite concrete, constructions of *complexes* beginning from an explicit list of ring elements, which we can later use and manipulate to obtain resolutions in some cases. The complexes we construct will also, as we shall see, carry quite a lot of information that is relevant to our long-term goal of understanding local cohomology.

In constructing various complexes from a sequence of elements, we will begin with the case of a single element, and inductively patch copies together to build the final product. This patching will be done by taking the tensor product of two or more complexes, a procedure we now define in general. Let R be an arbitrary commutative ring.

Definition 6.1. Let

$$F^\bullet : \ \cdots \longrightarrow F^i \xrightarrow{\varphi^i} F^{i+1} \longrightarrow \cdots$$

and

$$G^\bullet : \ \cdots \longrightarrow G^i \xrightarrow{\psi^i} G^{i+1} \longrightarrow \cdots$$

be (cohomologically indexed) complexes of R-modules. Then the *tensor product* of F and G is

$$F^\bullet \otimes_R G^\bullet : \ \cdots \longrightarrow \bigoplus_{i+j=k} F^i \otimes_R G^j \xrightarrow{\partial^k} \bigoplus_{i+j=k+1} F^i \otimes_R G^j \longrightarrow \cdots ,$$

where ∂^k is defined on simple tensors $x \otimes y \in F^i \otimes_R G^j$ by

$$\partial^k(x \otimes y) = \varphi^i(x) \otimes y + (-1)^i x \otimes \psi^j(y) .$$

An exactly similar definition applies to homologically-indexed complexes.

Remark 6.2. The sign in the definition of ∂^k is there precisely so that $F^\bullet \otimes_R G^\bullet$ is a complex. With this definition, it is straightforward to check that the tensor product defines an honest binary operation on complexes, which, if R is commutative, is both associative and commutative.

The Koszul complex

When we are handed a single element of a ring, there is one complex simply crying out to be constructed.

Definition 6.3. Let R be a ring and $x \in R$. The *Koszul complex on x* is

$$K_\bullet(x) : \quad 0 \longrightarrow R \xrightarrow{\ x\ } R \longrightarrow 0 ,$$

with R in degrees 1 and 0. For a sequence $\underline{x} = x_1, \dots, x_n$, the Koszul complex on $\underline{x}$ is defined by

$$K_\bullet(\underline{x}) = K_\bullet(x_1) \otimes_R \cdots \otimes_R K_\bullet(x_n) .$$

Example 6.4. Let $x, y \in R$. The Koszul complex on x is

$$K_\bullet(x): \quad 0 \longrightarrow R \xrightarrow{\ x\ } R \longrightarrow 0\,,$$

and that on y is

$$K_\bullet(y): \quad 0 \longrightarrow R \xrightarrow{\ y\ } R \longrightarrow 0\,.$$

The tensor product is

$$K_\bullet(x,y): \quad 0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]} R^2 \xrightarrow{[y\ \ x]} R \longrightarrow 0$$

where the three nonzero modules are in degrees 2, 1, and 0, left to right. Observe that $K_\bullet(x,y)$ is indeed a complex.

Remark 6.5. Let $\underline{x} = x_1, \ldots, x_n$. Then some simple counting using Definition 6.1 reveals that the r^{th} module in the Koszul complex $K_\bullet(\underline{x})$ is given by

$$K_r(\underline{x}) \cong R^{\binom{n}{r}}\,,$$

where $\binom{n}{r}$ is the appropriate binomial coefficient. The natural basis for this free module is the set $\{e_{i_1,\ldots,i_r}\}$, where $1 \le i_1 < \cdots < i_r \le n$. In terms of this basis, the r^{th} differential ∂_r is given by

$$\partial_r(e_{i_1,\ldots,i_r}) = \sum_{j=1}^{r} (-1)^{i-1} x_{i_j} e_{i_1,\ldots,\widehat{i_j},\ldots,i_r}\,.$$

Exercise 6.6. Construct the Koszul complex on a sequence of three elements $x, y, z \in R$. Compare with Example 3.14.

Exercise 6.7. For a sequence of any length, $\underline{x} = x_1, \ldots, x_n$, identify the maps ∂_1 and ∂_n in $K_\bullet(\underline{x})$.

The Koszul complex as defined above holds an enormous amount of information about the sequence $\underline{x} = x_1, \ldots, x_n$ and the ideal of R that they generate. In future lectures we'll see some of this information laid bare. For the best applications, however, we will want more *relative* information about $\underline{x}$ and its impact on various R-modules. We therefore define the Koszul complex on a module M, and introduce the Koszul homology groups.

Definition 6.8. Let R be a commutative ring, $\underline{x} = x_1, \ldots, x_n$ a sequence of elements of R, and M an R-module.

(1) The *Koszul complex of $\underline{x}$ on M* is $K_\bullet(\underline{x}, M) := K_\bullet(\underline{x}) \otimes_R M$.
(2) The *Koszul homology of $\underline{x}$ on M* is the homology of this complex, so $H_j(\underline{x}, M) := H_j(K_\bullet(\underline{x}, M))$ for $j = 0, \ldots, n$.

Example 6.9. Let $x \in R$ be a single element and M an R-module. Then the tininess of the Koszul complex $K_\bullet(x, M)$ makes computing the Koszul homology trivial:

$$H_0(x, M) = M/xM$$
$$H_1(x, M) = (0 :_M x) = \{m \in M \mid xm = 0\}.$$

In particular, we can make two immediate observations:

(1) If $xM \ne M$, that is, x does not act "like a unit" on M, then $H_0(x, M) \ne 0$. In particular, if M is finitely generated and x is in the Jacobson radical of R, then H_0 is nonzero by Nakayama's Lemma.

(2) If x is a *nonzerodivisor* on M, that is, $xm \neq 0$ for all nonzero $m \in M$, then $H_1(x, M) = 0$.

In order to put this example in its proper context, let us insert here a brief interlude on *regular sequences* and *depth*.

Regular sequences and depth: a first look

Definition 6.10. Let R be a ring and $x \in R$. We say that x is a *nonzerodivisor* if $xy \neq 0$ for all nonzero $y \in R$. If in addition x is a nonunit, say that x is a *regular element*.

Let moreover M be an R-module. Then x is a *nonzerodivisor on* M if $xm \neq 0$ for all nonzero $m \in M$, and a *regular element on* M (or M*-regular*) if in addition $xM \neq M$.

Remark 6.11. From Example 6.9 we see that $x \in R$ is M-regular if and only if $H_0(x, M) \neq 0$ and $H_1(x, M) = 0$.

Definition 6.12. A sequence of elements $\underline{x}$ of R is a *regular sequence on* M if

(1) x_1 is M-regular, and
(2) for each $i = 2, \ldots, n$, x_i is regular on $M/(x_1, \ldots, x_{i-1})M$.

Remark 6.13.

(1) Some authors allow the possibility that $xM = M$, and call such a sequence "weakly" M-regular.
(2) In such an inductive definition, the order of the x_i is essential. For example, the sequence $X - 1, XY, XZ$ is regular in $\mathbb{K}[X, Y, Z]$, while $XY, XZ, X - 1$ is not. If, however, $(R, \mathfrak{m})$ is local and $\underline{x}$ is contained in $\mathfrak{m}$, then we shall see below that the order of the x_i is immaterial.

The set of regular elements is easy to describe.

Lemma 6.14. *Let R be a Noetherian commutative ring. The set of zerodivisors on a finitely generated R-module M is the union of the associated primes of M.*

Proof. Exercise. □

We'll finish this interlude by smuggling in one more definition.

Definition 6.15. Let $\mathfrak{a} \subseteq R$ be an ideal and M an R-module. The *depth of* $\mathfrak{a}$ *on* M is the maximal length of an M-regular sequence contained in $\mathfrak{a}$, denoted $\operatorname{depth}_R(\mathfrak{a}, M)$.

Depth will reappear in Lectures 8 and 9.

Back to the Koszul complex

Let us return now to the Koszul complex. We computed above the Koszul homology of a single element, and now recognize it as determining regularity. We have high hopes for the case of two elements.

Example 6.16. Let $x, y \in R$ and let M be an R-module. Then the homology of the complex

$$K_\bullet(\{x,y\},M):\ 0 \longrightarrow M \xrightarrow{\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]} M^2 \xrightarrow{[\,y\ x\,]} M \longrightarrow 0$$

at the ends can be computed easily. We have

$$\begin{aligned} &H_0(\{x,y\},M) = M/(x,y)M, \text{ and} \\ &H_2(\{x,y\},M) = (0 :_M (x,y)) = \{m \in M \mid xm = ym = 0\}. \end{aligned}$$

What is the homology in the middle? Let $(a,b) \in M^2$ be such that $ya + bx = 0$. Then $ya = -xb$, so in particular $a \in (xM :_M y)$. Assume for the moment that x is regular on M, and take $a \in (xM :_M y)$. Then there exists some b so that $ya = -xb$, and since x is M-regular, there is *precisely one* such b. In other words, if we assume that x is a nonzerodivisor on M, then $\ker[x\ y] \cong (xM :_M y)$. Still assuming that x is M-regular, we can also identify the image of $\left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right]$ as

$$\{(xc, -yc) \mid c \in M\} \cong xM\,.$$

Therefore

$$H_1(\{x,y\},M) \cong (xM :_M y)/xM\,.$$

Exercise 6.17. Assume that x is M-regular, and prove that x, y is an M-regular sequence if and only if $(xM :_M y) = xM$.

Examples 6.16 and 6.11 are part of what is usually called the "depth-sensitivity" of the Koszul complex. See [114, 16.5] for a proof.

Theorem 6.18. *Let R be a commutative ring and $\underline{x} = x_1, \ldots, x_n$ a sequence of elements of R. If $\underline{x}$ is regular on M, then $H_j(\underline{x}, M) = 0$ for all $j > 0$ and $H_0(\underline{x}, M) = M/\underline{x}M \neq 0$. If either $(R, \mathfrak{m})$ is Noetherian local, $x \in \mathfrak{m}$, and M is nonzero finitely generated, or R is $\mathbb{N}$-graded, M is nonzero $\mathbb{N}$-graded, and the elements $\underline{x}$ are homogeneous of positive degree, then there is a strong converse: if $H_1(\underline{x}, M) = 0$, then $\underline{x}$ is an M-regular sequence.*

As a corollary, we can conclude that depth is a "geometric" property:

Corollary 6.19. *If $x_1, \ldots, x_n$ is an M-regular sequence, then $x_1^{a_1}, \ldots x_n^{a_n}$ is M-regular as well for any positive integers $a_1, \ldots, a_n$. In particular,* $\operatorname{depth}_R(\mathfrak{a}, M) = \operatorname{depth}_R(\sqrt{\mathfrak{a}}, M)$.

We also mention the following fact, which we won't need, but which motivates some of our results in Lecture 8. See [31, 17.4] for a proof.

Proposition 6.20. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal of R. Suppose that $\mathfrak{a}$ is minimally generated by n elements and that $\mathfrak{a}$ contains an M-regular sequence of length n. Then any minimal system of generators for $\mathfrak{a}$ is an M-regular sequence.*

For later applications, it will occasionally be useful to adjust the indexing of the Koszul complex.

Definition 6.21. Let R be a commutative ring and $x \in R$. The *cohomological Koszul complex on x* is

$$K^\bullet(x):\quad 0 \longrightarrow R \xrightarrow{\ x\ } R \longrightarrow 0\,,$$

which is identical to the usual Koszul complex, except with R in degrees 0 and 1. For a sequence $\underline{x} = x_1, \dots, x_n$, define $K^\bullet(\underline{x}) = K^\bullet(x_1) \otimes_R \cdots \otimes_R K^\bullet(x_n)$. Finally, for an R-module M, we put $K^\bullet(\underline{x}, M) = K^\bullet(\underline{x}) \otimes_R M$.

Exercise 6.22. Prove that $K^\bullet(\underline{x}) \otimes_R M$ is isomorphic to $\operatorname{Hom}_R(K_\bullet(\underline{x}), M)$.

The Čech complex

Given again a single element x in a ring R, it may seem like the only complex we can build from such meager information is the Koszul complex. If we insist on clinging to the world of finitely generated R-modules, this is essentially true. If, however, we allow some small amount of infinite generation, new vistas open to us.

The Čech complex attached to a sequence of ring elements, like the Koszul, is built inductively by tensoring together short complexes. Recall that for $x \in R$, the localization R_x, also sometimes written $R[\frac{1}{x}]$, is obtained by inverting the multiplicatively closed set $\{1, x, x^2, \dots\}$.

Definition 6.23. Let R be a commutative ring and $x \in R$. The *Čech complex on* x is

$$C^\bullet(x; R): \quad 0 \longrightarrow R \xrightarrow{\ \iota\ } R_x \longrightarrow 0\,,$$

with R in degree 0 and R_x in degree 1, and where ι is the canonical map sending each $r \in R$ to the fraction $\frac{r}{1} \in R_x$. For a sequence $\underline{x} = x_1, \dots, x_n$ in R, the Čech complex on $\underline{x}$ is $C^\bullet(\underline{x}; R) := C^\bullet(x_1; R) \otimes_R \cdots \otimes_R C^\bullet(x_n; R)$. For an R-module M, define $C^\bullet(\underline{x}; M) = C^\bullet(\underline{x}; R) \otimes_R M$. The j^{th} *Čech cohomology* is defined by $H^j(\underline{x}; M) := H^j(C^\bullet(\underline{x}; M))$.

Example 6.24. As with the Koszul complexes, the Čech complex is easy to describe for small n. In case $\underline{x} = x$ is a single element, $C^\bullet(x; R)$ is given by the definition. We note that

$$\begin{aligned} H^0(x; R) &= \{r \in R \mid \frac{r}{1} = 0 \text{ in } R_x\} \\ &= \{r \in R \mid x^a r = 0 \text{ for some } a \geq 0\} \\ &= \bigcup_{a \geq 0} 0 :_R x^a \end{aligned}$$

is the union of annihilators of x^a. This is sometimes written $0 :_R x^\infty$.

Meanwhile, $H^1(x; R) \cong R_x/R$. This expression for H^1 is ambiguous and not very satisfying (particularly if x is a zerodivisor); we'll correct for this shortly. For now, suppose that $R = \mathbb{K}[x]$ is the univariate polynomial ring over a field $\mathbb{K}$. Then $R_x \cong \mathbb{K}[x, x^{-1}]$ is the ring of *Laurent polynomials*. The quotient $\mathbb{K}[x, x^{-1}]/\mathbb{K}[x]$ is generated (over $\mathbb{K}$) by all the negative powers x^{-c}, $c \in \mathbb{N}$, and has R-module structure dictated by

$$x^a x^{-c} = \begin{cases} x^{a-c} & \text{if } a < c, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 6.25. Prove that $\mathbb{K}[x, x^{-1}]/\mathbb{K}[x]$ is isomorphic to the injective hull of the residue field of $\mathbb{K}[x]$.

Example 6.26. For $\underline{x} = \{x, y\}$ a sequence of two elements, we have the tensor product of

$$C^\bullet(x; R): \quad 0 \longrightarrow R \xrightarrow{r \mapsto \frac{r}{1}} R_x \longrightarrow 0$$

and

$$C^\bullet(y;R): \quad 0 \longrightarrow R \xrightarrow{r \mapsto \frac{r}{1}} R_y \longrightarrow 0\,,$$

which is

$$0 \longrightarrow R \otimes R \xrightarrow{\alpha} (R_x \otimes R) \oplus (R \otimes R_y) \xrightarrow{\beta} R_x \otimes R_y \longrightarrow 0\,.$$

The map α sends $1 \otimes 1$ to $(\frac{1}{1} \otimes 1, 1 \otimes \frac{1}{1})$. For β we have

$$\beta(\frac{1}{1} \otimes 1, 0) = (-1)\frac{1}{1} \otimes \frac{1}{1}\,, \quad \text{and} \quad \beta(0, 1 \otimes \frac{1}{1}) = \frac{1}{1} \otimes \frac{1}{1}\,.$$

Simplified, this becomes

$$C^\bullet(\{x,y\};R): \quad 0 \longrightarrow R \xrightarrow{1 \mapsto (1,1)} R_x \oplus R_y \xrightarrow[(0,1)\ \mapsto\ \ 1]{(1,0)\ \mapsto\ -1} R_{xy} \longrightarrow 0\,.$$

Let's try to compute the cohomology $H^j(\{x,y\};R)$ for $j = 0, 2$. If $r \in R$ maps to zero in $R_x \oplus R_y$, so that $(\frac{r}{1}, \frac{r}{1}) = 0$, then there exist integers $a, b \geq 0$ such that $x^a r = y^b r = 0$. Equivalently, the ideal $(x,y)^c$ kills r for some $c \geq 0$. Thus

$$H^0(\{x,y\};R) \cong \bigcup_{c \geq 0} 0 :_R (x,y)^c$$

is the union of all annihilators of the ideals $(x,y)^c$. On the other hand, $H^2(\{x,y\};R) \cong R_{xy}/(R_x + R_y)$, which again is a less than completely satisfactory answer. Here is a more useful one:

Exercise 6.27. Observe that an element of $H^2(\{x,y\};R)$ can be written $\eta = \left[\frac{r}{(xy)^c}\right]$, that is, as an equivalence class of fractions in R_{xy}. Then show that $\eta = 0$ iff there exists $d \geq 0$ such that

$$r(xy)^d \in (x^{c+d}, y^{c+d})\,.$$

Conclude that for $\{x,y\}$ a regular sequence, $\eta = \left[\frac{r}{(xy)^c}\right]$ represents the zero element if and only if $r \in (x,y)^c$. State and prove the analogous statements for $H^j(\{x_1, \ldots, x_n\};R)$, $J \leq n$.

Remark 6.28. Unlike the Koszul complex on a sequence of elements, in which all the modules are free, the Čech complex is made up of direct sums of localizations of R. Specifically, we can see that $C^0(\underline{x}) \cong R$, while $C^1(\underline{x}) \cong R_{x_1} \oplus \cdots \oplus R_{x_n}$, and in general $C^k(\underline{x})$ is the direct sum of all localizations $R_{x_{i_1} \cdots x_{i_k}}$, where $1 \leq i_1 < \cdots < i_k \leq n$. In particular, $C^n(\underline{x}) \cong R_{x_1 \cdots x_n}$. Note that in general $C^k(\underline{x})$ is not finitely generated over R, but that it is *flat*.

Lecture 7. Local cohomology - the basics (SI)

Let R be a Noetherian commutative ring and $\mathfrak{a}$ an ideal in R.

Definition 7.1. For each R-module M, set

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^t m = 0 \quad \text{for some } t \in \mathbb{N}\}.$$

This is the $\mathfrak{a}$-*torsion functor* on the category of R-modules. It extends to a functor on the category of complexes of R-modules: for each complex $I^\bullet$ of R-modules, $\Gamma_{\mathfrak{a}}(I^\bullet)$ is the complex whose component in degree n is $\Gamma_{\mathfrak{a}}(I^n)$, and the differential is that induced by $I^\bullet$.

It is an elementary exercise to check that the $\mathfrak{a}$-torsion functor is left exact; its n^{th} right derived functor is denoted $H^n_{\mathfrak{a}}(-)$, i.e.,

$$H^n_{\mathfrak{a}}(M) = H^n(\Gamma_{\mathfrak{a}}(I^\bullet)),$$

where $I^\bullet$ is an injective resolution of M. The R-module $H^n_{\mathfrak{a}}(M)$ is the n^{th} *local cohomology of M with support in* $\mathfrak{a}$—since $\Gamma_{\mathfrak{a}}(I^\bullet)$ is a complex of R-modules, each $H^n_{\mathfrak{a}}(M)$ is an R-module.

Here are a few basic properties of local cohomology:

Proposition 7.2. *Let M be an R-module.*

(1) *One has $H^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M)$, and $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion all n.*
(2) *If $\mathfrak{b}$ is an ideal with $\operatorname{rad}\mathfrak{b} = \operatorname{rad}\mathfrak{a}$, then $H^n_{\mathfrak{a}}(M) \cong H^n_{\mathfrak{b}}(M)$ for each n.*
(3) *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. Then, for each integer n, one has*

$$H^n_{\mathfrak{a}}\Big(\bigoplus_\lambda M_\lambda\Big) \cong \bigoplus_\lambda H^n_{\mathfrak{a}}(M_\lambda).$$

(4) *An exact sequence of R-modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ induces a long exact sequence in local cohomology*

$$\cdots \longrightarrow H^n_{\mathfrak{a}}(L) \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{\mathfrak{a}}(N) \longrightarrow H^{n+1}_{\mathfrak{a}}(L) \longrightarrow \cdots.$$

Proof. (1) Let $I^\bullet$ be an injective resolution of M. The left-exactness of $\Gamma_{\mathfrak{a}}(-)$ implies that

$$\Gamma_{\mathfrak{a}}(M) \cong H^0(\Gamma_{\mathfrak{a}}(I^\bullet)) = H^0_{\mathfrak{a}}(M).$$

Furthermore, the R-module $\Gamma_{\mathfrak{a}}(I^n)$ is $\mathfrak{a}$-torsion, so the same property carries over to its subquotient $H^n_{\mathfrak{a}}(M)$.

(2) This is immediate, once we note that $\Gamma_{\mathfrak{a}}(-) = \Gamma_{\mathfrak{b}}(-)$.

(3) Let $I^\bullet_\lambda$ be an injective resolution of M_λ, in which case $\bigoplus_\lambda I^\bullet_\lambda$ is an injective resolution of $\bigoplus_\lambda M_\lambda$. It is not hard to verify that $\Gamma_{\mathfrak{a}}\big(\bigoplus_\lambda I^\bullet_\lambda\big) = \bigoplus_\lambda \Gamma_{\mathfrak{a}}(I^\bullet_\lambda)$. Since homology commutes with direct sums, passing to homology yields the desired result.

(4) Let $G^\bullet$ and $J^\bullet$ be injective resolutions of L and N, respectively. One can then construct an injective resolution $I^\bullet$ of M which fits in an exact sequence of complexes of R-modules,

$$0 \longrightarrow G^\bullet \longrightarrow I^\bullet \longrightarrow J^\bullet \longrightarrow 0.$$

Since $G^\bullet$ consists of injective modules, this exact sequence is split in each degree, so induces an exact sequence of complexes of R-modules

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(G^\bullet) \longrightarrow \Gamma_{\mathfrak{a}}(I^\bullet) \longrightarrow \Gamma_{\mathfrak{a}}(J^\bullet) \longrightarrow 0.$$

The homology long exact sequence of this sequence is the one announced. $\square$

Example 7.3. Let $R = \mathbb{Z}$ and let p be a prime number. We want to compute the local cohomology with respect to the ideal (p) of finitely generated R-modules, that is to say, of finitely generated Abelian groups. Thanks to Proposition 7.2.3, it suffices to consider the case where the module is indecomposable. By the Fundamental Theorem of Abelian Groups, such a module is isomorphic to $\mathbb{Z}/(d)$ where either $d = 0$, or d is a prime power. For any integer d, the complex

$$0 \longrightarrow \mathbb{Q}/d\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

is an injective resolution of $\mathbb{Z}/(d)$. In computing local cohomology, one has two cases to consider; in what follows, $\mathbb{Z}_p$ denotes $\mathbb{Z}$ with the element p inverted.

Case (1). If $M = \mathbb{Z}/(p^e)$ for some integer $e \geqslant 1$, then

$$H^0_{(p)}(M) = \mathbb{Z}/(p^e) \quad \text{and} \quad H^1_{(p)}(M) = 0.$$

This is clear given the injective resolution above, as are the other cases:

Case (2). If $M = \mathbb{Z}/(d)$ with d a nonzero integer relatively prime to p, then

$$H^0_{(p)}(M) = 0 \quad \text{and} \quad H^1_{(p)}(M) = \mathbb{Z}_p/d\mathbb{Z}_p.$$

Case (3). If $M = \mathbb{Z}$, then

$$H^0_{(p)}(M) = 0 \quad \text{and} \quad H^1_{(p)}(M) = \mathbb{Z}_p/\mathbb{Z}.$$

The calculation of the local cohomology with respect to *any* ideal in $\mathbb{Z}$ is equally elementary; see also Theorem 7.13 below. One noteworthy feature of this example is that $H^n_{\mathfrak{a}}(-) = 0$ for $n \geqslant 2$ for any ideal $\mathfrak{a}$; confer Proposition 9.12.

Example 7.4. Let R be a ring and $\mathfrak{a}$ an ideal in R. If $\mathfrak{a}$ is *nilpotent*, that is say, if $\mathfrak{a}^e = 0$ for some integer $e \geqslant 0$, then

$$H^0_{\mathfrak{a}}(M) = M, \quad \text{while} \quad H^n_{\mathfrak{a}}(M) = 0 \quad \text{for } n \geqslant 1.$$

For some purposes, for example in the proof of Theorem 7.10, it is useful to know the local cohomology of injective modules. An injective R-module is a direct sum of modules $E_R(R/\mathfrak{p})$ for prime ideals $\mathfrak{p}$ of R, see Theorem A.22. Thus, by Proposition 7.2.3, it suffices to focus on indecomposable injectives $E_R(R/\mathfrak{p})$.

Example 7.5. Let R be a ring and $\mathfrak{a}$ an ideal in R. For each prime ideal $\mathfrak{p}$ in R, one has $H^n_{\mathfrak{a}}(E_R(R/\mathfrak{p})) = 0$ for $n \geqslant 1$, and

$$H^0_{\mathfrak{a}}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) & \text{if } \mathfrak{a} \subseteq \mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases}$$

This follows from the definition since $E_R(R/\mathfrak{p})$ is injective and $\mathfrak{p}$-torsion!

Here is one application; keep in mind Theorem A.25.

Exercise 7.6. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated R-module. Prove that the R-module $H^n_{\mathfrak{m}}(M)$ is Artinian for each integer n.

Next, we describe alternative methods for computing local cohomology.

Theorem 7.7. *For each R-module M, there is a natural isomorphism*

$$\varinjlim_t \operatorname{Ext}^n_R(R/\mathfrak{a}^t, M) \cong H^n_{\mathfrak{a}}(M) \quad \textit{for each } n \geqslant 0.$$

Proof. For each R-module I and integer t, one has a functorial identification

$$\operatorname{Hom}_R(R/\mathfrak{a}^t, I) \xrightarrow{\cong} \{x \in I \mid \mathfrak{a}^t x = 0\} \subseteq \Gamma_{\mathfrak{a}}(I), \qquad \text{where } f \mapsto f(1).$$

With this identification, one has a direct system

$$\operatorname{Hom}_R(R/\mathfrak{a}, I) \subseteq \cdots \subseteq \operatorname{Hom}_R(R/\mathfrak{a}^t, I) \subseteq \operatorname{Hom}_R(R/\mathfrak{a}^{t+1}, I) \subseteq \cdots$$

of submodules of $\Gamma_{\mathfrak{a}}(I)$. It is evident that its limit (that is to say, its union) equals $\Gamma_{\mathfrak{a}}(I)$; in other words, one has

$$\varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}^t, I) = \Gamma_{\mathfrak{a}}(I).$$

Let $I^\bullet$ be an injective resolution of M. The construction of the direct system above is functorial, so

$$\varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet) = \Gamma_{\mathfrak{a}}(I^\bullet).$$

Since $H^n(-)$ commutes with direct limits, Remark 4.30, the preceding identification results in a natural isomorphism

$$\varinjlim_t H^n(\operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet)) \cong H^n(\Gamma_{\mathfrak{a}}(I^\bullet)) = H^n_{\mathfrak{a}}(M).$$

It remains to note that $H^n(\operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet)) = \operatorname{Ext}^n_R(R/\mathfrak{a}^t, M)$. □

Remark 7.8. In the context of Theorem 7.7, let $\{\mathfrak{a}_t\}_{t\geqslant 0}$ be a decreasing chain of ideals *cofinal* with the chain $\{\mathfrak{a}^t\}_{t\geqslant 0}$, that is to say, for each integer $t \geqslant 0$, there exist positive integers c, d such that $\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t \subseteq \mathfrak{a}^{t-d}$. Then there is a functorial isomorphism

$$\varinjlim_t \operatorname{Ext}^n_R(R/\mathfrak{a}_t, M) \cong H^n_{\mathfrak{a}}(M).$$

Another viewpoint is that cofinal systems induce the same topology, so the local cohomology modules they define are the same. These considerations apply, for instance, when R is a ring of positive prime characteristic p. Then the system $\{\mathfrak{a}^{[p^e]}\}_{e\geqslant 0}$ of Frobenius powers of $\mathfrak{a}$ is cofinal with $\{\mathfrak{a}^t\}_{t\geqslant 0}$, so one obtains a functorial isomorphism of R-modules

$$\varinjlim_e \operatorname{Ext}^n_R(R/\mathfrak{a}^{[p^e]}, M) \cong H^n_{\mathfrak{a}}(M).$$

This expression for local cohomology was exploited by Peskine and Szpiro in their ground-breaking work on intersection theorems [127]; see also Lectures 21 and 22.

Next we express local cohomology in terms of Koszul complexes.

7.9. Let x be an element in R. For each integer t, consider the Koszul complex $K(x^t)$ on x^t,

$$0 \longrightarrow R \xrightarrow{x^t} R \longrightarrow 0.$$

This complex is concentrated in (cohomological) degrees -1 and 0, and is augmented to $R/(x^t)$, viewed as a complex concentrated in degree 0. The complexes $\{K(x^t)\}_{t\geqslant 1}$ form an inverse system, with structure morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{x^{t+1}} & R & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle x} & & \Big\| & & \\ 0 & \longrightarrow & R & \xrightarrow{x^t} & R & \longrightarrow & 0 \end{array} \tag{7.9.1}$$

compatible with the augmentations. Let $\boldsymbol{x} = x_1, \ldots, x_c$ be elements in R, and set $\boldsymbol{x}^t = x_1^t, \ldots, x_c^t$. The Koszul complex on $\boldsymbol{x}^t$ is the complex of R-modules

$$K(\boldsymbol{x}^t) = K(x_1^t) \otimes_R \cdots \otimes_R K(x_c^t)$$

concentrated in degrees $[-c, 0]$. One has an augmentation $\epsilon_t\colon K(\boldsymbol{x}^t) \longrightarrow R/(\boldsymbol{x}^t)$, which we view as a (degree zero) morphism of complexes. Associated to each x_i, there is an inverse system as in (7.9.1); tensoring these componentwise yields an inverse system of complexes of R-modules

$$\cdots \longrightarrow K(\boldsymbol{x}^{t+1}) \longrightarrow K(\boldsymbol{x}^t) \longrightarrow \cdots \longrightarrow K(\boldsymbol{x}), \tag{7.9.2}$$

compatible with the augmentations ϵ_t.

Let M be an R-module and $\eta\colon M \longrightarrow I^\bullet$ an injective resolution of M. The maps ϵ_t and η induce morphisms of complexes of R-modules

$$\mathrm{Hom}_R(R/(\boldsymbol{x}^t), I^\bullet) \longrightarrow \mathrm{Hom}_R(K(\boldsymbol{x}^t), I^\bullet) \longleftarrow \mathrm{Hom}_R(K(\boldsymbol{x}^t), M),$$

where the map on the left is $\mathrm{Hom}_R(\epsilon_t, I^\bullet)$, and the right is $\mathrm{Hom}_R(K(\boldsymbol{x}^t), \eta)$. The latter is a quasi-isomorphism because $K(\boldsymbol{x}^t)$ is a bounded complex of free R-modules. Thus, passing to homology yields, for each integer n, a diagram of homomorphisms of R-modules

$$\mathrm{Ext}_R^n(R/(\boldsymbol{x}^t), M) \longrightarrow H^n(\mathrm{Hom}_R(K(\boldsymbol{x}^t), I^\bullet)) \xleftarrow{\cong} H^n(\mathrm{Hom}_R(K(\boldsymbol{x}^t), M)),$$

and hence a homomorphism of R-modules

$$\theta_t^n\colon \mathrm{Ext}_R^n(R/(\boldsymbol{x}^t), M) \longrightarrow H^n(\mathrm{Hom}_R(K(\boldsymbol{x}^t), M)).$$

It is not hard to verify that this homomorphism is compatible with the inverse system in (7.9.2). Because $\mathrm{Hom}_R(-, M)$ is contravariant, one gets a compatible *direct* system of R-modules

$$\begin{tikzcd}
\mathrm{Ext}_R^n(R/(\boldsymbol{x}^t), M) \arrow[r, "\theta_t^n"] \arrow[d] & H^n(\mathrm{Hom}_R(K(\boldsymbol{x}^t), M)) \arrow[d] \\
\mathrm{Ext}_R^n(R/(\boldsymbol{x}^{t+1}), M) \arrow[r, "\theta_{t+1}^n"] & H^n(\mathrm{Hom}_R(K(\boldsymbol{x}^{t+1}), M)).
\end{tikzcd}$$

In the limit, this gives us a homomorphism of R-modules

$$\theta^n(M)\colon \varinjlim_t \mathrm{Ext}_R^n(R/(\boldsymbol{x}^t), M) \longrightarrow \varinjlim_t H^n(\mathrm{Hom}_R(K(\boldsymbol{x}^t), M)).$$

The module on the left is $H^n_{(\boldsymbol{x})}(M)$; this follows from the discussion in Remark 7.8, because the system of ideals $\{(\boldsymbol{x}^t)\}_{t\geqslant 1}$ is cofinal with the system $\{(\boldsymbol{x})^t\}_{t\geqslant 1}$. It is an important point that the $\theta^n(-)$ are functorial in M, and also compatible with connecting homomorphisms: each exact sequence of R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

induces a commutative diagram of R-modules

$$\begin{tikzcd}
{} \arrow[r] & H^n_{(\boldsymbol{x})}(L) \arrow[r] \arrow[d, "\theta^n(L)"] & H^n_{(\boldsymbol{x})}(M) \arrow[r] \arrow[d, "\theta^n(M)"] & H^n_{(\boldsymbol{x})}(N) \arrow[r] \arrow[d, "\theta^n(N)"] & H^{n+1}_{(\boldsymbol{x})}(L) \arrow[r] \arrow[d, "\theta^{n+1}(L)"] & {} \\
{} \arrow[r] & F^n_{(\boldsymbol{x})}(L) \arrow[r] & F^n_{(\boldsymbol{x})}(M) \arrow[r] & F^n_{(\boldsymbol{x})}(N) \arrow[r] & F^{n+1}_{(\boldsymbol{x})}(L) \arrow[r] & {}
\end{tikzcd}$$

where $F^n_{(\boldsymbol{x})}(-) = \varinjlim_t H^n(\mathrm{Hom}_R(\Sigma^c K(\boldsymbol{x}^t), -))$. All these claims are easy to verify given the construction of $F^n_{(\boldsymbol{x})}(-)$ and the $\theta^n(-)$. In category-theory language, what we are saying is that $\{\theta^n(-)\}_{n\geqslant 0}$ define a natural transformation between δ-functors.

Theorem 7.10. *Let $\boldsymbol{x} = x_1, \dots, x_c$ be a set of generators for an ideal $\mathfrak{a}$. For each R-module M and integer n, the homomorphism*

$$\theta^n \colon H^n_{\mathfrak{a}}(M) \longrightarrow \varinjlim_t H^n(\operatorname{Hom}_R(K(\boldsymbol{x}^t), M))$$

constructed above is bijective.

Proof. Set $F^n_{\boldsymbol{x}}(-) = \varinjlim_t H^n(\operatorname{Hom}_R(K(\boldsymbol{x}^t), -))$. The argument is a standard one for proving that a natural transformation between δ-functors is an equivalence:

(1) prove that $\theta^0(M)$ is an isomorphism for any R-module M;
(2) prove that $H^n_{\mathfrak{a}}(I) = 0 = F^n_{\boldsymbol{x}}(I)$ for each injective R-module I and $n \geqslant 1$;
(3) use induction on n to verify that $\theta^n(M)$ is an isomorphism for each n.

Here is how these steps are executed:

Step (1). We claim that

$$F^0_{\boldsymbol{x}}(M) = \varinjlim_t \operatorname{Hom}_R(R/(\boldsymbol{x}^t), M) = \Gamma_{\mathfrak{a}}(M).$$

The first equality holds because $\operatorname{Hom}_R(-, M)$ is left exact, while the second holds because the system of ideals $\{(\boldsymbol{x}^t)\}_{t\geqslant 1}$ is cofinal with the system $\{(\boldsymbol{x})^t\}_{t\geqslant 1}$.

Step (2). Since I is injective, $H^n_{\mathfrak{a}}(I) = 0$ for $n \geqslant 1$. As to the vanishing of $F^n_{\boldsymbol{x}}(I)$, it is not hard to check that $F^n_{\boldsymbol{x}}(-)$ commutes with direct sums, so it suffices to verify the claim for an indecomposable injective $E = E_R(R/\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of R. In this case, as E is naturally an $R_{\mathfrak{p}}$-module, one has

$$\operatorname{Hom}_R(K(\boldsymbol{x}^t), E) \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}} \otimes_R K(\boldsymbol{x}^t), E).$$

If $\mathfrak{p} \not\supseteq (\boldsymbol{x})$, one of the x_i must be invertible in $R_{\mathfrak{p}}$, so $R_{\mathfrak{p}} \otimes_R K(\boldsymbol{x}^t)$ is acyclic for each t. It follows that $F^n_{\boldsymbol{x}}(E) = 0$ for all $n \geqslant 1$.

In the case $\mathfrak{p} \supseteq (\boldsymbol{x})$, the R-module E, being $\mathfrak{p}$-torsion, is also $(\boldsymbol{x})$-torsion. In particular, any homomorphism $L \longrightarrow E$, where L is a finitely generated R-module, is $(\boldsymbol{x})$-torsion. For $u \geqslant t \geqslant 1$, the homomorphism

$$\alpha^n_{ut} \colon \operatorname{Hom}_R(K(\boldsymbol{x}^t), E)^n \longrightarrow \operatorname{Hom}_R(K(\boldsymbol{x}^u), E)^n$$

in the direct system defining $F^n_{\boldsymbol{x}}(E)$ is induced by $K^{-n}(\boldsymbol{x}^u) \xrightarrow{\wedge^n A} K^{-n}(\boldsymbol{x}^t)$, where $A = (a_{ij})$ is the $c \times c$ matrix with

$$a_{ij} = \begin{cases} x_i^{u-t} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Therefore, for $n \geqslant 1$, the matrix $\wedge^n A$ has coefficients in $(\boldsymbol{x})$. The upshot is that for a fixed integer $t \geqslant 1$ and homomorphism $f \colon K^{-n}(\boldsymbol{x}^t) \longrightarrow E$, since f is $(\boldsymbol{x})$-torsion, there exists $u \geqslant t$ with $\alpha^n_{ut}(f) = 0$. Thus cycles in $\operatorname{Hom}_R(K(\boldsymbol{x}^t), E)^n$ do not survive in the limit, so $F^n_{\boldsymbol{x}}(E) = 0$, as claimed.

Step (3). We argue by an induction on n that $\theta^n(-)$ is an isomorphism for all R-modules; the basis of the induction is Step (1). Assume that $\theta^{n-1}(-)$ is an isomorphism for some integer $n \geqslant 1$. Given an R-module M, embedded it into an injective module I to get an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow N \longrightarrow 0.$$

The functoriality of the $\theta^n(-)$, discussed in (7.9), yields a commutative diagram

$$\begin{tikzcd} H^{n-1}_{\mathfrak{a}}(I) \arrow[r] \arrow[d, "\theta^{n-1}(I)"', "\cong"] & H^{n-1}_{\mathfrak{a}}(N) \arrow[r] \arrow[d, "\theta^{n-1}(N)"', "\cong"] & H^{n}_{\mathfrak{a}}(M) \arrow[r] \arrow[d, "\theta^{n}(M)"'] & H^{n}_{\mathfrak{a}}(I) = 0 \\ F^{n-1}_{\mathfrak{a}}(I) \arrow[r] & F^{n-1}_{\mathfrak{a}}(N) \arrow[r] & F^{n}_{\mathfrak{a}}(M) \arrow[r] & F^{n}_{\mathfrak{a}}(I) = 0, \end{tikzcd}$$

where the isomorphisms are from the induction hypothesis, and the vanishing assertions follow from Step (2). Therefore, by the three-lemma (if there is such a thing), $\theta^n(M)$ is bijective. □

Next we provide an alternative, and more useful, formulation of Theorem 7.7.

7.11. The stable Koszul complex. As before, let $\boldsymbol{x} = x_1, \dots, x_c$ be elements in R, and $t \geqslant 1$ an integer. There is a canonical isomorphism

$$\mathrm{Hom}_R(K(\boldsymbol{x}^t), M) \cong \Sigma^{-c} K(\boldsymbol{x}^t) \otimes_R M$$

of complexes of R-modules, where $\Sigma^{-c} K(\boldsymbol{x}^t)$ denotes the complex $K(\boldsymbol{x}^t)$ shifted c steps to the right. Set $C_{\boldsymbol{x}}(M) = \varinjlim_t (\Sigma^{-c} K(\boldsymbol{x}^t) \otimes_R M)$. Since direct limits commute with tensor products, we have

$$C_{\boldsymbol{x}}(M) = \big(\varinjlim_t \Sigma^{-c} K(\boldsymbol{x}^t)\big) \otimes_R M = C_{\boldsymbol{x}}(R) \otimes_R M. \tag{7.11.1}$$

We want to analyze $C_{\boldsymbol{x}}(R)$. The tensor product decomposition of $K(\boldsymbol{x}^t)$ implies

$$C_{\boldsymbol{x}}(R) \cong \big(\varinjlim_t \Sigma^{-1} K(x_1^t)\big) \otimes_R \cdots \otimes_R \big(\varinjlim_t \Sigma^{-1} K(x_c^t)\big). \tag{7.11.2}$$

Thus, it suffices to examine $C_x(R)$ for an element $x \in R$. The limit system in question is obtained by applying $\mathrm{Hom}_R(-, R)$ to the one in (7.9.1), and looks like

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "x^t"] \arrow[d, equal] & R \arrow[r] \arrow[d, "x"] & 0 \\ 0 \arrow[r] & R \arrow[r, "x^{t+1}"] & R \arrow[r] & 0. \end{tikzcd}$$

The direct limit of the system $R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots$ is R_x; see Exercise 7.12. With this in hand, it is easy to see that the direct limit $C_x(R)$ of the system above is the complex

$$0 \longrightarrow R \longrightarrow R_x \longrightarrow 0,$$

sitting in degrees 0 and 1, where the map $R \longrightarrow R_x$ is the canonical localization map. Feeding this into (7.11.2), one obtains that $C_{\boldsymbol{x}}(R)$ is the complex with

$$C^n_{\boldsymbol{x}} = \bigoplus_{1 \leqslant i_1 < \cdots < i_n \leqslant c} R_{x_{i_1} \cdots x_{i_n}},$$

and differential $C^n_{\boldsymbol{x}} \longrightarrow C^{n+1}_{\boldsymbol{x}}$ defined to be the alternating sums of maps

$$\partial\left(R_{x_{i_1} \cdots x_{i_n}}\right)_{j_1, \dots, j_{n+1}} = \begin{cases} 1 & \text{if } \{i_1, \dots, i_n\} \subset \{j_1, \dots, j_{n+1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

This is the *stable Koszul complex* or the *extended Čech complex* associated with $\boldsymbol{x}$. It has the form

$$0 \longrightarrow R \longrightarrow \bigoplus_{1 \leqslant i \leqslant c} R_{x_i} \longrightarrow \bigoplus_{1 \leqslant i < j \leqslant c} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_c} \longrightarrow 0.$$

Exercise 7.12. Prove that the localization R_x is isomorphic to the direct limit

$$\varinjlim \left(R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots \right).$$

Using (7.11.1) and the description of $C_{\boldsymbol{x}}(R)$, Theorem 7.10 translates to:

Theorem 7.13. *Let $\boldsymbol{x} = x_1, \dots, x_c$ be a set of generators for an ideal $\mathfrak{a}$. For each R-module M and integer n, there is a natural isomorphism*

$$H^n_{\mathfrak{a}}(M) \cong H^n\left(C_{\boldsymbol{x}}(R) \otimes_R M\right).$$

One can now "calculate" the last possible local cohomology module; compare this result with Exercise 9.7.

Corollary 7.14. *If $\boldsymbol{x} = x_1, \dots, x_c$ is a set of generators for an ideal $\mathfrak{a}$, then, for each R-module M, one has $H^n_{\mathfrak{a}}(M) = 0$ for $n \geqslant c+1$, and*

$$H^c_{\mathfrak{a}}(M) = \frac{M_{x_1 \cdots x_c}}{\sum_{i=1}^{c} M_{x_1 \cdots \widehat{x}_i \cdots x_c}},$$

where the localization $M_{x_1 \cdots \widehat{x}_i \cdots x_c}$ is identified with its image in $M_{x_1 \cdots x_c}$. □

Here is something else to keep in mind about the stable Koszul complex:

Remark 7.15. For each integer n, the R-module $C^n_{\boldsymbol{x}}(R)$ is flat, since it is a sum of localizations of R. Therefore $C_{\boldsymbol{x}}(R)$ is a bounded complex of flat modules.

The next item adds to the list of properties of local cohomology stated in Proposition 7.2. These are all straightforward applications of Theorem 7.13, and are left as exercises. By the way, try to verify properties in Proposition 7.2 using Theorem 7.13, and the ones below without taking recourse to it!

Proposition 7.16. *Let R be a ring, $\mathfrak{a}$ an ideal in R, and M an R-module.*

(1) *If U is a multiplicatively closed subset of R, then*

$$H^n_{\mathfrak{a}}(U^{-1}M) \cong U^{-1} H^n_{\mathfrak{a}}(M).$$

(2) *If $R \longrightarrow S$ is a homomorphism of rings and N is an S-module, then*

$$H^n_{\mathfrak{a}}(N) \cong H^n_{\mathfrak{a}S}(N).$$

(3) *If a homomorphism of rings $R \longrightarrow S$ is flat, then there is a natural isomorphism of S-modules*

$$S \otimes_R H^n_{\mathfrak{a}}(M) \cong H^n_{\mathfrak{a}S}(S \otimes_R M).$$

The following calculation illustrates Corollary 7.14:

Example 7.17. Let $\mathbb{K}$ be a field, $R = \mathbb{K}[x_1, \dots, x_c]$ a polynomial ring over $\mathbb{K}$, and let $\mathfrak{a} = (x_1, \dots, x_c)$. Then

$$H^n_{\mathfrak{a}}(R) = \begin{cases} \mathbb{K}[x_1^{-1}, \dots, x_c^{-1}](x_1^{-1} \cdots x_c^{-1}) & \text{if } n = c, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{K}[x_1^{-1}, \dots, x_c^{-1}]$ denotes the $\mathbb{K}$-vector space of polynomials in $x_1^{-1}, \dots, x_c^{-1}$ with the R-action defined by

$$x_i \cdot (x_1^{a_1} \cdots x_c^{a_c}) = \begin{cases} x_1^{a_1} \cdots x_i^{a_i+1} \cdots x_c^{a_c} & \text{if } a_i \leqslant -2, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, the claim about $H^n_{\mathfrak{a}}(R)$ for $n \geqslant c$ follows from Corollary 7.14. As to the other values of n, for each t the sequence $x_1^t, \ldots, x_c^t$ is R-regular. Thus, the depth sensitivity of Koszul complexes implies that $H^n(K(\boldsymbol{x}^t)) = 0$ for $0 \leqslant n \leqslant c-1$. Since $C_{\boldsymbol{x}}(R)$ is a direct limit of these Koszul complexes, and homology commutes with direct limits, one obtains that $H^n(C_{\boldsymbol{x}}(R)) = 0$ for $n < c$; see Theorem 9.1.

Exercise 7.18. Compute $H^1_{(xy,xz)}(R)$ where $R = \mathbb{K}[x,y,z]$.

Exercise 7.19. Compute $H^2_{\mathfrak{m}}(R)$ where $R = \mathbb{K}[x,y,z]/(xz-y^2)$ and $\mathfrak{m} = (x,y,z)$.

Lecture 8. Hilbert Syzygy Theorem and Auslander-Buchsbaum Theorem (GL)

In this lecture we consider the top rung of the celebrated "hierarchy of rings": regular local rings. More generally, we will examine finitely generated modules of *finite projective dimension* over (local) rings. Regularity is characterized by finiteness of the projective dimension for every finitely generated module. Along the way, we will need to reconsider regular sequences, and prove the Auslander-Buchsbaum Theorem, which relates the existence of regular sequences to the finiteness of projective dimension.

In this lecture, we generally are concerned with a *local* ring $(R, \mathfrak{m}, \mathbb{K})$. This means that R is a Noetherian commutative ring with unique maximal ideal $\mathfrak{m}$, and $\mathbb{K} = R/\mathfrak{m}$.

8.1. Recall from Lecture 3 that the projective dimension $\operatorname{pd}_R M$ of a module M over a commutative ring R is by definition the minimal length of an R-projective resolution of M. Our first task is to give a homological characterization of projective dimension, at least in the case of a finitely generated module over a local ring. As we know, finitely generated projective modules over local rings are free, so a projective resolution has the form

$$\cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0 \tag{8.1.1}$$

with each F_i free of finite rank. In this case we also call (8.1.1) a *free resolution.* Note that by choosing bases for each F_i, we can write each φ_i as a matrix with entries from R.

Definition 8.2. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. A free resolution (8.1.1) is *minimal* if for each i, $\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. Equivalently, the entries of matrices representing the maps φ_i are all contained in $\mathfrak{m}$.

Proposition 8.3. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated nonzero R-module. Then $\operatorname{pd}_R M$ is the length of every minimal free resolution of M. Specifically, that value is given by*

$$\operatorname{pd}_R M = \inf\{i \geq 0 \mid \operatorname{Tor}_{i+1}^R(\mathbb{K}, M) = 0\}\,.$$

Proof. Let

$$\cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0 \tag{8.3.1}$$

be a minimal free resolution of M. Let β_i be the rank of the i^{th} free module F_i. We claim that $\beta_i = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)$. To see this, apply $R/\mathfrak{m} \otimes_R -$ to (8.3.1), and consider the truncation at $F_0/\mathfrak{m}F_0$. The maps φ_i then give homomorphisms between free modules over $\mathbb{K}$. Note that $F_i/\mathfrak{m}F_i \cong \mathbb{K}^{\beta_i}$. Since (8.3.1) was chosen minimal, the entries of φ_i were in $\mathfrak{m}$, and as in Example 3.14 the induced map $\overline{\varphi_i} : F_i/\mathfrak{m}F_i \longrightarrow F_{i-1}/\mathfrak{m}F_{i-1}$ is the zero map. Now, $\operatorname{Tor}_i^R(\mathbb{K}, M)$ is the homology in the i^{th} position of the complex with zero differentials

$$\cdots \xrightarrow{0} \mathbb{K}^{\beta_n} \xrightarrow{0} \mathbb{K}^{\beta_{n-1}} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{K}^{\beta_1} \xrightarrow{0} \mathbb{K}^{\beta_0} \longrightarrow 0\,,$$

which is $\mathbb{K}^{\beta_i}$, as claimed.

Now it is clear that $\operatorname{pd}_R M$ is the least i such that $\beta_{i+1} = 0$ in some free resolution, which is the least i such that $\beta_{i+1} = 0$ in every minimal free resolution. $\square$

Definition 8.4. The numbers $\beta_i = \beta_i^R(M) = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(\mathbb{K}, M)$ appearing in the proof of Proposition 8.3 are the *Betti numbers of M over R.*

Corollary 8.5. *The* global dimension *of a local ring* $(R, \mathfrak{m}, \mathbb{K})$ *is* $\operatorname{pd}_R \mathbb{K}$*. In particular,* $\operatorname{pd}_R \mathbb{K} < \infty$ *if and only if* $\operatorname{pd}_R M < \infty$ *for every finitely generated R-module M, and in this case* $\operatorname{pd}_R M \leq \operatorname{pd}_R \mathbb{K}$*.*

Proof. For any R-module M, we can compute $\operatorname{Tor}_i^R(\mathbb{K}, M)$ from a free resolution of $\mathbb{K}$. If $\operatorname{pd}_R \mathbb{K} < \infty$ then $\operatorname{Tor}_i^R(\mathbb{K}, M) = 0$ for $i > \operatorname{pd}_R \mathbb{K}$, and it follows that $\operatorname{pd}_R M \leq \operatorname{pd}_R \mathbb{K}$. □

The next definition is historically correct, but seems out of sequence here. Luckily, we will shortly prove that this is exactly the right place for it.

Definition 8.6. A local ring $(R, \mathfrak{m}, \mathbb{K})$ is *regular* if $\mathfrak{m}$ can be generated by $\dim R$ elements.

Remark 8.7. Recall that the minimal number of generators of $\mathfrak{m}$ is called the *embedding dimension of R.* It follows from Krull's (Generalized) Principal Ideal Theorem that $\mu(\mathfrak{m}) \geq \operatorname{height} \mathfrak{m} = \dim R$; regular rings are those for which equality obtains.

Geometrically, over a field of characteristic zero, say, regular local rings correspond to smooth (or "nonsingular") points on algebraic varieties. They are those for which the *tangent space* to the variety (at the specified point) has dimension no greater than that of the variety itself. It also turns out that $(R, \mathfrak{m})$ is a regular local ring if and only if the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ is a polynomial ring over the field $R/\mathfrak{m}$. We won't use this fact here, but it lends credence to the idea that all regular rings look more or less like polynomial rings over a field.

It will turn out below that a minimal generating set for the maximal ideal of a regular local ring is a regular sequence (as defined in Lecture 6). The regularity of this sequence is the key to our homological characterization of regular rings. Let us therefore consider regular sequences more carefully on their own terms.

Regular sequences and depth redux

The basic question we must address is how to establish the existence of a regular sequence, short of actually specifying elements. Specifically, given a ring R, an ideal $\mathfrak{a}$, and a module M, how can we tell whether $\mathfrak{a}$ contains an M-regular element? More generally, can we get a lower bound on $\operatorname{depth}_R(\mathfrak{a}, M)$?

Lemma 8.8. *Let R be a Noetherian ring and M, N finitely generated R-modules. Set* $\mathfrak{a} = \operatorname{ann} M$*. Then* $\mathfrak{a}$ *contains an N-regular element if and only if* $\operatorname{Hom}_R(M, N) = 0$*.*

Proof. ($\Longrightarrow$) We leave this direction as an easy exercise.
($\Longleftarrow$) Assume that $\mathfrak{a}$ consists entirely of zerodivisors on N. Then by Lemma 6.14, $\mathfrak{a}$ is contained in the union of the associated primes of N and, using prime avoidance, we can find $\mathfrak{p} \in \operatorname{Ass} N$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Localize at $\mathfrak{p}$ and reset notation to assume that $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m} \in \operatorname{Ass} N$. (Since $\operatorname{Hom}_R(M, N)_{\mathfrak{p}} = \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$, it suffices to show that the localized module is nonzero.) Then we have a surjection $M \longrightarrow M/\mathfrak{m}M \longrightarrow R/\mathfrak{m}$, and a monomorphism $R/\mathfrak{m} \hookrightarrow N$. The composition gives a nonzero homomorphism $M \longrightarrow N$. □

In combination with the following easy consequence of the long exact sequence of Ext, Lemma 8.8 will allow us to compute depths.

Proposition 8.9. *Let R be a ring and M, N R-modules. Suppose that there is an N-regular sequence $\underline{x} = x_1, \dots, x_n$ in $\mathfrak{a} := \operatorname{ann} N$. Then*

$$\operatorname{Ext}^n_R(M, N) \cong \operatorname{Hom}_R(M, N/\underline{x}N) \cong \operatorname{Hom}_{R/(\underline{x})}(M, N/\underline{x}N)\,.$$

Definition 8.10. We will say that a sequence of elements $x_1, \dots, x_n$ in an ideal $\mathfrak{a}$ of R is a *maximal regular sequence* on a module M (or *maximal M-regular sequence*) if $x_1, \dots, x_n$ is regular on M, and $x_1, \dots, x_n, y$ is not a regular sequence for any $y \in \mathfrak{a}$.

It's an easy exercise to show that in a Noetherian ring, every regular sequence can be lengthened to a maximal one. What's less obvious is that every regular sequence can be extended to one of the maximum possible length.

Theorem 8.11 (Rees)**.** *Let R be a Noetherian ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal of R such that $\mathfrak{a}M \neq M$. Then any two maximal M-regular sequences in $\mathfrak{a}$ have the same length, namely*

$$\operatorname{depth}_R(\mathfrak{a}, M) = \min\{i \geq 0 \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}\,.$$

One special case arises so often that we single it out. When $(R, \mathfrak{m}, \mathbb{K})$ is a local ring, we write simply $\operatorname{depth} M$ for $\operatorname{depth}_R(\mathfrak{m}, M)$.

Corollary 8.12. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated R-module. Then*

$$\operatorname{depth} M = \min\{i \mid \operatorname{Ext}^i_R(\mathbb{K}, M) \neq 0\}\,.$$

This fortuitous coincidence of an "elementary" property with a homological one accounts for the great power of the concept of depth. We note three immediate consequences; the first follows from the long exact sequence of Ext, and the second from computation of Ext via a projective resolution of the first argument.

Corollary 8.13 (The Depth Lemma)**.** *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of finitely generated modules over a ring R. Then for any ideal $\mathfrak{a}$,*

(1) $\operatorname{depth}_R(\mathfrak{a}, M) \geq \min\{\operatorname{depth}_R(\mathfrak{a}, M'), \operatorname{depth}_R(\mathfrak{a}, M'')\}$ *,*
(2) $\operatorname{depth}_R(\mathfrak{a}, M') \geq \min\{\operatorname{depth}_R(\mathfrak{a}, M), \operatorname{depth}_R(\mathfrak{a}, M'') + 1\}$ *, and*
(3) $\operatorname{depth}_R(\mathfrak{a}, M'') \geq \min\{\operatorname{depth}_R(\mathfrak{a}, M') - 1, \operatorname{depth}_R(\mathfrak{a}, M)\}$ *.*

Corollary 8.14. *For nonzero finitely generated M,* $\operatorname{depth}_R(\mathfrak{a}, M) \leq \operatorname{pd}_R R/\mathfrak{a}$.

Ideals so that equality is attained in Corollary 8.14, $\operatorname{depth}_R(\mathfrak{a}, R) = \operatorname{pd}_R R/\mathfrak{a}$, are called *perfect*. Our next main result is a substantial sharpening of this Corollary in a special case, the Auslander-Buchsbaum formula.

Theorem 8.15 (Auslander-Buchsbaum)**.** *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a nonzero finitely generated R-module of finite projective dimension. Then*

$$\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{depth} R\,.$$

Proof. If $\operatorname{pd}_R M = 0$, then M is free, and $\operatorname{depth} M = \operatorname{depth} R$. We may therefore assume that $h := \operatorname{pd}_R M \geq 1$. If $h = 1$, let

$$0 \longrightarrow R^n \xrightarrow{\ \varphi\ } R^m \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of M. We consider φ as an $m \times n$ matrix over R, with entries in $\mathfrak{m}$ by minimality. Apply $\mathrm{Hom}_R(\mathbb{K}, -)$ to obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_R(\mathbb{K}, R^n) \xrightarrow{\varphi_*} \mathrm{Ext}^i_R(\mathbb{K}, R^m) \longrightarrow \mathrm{Ext}^i_R(\mathbb{K}, M) \longrightarrow \cdots .$$

The entries of φ_* are the same as those of φ, after the identification $\mathrm{Ext}^i_R(\mathbb{K}, R^n) \cong \mathrm{Ext}^i_R(\mathbb{K}, R)^n$. Since each $\mathrm{Ext}^i_R(\mathbb{K}, R^n)$ is a vector space over $\mathbb{K}$, φ_* is identically zero. For each i, then, we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^i_R(\mathbb{K}, R)^m \longrightarrow \mathrm{Ext}^i_R(\mathbb{K}, M) \longrightarrow \mathrm{Ext}^{i+1}_R(\mathbb{K}, R)^n \longrightarrow 0 .$$

It follows that $\operatorname{depth} M = \operatorname{depth} R - 1$, and we are done in this case.

If $h > 1$, take any exact sequence $0 \longrightarrow M' \longrightarrow R^m \longrightarrow M \longrightarrow 0$. Then $\mathrm{pd}_R M' = \mathrm{pd}_R M - 1$. By induction, $\operatorname{depth} M' = \operatorname{depth} R - h + 1$. But by the Depth Lemma, $\operatorname{depth} M' = \operatorname{depth} M + 1$, and the result follows. □

Corollary 8.16. *Over a local ring R, a module M of finite projective dimension has* $\mathrm{pd}_R M \leq \operatorname{depth} R$.

Remark 8.17. For noncommutative rings, this result is quite false. In fact, it is an open question in the theory of noncommutative Artin rings (the so-called *finitistic dimension conjecture*) whether the number

$$\mathrm{fin.\,dim.}\, R = \sup\{\mathrm{pd}_R M \mid \mathrm{pd}_R M < \infty\}$$

is finite.

Here are two amusing applications of the Auslander-Buchsbaum formula. You may need to look ahead to Lectures 10 and 11 for the relevant definitions.

Exercise 8.18. Let S be a regular local ring and $\mathfrak{a}$ an ideal such that $R = S/\mathfrak{a}$ is a Cohen-Macaulay ring with $\dim(R) = \dim(S) - 1$. Prove that $\mathfrak{a}$ is principal.

Exercise 8.19. Let S be a regular local ring and $\mathfrak{a}$ an ideal such that $R = S/\mathfrak{a}$ is a Gorenstein ring with $\dim(R) = \dim(S) - 2$. Prove that $\mathfrak{a}$ is a complete intersection (*i.e.,* generated by two elements).

Let us return to the singular world of regular rings. Here are two false statements that are nonetheless useful: every regular ring looks like a polynomial ring over a field, and regular sequences behave like polynomial indeterminates. Our next goal is to revise these statements so that they make sense, and then to prove them. To get an idea where we're headed, observe the following: For $R = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring, the sequence $\underline{x} = x_1, \ldots, x_n$ is a regular sequence. (This falls out immediately upon induction on n.) In particular, the Koszul complex on $\underline{x}$ is exact, so that $R/(\underline{x}) = \mathbb{K}$ has finite projective dimension. From Corollary 8.5 it follows that R has finite global dimension. All we need do is to replace $\mathbb{K}[x_1, \ldots, x_n]$ by an arbitrary regular local ring.

Here is a first easy lemma, the proof of which we leave as an exercise.

Lemma 8.20. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. If $\mathfrak{a}$ contains a regular sequence of length n, then $\mathfrak{a}$ has height at least n.*

Caution: the converse is quite false! (But see Lecture 10.)

Lemma 8.21. *Let $(R, \mathfrak{m})$ be a local ring and x a minimal generator of $\mathfrak{m}$ (so that $x \in \mathfrak{m} \setminus \mathfrak{m}^2$). Then R is a regular local ring if and only if $R/(x)$ is so.*

Proof. Extend x to a full system of parameters $x = x_1, x_2, \ldots, x_n$. Then the maximal ideal of $R/(x)$ is generated by the images of $x_2, \ldots, x_n$, and has dimension one less than that of R. $\square$

Lemma 8.22. *A regular local ring is a domain.*

Proof. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d. The case $d = 0$ being trivial, assume first that $d = 1$. Then $\mathfrak{m}$ is a principal ideal, generated by some element $x \in R$. As $\dim R > 0$, $\mathfrak{m}$ is not nilpotent, but by Krull's Intersection Theorem, the intersection $\bigcap_{j \geq 0} x^j R$ is trivial. It follows that any element $a \in R$ can be written uniquely as a product of a unit times a power of x. If, then, $a = ux^p$ and $b = vx^q$ are such that $ab = 0$, with u and v units, we have $uvx^{p+q} = 0$, so that $x^{p+q} = 0$, a contradiction.

In the general case $d \geq 2$, use prime avoidance to find an element $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ outside the minimal primes of R. By induction and Lemma 8.21, $R/(x)$ is a domain, so (x) is a prime ideal. Now apply the same argument as above. $\square$

Proposition 8.23. *Let $\underline{x} = x_1, \ldots, x_n$ be a sequence of elements of a local ring $(R, \mathfrak{m})$. Consider the statements*

(1) $\underline{x}$ is an R-regular sequence.
(2) $\operatorname{height}(x_1, \ldots, x_i) = i$ *for* $i = 1, \ldots, n$.
(3) $\operatorname{height}(x_1, \ldots, x_n) = n$.
(4) $\underline{x}$ is part of a system of parameters for R.

Then (1) $\implies$ (2) $\implies$ (3) $\implies$ (4). *If R is a regular local ring, then each implication is an equivalence.*

In fact, one can get by with much less than regularity; the last statement of the Lemma remains true if R is only *Cohen-Macaulay*. See Lecture 10.

Proof. (1) $\implies$ (2). By the definition of a regular sequence and Lemma 6.14, we have $\operatorname{height}(x_1) < \operatorname{height}(x_1, x_2) < \cdots$; now use Lemma 8.20.

(2) $\implies$ (3). This one is obvious.

(3) $\implies$ (4). If R has dimension n, we are done. If $\dim R > n$, then $\mathfrak{m}$ is not a minimal prime of $(\underline{x})$. It follows that there exists $x_{n+1} \in \mathfrak{m} \setminus (\underline{x})$ so that $\operatorname{height}(x_1, \ldots, x_n, x_{n+1}) = n + 1$. Continuing in this way, we obtain a system of parameters for R, as desired.

Now assume that R is regular. Take a system of parameters $\underline{x}$ such that $\underline{x}$ generate $\mathfrak{m}$. In particular, each x_i is a minimal generator of $\mathfrak{m}$. As R is a domain by Lemma 8.22, x_1 is certainly a nonzerodivisor. As $R/(x_1)$ is again a regular local ring by Lemma 8.21, we are done by induction. $\square$

Putting the pieces together, we have shown that if $(R, \mathfrak{m})$ is a regular local ring, then $\mathfrak{m}$ is generated by a regular sequence, so $R/\mathfrak{m}$ has finite projective dimension. This leads us to the celebrated theorem of Serre.

Theorem 8.24 (Serre)**.** *The following are equivalent for a local ring* $(R, \mathfrak{m})$.

(1) R is regular.
(2) The global dimension of R is equal to $\dim R$.
(3) R has finite global dimension.

Proof. We have already established (1) $\implies$ (2), and (2) $\implies$ (3) is clear. For (3) $\implies$ (1), we go again by induction, this time on t, the minimal number of generators of $\mathfrak{m}$. If $t = 0$, then the zero ideal is maximal in R, so R is a field. Assume then that $t \geq 1$ and R has global dimension $g < \infty$. We first note that $\mathfrak{m} \notin \mathrm{Ass}(R)$: the finite free resolution of $R/\mathfrak{m}$ has all its matrices taking entries from $\mathfrak{m}$, so the final nonzero free module F_n is contained in $\mathfrak{m}F_{n-1}$. If $\mathfrak{m} \in \mathrm{Ass}(R)$, then $\mathfrak{m}$ is the annihilator of an element $a \in R$, so that $aF_n = 0$, a contradiction. By prime avoidance, then, we may take an element $x \in \mathfrak{m}$ outside of $\mathfrak{m}^2$ and the associated primes of R. The long exact sequence of Ext shows that $R/(x)$ has global dimension $g-1$, so $R/(x)$ is regular by induction. Finally, Lemma 8.21 implies that R is regular as well. □

Remark 8.25. So far we have clung to the case of local rings. A little care, however, allows one to generalize everything in this lecture to the case of graded rings, homogeneous elements, and homogeneous resolutions. In particular, we have the following theorem, for which an argument could be made that it is the second[9] theorem of commutative algebra. See [27] for a proof due to Schreyer. In particular, the proof given there, like Hilbert's original proof, does indeed produce a *free* resolution rather than merely a projective one.

Theorem 8.26 (Hilbert Syzygy Theorem)**.** *Let $\mathbb{K}$ be a field. Then every finitely generated module over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ has a free resolution of length at most n. If M is graded (with respect to any grading on $\mathbb{K}[x_1, \ldots, x_n]$) then the resolution can be chosen to be graded as well.*

[9] Since Hilbert's proof of the Syzygy Theorem (1890) uses his Basis Theorem (1888), there is at least one older.

Lecture 9. Depth and cohomological dimension (SI)

Given a cohomology theory, a basic problem is to relate its vanishing to properties of the object under consideration. For example, given a module M over a ring R, the functor $\operatorname{Ext}^1_R(M,-)$ is zero if and only if M is projective, while $\operatorname{Tor}^R_1(M,-)$ is zero if and only if M is flat. This section provides (partial) answers in the case of local cohomology.

Depth

Recall from Definition 6.15 that $\operatorname{depth}_R(\mathfrak{a}, M)$ denotes the length of the longest M-regular sequence contained in the ideal $\mathfrak{a}$. In Theorem 8.11 it was proved that when the R-module M is finitely generated, its depth with respect to $\mathfrak{a}$ can be measured in terms of the vanishing of $\operatorname{Ext}^*_R(R/\mathfrak{a}, M)$. One consequence of the following theorem is that depth is detected also by local cohomology modules; perhaps not a surprise, given Theorem 7.7.

Theorem 9.1. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal in R, and K the Koszul complex on a finite generating set for $\mathfrak{a}$. For each R-module M, the numbers*

$$\begin{aligned}&\inf\{n \mid \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \neq 0\},\\ &\inf\{n \mid H^n_{\mathfrak{a}}(M) \neq 0\}, \textit{ and}\\ &\inf\{n \mid H^n(\operatorname{Hom}_R(K, M)) \neq 0\}\end{aligned}$$

coincide. In particular, when M is finitely generated,

$$\operatorname{depth}_R(\mathfrak{a}, M) = \inf\{n \mid H^n_{\mathfrak{a}}(M) \neq 0\}.$$

Sketch of proof. Denote the three numbers in question e, l, and k, respectively; assume that each of these is finite, that is to say, the cohomology modules in question are nonzero (in some degree). The argument is more delicate when we do not assume a priori that these numbers are all finite; see the proof of [42, Theorem 2.1].

Let $I^\bullet$ be an injective resolution of M, in which case $H^*_{\mathfrak{a}}(M)$ is the cohomology of the complex $\Gamma_{\mathfrak{a}}(I^\bullet)$. It is not hard to verify that

$$H^n\big(\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(I^\bullet))\big) = \begin{cases} 0 & \text{for } n < l,\\ \operatorname{Hom}_R(R/\mathfrak{a}, H^l_{\mathfrak{a}}(M)) & \text{for } n = l.\end{cases}$$

By Proposition 7.2.1, the R-module $H^n(\Gamma_{\mathfrak{a}}(I^\bullet)) = H^n_{\mathfrak{a}}(M)$, is $\mathfrak{a}$-torsion, so the module $\operatorname{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ is nonzero. On the other hand, it is clear that

$$\operatorname{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(I^\bullet)) = \operatorname{Hom}_R(R/\mathfrak{a}, I^\bullet).$$

The preceding displays now yield $e = l$.

To prove that $l = k$, one first proves that the inclusion $\Gamma_{\mathfrak{a}}(I^\bullet) \subseteq I^\bullet$ induces a quasi-isomorphism $\operatorname{Hom}_R(K, I^\bullet) \simeq \operatorname{Hom}_R(K, \Gamma_{\mathfrak{a}}(I^\bullet))$. Since $H^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion for each n, the desired result follows from a repeated application of the following claim:

Let $C^\bullet$ be a complex of R-modules such that $H^n(C^\bullet)$ is $\mathfrak{a}$-torsion for each integer n, and zero for $n \ll 0$. For each $a \in \mathfrak{a}$, one has

$$\inf\{n \mid H^n(\operatorname{Hom}_R(K(a), C^\bullet)) \neq 0\} = \inf\{n \mid H^n(C^\bullet) \neq 0\}.$$

Indeed, this is immediate from the long exact sequence that results when we apply $\operatorname{Hom}_R(-, C^\bullet)$ to the exact sequence of complexes $0 \longrightarrow R \longrightarrow K(a) \longrightarrow \Sigma R \longrightarrow 0$, and pass to homology. $\square$

The preceding result suggests that when dealing with an arbitrary (that is to say, not necessarily finitely generated) module M, the 'right' notion of depth is the one introduced via any one of the formulae in the theorem above. Such an approach also has the merit that it immediately extends to the case where M is a *complex* of modules. What is more, Foxby and Iyengar [42] have proved that Theorem 9.1 extends to all complexes, with no restrictions on their homology. Thus, all (homological) notions of depth lead to the same invariant.

Remark 9.2. It turns out that for $d = \operatorname{depth}_R(\mathfrak{a}, M)$, one has

$$\operatorname{Ass}_R \operatorname{Ext}^R_d(R/\mathfrak{a}, M) = \operatorname{Ass}_R H^d_{\mathfrak{a}}(M) = \operatorname{Ass}_R H^d(\operatorname{Hom}_R(K, M)),$$

see [58, ??] or the discussion in [85, page 564]. When the R-module M is finitely generated, so is $H^d(\operatorname{Hom}_R(K, M))$; in particular, the latter has only finitely many associated primes. The equalities above now imply that the R-module $H^d_{\mathfrak{a}}(M)$ has only finitely many associated primes. This suggests a natural question: does each local cohomology module have finitely many associated primes? This is not the case, as we will see in Lecture 22

Now we know in which degree the nonzero local cohomology modules of an R-module M begin to appear. Theorem 7.13 tells us that they disappear eventually, so the next natural step is to determine in which degree the last nonvanishing cohomology module occurs; what one has in mind is a statement akin to Theorem 9.1. This has proved to be a rather difficult endeavour, and every result we know of relates this number to the topology of the support of the module M. The first of these is due to Grothendieck:

Theorem 9.3. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and M a finitely generated R-module. Then*

$$\sup\{n \mid H^n_{\mathfrak{m}}(M) \neq 0\} = \dim_R M.$$

The proof we present uses the local duality theorem, covered later in Lecture 11.

Proof. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R; it is a local ring with maximal ideal $\mathfrak{m}\widehat{R}$, and residue field $\mathbb{K}$. The $\widehat{R}$-module $\widehat{R} \otimes_R M$ is finite, with

$$\dim_{\widehat{R}} \widehat{M} = \dim_R M \quad \text{and} \quad H^*_{\mathfrak{m}\widehat{R}}(\widehat{R} \otimes_R M) \cong H^*_{\mathfrak{m}}(M),$$

where the first equality is essentially [4, Corollary 11.19], and the second follows from Proposition 7.16.3. Thus, substituting $\widehat{R}$ and $\widehat{R} \otimes_R M$ for R and M respectively, we may assume that R is $\mathfrak{m}$-adically complete. Cohen's Structure Theorem now provides a surjective homomorphism $(Q, \mathfrak{n}, \mathbb{K}) \longrightarrow R$, with Q a regular local ring. According to Proposition 7.16.2, viewing M as a Q-module through R, one has $H^*_{\mathfrak{n}}(M) \cong H^*_{\mathfrak{m}}(M)$, so we may replace R by Q and assume that R is a complete regular local ring.

We are now in a position to apply Theorem 11.32 which yields, for each integer n, an isomorphism of R-modules

$$H^n_{\mathfrak{m}}(M) \cong \operatorname{Ext}_R^{\dim R - n}(M, R)^\vee,$$

with $(-)^\vee = \operatorname{Hom}_R(-, E)$, where E is the injective hull of $\mathbb{K}$. Since $(-)^\vee$ is faithful (that is, it takes nonzero modules to nonzero modules), the preceding isomorphisms imply the first equality below

$$\begin{aligned}\sup\{n \mid H^n_{\mathfrak{m}}(M) \neq 0\} &= \dim R - \inf\{l \mid \operatorname{Ext}^l_R(M, R) \neq 0\},\\ &= \dim R - \operatorname{grade}_R M.\end{aligned}$$

The second equality is given by Theorem **??**. Finally, $\dim R - \operatorname{grade}_R M = \dim M$, since R is Cohen-Macaulay; see [16, (2.1.2)].

Grade has not been defined; also, let us not have to rely on BH here. □

Remark 9.4. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. For any finitely generated R-module M, we know from Theorems 9.1 and 9.3 that $H^n_{\mathfrak{m}}(M) = 0$ for $n \notin [\operatorname{depth} M, \dim M]$, and that it is nonzero for $n = \operatorname{depth} M$ and $\dim M$. In general, nothing can be said about the vanishing of local cohomology for intermediate values of n: given any sequence of non-negative integers $n_0 < \cdots < n_s$, there exists a local ring $(R, \mathfrak{m}, \mathbb{K})$ with $\operatorname{depth} R = n_0$ and $\dim R = n_s$, and such that $H^n_{\mathfrak{m}}(R)$ is nonzero exactly when n in one of the integers n_i, see Evans and Griffith [35].

The search for a meaning to the top degree of nonvanishing of local cohomology (with respect to an arbitrary ideal) leads to the following considerations:

Cohomological dimension

Definition 9.5. Let R be a Noetherian ring and $\mathfrak{a}$ an ideal in R. For each R-module M, set

$$\operatorname{cd}_R(\mathfrak{a}, M) = \inf\{s \in \mathbb{N} \mid H^n_{\mathfrak{a}}(M) = 0 \text{ for each } n \geqslant s + 1\}.$$

The *cohomological dimension* of $\mathfrak{a}$ in R is the integer

$$\operatorname{cd}_R(\mathfrak{a}) = \sup\{\operatorname{cd}_R(\mathfrak{a}, M) \mid M \text{ an } R\text{-module}\}.$$

It is immediate from Theorem 9.3 that for a local ring R with maximal ideal $\mathfrak{m}$, one has $\operatorname{cd}_R(\mathfrak{m}) = \dim R$.

It turns out that the cohomological dimension has a 'test module':

Theorem 9.6. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Then $\operatorname{cd}_R(\mathfrak{a}) = \operatorname{cd}_R(\mathfrak{a}, R)$.*

Proof. Set $d = \operatorname{cd}_R(\mathfrak{a})$; by Corollary 7.14, this number is finite. Thus, $H^n_{\mathfrak{a}}(-) = 0$ for $n \geqslant d + 1$, and there is an R-module M with $H^d_{\mathfrak{a}}(M) \neq 0$. Pick a surjective homomorphism $F \twoheadrightarrow M$, with F a free R-module, and complete to an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0.$$

From the resulting long exact sequence, Proposition 7.2.4, one obtains an exact sequence

$$H^d_{\mathfrak{a}}(K) \longrightarrow H^d_{\mathfrak{a}}(F) \longrightarrow H^d_{\mathfrak{a}}(M) \longrightarrow H^{d+1}_{\mathfrak{a}}(K) = 0.$$

We conclude that $H^d_{\mathfrak{a}}(F) \neq 0$, and therefore $H^d_{\mathfrak{a}}(R) \neq 0$ by Proposition 7.2.3. □

The result above can be enhanced to a precise expression relating the local cohomology of M and R in high degrees:

Exercise 9.7. Let R be a Noetherian ring, $\mathfrak{a}$ an ideal in R, and set $d = \mathrm{cd}_R(\mathfrak{a})$. Prove that for any R-module M, one has a natural isomorphism

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{a}}(R) \otimes_R M.$$

Hint: Theorem 9.6 implies that the functor $H^d_{\mathfrak{a}}(-)$ is right exact. (By the way, what are its left derived functors?)

Exercise 9.8. Let $\mathfrak{a}$ an ideal in a Noetherian ring R, and let M be a finitely generated R-module. Prove the following:

(1) $\mathrm{cd}_R(\mathfrak{a}, M) = \mathrm{cd}_R(\mathfrak{a}, R/\operatorname{ann}_R M)$.

(2) If N is a finitely generated R-module and $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(N)$, then

$$\mathrm{cd}_R(\mathfrak{a}, M) \leqslant \mathrm{cd}_R(\mathfrak{a}, N).$$

(3) Find examples to show that the analogues of (1) and (2) fail for $\operatorname{depth}_R(\mathfrak{a}, -)$.

One way to do approach these exercises is via the following, independently relevant, exercise:

Exercise 9.9. Let R be a Noetherian ring and M a finitely generated R-module. If $\operatorname{Supp}_R(M) = \operatorname{Spec} R$, show that for each nonzero R-module H, the module $H \otimes_R M$ is nonzero.

Arithmetic rank

Recall that if ideals $\mathfrak{a}$ and $\mathfrak{b}$ have the same radical, then $H^n_{\mathfrak{a}}(-) = H^n_{\mathfrak{b}}(-)$; see Proposition 7.2.2. This suggests the following definition.

Definition 9.10. Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. The *arithmetic rank* of $\mathfrak{a}$ is the number

$$\operatorname{ara}\mathfrak{a} = \inf\{\nu(\mathfrak{b}) \mid \mathfrak{b} \text{ an ideal with } \operatorname{rad}\mathfrak{b} = \operatorname{rad}\mathfrak{a}\},$$

where $\nu(\mathfrak{b})$ stands for the minimal number of generators of the ideal $\mathfrak{b}$. Evidently, $\operatorname{ara}\mathfrak{a} \leqslant \nu(\mathfrak{a})$; however, the arithmetic rank of $\mathfrak{a}$ can be a lot smaller than $\nu(\mathfrak{a})$; consider, for example, that $\operatorname{ara}(\mathfrak{a}^n) = \operatorname{ara}\mathfrak{a}$ for each integer $n \geqslant 1$.

Remark 9.11. The arithmetic rank is also pertinent from a geometric perspective. For instance, when R is a polynomial ring over an algebraically closed field, $\operatorname{ara}\mathfrak{a}$ equals the minimal number of hypersurfaces needed to cut out the algebraic set $\operatorname{Var}(\mathfrak{a})$ in affine space. By the way, the radical of $\mathfrak{a}$ is not necessarily the 'best' ideal defining $\operatorname{Var}(\mathfrak{a})$, see Example 9.21.

Now we return to cohomological dimensions:

Proposition 9.12. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. Then*

$$\operatorname{height}\mathfrak{a} \leqslant \mathrm{cd}_R(\mathfrak{a}) \leqslant \operatorname{ara}\mathfrak{a}.$$

Proof. For any ideal $\mathfrak{b}$ of R with $\operatorname{rad}\mathfrak{b} = \operatorname{rad}\mathfrak{a}$, we have $\mathrm{cd}_R(\mathfrak{a}) = \mathrm{cd}_R(\mathfrak{b})$ by Proposition 7.2.2, and $\mathrm{cd}_R(\mathfrak{a}) \leqslant \nu(\mathfrak{b})$ by Corollary 7.14. This proves the inequality on the right.

Let $h = \operatorname{height}\mathfrak{a}$, and pick a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ with $\dim R_{\mathfrak{p}} = h$. Then

$$H^h_{\mathfrak{a}}(R)_{\mathfrak{p}} \cong H^h_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \cong H^h_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}),$$

where the first isomorphism is by Proposition 7.16.3, and the second by Proposition 7.2.2. But $H^h_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$ by Grothendieck's Theorem 9.3, so the above isomorphisms imply that $H^h_{\mathfrak{a}}(R) \neq 0$, which settles the inequality on the left. □

This proposition leads us to an important result on the arithmetic rank of ideals in local rings; in its original form, it is due to Kronecker [91], and has been extended and improved on by several people, notably Forster [40]; see also [103].

Theorem 9.13 (Kronecker-Foster)**.** *Let $\mathfrak{a}$ be an ideal in a local ring R. Then* $\operatorname{ara}\mathfrak{a} \leqslant \dim R$*.*

We provide a proof of this result; it uses a weak form of a result commonly known as "prime avoidance" [4, Proposition 1.11].

Lemma 9.14 (Prime avoidance)**.** *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals in a commutative ring R. If an ideal $\mathfrak{a}$ is such that $\mathfrak{a} \subseteq \bigcup_{i=1}^n \mathfrak{p}_i$, then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.*

Proof of Theorem 9.13. Set $\mathcal{P} = \operatorname{Spec} R \setminus V(\mathfrak{a})$; these are the prime ideals of R not containing $\mathfrak{a}$. For each $n \geqslant 0$, set $\mathcal{P}(n) = \{\mathfrak{p} \in \mathcal{P} \mid \operatorname{height}\mathfrak{p} = n\}$, so that

$$\mathcal{P} = \bigcup_{i=0}^{d-1} \mathcal{P}(n),$$

where $d = \dim R$. The idea of the proof is to pick elements $r_0, \dots, r_d$ in $\mathfrak{a}$ such that for $0 \leqslant i \leqslant d-1$, the ideal $\mathfrak{b}_i = (r_0, \dots, r_i)$ satisfies the following condition:

$$(*) \qquad \text{if } \mathfrak{p} \in \mathcal{P}(i), \text{ then } \mathfrak{b}_i \not\subseteq \mathfrak{p}.$$

Once this is accomplished, we have $\operatorname{rad}\mathfrak{b}_{d-1} = \operatorname{rad}\mathfrak{a}$ giving the desired result.

The choice of the r_i is iterative: for $i = 0$, since $\mathcal{P}(0)$ is a subset of the minimal primes of R, its cardinality is finite, so we may pick an element $r_0 \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p}\in\mathcal{P}(0)} \mathfrak{p}$ by prime avoidance, Lemma 9.14. Evidently, $\mathfrak{b}_0$ satisfies $(*)$ for $i = 0$.

Suppose that for some $0 \leqslant i \leqslant d-2$, elements $r_0, \dots, r_i$, have been chosen such that condition $(*)$ is satisfied for i. Another use of prime avoidance allows us to pick an element

$$r_{i+1} \in \mathfrak{a} \ \setminus \bigcup_{\mathfrak{p}\in \operatorname{Min}(\mathfrak{b}_i)\cap\mathcal{P}(i+1)} \mathfrak{p}.$$

We claim that the ideal $\mathfrak{b}_{i+1} = \mathfrak{b} + (r_{i+1})$ satisfies condition $(*)$ for $i+1$. Indeed, suppose there exists $\mathfrak{p}$ in $\mathcal{P}(i+1)$ containing $\mathfrak{b}_{i+1}$, and hence also $\mathfrak{b}_i$. Since $\operatorname{height}\mathfrak{p} = i+1$, if there is a prime ideal $\mathfrak{p}'$ with $\operatorname{height}\mathfrak{p}' = i$ such that $\mathfrak{b}_i \subseteq \mathfrak{p}' \subseteq \mathfrak{p}$, then $\mathfrak{p}' \in \mathcal{P}(i)$, which contradicts condition $(*)$. Thus $\mathfrak{p}$ is minimal over $\mathfrak{b}_i$, that is to say, $\mathfrak{p} \in \operatorname{Min}(\mathfrak{b}_i) \cap \mathcal{P}(i+1)$. Therefore $r_{i+1} \in \mathfrak{b}_{i+1} \subseteq \mathfrak{p}$, which is a contradiction. This completes the induction argument, and hence the proof of the result. □

Remark 9.15. Suppose R is a Noetherian ring of dimension d, but is not local. Then, following the above construction, we obtain an ideal $\mathfrak{b}_{d-1}$ that satisfies condition $(*)$ for all prime ideals of height less than $\dim R$. Moreover, there is at most a finite set of prime (indeed, maximal) ideals of height d for which the condition fails. Picking a final element r_d in $\mathfrak{a}$ but outside these finitely many maximal ideals, gives an ideal $\mathfrak{b} = (r_0, \dots, r_d) \subseteq \mathfrak{a}$ with the same radical as $\mathfrak{a}$. In particular, all varieties in affine d-space over a zero-dimensional ring can be defined by $d+1$ equations.

Moreover, if R is a standard graded polynomial ring in d variables over a local ring of dimension zero, and if $\mathfrak{a}$ is homogeneous, then the construction in the proof of Theorem 9.13 shows how to obtain a homogeneous ideal $\mathfrak{b}_{d-1}$ that satisfies condition $(*)$ for all primes of height less than d, *and* for the homogeneous maximal ideal. In particular, all varieties in projective $(d-1)$-space over a field can be defined by d equations.

Proposition 9.16. *Let $\mathfrak{a}$ be an ideal in a Noetherian ring R. For each finitely generated R-module M, one has that $H^n_{\mathfrak{a}}(M) = 0$ for $n \geqslant \dim M + 1$.*

In particular, $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R$.

Proof. Note that for any R-module N, if $N_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Spec} R$, then $N = 0$. For any prime ideal $\mathfrak{p}$, one has $H^n_{\mathfrak{a}}(M)_{\mathfrak{p}} = H^n_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for each n, see Proposition 7.16.1. Since $\dim M_{\mathfrak{p}} \leqslant \dim M$, it suffices to consider the case where R is local.

Now M is a module over the ring $S = R/\operatorname{ann}_R(M)$, and Proposition 7.16.3 implies that $H^n_{\mathfrak{a}}(M) \cong H^n_{\mathfrak{a}S}(M)$ for each n. It remains to note that $\operatorname{cd}_S(\mathfrak{a}S) \leqslant \dim S$ by Theorem 9.13, and that $\dim S = \dim M$. $\square$

Exercise 9.17. Let R be a Noetherian ring of dimension d. Show that $H^d_{\mathfrak{a}}(M) = H^d_{\mathfrak{a}}(R) \otimes_R M$ for any R-module M.

It is clear that Proposition 9.16 is not optimal; for example, the arithmetic rank of $\mathfrak{a}$ could be smaller than $\dim R$. The question arises: what is the import of the nonvanishing of $H^{\dim R}_{\mathfrak{a}}(R)$? In the lectures ahead we will encounter a number of answers, which cover different contexts. Here is a prototype, due to Hartshorne and Lichtenbaum; its proof is given in Lecture 14.

Theorem 9.18. *Let R be a d-dimensional complete local domain, and let $\mathfrak{a}$ be an ideal in R. Then $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R - 1$ if and only if $\dim R/\mathfrak{a} \geqslant 1$.*

Proposition 9.12 may also be used to obtain lower bound on arithmetic ranks. Here is an beautiful example, due to Hartshorne, that illustrates this particular use of cohomological dimensions:

Example 9.19. Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x, y, u, v]$. Consider the ideals $\mathfrak{b}' = (x, y)$ and $\mathfrak{b}'' = (u, v)$, and set $\mathfrak{a} = \mathfrak{b}' \cap \mathfrak{b}''$. Note that $\operatorname{height}_R(\mathfrak{a}) = 2$. We claim that $\operatorname{ara}\mathfrak{a} \geqslant 3$.

Indeed, the Mayer-Vietoris sequence 15.2 arising from the ideals $\mathfrak{b}'$ and $\mathfrak{b}''$ yields an exact sequence

$$\longrightarrow H^3_{\mathfrak{b}'}(R) \oplus H^3_{\mathfrak{b}''}(R) \longrightarrow H^3_{\mathfrak{a}}(R) \longrightarrow H^4_{\mathfrak{b}'+\mathfrak{b}''}(R) \longrightarrow H^4_{\mathfrak{b}'}(R) \oplus H^4_{\mathfrak{b}''}(R) \longrightarrow .$$

From Corollary 7.14 one obtains that the first and the last displayed terms of this exact sequence are zero, and from Theorem 9.3 one obtains that the second term is nonzero, since $\mathfrak{b}' + \mathfrak{b}'' = (x, y, u, v)$ is a maximal ideal of height 4. Thus, $H^3_{\mathfrak{a}}(R) \neq 0$, so that $\operatorname{ara}\mathfrak{a} \geqslant 3$ by Proposition 9.12. The following exercise shows that $\operatorname{ara}\mathfrak{a} = 3$.

Exercise 9.20. Let $R = \mathbb{K}[x, y, u, v]$. Find elements $f, g, h \in R$ with

$$\operatorname{rad}(f, g, h) = (x, y) \cap (u, v).$$

The ideal $\mathfrak{a}$ in Example 9.19 is not a set-theoretic complete intersection: the variety that it defines has codimension two, but it cannot be defined by two equations. It is an open question whether every *irreducible* curve is a set theoretic complete intersection; Cowsik and Nori [24] have proved that this is so over fields of positive characteristic.

The last item in this lecture is an example, promised in Remark 9.11, which shows that the radical of an ideal does not necessarily provide the most optimal set of generators for the variety it defines.

Example 9.21. Let $f \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial of degree three. Assume that the singular locus of $\operatorname{Var}(f) \subset \mathbb{A}^3$ is precisely the origin. In this case, $f = 0$ defines a smooth projective elliptic curve $E = \operatorname{Var}(f)$ in $\operatorname{Proj}\mathbb{C}[x, y, z] = \mathbb{P}^2_{\mathbb{C}}$.

Consider the Segre embedding $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^5_{\mathbb{C}}$ which, in homogeneous coordinates, corresponds to the map

$$(x, y, z) \times (s, t) \mapsto (xs, ys, zs, xt, yt, zt).$$

Let $u_1, u_2, u_3, v_1, v_2, v_3$ be homogeneous coordinates in $\mathbb{P}^5_{\mathbb{C}}$, and $R = \mathbb{C}[u_1, \dots, v_3]$. Let $\Delta_1, \Delta_2, \Delta_3$ be the maximal minors of the matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

Then the image of $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ equals $\operatorname{Var}(\Delta_1, \Delta_2, \Delta_3)$. We will see in Example 19.30 that even though the image is only codimension two, one cannot get away with only two defining equations. However, let us study the image of $E \times \mathbb{P}^1_{\mathbb{C}}$ in $\mathbb{P}^5_{\mathbb{C}}$. The (full) defining ideal of $E \times \mathbb{P}^1_{\mathbb{C}}$ is generated by the three minors, and all 'bihomogeneous consequences' of the equation $f(x, y, z) = 0$, where $\deg(u_i) = (1, 0)$ and $\deg(v_j) = (0, 1)$; these other equations arise from expressing in terms of the u_i, v_j the equations

$$s^3 \cdot f = s^2 t \cdot f = st^2 \cdot f = t^3 \cdot f = 0.$$

There is a certain ambiguity here. For example, if $f = x^3 + y^3 + z^3 + xyz$ then $s^2 tf = u_1^2 v_1 + u_2^2 v_2 + u_3^2 v_3 + u_1 u_2 v_3 = u_1^2 v_1 + u_2^2 v_2 + u_3^2 v_3 + u_1 u_3 v_2$. Of course, these ways of rewriting differ simply by an expression in the ideal $(\Delta_1, \Delta_2, \Delta_3)$. However, there are four equations that are naturally associated to the situation. These are $F_1 = f(u_1, u_2, u_3)$ and F_2, F_3, F_4 which arise by the rule

$$F_{i+1} = \frac{1}{4 - i} \sum_{j=1}^{3} v_j \frac{\partial F_i}{\partial u_j}.$$

Note that $F_4 = f(v_1, v_2, v_3)$. The ideal of R defining $E \times \mathbb{P}^1_{\mathbb{C}}$ is the prime ideal

$$\mathfrak{a} = (\Delta_1, \Delta_2, \Delta_3, F_1, F_2, F_3, F_4).$$

One cannot define the image of $E \times \mathbb{P}^1_{\mathbb{C}}$ by fewer than four equations since the local cohomology module $H^4_{\mathfrak{a}}(R)$ is nonzero by an argument involving the topology of the elliptic curve. Parts of this argument are treated in Lecture 19. On the other hand, using the group law of the elliptic curve, one can show that every point in $\mathbb{P}^4$ that lies on the common zero locus of all four F_j also lies on the three hypersurfaces described by the Δ_i. It follows that the 7-generated prime ideal $\mathfrak{a}$ is the radical of the 4-generated ideal (F_1, F_2, F_3, F_4), and that $\operatorname{ara}\mathfrak{a} = 4$. For details, the reader is invited to take a look at [147].

Definition 10.1. A local ring $(R, \mathfrak{m})$ is *Cohen-Macaulay* if some (equivalently, every) system of parameters for R is a regular sequence on R. A ring R is said to be *Cohen-Macaulay* if $R_{\mathfrak{m}}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$ of R.

If M is a module over a local ring $(R, \mathfrak{m})$, the depth of $\mathfrak{m}$ on M is often abbreviated as $\operatorname{depth} M$. Recall that this is the length of a maximal sequence of elements in the ideal $\mathfrak{m}$ which form a regular sequence on M. Consequently a local ring R is Cohen-Macaulay if and only if $\operatorname{depth} R = \dim R$.

If R is an $\mathbb{N}$-graded ring, finitely generated over a field $R_0 = \mathbb{K}$, then R is Cohen-Macaulay if and only if some (equivalently, every) homogeneous system of parameters is a regular sequence on R.

Example 10.2. Rings of dimension 0 are trivially Cohen-Macaulay. A domain of dimension 1 is Cohen-Macaulay.

Example 10.3. By [4, Theorem 11.22, Lemma 11.23], a regular local ring is a domain. Let $(A, \mathfrak{m})$ be a regular local ring of dimension d, and $x_1, \dots, x_d \in A$ be elements which generate the maximal ideal $\mathfrak{m}$. Since A is a domain, x_1 is not a zerodivisor on A. For all $2 \leqslant i \leqslant d$, the ring $A/(x_1, \dots, x_{i-1})$ is a regular local ring, so x_i is not a zerodivisor on $A/(x_1, \dots, x_{i-1})$. It follows that a regular local ring is Cohen-Macaulay.

Example 10.4. Let A be a regular local ring of dimension d, and $f_1, \dots, f_r \in A$ be elements such that the ring $R = A/(f_1, \dots, f_r)$ has dimension $d - r$. If a ring R (or, more generally, its completion) has this form, then R is said to be a *complete intersection*. The elements $f_1, \dots, f_r \in A$ can be extended to a system of parameters $f_1, \dots, f_r, y_1, \dots, y_{d-r}$ for A, in which case the images of $y_1, \dots, y_{d-r}$ in R form a system of parameters for R. Since A is Cohen-Macaulay, $f_1, \dots, f_r, y_1, \dots, y_{d-r}$ is a regular sequence on A, but then $y_1, \dots, y_{d-r}$ is a regular sequence on $R = A/(f_1, \dots, f_r)$. Hence a complete intersection is Cohen-Macaulay.

Example 10.5. Let $R = \mathbb{K}[x, y]/(x^2, xy)$. Then R has dimension 1, and the element y constitutes a homogeneous system of parameters for R. However y is a zerodivisor, so R is not Cohen-Macaulay.

Example 10.6. Let R be the subring of the polynomial ring $\mathbb{K}[s, t]$ generated, as a $\mathbb{K}$-algebra, by the monomials s^4, s^3t, st^3, t^4. Then R has dimension 2, and s^4, t^4 is a homogeneous system of parameters for R. Since R is a domain, s^4 is a nonzerodivisor. However t^4 is a zerodivisor on R/s^4R since

$$t^4(s^3t)^2 = s^4(st^3)^2$$

and $(s^3t)^2 \notin s^4R$. It follows that R is not Cohen-Macaulay.

Example 10.7. Let R be a subring of a polynomial ring $\mathbb{K}[x_1, \dots, x_d]$ which is generated, as a $\mathbb{K}$-algebra, by monomials in the variables $x_1, \dots, x_d$. Such *affine semigroup rings* are discussed in Lecture 20. If R is a normal ring, then it is Cohen-Macaulay by a theorem of Hochster, [69, Theorem 1]. Note that the ring in Example 10.6 is not normal. For a proof of Hochster's theorem using $\mathbb{Z}^d$-graded homological algebra and convex polyhedral geometry, see Exercise 20.35.

We shall see next how the Cohen-Macaulay property arises quite naturally in different situations; for more on the question "What does it really mean for a ring to Cohen-Macaulay?" see Hochster's beautiful survey article [71].

Noether normalization

We recall the Noether normalization theorem in its graded form:

Theorem 10.8. *Let R be an $\mathbb{N}$-graded ring, which is finitely generated over a field $R_0 = \mathbb{K}$. If $x_1, \dots, x_d$ is a homogeneous system of parameters for R, then $x_1, \dots, x_d$ are algebraically independent over $\mathbb{K}$, and R is a finitely generated module over the subring $\mathbb{K}[x_1, \dots, x_d]$.*

In the situation above, a natural question arises: when is R a free module over the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$? Before giving the answer, we look at a few examples.

Example 10.9. Let S_n be the symmetric group on n symbols acting on the polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ by permuting the variables. Then the ring of invariants is $R^{S_n} = \mathbb{K}[e_1, \dots, e_n]$, where e_i is the elementary symmetric function of degree i in the variables $x_1, \dots, x_n$. The ring R is a free R^{S_n}-module with basis

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}, \quad \text{where } 0 \leqslant m_i \leqslant i-1 \text{ for } 1 \leqslant i \leqslant n,$$

see, for example, [3, Chapter II.G].

Example 10.10. Fix a positive integer d, and let R be the subring of the polynomial ring $\mathbb{K}[x, y]$ which is generated, as a $\mathbb{K}$-algebra, by the monomials of degree d, i.e., by the elements $x^d, x^{d-1}y, \dots, xy^{d-1}, y^d$. As a homogeneous system of parameters for R, we take x^d, y^d. Then Theorem 10.8 implies that R is a finitely generated module over the polynomial ring $A = \mathbb{K}[x^d, y^d]$; indeed, the monomials $x^i y^j$ with $0 \leqslant i, j \leqslant d-1$ are a generating set for R as an A-module. It is a straightforward exercise to prove that R is a free A-module.

Example 10.11. Let $R = \mathbb{K}[s^4, s^3t, st^3, t^4]$, which is the ring encountered in Example 10.6. The elements s^4, t^4 form a homogeneous system of parameters for R, and so R is a finitely generated module over the polynomial subring $A = \mathbb{K}[s^4, t^4]$. The monomials $1, s^3t, st^3, s^6t^2, s^2t^6$ are a minimal generating set for R as an A-module. However R is not a free module on this minimal generating set, since we have a relation

$$t^4(s^6t^2) = s^4(s^2t^6).$$

Note that this is precisely the relation we used earlier to demonstrate that R is not Cohen-Macaulay.

Theorem 10.12. *Let R be an $\mathbb{N}$-graded ring which is finitely generated over a field $R_0 = \mathbb{K}$, and $x_1, \dots, x_d$ be a homogeneous system of parameters for R. Then R is Cohen-Macaulay if and only if it is a free module over the polynomial subring $\mathbb{K}[x_1, \dots, x_d]$.*

Proof. Consider R as a module over the polynomial ring $A = \mathbb{K}[x_1, \dots, x_d]$. The Hilbert syzygy theorem implies that R has finite projective dimension over A. The Auslander-Buchsbaum formula then gives us

$$\operatorname{depth} R + \operatorname{pd}_A R = \operatorname{depth} A.$$

Since $\operatorname{depth} A = d = \dim R$, the ring R is Cohen-Macaulay if and only if $\operatorname{pd}_R A = 0$, i.e., if and only if R is a projective A-module. Since R is a finitely generated graded module over the graded ring A, it follows that R is a projective module if and only if it is free. $\square$

Exercise 10.13. Let $\mathbb{K}$ denote a field. For each of the following, find a homogeneous systems of parameters and determine whether the ring is Cohen-Macaulay.

(1) $R = \mathbb{K}[x, y, z]/(xy, yz)$.
(2) $R = \mathbb{K}[x, y, z]/(xy, yz, zx)$.
(3) $R = \mathbb{K}[s, t, x, y]/(sx, sy, tx, ty)$.

Exercise 10.14. Let R be an $\mathbb{N}$-graded ring finitely generated over a field $R_0 = \mathbb{K}$. If R is Cohen-Macaulay with a homogeneous system of parameters $f_1, \ldots, f_d$, prove that the Hilbert-Poincaré series of R has the form

$$P(R, t) = \frac{g(t)}{(1 - t^{e_1}) \cdots (1 - t^{e_d})}$$

where $\deg f_i = e_i$ and $g(t)$ is a polynomial with **nonnegative** integer coefficients.

Intersection multiplicities

Let $f, g \in \mathbb{K}[x, y]$ be two polynomials without a common factor. Then $\mathrm{Var}(f)$ and $\mathrm{Var}(g)$ are plane curves with isolated points of intersection. Suppose that the origin $p = (0, 0)$ is one of these intersection points, and we wish to compute the *intersection multiplicity* or *order of tangency* of the curves at the point p, this can be achieved by working in the local ring $R = \mathbb{K}[x, y]_{(x,y)}$ and taking the length of the module

$$R/(f, g) \cong R/(f) \otimes_R R/(g).$$

Example 10.15. The intersection multiplicity of the parabola $\mathrm{Var}(y - x^2)$ with the x-axis $\mathrm{Var}(y)$ is

$$\ell\left(\frac{\mathbb{K}[x, y]_{(x,y)}}{(y - x^2, y)}\right) = 2,$$

and with the y-axis $\mathrm{Var}(x)$ is

$$\ell\left(\frac{\mathbb{K}[x, y]_{(x,y)}}{(y - x^2, x)}\right) = 1.$$

Example 10.16. The intersection multiplicity of $\mathrm{Var}(y^2 - x^2 - x^3)$ with the x-axis $\mathrm{Var}(y)$ is

$$\ell\left(\frac{\mathbb{K}[x, y]_{(x,y)}}{(y^2 - x^2 - x^3, y)}\right) = \ell\left(\frac{\mathbb{K}[x, y]_{(x,y)}}{(y, x^2(1 + x))}\right) = \ell\left(\frac{\mathbb{K}[x, y]_{(x,y)}}{(x^2, y)}\right) = 2.$$

This illustrates the need to work with the *local* ring $\mathbb{K}[x, y]_{(x,y)}$ to measure the intersection multiplicity at $(0, 0)$; the length of $\mathbb{K}[x, y]/(y^2 - x^2 - x^3, y)$ is 3 since it also counts the other intersection point of the curves, $(-1, 0)$.

Remark 10.17. For plane curves $\mathrm{Var}(f)$ and $\mathrm{Var}(g)$, the length of $\mathbb{C}[x, y]_{(x,y)}/(f, g)$ gives the *correct* intersection multiplicity of the curves at the origin, in the sense that for typical small complex numbers ϵ, this intersection multiplicity is precisely the number of distinct intersection points of $\mathrm{Var}(f)$ and $\mathrm{Var}(g + \epsilon)$ that lie in a small neighbourhood of the origin.

To generalize from plane curves to arbitrary algebraic sets, let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathbb{K}[x_1, \ldots, x_n]$ defining algebraic sets $\mathrm{Var}(\mathfrak{a})$ and $\mathrm{Var}(\mathfrak{b})$ with an isolated point of

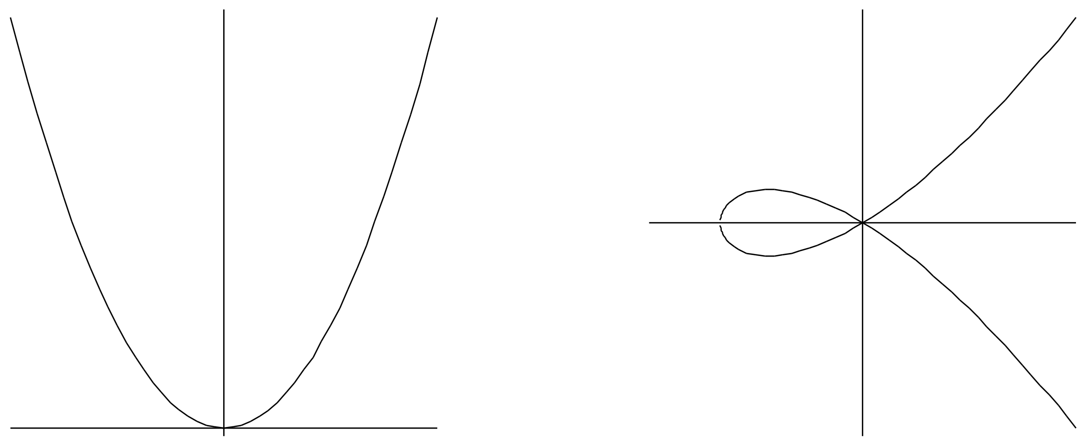


FIGURE 3. The curves $y = x^2$ and $y^2 = x^2 + x^3$

intersection, $p = (0, \dots, 0)$. Working in the local ring $R = \mathbb{K}[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ and taking the length of the module

$$R/(\mathfrak{a} + \mathfrak{b}) \cong R/\mathfrak{a} \otimes_R R/\mathfrak{b}$$

may or may not give the correct answer in the sense of perturbing the equations and counting distinct points. Serre's definition in [143] gives the correct answer:

$$\chi(R/\mathfrak{a}, R/\mathfrak{b}) = \sum_{i=0}^{\dim R} (-1)^i \ell(\mathrm{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b})).$$

Example 10.18. Let $\mathfrak{a} = (x^3 - w^2y, x^2z - wy^2, xy - wz, y^3 - xz^2)$ and $\mathfrak{b} = (w, z)$ be ideals of the polynomial ring $\mathbb{C}[w, x, y, z]$. Then the ideal $\mathfrak{a} + \mathfrak{b} = (w, z, x^3, xy, y^3)$ has radical (w, x, y, z), so the algebraic sets $\mathrm{Var}(\mathfrak{a})$ and $\mathrm{Var}(\mathfrak{b})$ have a unique point of intersection, namely the origin in $\mathbb{C}^4$. Let $R = K[w, x, y, z]_{(w,x,y,z)}$. Note that

$$\ell(R/(\mathfrak{a} + \mathfrak{b})) = \ell(R/(w, z, x^3, xy, y^3)) = 5.$$

However we claim that the intersection multiplicity of $\mathrm{Var}(\mathfrak{a})$ and $\mathrm{Var}(\mathfrak{b})$ should be 4. To see this, we perturb the linear space $\mathrm{Var}(w, z)$ and count the number of points in the intersection

$$\mathrm{Var}(\mathfrak{a}) \cap \mathrm{Var}(w - \delta, z - \epsilon)$$

for typical small complex numbers δ and ϵ, i.e., we determine the number of elements of $\mathbb{C}^4$ which are solutions of the equations

$$w = \delta,\ z = \epsilon,\ x^3 - w^2y = 0,\ x^2z - wy^2 = 0,\ xy - wz = 0,\ y^3 - xz^2 = 0.$$

It is easily seen that the value of x-coordinate is a fourth root of $\delta^3\epsilon$, and that the choice of x uniquely determines the value of the y-coordinate. Hence for nonzero δ and ϵ, there are four distinct intersection points.

To determine the Serre intersection multiplicity $\chi(R/\mathfrak{a}, R/\mathfrak{b})$, first note that the Koszul complex

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -z \\ w \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} w & z \end{pmatrix}} R \longrightarrow 0.$$

gives a projective resolution of $R/\mathfrak{b}$. To compute $\operatorname{Tor}_i^R(R/\mathfrak{a}, R/\mathfrak{b})$, we tensor this complex with

$$R/\mathfrak{a} \cong \mathbb{C}[s^4, s^3t, st^3, t^4],$$

and take the homology of the resulting complex

$$0 \longrightarrow R/\mathfrak{a} \xrightarrow{\begin{pmatrix} -t^4 \\ s^4 \end{pmatrix}} (R/\mathfrak{a})^2 \xrightarrow{\begin{pmatrix} s^4 & t^4 \end{pmatrix}} R/\mathfrak{a} \longrightarrow 0.$$

In this notation, the module $\operatorname{Tor}_1^R(R/\mathfrak{a}, R/\mathfrak{b})$ is the $\mathbb{C}$-vector space spanned by the element

$$\begin{pmatrix} -s^2t^6 \\ s^6t^2 \end{pmatrix} \in (R/\mathfrak{a})^2.$$

Since $\operatorname{Tor}_2^R(R/\mathfrak{a}, R/\mathfrak{b}) = 0$ and $\ell(\operatorname{Tor}_0^R(R/\mathfrak{a}, R/\mathfrak{b})) = \ell(R/(\mathfrak{a}+\mathfrak{b})) = 5$, we see that

$$\chi(R/\mathfrak{a}, R/\mathfrak{b}) = 5 - 1 + 0 = 4.$$

As the following theorem illustrates, the issue is precisely that the ring $R/\mathfrak{a} \cong \mathbb{C}[s^4, s^3t, st^3, t^4]$ is not Cohen-Macaulay.

Theorem 10.19. [143, page 111] *Let $(R, \mathfrak{m})$ be a regular local ring, and $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary. Then the Serre intersection multiplicity $\chi(R/\mathfrak{a}, R/\mathfrak{b})$ equals*

$$\ell(R/(\mathfrak{a}+\mathfrak{b})) = \ell(\operatorname{Tor}_0^R(R/\mathfrak{a}, R/\mathfrak{b}))$$

if and only if $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are Cohen-Macaulay rings.

The naïve attempt $\ell(R/(f,g))$ gives the correct intersection multiplicity for plane curves $\operatorname{Var}(f)$ and $\operatorname{Var}(g)$ since the rings $R/(f)$ and $R/(g)$ are Cohen-Macaulay.

Exercise 10.20. Let $R = \mathbb{C}[w,x,y,z]$, $\mathfrak{a} = (w^3 - x^2, wy - xz, y^2 - wz^2, w^2z - xy)$, and $\mathfrak{b} = (w, z)$.

(1) Check that the ring $R/\mathfrak{a} \cong \mathbb{C}[s^2, s^3, st, t]$ is not Cohen-Macaulay.
(2) Compute the length of $R/(\mathfrak{a}+\mathfrak{b})$.
(3) Compute the intersection multiplicity of the algebraic sets $\operatorname{Var}(\mathfrak{a})$ and $\operatorname{Var}(\mathfrak{b})$ at the origin in $\mathbb{C}^4$.

Invariant theory

Now let G be a group acting on a polynomial ring T. We use T^G to denote the *ring on invariants*, i.e., the ring

$$T^G = \{x \in T : g(x) = x \text{ for all } g \in G\}.$$

Example 10.21. Let $T = \mathbb{K}[a,b,c,s,t]$ be a polynomial ring over an infinite field $\mathbb{K}$. Consider the action of the multiplicative group $G = \mathbb{K} \setminus \{0\}$ on T under which an element $\lambda \in G$ sends a polynomial $f(a,b,c,s,t) \in T$ to the polynomial

$$f(\lambda a, \lambda b, \lambda c, \lambda^{-1}s, \lambda^{-1}t).$$

Note that under this action, every monomial in T is taken to a scalar multiple. Let $f \in T$ be a polynomial which is fixed by the group action. If a monomial $a^i b^j c^k s^m t^n$ occurs in f with nonzero coefficient, comparing coefficients of this monomial in f and $\lambda(f)$ gives us

$$\lambda^{i+j+k-m-n} = 1 \quad \text{for all} \quad \lambda \in G.$$

Since G is infinite, we must have $i + j + k = m + n$. It follows that the ring of invariants is the monomial ring

$$T^G = \mathbb{K}[as, bs, cs, at, bt, ct].$$

The polynomial ring $R = \mathbb{K}[u, v, w, x, y, z]$ maps onto T^G via the $\mathbb{K}$-algebra homomorphism φ where

$$\varphi(u) = as, \quad \varphi(v) = bs, \quad \varphi(w) = cs, \quad \varphi(x) = at, \quad \varphi(y) = bt, \quad \varphi(z) = ct.$$

By Exercise 1.34, $\ker(\varphi)$ is the prime ideal $\mathfrak{p} = (\Delta_1, \Delta_2, \Delta_3)$ where $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, $\Delta_3 = uy - vx$. We would like to obtain a *graded* resolution of $R/\mathfrak{p} \cong T^G$ over R, i.e., one where the maps in the resolution preserve degree. Towards this end, we use $R(m)$ to denote the module R with the shifted grading where $[R(m)]_n = [R]_{m+n}$. The graded resolution of $R/\mathfrak{p}$ is

$$0 \longrightarrow R^2(-3) \xrightarrow{\begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix}} R^3(-2) \xrightarrow{\begin{pmatrix} \Delta_1 & \Delta_2 & \Delta_3 \end{pmatrix}} R \longrightarrow 0.$$

Such a resolution can be used to compute the Hilbert-Poincaré series of $R/\mathfrak{p}$ as follows. For each integer n, we have an exact sequence of $\mathbb{K}$-vector spaces

$$0 \longrightarrow [R^2(-3)]_n \longrightarrow [R^3(-2)]_n \longrightarrow [R]_n \longrightarrow [R/\mathfrak{p}]_n \longrightarrow 0.$$

The alternating sum of the vector space dimensions must be zero, so

$$P(R/\mathfrak{p}, t) = P(R, t) - 3P(R(-2), t) + 2P(R(-3), t).$$

Since $P(R(-m), t) = t^m P(R, t)$ and $P(R, t) = (1 - t)^{-6}$, we see that

$$P(R/\mathfrak{p}, t) = \frac{1 - 3t^2 + 2t^3}{(1 - t)^6} = \frac{1 + 2t}{(1 - t)^4},$$

which is, of course, precisely what we obtained earlier in Example 1.32.

Remark 10.22. Given an action of G on a polynomial ring T, the *first fundamental problem of invariant theory*, according to Hermann Weyl [160], is to find generators for the ring of invariants T^G, in other words to find a polynomial ring R with a surjection $\varphi : R \longrightarrow T^G$. The *second fundamental problem* is to find relations amongst these generators, i.e., to find a free R-module R^{b_1} which surjects onto $\ker \varphi$. Continuing this sequence of fundamental problems, one would like to determine the resolution of T^G as an R-module, i.e., to find an exact complex

$$\cdots \longrightarrow R^{b_3} \longrightarrow R^{b_2} \longrightarrow R^{b_1} \longrightarrow R \xrightarrow{\varphi} T^G \longrightarrow 0.$$

In Example 10.21, we obtained this for the given group action, and saw how the resolution provides information such as the Hilbert-Poincaré series (and hence the dimension, multiplicity, etc.) of the ring of invariants T^G. Another fundamental question then arises: what is the length of the minimal resolution of T^G as an R-module, i.e., what is the projective dimension $\operatorname{pd}_R T^G$? The Cohen-Macaulay property appears once again:

Theorem 10.23. *Let G be a group acting on a polynomial ring $T = \mathbb{K}[x_1, \ldots, x_d]$ by degree preserving $\mathbb{K}$-algebra automorphisms. Assume that T^G is a finitely generated $\mathbb{K}$-algebra, and let R be a polynomial ring mapping onto T^G. Then*

$$\operatorname{pd}_R T^G \geqslant \dim R - \dim T^G,$$

and equality holds precisely if T^G is Cohen-Macaulay.

Proof. By the Auslander-Buchsbaum formula,

$$\operatorname{pd}_R T^G = \operatorname{depth} R - \operatorname{depth} T^G.$$

The polynomial ring R is Cohen-Macaulay and $\operatorname{depth} T^G \leqslant \dim T^G$, so we get the asserted inequality. Equality holds if and only if $\operatorname{depth} T^G = \dim T^G$, i.e., precisely when T^G is Cohen-Macaulay. □

Exercise 10.24. Let p be a prime and let $T = \mathbb{Z}/p\mathbb{Z}[a, b, c, s, t]$ be a polynomial ring. Consider the action of the multiplicative group $G = \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ on T under which an element $\lambda \in G$ sends a polynomial $f(a, b, c, s, t) \in T$ to the polynomial

$$f(\lambda a, \lambda b, \lambda c, \lambda^{-1}s, \lambda^{-1}t).$$

Determine the ring of invariants T^G.

Exercise 10.25. Let $T = \mathbb{C}[x_1, \ldots, x_d]$ be a polynomial ring, n be a positive integer, and σ be the $\mathbb{C}$-linear automorphism of T with

$$\sigma(x_i) = e^{2\pi i/n} x_i \quad \text{for all } 1 \leqslant i \leqslant d.$$

Determine the ring of invariants T^G where G is the cyclic group generated by σ.

We record some results which imply that the rings of invariants arising in several important situations are indeed Cohen-Macaulay.

Theorem 10.26 (Hochster-Eagon, [73])**.** *Let T be a polynomial ring over a field $\mathbb{K}$, and let G be a finite group acting on T by degree preserving $\mathbb{K}$-algebra automorphisms. If $|G|$ is invertible in $\mathbb{K}$, then T^G is Cohen-Macaulay.*

Proof. Consider the map $\rho : T \longrightarrow T^G$ given by

$$\rho(t) = \frac{1}{|G|} \sum_{g \in G} g(t).$$

It is easily verified that $\rho(t) = t$ for all $t \in T^G$, and that ρ is a T^G-module homomorphism. Hence T^G is a direct summand of T as a T^G-module, i.e., $T \cong T^G \oplus M$ for some T^G-module M.

Let $x_1, \ldots, x_d$ be a homogeneous system of parameters for T^G. Since G is finite, T is an integral extension of T^G, so $x_1, \ldots, x_d$ is a system of parameters for T as well. The ring T is Cohen-Macaulay, so $x_1, \ldots, x_d$ is a regular sequence on T. But then it is also a regular sequence on its direct summand T^G. □

The proof of Theorem 10.26 shows, more generally, that a direct summand S of a Cohen-Macaulay ring T is Cohen-Macaulay, provided that a system of parameters for S forms part of a system of parameters for T. In general, a direct summand of a Cohen-Macaulay ring need not be Cohen-Macaulay as we see in the next example.

Example 10.27. Let $\mathbb{K}$ be an infinite field, and let T be the hypersurface
$$T = \mathbb{K}[x_0, x_1, x_2, y_0, y_1]/(x_0^3 + x_1^3 + x_2^3).$$
The multiplicative group $G = \mathbb{K} \setminus \{0\}$ acts $\mathbb{K}$-linearly on T where
$$\lambda : \begin{cases} x_i \longmapsto \lambda x_i \\ y_j \longmapsto \lambda^{-1} y_j \end{cases} \qquad \text{for } \lambda \in G.$$
As in Example 10.21, the ring of invariants T^G is the $\mathbb{K}$-algebra generated by the elements $x_i y_j$. The ring T is a complete intersection, and hence is Cohen-Macaulay. Also, it is easy to see that T^G is a direct summand of T. However T^G is not Cohen-Macaulay: the elements $x_0y_0, x_1y_1, x_1y_0 + x_0y_1$ form a homogeneous system of parameters for T^G (verify!) and satisfy the relation
$$x_2y_0x_2y_1(x_1y_0 + x_0y_1) = (x_2y_0)^2 x_1y_1 + (x_2y_1)^2 x_0y_0,$$
which shows that $x_1y_0+x_0y_1$ is a zerodivisor on $T^G/(x_0y_0, x_1y_1)T^G$. For a different proof that T^G is not Cohen-Macaulay, see Example 22.5.

Remark 10.28. A *linear algebraic group* is Zariski closed subgroup of a general linear group $GL_n(\mathbb{K})$. A linear algebraic group G is *linearly reductive* if every finite dimensional G-module is a direct sum of irreducible G-modules, equivalently, if every G-submodule has a G-stable complement.

Linearly reductive groups in characteristic zero include finite groups, algebraic tori (i.e., products of copies of the multiplicative group of the field), and the classical groups $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$, $Sp_{2n}(\mathbb{K})$, $O_n(\mathbb{K})$ and $SO_n(\mathbb{K})$.

If a linearly reductive group acts on a finitely generated $\mathbb{K}$-algebra T by degree preserving $\mathbb{K}$-algebra automorphisms, then there is a T^G-linear map, the *Reynolds operator* $\rho : T \longrightarrow T^G$, which makes T^G a direct summand of T.

Theorem 10.29 (Hochster-Roberts, [76])**.** *Let G be a linearly reductive group acting linearly on a polynomial ring T. Then T^G is Cohen-Macaulay. More generally, a direct summand of a polynomial ring is Cohen-Macaulay.*

We record a few examples of rings of invariants which, by the Hochster-Roberts theorem, are Cohen-Macaulay.

Example 10.30. Let $n \leqslant d$ be positive integers, $X = (x_{ij})$ be an $n \times d$ matrix of variables over a field $\mathbb{K}$, and consider the polynomial ring $T = \mathbb{K}[X]$, i.e., T is a polynomial ring in nd variables. Let $G = SL_n(\mathbb{K})$ be the special linear group acting on T as follows:
$$M : x_{ij} \longrightarrow (MX)_{ij},$$
i.e., an element $M \in G$ send x_{ij}, the (i, j) entry of the matrix X, to the (i, j) entry of the matrix MX. Since $\det M = 1$, it follows that the size n minors of X are fixed by the group action. It turns out whenever $\mathbb{K}$ is infinite, T^G is the $\mathbb{K}$-algebra generated by these size n minors. The ring T^G is the homogeneous coordinate ring of the Grassmann variety of n dimensional subspaces of a d-dimensional vector space. The relations between the minors are the well-known Plücker relations.

Example 10.31. Let $X = (x_{ij})$ and $Y = (y_{jk})$ be $r \times n$ and $n \times s$ matrices of variables over an infinite field $\mathbb{K}$, and consider the polynomial ring $T = \mathbb{K}[X, Y]$ of dimension $rn+ns$. Let $G = GL_n(\mathbb{K})$ be the general linear group acting on R where $M \in G$ maps the entries of X to corresponding entries of XM^{-1} and the entries of Y to those of MY. Then T^G is the $\mathbb{K}$-algebra generated by the entries of the product matrix XY. If $Z = (z_{ij})$ is an $r \times s$ matrix of new variables mapping onto the entries of XY, the kernel of the induced $\mathbb{K}$-algebra surjection $\mathbb{K}[Z] \longrightarrow T^G$ is the ideal generated by the size $n+1$ minors of the matrix Z. These *determinantal rings* are the subject of [17]. The case where $r = 2$, $s = 3$, $n = 1$ was earlier encountered in Example 10.21.

Exercise 10.32. Let $X = (x_{ij})$ be a $n \times n$ matrix of variables over a field $\mathbb{K}$, and take the polynomial ring

$$A = \mathbb{K}[x_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n].$$

Consider the hypersurface $R = A/(\det X)$. If Δ is any size $(n-1)$ minor of X, show that R_Δ, i.e., the localization of R at the element Δ, is a regular ring.

Exercise 10.33. Let G be a group acting by ring automorphisms on a domain R.

(1) Show that the action of G on R extends to an action of G on the fraction field $\mathbb{L}$ of R.
(2) If R is normal, show that the ring of invariants R^G is normal.
(3) If G is finite, prove that $\mathbb{L}^G$ is the fraction field of R^G.

Local cohomology

We have seen how Cohen-Macaulay rings come up in the study of intersection multiplicities, and in studying resolution of rings of invariants. They also arise in a natural way when considering local cohomology:

Theorem 10.34. *Let $(R, \mathfrak{m})$ be a local ring of dimension d. Then R is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(R) = 0$ for all $i \neq d$.*

Proof. Let $x_1, \dots, x_d$ be a system of parameters for R. Then $H^i_{\mathfrak{m}}(R)$ can be computed as the i^{th} cohomology module of the complex

$$0 \longrightarrow R \longrightarrow \bigoplus R_{x_i} \longrightarrow \bigoplus R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_d} \longrightarrow 0,$$

hence $H^i_{\mathfrak{m}}(R) = 0$ for all $i > d$.

By Theorem 9.1, the depth of R is the least integer i such that $H^i_{\mathfrak{m}}(R)$ is nonzero. Since R is Cohen-Macaulay if and only if $\operatorname{depth} R = d$, the result follows. $\square$

Lecture 11. Gorenstein Rings (CM)

In this lecture we introduce Gorenstein rings, which, among the class of commutative Noetherian rings, are remarkable for their 'duality properties'. The definition we adopt, however, is perhaps not too illuminating:

Definition 11.1. A Noetherian ring R is said to be *Gorenstein* if $\operatorname{injdim}_R R < \infty$.

Note that when $\operatorname{injdim}_R R$ is finite, so is $\operatorname{injdim}_{R_\mathfrak{p}}(R_\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ in R. Hence when R is Gorenstein, so is $R_\mathfrak{p}$. As to the converse:

Exercise 11.2. Prove that a Noetherian R is Gorenstein if and only if $\dim R$ is finite and $R_\mathfrak{m}$ is Gorenstein for each maximal ideal $\mathfrak{m}$ in R.

In the literature, you may find that a Noetherian ring R is defined to be 'Gorenstein' if it is locally Gorenstein; the finiteness of Krull dimension has been sacrificed. However, from the perspective of these lectures, where we focus on Gorenstein rings for their duality properties, the more stringent definition is the 'right one'.

Here is one source of Gorenstein rings; recall that a regular ring is one which has finite global dimension.

Proposition 11.3. *A regular local ring is Gorenstein.*

Proof. Since R has finite global dimension, $\operatorname{Ext}_R^i(-, R) = 0$ for $i \gg 0$, and so R has finite injective dimension. □

It turns out when R is Gorenstein, the injective resolution of R (over itself) can be described completely; see Remark 11.28. This is a consequence of general results concerning the nature of injective resolution, discussed below.

Bass numbers

Let R be a Noetherian ring and let M be a finitely generated R-module. Let $I^\bullet$ be the minimal injective resolution of M. For each prime ideal $\mathfrak{p}$ and integer i, the number

$$\mu_R^i(\mathfrak{p}, M) = \text{number of copies of } E_R(R/\mathfrak{p}) \text{ in } I^i,$$

is the i^{th} *Bass number* of M with respect to $\mathfrak{p}$, see Definition A.24. When R is local with maximal ideal $\mathfrak{m}$, we sometimes write $\mu_R^i(M)$ for $\mu_R^i(\mathfrak{m}, M)$. By Theorem A.25, the Bass numbers can be calculated as

$$\mu_R^i(\mathfrak{p}, M) = \operatorname{rank}_{\mathbb{K}(\mathfrak{p})} \operatorname{Hom}_{R_\mathfrak{p}}(\mathbb{K}(\mathfrak{p}), I^i_\mathfrak{p}) = \operatorname{rank}_{\mathbb{K}(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(\mathbb{K}(\mathfrak{p}), M_\mathfrak{p}),$$

where $\mathbb{K}(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. These formulae also show that Bass numbers are finite, and that they can be calculated locally, that is to say, if U is a multiplicatively closed subset of R with $U \cap \mathfrak{p} = \varnothing$, then

$$\mu_R^i(\mathfrak{p}, M) = \mu^i_{U^{-1}R}(U^{-1}\mathfrak{p}, U^{-1}M).$$

The following exercise is a first step towards 'understanding' the structure of minimal injective resolutions.

Exercise 11.4. Let M be an R-module. Prove that

$$\operatorname{Ass}_R(M) = \{\mathfrak{p} \mid \mu_R^0(\mathfrak{p}, M) \neq 0\} = \operatorname{Ass} E_R(M).$$

Definition 11.5. A functor F on R-modules is *half-exact* if, given an exact sequence of R-modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$, the sequence

$$F(L) \longrightarrow F(M) \longrightarrow F(N)$$

is exact.

Exercise 11.6. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and let F be a half-exact functor. If $F(\mathbb{K}) = 0$, prove that $F(L) = 0$ for any finite length R-module L.

Bass numbers propagate along chains of prime ideals in the following sense:

Lemma 11.7. *Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of R with $\operatorname{height}(\mathfrak{q}/\mathfrak{p}) = 1$. For any finitely generated R-module M, if $\mu_R^i(\mathfrak{p}, M) \neq 0$, then $\mu_R^{i+1}(\mathfrak{q}, M) \neq 0$.*

Proof. Localizing at $\mathfrak{q}$, we may assume that R is local with maximal ideal $\mathfrak{m}$, and that $\dim R/\mathfrak{p} = 1$. Suppose that $\mu_R^{i+1}(M) = 0$, in other words, that

$$\operatorname{Ext}_R^{i+1}(\mathbb{K}(\mathfrak{m}), M) = 0.$$

Since $\operatorname{Ext}_R^{i+1}(-, M)$ is a half-exact functor, $\operatorname{Ext}_R^{i+1}(L, M) = 0$ for any R-module L of finite length; see Exercise 11.6. In particular, $\operatorname{Ext}_R^{i+1}(R/(\mathfrak{p} + xR), M) = 0$ for any element $x \in R \setminus \mathfrak{p}$, since the length of the R-module $R/(\mathfrak{p} + xR)$ is finite. Now the short exact sequence

$$0 \longrightarrow R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p} + xR) \longrightarrow 0$$

yields an exact sequence of finitely generated modules

$$\cdots \longrightarrow \operatorname{Ext}_R^i(R/\mathfrak{p}, M) \xrightarrow{x} \operatorname{Ext}_R^i(R/\mathfrak{p}, M) \longrightarrow \operatorname{Ext}_R^{i+1}(R/(\mathfrak{p} + xR), M) = 0.$$

Nakayama's Lemma yields $\operatorname{Ext}_R^i(R/\mathfrak{p}, M) = 0$, and hence $\mu_R^i(\mathfrak{p}, M) = 0$. □

Here is an immediate corollary:

Corollary 11.8. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring, and let M be a finitely generated R-module. Then*

$$\operatorname{injdim}_R M = \sup\{i \mid \operatorname{Ext}_R^i(\mathbb{K}, M) \neq 0\}.$$

There is a better result, proved by Fossum, Foxby, Griffith, and Reiten [41], and also by P. Roberts [130]:

Theorem 11.9. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and let M be a finitely generated R-module. Then $\operatorname{Ext}_R^i(\mathbb{K}, M) \neq 0$ for $\operatorname{depth}_R M \leqslant i \leqslant \operatorname{injdim}_R M$.* □

Using Exercise 11.4 and the Lemma 11.7, solve:

Exercise 11.10. For any finitely generated R-module M, one has an inequality:

$$\operatorname{injdim} M \geqslant \dim M.$$

In particular, if R admits a nonzero finitely generated injective module, then R is Artinian. (See also Remark 11.13.)

This exercise is a little misleading in that the injective dimension of M is either infinite, or depends only on R. More precisely:

Proposition 11.11. *Let R be a local ring and M a finitely generated R-module. If $\operatorname{injdim}_R M$ is finite, then*

$$\operatorname{injdim} M = \operatorname{depth} R.$$

Proof. Set $d = \operatorname{depth} R$, and choose a maximal R-sequence $\boldsymbol{x} = x_1, \dots, x_d \in \mathfrak{m}$. Computing via the Koszul resolution of $R/\boldsymbol{x}R$ over R, one sees that

$$\operatorname{Ext}^d_R(R/\boldsymbol{x}R, M) \cong M/\boldsymbol{x}M,$$

which is nonzero by Nakayama's Lemma. Thus $\operatorname{injdim} M \geqslant d = \operatorname{depth} R$. Suppose that $e = \operatorname{injdim} M > d$. Since $\operatorname{depth} R/\boldsymbol{x}R = 0$, there is an exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow R/\boldsymbol{x}R \longrightarrow C \longrightarrow 0.$$

Since $\operatorname{Ext}^{e+1}_R(-, M) = 0$, the induced long exact sequence has the form

$$\cdots \longrightarrow \operatorname{Ext}^e_R(R/\boldsymbol{x}R, M) \longrightarrow \operatorname{Ext}^e_R(\mathbb{K}, M) \longrightarrow 0.$$

Since $e > d = \operatorname{pd}_R(R/\boldsymbol{x}R)$, one has $\operatorname{Ext}^e_R(R/\boldsymbol{x}R, M) = 0$, and so $\operatorname{Ext}^e_R(\mathbb{K}, M) = 0$. However, $\operatorname{Ext}^i_R(\mathbb{K}, M) = 0$ for $i > e$ since $\operatorname{injdim}_R(M) = e$, so Corollary 11.8 yields $\operatorname{injdim}_R(M) < e$; this is a contradiction. $\square$

The proof of the following result is now clear:

Corollary 11.12. *A Gorenstein ring is Cohen-Macaulay.*

Remark 11.13. More generally, if a Noetherian local ring R has a finitely generated module of finite injective dimension, then R is Cohen-Macaulay. This is known as Bass's Conjecture, now a theorem as Peskine and Szpiro showed that it follows from the the Intersection Theorem [127], which in turn they had proved for a large class of rings containing a field and which Roberts subsequently proved for all local rings [133].

Recognizing Gorenstein rings

Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring. Recall that the *socle* of an R-module M, denoted $\operatorname{soc}(M)$, is the submodule $(0 :_M \mathfrak{m})$ of M; it may be identified with $\operatorname{Hom}_R(\mathbb{K}, M)$. It is thus a $\mathbb{K}$-vector space, and indeed, the largest $\mathbb{K}$-vector space contained in M. (Convince yourself that the terminology is particularly well-chosen!)

Theorem 11.14. *Let R be a zero-dimensional local ring. The following conditions are equivalent:*

(1) *R is Gorenstein;*
(2) *R is injective as an R-module;*
(3) $\operatorname{rank}_{\mathbb{K}} \operatorname{soc}(R) = 1$*;*
(4) *$E_R(\mathbb{K}) \cong R$;*
(5) *the ideal* (0) *of R is irreducible.*

The equivalence of (1) and (2) is immediate from Proposition 11.11, and that of (2) and (3) was proved in Theorem A.33. The equivalence of these with the other conditions are all consequences of the following exercise; do fill in the details.

Exercise 11.15. Let R be a local ring and M an Artinian R-module. Prove that the inclusion $\operatorname{soc}(M) \subseteq M$ is an essential extension. Does this statement hold when M is not Artinian?

Condition (3) in the result above is a particularly simple test for identifying Gorenstein rings:

Example 11.16. Let $\mathbb{K}$ be a field. The ring $\mathbb{K}[x,y]/(x^2,y^2)$ is Gorenstein since its socle is generated by a single element xy, while the ring $\mathbb{K}[x,y]/(x,y)^2$ is not, since its socle is minimally generated by the elements x and y.

However, one must be careful: this test applies only to zero dimensional rings. The local ring $R = \mathbb{K}[[x,y]]/(x^2,xy)$ is not Gorenstein (check!) and yet $\operatorname{soc}(R) \cong \mathbb{K}$.

How can one identify higher dimensional Gorenstein rings? One way is to use following result:

Proposition 11.17. *Let $\boldsymbol{x}$ be a regular sequence on a ring R.*

(1) *If R is Gorenstein, then so is $R/(\boldsymbol{x})$.*
(2) *The converse holds if R is local.*

The proof of this result is almost trivial, once we use Corollary 11.8, and the following theorem due to Rees:

Theorem 11.18. *Let M and N be R-modules. If $x \in \operatorname{ann}_R N$ is a nonzerodivisor on R and on M, then*

$$\operatorname{Ext}_R^{i+1}(N,M) \cong \operatorname{Ext}_{R/xR}^{i}(N, M/xM) \qquad \textit{for each } i.$$

Here is one use of Proposition 11.17.

Example 11.19. Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x^3,x^5,x^7]$, viewed as a subring of $\mathbb{K}[x]$. Evidently R is a domain of dimension 1, and hence Cohen-Macaulay. However, $\operatorname{soc}(R/x^3R)$ has rank two. Therefore the ring R/x^3R is not Gorenstein, and so neither is R itself.

We now enlarge our supply of Gorenstein rings; first, a technical note:

Lemma 11.20. *A local ring $(R,\mathfrak{m},\mathbb{K})$ is Gorenstein if and only if its $\mathfrak{m}$-adic completion $\widehat{R}$ is Gorenstein.*

Proof. This follows from the isomorphism

$$\operatorname{Ext}_{\widehat{R}}^{i}(\mathbb{K},\widehat{R}) \cong \operatorname{Ext}_R^i(\mathbb{K},R) \otimes_R \widehat{R}$$

and Corollary 11.8. □

Proposition 11.21. *If a local ring is a complete intersection, then it is Gorenstein.*

Proof. The completion $\widehat{R}$ must be a complete intersection, so, by the previous lemma, we may assume that R is complete. Then $R \cong Q/(\boldsymbol{x})$ for some regular local ring Q and a Q-regular sequence $\boldsymbol{x}$. Now Q is Gorenstein by Proposition 11.3, hence so is R by Proposition 11.17. □

There are more Gorenstein rings than complete intersections:

Example 11.22. Let $\mathbb{K}$ be a field, and let

$$R = \mathbb{K}[x,y,z]/(x^2-y^2, y^2-z^2, xy, yz, xz).$$

Note that R is 0-dimensional; its socle is generated by x^2, and hence it is Gorenstein. However, the ideal of relations defining R can easily be seen to be minimally generated by the 5 elements listed, which do not form a regular sequence on $\mathbb{K}[x,y,z]$.

Remark 11.23. In the previous example, we used the fact that the property of being complete intersection can be checked using any presentation of the ring R: if $R \cong Q/\mathfrak{a}$ for some regular ring Q and ideal $\mathfrak{a}$ of Q, then R is complete intersection if and only if $\mathfrak{a}$ is generated by a regular sequence.

Determinantal rings are another (potential) source of Gorenstein rings.

Example 11.24. Let $\mathbb{K}$ be a field, and $X = (x_{ij})$ an $m \times n$ matrix of indeterminates. Let $\mathbb{K}[X]$ denote the polynomial ring in the mn indeterminates x_{ij}. Fix an integer $r \geqslant 1$, and set

$$R = \mathbb{K}[X]/I_r(X),$$

where $I_r(X)$ denotes the ideal generated by the $r \times r$ minors of the matrix X. This is the coordinate ring of the algebraic set of $m \times n$ matrices over $\mathbb{K}$ of rank less than r as we saw in Example 1.5. The ring R is a Cohen-Macaulay normal domain of dimension $(m + n - r + 1)(r - 1)$. However, for $r > 1$, the ring R is Gorenstein if and only if $m = n$.

Let us verify this last statement when $m = 2$, $n = 3$, $r = 2$, that is, for the ring

$$R = \mathbb{K}[u, v, w, x, y, z]/(vz - wy, wx - uz, uy - vx).$$

Recall from Example 1.32 that the elements $u, v - x, w - y, z$ form a system of parameters for R. Now

$$R/(u, v - x, w - y, z) = \mathbb{K}[x, y]/(x^2, xy, y^2).$$

is not Gorenstein (look at its socle), and hence neither is R.

One is now in a position to extend Theorem 11.14 to arbitrary local rings.

Theorem 11.25. *Let* $(R, \mathfrak{m}, \mathbb{K})$ *be a local ring with* $\dim R = d$. *The following conditions are equivalent.*

(1) *R is Gorenstein;*
(2) $\operatorname{injdim}_R R = d$*;*
(3) *R is Cohen-Macaulay and* $\operatorname{rank}_{\mathbb{K}} \operatorname{Ext}^d_R(\mathbb{K}, R) = 1$*;*
(4) *some (equivalently, every) system of parameters for R generates an irreducible ideal.*

Remark 11.26. P. Roberts proved that in condition (3) above, one can drop the requirement that R is Cohen-Macaulay; thus, $\operatorname{Ext}^d_R(\mathbb{K}, R) \cong \mathbb{K}$ already implies that R is Gorenstein, [132]. The *type* of an finitely generated R-module M is the number

$$\operatorname{type}_R(M) = \dim_{\mathbb{K}} \operatorname{Ext}^n_R(\mathbb{K}, R),$$

where $n = \operatorname{depth}_R M$. Thus, the equivalence of (1) and (3) is the statement that Gorenstein rings are precisely Cohen-Macaulay rings of type 1.

Proof of Theorem 11.25. Conditions (1) and (2) are equivalent by Proposition 11.11 and Corollary 11.12.

(1) $\iff$ (3): Corollary 11.12 lets us assume that R is Cohen-Macaulay. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a maximal R-sequence. Rees' theorem 11.18 yields isomorphisms

$$\operatorname{Ext}^d_R(\mathbb{K}, R) \cong \operatorname{Ext}^{d-1}_R(\mathbb{K}, R/x_1R) \cong \cdots \cong \operatorname{Hom}_R(\mathbb{K}, R/(\boldsymbol{x})).$$

Now apply Proposition 11.17 and Theorem 11.14.

(1) $\implies$ (4): Again, reduce to dimension 0 and then apply Theorem 11.14.

(4) $\implies$ (1): Induce on $\dim R$. If $\dim R = 0$, apply Theorem 11.14. If $\dim R \geqslant 1$, we find a nonzerodivisor as follows: Let $\boldsymbol{x} = x_1, \dots, x_d$ be a system of parameters for R, and note that $\boldsymbol{x}^t = x_1^t, \dots, x_d^t$ is a system of parameters for all $t \geqslant 1$. By Theorem 11.14, each $R_t = R/(\boldsymbol{x}^t)$ is Gorenstein, and so its socle, being 1-dimensional, is contained in every ideal, and hence in the ideal $(\boldsymbol{x}^{t-1})/(\boldsymbol{x}^t)$. Thus

$$((\boldsymbol{x}^t) : \mathfrak{m}) \subseteq (\boldsymbol{x}^{t-1})$$

for each $t \geqslant 1$. Since $\operatorname{soc}(R)$ is contained in $(\boldsymbol{x}^t) : \mathfrak{m}$ for each t, we conclude that

$$\operatorname{soc} R \subseteq \bigcap_t (\boldsymbol{x}^{t-1}) = 0.$$

Hence depth $R > 0$. Choose a nonzerodivisor x, and apply the induction hypothesis to R/xR. □

Injective resolutions of Gorenstein rings

We now turn to the structure of minimal resolutions of Gorenstein rings.

Theorem 11.27. *Let R be a Noetherian ring and $\mathfrak{p}$ a prime ideal in R. The following conditions are equivalent:*

(1) *$R_{\mathfrak{p}}$ is Gorenstein;*
(2) *$\mu_R^i(\mathfrak{p}, R) = 0$ for each integer $i >$ height $\mathfrak{p}$;*
(3) *$\mu_R^i(\mathfrak{p}, R) = 0$ for some integer $i >$ height $\mathfrak{p}$;*
(4) $\mu_R^i(\mathfrak{p}, R) = \begin{cases} 0 & \text{if } i < \operatorname{height} \mathfrak{p}, \\ 1 & \text{if } i = \operatorname{height} \mathfrak{p}. \end{cases}$

Proof. We may assume that R is local with $\mathfrak{p} = \mathfrak{m}$.

The equivalence of (1) and (4) follows from Theorem 11.25. Proposition 11.11 yields (1) $\implies$ (2), while it is clear that (2) $\implies$ (3). Finally, (3) $\implies$ (1) is an immediate consequence of Theorem 11.9. □

Remark 11.28. Let R be a Gorenstein ring and $I^\bullet$ a minimal injective resolution of R. Since, for all prime ideals $\mathfrak{p}$, the ring $R_{\mathfrak{p}}$ is Gorenstein and furthermore $I^\bullet_{\mathfrak{p}}$ is a minimal resolution of $R_{\mathfrak{p}}$ by Proposition A.23, Theorem 11.27 implies that

$$I^i \cong \bigoplus_{\operatorname{height} \mathfrak{p} = i} E_R(R/\mathfrak{p})$$

for each i. Thus each $E_R(R/\mathfrak{p})$ appears exactly once in the complex $I^\bullet$, namely in degree height $\mathfrak{p}$.

Local duality for Gorenstein rings

One direction of the result below is called the ‘Grothendieck duality theorem’ for Gorenstein rings; ‘Grothendieck’ because it was proved by the man himself, and ‘duality’ for reasons explained in Lecture 18.

Theorem 11.29. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring of dimension d. Then R is Gorenstein if and only if*

$$H^i_{\mathfrak{m}}(R) = \begin{cases} 0 & \text{for } i \neq d, \\ E_R(\mathbb{K}) & \text{for } i = d. \end{cases}$$

Proof. When R is Gorenstein, its local cohomology with respect to $\mathfrak{m}$ is evident from Example 7.5 and the minimal injective resolution of R, see Remark 11.28. As to the converse, since $H^i_{\mathfrak{m}}(R) = 0$ for $i \neq d$, the ring R is Cohen-Macaulay, see Theorem 9.1. Now solve Exercise 11.30 below and apply Proposition 11.14. □

Exercise 11.30. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring, M a finitely generated R-module, and $\boldsymbol{x} = x_1, \dots, x_n$ a regular sequence on M. Set $d = \operatorname{depth} M$. Prove that there is a natural isomorphism

$$H_{\mathfrak{m}}^{d-n}(M/\boldsymbol{x}M) \cong \operatorname{Ext}_R^n(R/\boldsymbol{x}R, H_{\mathfrak{m}}^d(M)).$$

Hint: induce on n.

Using the preceding theorem and Theorem 7.10, one can obtain an explicit description of $E_R(\mathbb{K})$ for any local Gorenstein ring:

Remark 11.31. Let $\boldsymbol{x} = x_1, \dots, x_d$ be a system of parameters for R. Note that $H_{\mathfrak{m}}^n(-) = H_{(\boldsymbol{x})}^n(-)$ for each n. For $t \geqslant 1$, set $\boldsymbol{x}^t = x_1^t, \dots, x_d^t$. It is easy to see that

$$H^d(\operatorname{Hom}_R(K(\boldsymbol{x}^t), R)) = R/(\boldsymbol{x}^t),$$

and that, in the direct system in Theorem 7.10, the induced homomorphism

$$R/(\boldsymbol{x}^t) = H^d(\operatorname{Hom}_R(K(\boldsymbol{x}^t), R)) \longrightarrow H^d(\operatorname{Hom}_R(K(\boldsymbol{x}^{t+1}), R)) = R/(\boldsymbol{x}^{t+1})$$

is given by multiplication by the element $x = \prod_{i=1}^d x_i$. Thus, according to Theorem 11.29, when R is Gorenstein, one has

$$E_R(\mathbb{K}) = \varinjlim \left(R/(\boldsymbol{x}) \xrightarrow{x} R/(\boldsymbol{x}^2) \xrightarrow{x} R/(\boldsymbol{x}^3) \xrightarrow{x} \cdots \right).$$

Now we come to one of the principal results on Gorenstein rings: local duality. There is a version of the local duality theorem for Cohen-Macaulay rings with a canonical modules, and, better still, for any local ring with a dualizing complex. For the moment, however, we focus on the case of Gorenstein rings. The connection to Serre duality on projective space is explained in Lecture 18. It is an important point that the isomorphisms below are 'natural'; once again, this is explained in Lecture 18.

Theorem 11.32. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d. For each finitely generated R-module M, there are isomorphisms*

$$H_{\mathfrak{m}}^i(M) \cong \operatorname{Ext}_R^{d-i}(M, R)^\vee \qquad \text{for } 0 \leqslant i \leqslant d,$$

where $(-)^\vee = \operatorname{Hom}_R(-, E_R(\mathbb{K}))$.

Proof. Set $E = E_R(\mathbb{K})$, and let $\boldsymbol{x} = x_1, \dots, x_d$ be a system of parameters for R. By Theorem 7.13, for each integer i, the R-module $H_{\mathfrak{m}}^i(M)$ is the i^{th} cohomology of the complex $C_{\boldsymbol{x}}(R) \otimes_R M$, where $C_{\boldsymbol{x}}(R)$ is stable Koszul complex associated with the sequence $\boldsymbol{x}$. Since R is Gorenstein, Theorem 11.29 yields

$$H^i(C_{\boldsymbol{x}}(R)) = \begin{cases} 0 & \text{if } i < d, \\ E & \text{if } i = d. \end{cases}$$

Therefore, $C_{\boldsymbol{x}}(R)$ is a finite resolution of E by flat modules, and hence

$$H_{\mathfrak{m}}^i(M) = H^i(C_{\boldsymbol{x}}(R) \otimes_R M) = \operatorname{Tor}_{d-i}^R(E, M).$$

We claim that the module on the right is precisely $\operatorname{Ext}_R^{d-i}(M, R)^\vee$. To see this, let $I^\bullet$ be a minimal injective resolution of R; note that $I^n = 0$ for $n > d$ since R is Gorenstein. Thus, the canonical morphism of complexes

$$\operatorname{Hom}_R(I^\bullet, E) \otimes_R M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, I^\bullet), E) \tag{11.32.1}$$

is bijective. Moreover, $\operatorname{Hom}_R(I^\bullet, E)$ is a bounded complex of flat R-modules with

$$H_n(\operatorname{Hom}_R(I^\bullet, E)) = \begin{cases} 0 & \text{if } n \neq 0, \\ E & \text{if } n = 0. \end{cases}$$

Thus, $\operatorname{Hom}_R(I^\bullet, E)$ is a flat resolution of E. Hence taking homology in (11.32.1) yields, for each integer n, an isomorphism of R-modules

$$\operatorname{Tor}_n^R(E, M) \cong \operatorname{Hom}_R(\operatorname{Ext}_R^n(M, R), E).$$

□

The proof is admittedly terse. Exercise: fill in the details! The preceding theorem gives a description of local cohomology over *any* complete local ring: Any such ring is a surjective image of a regular local ring, and then the following corollary applies:

Corollary 11.33. *If $(R, \mathfrak{m}, \mathbb{K})$ is a homomorphic image of a Gorenstein local ring Q of dimension c, and M is a finitely generated R-module, then*

$$H_{\mathfrak{m}}^n(M) \cong \operatorname{Ext}_Q^{c-i}(M, Q)^\vee$$

where $(-)^\vee = \operatorname{Hom}_Q(-, E_Q(\mathbb{K}))$. □

But this is not so satisfactory: One would like is an intrinsic duality theorem, involving only cohomology groups over R itself. This is given by Serre-Grothendieck duality, Theorem 18.14, for which one needs canonical modules.

Canonical modules

The following exposition of canonical modules follows that of [16]. The graded and non-local cases are covered in Lectures 13 and 18, respectively. The latter also explains the duality theorem for Cohen-Macaulay rings with canonical modules.

Definition 11.34. Let R be a Cohen-Macaulay local ring. A maximal Cohen-Macaulay R-module C of finite injective dimension and type one is called a *canonical module* for R. Thus, a finitely generated R-module C is a canonical module for R if and only if

$$\mu_R^n(C) = \begin{cases} 0 & \text{for } n \neq \dim R, \\ 1 & \text{for } n = \dim R. \end{cases}$$

We have already seen examples of canonical modules: When R is Gorenstein, R itself is a canonical module, by Theorem 11.27. When R is Artinian, $E_R(\mathbb{K})$ is a canonical module for R. The following theorem establishes, among other things, that when a canonical module exists, it is unique; you may consult [16] for a proof.

Theorem 11.35. *Let R be a Cohen-Macaulay local ring. For any two canonical modules C and C' for R, one has*

(1) $C/\boldsymbol{x}C \cong E_{R/\boldsymbol{x}R}(\mathbb{K})$;
(2) $C \cong C'$;
(3) *The canonical homomorphism $R \longrightarrow \operatorname{Hom}_R(C, C)$ is bijective.* □

Henceforth, we use ω_R to denote *the* canonical module of R. Theorem 11.35 implies the following change of rings statements:

Corollary 11.36. *Let R be a Cohen-Macaulay local ring with canonical module ω_R. Then there are isomorphisms*

(1) $\omega_{R/\boldsymbol{x}R} \cong \omega_R/\boldsymbol{x}\omega_R$ *for any R-sequence $\boldsymbol{x}$;*
(2) $\omega_{R_\mathfrak{p}} \cong (\omega_R)_\mathfrak{p}$ *for any prime ideal $\mathfrak{p}$;*
(3) $\omega_{\widehat{R}} \cong \widehat{\omega_R}$.

We turn next to the question of its existence. A first result is provided by Theorem 11.27.

Proposition 11.37. *Let R be a Cohen-Macaulay local ring. The following conditions are equivalent.*

(1) *R is Gorenstein;*
(2) *ω_R exists and $\omega_R \cong R$.*

The following change of rings result is useful in finding canonical modules.

Theorem 11.38. *Let $\varphi\colon R \longrightarrow S$ be a local homomorphism of Cohen-Macaulay local rings such that S is a finitely generated module over the image of φ. If R has a canonical module, then S has a canonical module, and, setting $t = \dim R - \dim S$, there is an isomorphism*

$$\omega_S \cong \operatorname{Ext}_R^t(S, \omega_R).$$

Sketch of Proof. Let $\mathbb{K} = R/\mathfrak{m}$ and $\mathbb{L} = S/\mathfrak{n}$ denote the respective residue fields. One can choose an R-sequence $x_1, \ldots, x_t$ of length $t = \dim R - \dim S$ in $\ker \varphi$. Lemma 11.18 and Corollary 11.36 (1) yield that

$$\operatorname{Ext}_R^t(S, \omega_R) \cong \operatorname{Hom}_{R/\boldsymbol{x}R}(S, \omega_{R/\boldsymbol{x}R}),$$

and so we may assume that $\dim R = \dim S$.

Next, one can choose a maximal R-sequence $y_1, \ldots, y_d$ and proceed to reduce in a straightforward way to the case where $\dim R = \dim S = 0$. In this case, since ω_R is an injective R-module, $\operatorname{Hom}_R(S, \omega_R)$ is an injective S-module. Since S is zero-dimensional, $\operatorname{Hom}_R(S, \omega_R)$ must be isomorphic to a finite sum of copies of $E_S(\mathbb{L}) \cong \omega_S$. By length counting using Lemma A.30, one can see that it must be a single copy, that is, $\operatorname{Hom}_R(S, \omega_R) \cong \omega_S$. $\square$

Note one consequence of the theorem above: if a ring is a homomorphic image of a Gorenstein local ring, then it has a canonical module. The converse is true as well, and its proof uses the following construction.

Remark 11.39. Let R be a ring and let M be an R-module. Then the *trivial extension* of R by M, written $R \ltimes M$, is the R-algebra formed by endowing the direct sum $R \oplus M$ with the following multiplication:

$$(r, m)(s, n) = (rs, rn + sm).$$

The submodule $0 \oplus M$ is an ideal whose square is zero, and taking the quotient by this ideal yields R again. If R is Noetherian and M finitely generated, then $R \ltimes M$ is Noetherian and $\dim(R \ltimes M) = \dim R$. If R is local with maximal ideal $\mathfrak{m}$, then $R \ltimes M$ is local with maximal ideal $\mathfrak{m} \oplus M$.

Exercise 11.40. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and let M be a finitely generated R-module. Prove that

$$\operatorname{soc}(R \ltimes M) = \{(r, m) \mid r \in \operatorname{soc}(R) \cap \operatorname{ann}_R M, \text{ and } m \in \operatorname{soc}(M)\}.$$

Here is the definitive result on the existence of canonical modules:

Theorem 11.41. *Let R be a Cohen-Macaulay local ring. The following conditions are equivalent.*

(1) *R has a canonical module;*
(2) *R is a homomorphic image of a Gorenstein local ring.*

Proof. The implication (2) $\implies$ (1) follows from Theorem 11.38.

Suppose that R has a canonical module ω, set $S = R \ltimes \omega$. We prove that S is Gorenstein; R is its image under the canonical surjection $S \twoheadrightarrow R$. Pick a maximal R-regular sequence $\boldsymbol{x}$ that is also a regular sequence on ω; this can be done because ω is maximal Cohen-Macaulay; also, see the exercise below. Thus, (the image of) $\boldsymbol{x}$ in S is an S-regular sequence, and

$$S/\boldsymbol{x}S \cong (R/\boldsymbol{x}R) \ltimes (\omega_R/\boldsymbol{x}\omega_R).$$

Since it suffices to prove that $S/\boldsymbol{x}S$ is Gorenstein by Proposition 11.17, and $\omega_R/\boldsymbol{x}\omega_R$ is the canonical module of $R/\boldsymbol{x}R$ by Corollary 11.36, passing to $R/\boldsymbol{x}R$, we may assume that R is Artinian. Now use Exercise 11.40 and Theorem 11.14 to conclude that S is Gorenstein. $\square$

Exercise 11.42. Let R be a Cohen-Macaulay ring and ω a canonical module for R. Prove that any R-regular sequence is also regular on ω. Is this true of any maximal Cohen-Macaulay R-module?

Theorem 11.43. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a d-dimensional Cohen-Macaulay local ring with a canonical module ω. Then there is an isomorphism*

$$H^d_{\mathfrak{m}}(R) \cong \operatorname{Hom}_R(\omega, E_R(\mathbb{K})).$$

Proof. This is left as an exercise using Theorem 11.41, Theorem 11.33, and Theorem 11.38. $\square$

Thought exercise: in the definition of canonical modules, why we do we begin with Cohen-Macaulay rings?

Lecture 12. Connections with sheaf cohomology (GL)

You may have noticed that in previous lectures we have attached two meanings to the phrase "Čech complex". In Lecture 2, a complex $\check{C}^\bullet(\mathfrak{U}, \mathcal{F})$ was defined for any sheaf $\mathcal{F}$ on a topological space and any open cover $\mathfrak{U}$ of X. Later, in lecture 5, we defined a complex $C^\bullet(\underline{x}; R)$ for any commutative ring R and sequence of elements $\underline{x}$. In one sense, the goal of this lecture is to reconcile this apparent overload of meaning: the two Čech complexes are "really" the same thing, at least up to a shift. In particular, the Čech cohomology of a ring is an invariant of the scheme structure. Specifically, we will prove that for an R-module M,

$$H^{j+1}_{\mathfrak{a}}(M) \cong H^j(\operatorname{Spec} R \setminus V(\mathfrak{a}), \widetilde{M})$$

for all $j \geq 1$. See Theorem 12.28 for the precise statement and notation.

Our goal will take us fairly far afield, through the dense thickets of scheme theory and sheaf cohomology, flasque resolutions and cohomology with supports. We'll only have time to point out the windows at the menagerie of topics in this area. A more careful inspection is well worth your time. In particular, nearly everything in this lecture is covered more thoroughly in [63] and more intuitively in [33]. See also the prologue and epilogue of [110].

We begin with some background on sheaves.

Sheaf Theory from definitions to cohomology

Sheaves are *global objects* that are completely determined by *local data*. In fact, one of the points of sheaf theory is that sheaves make it possible to speak sensibly of "local properties". Historically, the notion of a sheaf seems to go back to complex analysis at the end of the 19th century, under the guise of *analytic continuation*, and was developed further in Weyl's 1913 book [159]. The first rigorous definition is due to Leray, and the basic properties were worked out in the Cartan seminar in the 1940s and 50s. Sheaves were imported to algebraic geometry by Serre in 1955 [140], when he also established the basics of sheaf cohomology. Grothendieck, in his 1955 Kansas lectures, introduced presheaves and the categorical approach to cohomology.

Here is the definition.

Definition 12.1. Let X be a topological space. A *sheaf* $\mathcal{F}$ of abelian groups on X consists of the data:

- an abelian group $\mathcal{F}(U)$ for every open set $U \subseteq X$, and
- a *restriction map* $\rho_{VU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ for each pair of open sets $V \subseteq U$, also sometimes written $|_V$,

subject to the following restrictions.

(0) (Sanity) $\mathcal{F}(\emptyset) = 0$ and $\rho_{UU} = \mathrm{id}_U$ for all open $U \subseteq X$.
(1) (Functoriality) If $W \subseteq V \subseteq U$ are open sets, then composition of restriction maps behaves well, *i.e.,* the diagram

$$\begin{tikzcd} \mathcal{F}(W) \arrow[rr, "\rho_{WU}"] \arrow[dr, "\rho_{WV}"'] & & \mathcal{F}(U) \\ & \mathcal{F}(V) \arrow[ur, "\rho_{VU}"'] & \end{tikzcd}$$

commutes.

(2) (Locally zero implies zero) If $s, t \in \mathcal{F}(U)$ become equal after restriction to each V_α in an open cover $U = \bigcup_\alpha V_\alpha$, then they are equal.
(3) (Gluing) For any open cover $U = \bigcup_\alpha V_\alpha$, and any collection of elements $\{s_\alpha \in \mathcal{F}(V_\alpha)\}$ satisfying

$$s_\alpha|_{V_\alpha \cap V_\beta} = s_\beta|_{V_\alpha \cap V_\beta}$$

for all α, β, there exists $s \in \mathcal{F}(U)$ so that $s|_{V_\alpha} = s_\alpha$.

The first two axioms are fairly straightforward; they could be fancied up as, "$\mathcal{F}$ is a contravariant functor from the topology on X to abelian groups," or simplified as, "$\mathcal{F}$ is a sensible generalization of sending a set S to the continuous maps $S \longrightarrow \mathbb{C}$." The axiom of local zeroness and the gluing axiom codify the fact that sheaves are determined by their local data, and that local data varies "smoothly" over the space X. Note that (3) is an existence statement, while (2) asserts uniqueness. The two are sometimes combined into a single "sheaf axiom."

Before muddying the waters further with more definitions, let us have a concrete example which is also dear to our hearts.

Example 12.2. Let $X = \operatorname{Spec} R$ for some (Noetherian and commutative, like all rings in this lecture) ring R. Give X the Zariski topology (see Lecture 1), so that X has a base of open sets of the form

$$\begin{aligned} U_f &= \operatorname{Spec} R \setminus V(f) \\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \text{ does not contain } f\} \end{aligned}$$

for each $f \in R$. Define a sheaf *of rings* on X, called the *structure sheaf* and denoted $\mathcal{O}_X$, by

$$\mathcal{O}_X(U_f) = R_f\,.$$

In other symbols, $\mathcal{O}_X$ assigns to every distinguished open set the ring of fractions $R[\frac{1}{f}]$. In particular, $\mathcal{O}_X(X) = R$. For two elements $f, g \in R$, we have

$$\begin{aligned} U_g \subseteq U_f &\Longleftrightarrow V(f) \subseteq V(g) \\ &\Longleftrightarrow f \mid g \\ &\Longleftrightarrow \text{there is a natural localization map } R_f \longrightarrow R_g\,. \end{aligned}$$

Exercise 12.3. Check that $\mathcal{O}_X$ really is a sheaf. Specifically, check the axioms $(0) - (3)$ for the open sets U_f. Then show that any collection $\{\mathcal{F}(U_\alpha)\}$ satisfying the axioms and such that the $\{U_\alpha\}$ form a base for the topology of X defines a unique sheaf on X.

Example 12.4. If $\mathbb{K}$ is a field, then $\operatorname{Spec} \mathbb{K}$ is a single point, (0), which is both open and closed, and $\mathcal{O}_{\operatorname{Spec} \mathbb{K}}((0)) = \mathbb{K}$.

Example 12.5. Let $R = \mathbb{Z}$. Then $\operatorname{Spec} R$ consists of one point for each prime $p \in \mathbb{Z}$, plus a "generic" point corresponding to the zero ideal, whose closure is all of $\operatorname{Spec} \mathbb{Z}$. The distinguished open sets are the *cofinite* sets of primes, of the form

$$U_n = \{p \in \operatorname{Spec} \mathbb{Z} \;\mid\; p \not| \, n\}$$

for $n \in \mathbb{Z} \setminus \{0\}$. For each nonzero n, we have $\mathcal{O}_{\mathbb{Z}}(U_n) = \mathbb{Z}[\frac{1}{n}]$.

Notation 12.6. We write $\Gamma(U, \mathcal{F})$ as a synonym[10] for $\mathcal{F}(U)$. The elements of the abelian group, or module, or ring, $\Gamma(U, \mathcal{F})$ are the *sections of $\mathcal{F}$ over U*. In particular, the elements of $\Gamma(X, \mathcal{F})$ are the *global sections of $\mathcal{F}$*.

Example 12.7. Let $X = \operatorname{Spec} R$. The sections of the structure sheaf $\mathcal{O}_X$ over a distinguished open set U_f are just fractions $\frac{r}{f^n}$ with $r \in R$ and $n \geq 0$. In particular, global sections of $\mathcal{O}_X$ are just elements of R.

A word of caution: while the sections of $\mathcal{O}_X$ over a distinguished open set are easy to identify from the definition, unexpected sections can crop up. In particular, not every section over an open set comes from restriction of a global section. Here are two examples.

Example 12.8. Let $X = \mathbb{C}$ with the usual complex topology, and let $\mathcal{F}$ be the sheaf of bounded holomorphic functions on X. Then the global sections of $\mathcal{F}$ are the bounded entire functions $\mathbb{C} \longrightarrow \mathbb{C}$. By Liouville's theorem, there are hardly any of these – only the constant functions qualify! For proper open sets, like discs, there are of course many nonconstant sections, which can't possibly be restrictions of global ones.

Example 12.9. Let $R = \mathbb{K}[s^4, s^3t, st^3, t^4]$ (see Example 10.6), where $\mathbb{K}$ is some field. Set $X = \operatorname{Spec} R$ and let U be the open set $X \setminus \{(s^4, s^3t, st^3, t^4)\}$. As in Example 12.7, the global sections of $\mathcal{O}_X$ are just the elements of R. In particular, s^2t^2 is not a global section. The sections over U, however, include something new. To see this, we can use the surjection $\mathbb{K}[a, b, c, d] \longrightarrow R$ to think of X as an algebraic set embedded in $\mathbb{K}^4$. The open set U then corresponds to X with the origin deleted; equivalently, U is the subset of X where not all the coordinate functions a, b, c, d vanish simultaneously. If $a \neq 0$, then b^2/a represents $(s^3t)^2/s^4 = s^2t^2$, while if $d \neq 0$, then c^2/d represents s^2t^2. Since $a = d = 0$ forces $b = c = 0$, the two open sets defined by $a \neq 0$ and $b \neq 0$ cover all of U, and the sections b^2/a and c^2/d glue together to give $s^2t^2 \in \Gamma(U, \mathcal{F})$.

Being essentially categorical notions, sheaves of course come equipped with a notion of *morphisms*. These are the only reasonable thing: A morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of (group, module, ring...) homomorphisms $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ commuting with the restriction maps of $\mathcal{F}$ and $\mathcal{G}$.

Definition 12.10. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Define the kernel, image, and cokernel of φ by

1. $\ker \varphi(U) = \ker(\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U))$;
2. $\operatorname{image} \varphi(U) = \operatorname{image}(\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U))$;
3. $\operatorname{coker} \varphi(U) = \operatorname{coker}(\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U))$;

As benign as this definition is, it leads to serious difficulties very quickly.

Exercise 12.11. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves.

1. Check that $\ker \varphi$ is a sheaf.
2. Show that $\operatorname{image} \varphi$ and $\operatorname{coker} \varphi$ satisfy the Sanity and Functoriality axioms to be sheaves.
3. Try (for a little while) to show that $\operatorname{image} \varphi$ and $\operatorname{coker} \varphi$ are sheaves.

[10] This apparently unnecessary proliferation of symbols to represent the same thing is supposed to hint at the connections to come; see Lecture 7.

This is where the smooth landscape of sheaf theory begins to show some wrinkles, not to say crevasses. Since it's the wellspring of much of what follows, we emphasize:

The image and cokernel of a morphism of sheaves need not be a sheaf.

Here are two examples to indicate what goes wrong. One should be familiar from basic complex analysis, while one is from closer to home.

Example 12.12. Let $X = \mathbb{C}$ again, and define two sheaves over X: $\mathcal{F}(U)$ is the (additive) group of holomorphic functions $U \longrightarrow U$, and $\mathcal{G}(U)$ is the (multiplicative) group of nowhere vanishing holomorphic functions $U \longrightarrow U$. Define $\exp : \mathcal{F} \longrightarrow \mathcal{G}$ by

$$\exp(f)(z) = e^{2\pi i f(z)} .$$

Then the image of exp is not a sheaf. For any choice of a branch of the logarithm function, $f(z) = z$ is in the image of exp on the open subset of $\mathbb{C}$ defined by the branch. These open sets cover $\mathbb{C}$, but $f(z) = z$ has no global preimage on all of $\mathbb{C}$, so is not in the image of exp. In other words, $e^{2\pi i z}$ is locally invertible, but has no analytic global inverse.

Example 12.13. Let $(R, \mathfrak{m})$ be a local ring and let $\{x_1, \ldots, x_n\}$ be a set of generators for $\mathfrak{m}$. Set $X = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, the punctured spectrum of R. Let $\mathcal{O}_U$ be the sheaf obtained by restricting the structure sheaf of $\operatorname{Spec} R$ to U. Define a morphism $\varphi : \mathcal{O}_U^n \longrightarrow \mathcal{O}_U$ by $\varphi(s_1, \ldots, s_n) = \sum_i s_i x_i$. Then the image of φ is not a sheaf. To see this, put $U_i = U_{x_i}$ and note that $U_1, \ldots, U_n$ cover U. On each U_i we have $\varphi(0, \ldots, \frac{1}{x_i}, \ldots, 0) = 1$. Thus we have an open cover of U and a section of image φ on each constituent so that the sections agree on the overlaps, but the sections cannot be glued together.

Routing around the failure of the category of sheaves to be closed under the operations of taking images and cokernels involves two definitions: *presheaves* and *sheafification*.

Definition 12.14. A collection $\{\mathcal{F}(U), \rho_{VU}\}$ satisfying the Sanity and Functoriality axioms of Definition 12.1 is a *presheaf*.

Example 12.15. Let $X = \{a, b\}$ be the two-point space with the discrete topology. Define $\mathcal{F}(\{a\}) = \mathcal{F}(\{b\}) = 0$ and $\mathcal{F}(X) = \mathbb{Z}$. Check that $\mathcal{F}$ is a presheaf (of abelian groups) and not a sheaf.

It follows from Example 12.11(b) that the image and cokernel of a morphism of sheaves are presheaves. Even if this weren't enough reason to consider them, we'll see that one can do cohomology with presheaves as well. First, we mention the procedure for obtaining a sheaf from a presheaf. This requires one preliminary definition, which is the counterpart in sheaf theory of the local notion of a *germ* of functions.

Definition 12.16. Let $x \in X$ and let $\mathcal{F}$ be a presheaf on X. The *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_{x,X} := \varinjlim \mathcal{F}(U) ,$$

where the direct limit is taken over the directed system of all open sets U containing x, partially ordered by inclusion.

Exercise 12.17. Check that for $\mathfrak{p} \in X = \operatorname{Spec} R$, the stalk of the structure sheaf $\mathcal{O}_X$ over $\mathfrak{p}$ is the local ring $R_{\mathfrak{p}}$. We say that $(X, \mathcal{O}_X)$ is a *locally ringed space.*

Now we define the sheafification of a presheaf.

Definition 12.18. Let $\mathcal{F}$ be a presheaf on X. The *sheafification* of $\mathcal{F}$ is the unique sheaf $\widetilde{\mathcal{F}}$ and morphism of presheaves $\mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ so that the stalk $\mathcal{F}_{x,X} \longrightarrow \widetilde{\mathcal{F}}_{x,X}$ is an isomorphism for all $x \in X$.

We remark that sheafifications always exist ([63, II.1.2]).

We can now remedy our embarrassing lack of images and cokernels.

Definition 12.19. Let $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of (pre)sheaves.

(1) The *image sheaf of* φ is the sheafification of the presheaf image φ.
(2) The *cokernel sheaf of* φ is the sheafification of the presheaf coker φ.
(3) We say that φ is *surjective* if the image sheaf of φ is equal to $\mathcal{G}$.
(4) A sequence of morphisms of sheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ is *exact at* $\mathcal{G}$ if $\ker \psi$ is equal to the image sheaf of φ.

Lemma 12.20. *A morphism of sheaves $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ over X is surjective (bijective) if and only if the stalk $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is so for each $x \in X$. In particular, a sequence $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$ is exact if and only if the sequence of stalks $\mathcal{F}_{x,X} \longrightarrow \mathcal{G}_{x,X} \longrightarrow \mathcal{H}_{x,X}$ is exact for every $x \in X$.*

Exercise 12.21. Check that in Exercise 12.13 above, φ is surjective. Note that a surjective sheaf morphism need not restrict to surjective maps over open sets! In each case, however, φ restricts to a surjective map over "small enough" open sets. What about Exercise 12.12?

We also define the important notion of *sheaves associated to modules*.

Definition 12.22. Let R be a ring and $X = \operatorname{Spec} R$. For an R-module M we define *the sheafification of* M to be the unique sheaf $\widetilde{M}$ with $\widetilde{M}(U_f) \cong M_f$ for all $f \in R$. A sheaf obtained by sheafifying an R-module is called *quasi-coherent*; if the module is finitely generated over R, then the sheaf is *coherent*.

Note that this definition must be modified slightly in the graded case; see Lecture 13.

The sheafification of an R-module actually gives rise to a *sheaf of $\mathcal{O}_X$-modules*, that is, $\Gamma(U, \widetilde{M})$ is a $\Gamma(U, \mathcal{O}_X)$-module for every U.

Sheafification of R-modules is an exact functor, since by Lemma 12.20 we can measure exactness on stalks. We do, however, lose some information. For example, our best candidate for a projective object is the sheafification of the free module R, that is, $\mathcal{O}_X$ itself. It turns out, though, that there are surjective maps to $\mathcal{O}_X$ that are not split! See the next lecture. In particular, it's not at all clear how to take a projective resolution of an $\mathcal{O}_X$-module. We'll deal with this shortly.

First, let's investigate the functor that goes in the opposite direction: "take global sections."

Proposition 12.23. *The global sections functor $\Gamma(X, -)$ is left-exact. Specifically, if*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence of sheaves, then

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{G}) \longrightarrow \Gamma(X,\mathcal{H})$$

is an exact sequence of abelian groups.

This proposition points toward the solution of the problem of the missing projectives, and we finally reach the object of this lecture.

Definition 12.24. Let R be a ring and $\mathcal{F}$ a sheaf of modules over $X = \operatorname{Spec} R$ (i.e., an $\mathcal{O}_X$-module). Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^1} \mathcal{I}^1 \xrightarrow{d^2} \dots$$

be an *injective* resolution of $\mathcal{F}$. Apply the global sections functor to the truncation of this resolution to obtain

$$0 \longrightarrow \Gamma(X,\mathcal{I}^0) \xrightarrow{\Gamma d^1} \Gamma(X,\mathcal{I}^1) \xrightarrow{\Gamma d^2} \dots ,$$

a complex of R-modules. Then the j^{th} *sheaf cohomology of* $\mathcal{F}$ is then

$$H^j(X,\mathcal{F}) = \ker \Gamma d^{j+1} / \operatorname{image} \Gamma d^j .$$

In particular, we have $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$.

Remark 12.25. In order for this definition to make sense, we must be able to take injective resolutions of $\mathcal{O}_X$-modules. Given our hardships with projectives noted above, this is cause for trepidation. Luckily, the category of modules over any locally ringed space $(X,\mathcal{O}_X)$ has *enough injectives*. We can see this as follows: for any $\mathcal{O}_X$-module $\mathcal{F}$, the stalk $\mathcal{F}_x$ at a point $x \in X$ embeds in an injective module $\mathcal{E}_x$ over the local ring $\mathcal{O}_{X,x}$. Set $\mathcal{E} = \prod_x \mathcal{E}_x$. Then the natural composition $\mathcal{F} \longrightarrow \mathcal{E}$ is an embedding and $\mathcal{E}$ is an injective $\mathcal{O}_X$-module.

Exercise 12.26. Why doesn't Remark 12.25 work for projectives?

The definition of sheaf cohomology given here has two slight problems. It's essentially impossible to compute, and it lengthens our already lengthy list of cohomology theories to keep track of. We can solve both these problems at once.

12.27. Let R be a Noetherian ring and $\mathfrak{a} = (x_1,\dots,x_n)$ an ideal of R. Put $X = \operatorname{Spec} R$, $V(\mathfrak{a})$ the closed set of X defined by $\mathfrak{a}$, and $U = X \setminus V(\mathfrak{a})$. Let $\mathfrak{U} = \{U_i\}$ be the open cover of U given by $U_i = X \setminus V(x_i)$. Let M be an arbitrary R-module and $\widetilde{M}$ the sheafification. Then $\Gamma(U_i,\widetilde{M}) \cong M_{x_i}$.

We have two complexes associated to this data: the Čech complex $C^\bullet(\underline{x};M)$ of the sequence $\underline{x} = x_1,\dots,x_n$ and the module M, which has the R-module

$$\bigoplus_{1 \le i_1, < \cdots < i_k \le n} M_{x_{i_1} x_{i_2} \cdots x_{i_k}}$$

in the k^{th} position, and the topological Čech complex $\check{C}^\bullet(\mathfrak{U},\widetilde{M}|_U)$ associated to the open cover $\mathfrak{U}$ and sheaf $\widetilde{M}|_U$, which has

$$\prod_{1 \le i_1 < \cdots < i_{k+1} \le n} \Gamma(U_{i_1} \cap \cdots \cap U_{i_{k+1}}, \widetilde{M}|_U)$$

in the k^{th} spot. Recall that $\Gamma(U,\widetilde{M}|_U)$ does not appear in the topological complex, but (see Exercise 2.9) is naturally isomorphic to the kernel of the zeroth differential.

Theorem 12.28. *In the situation of 12.27, we have an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow H^0(X, \widetilde{M}) \longrightarrow H^0(U, \widetilde{M}|_U) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

and isomorphisms for all $j \geq 1$

$$H^{j+1}_{\mathfrak{a}}(M) \cong H^j(U, \widetilde{M}|_U)$$

between the local cohomology of M with support in $\mathfrak{a}$ and the sheaf cohomology of $\widetilde{M}$ over U.

Remark 12.29. In the exact sequence of the Theorem, the inclusion of $H^0_{\mathfrak{a}}(M)$ into $H^0(X, \widetilde{M}) = \Gamma(X, \widetilde{M}) \cong M$ is the natural one; its elements correspond to sections s of $\widetilde{M}$ supported only on $V(\mathfrak{a})$, so that $s_{\mathfrak{p}} = 0$ unless $\mathfrak{p} \supseteq \mathfrak{a}$. Such a section dies when restricted to $U = X \setminus V(\mathfrak{a})$. The cokernel $H^1_I(M)$ measures the obstruction to extending a section of $\widetilde{M}$ over U to a global one. In particular, this has the following useful consequence.

Corollary 12.30. *Let R be a Noetherian ring, M a finitely generated R-module, and $\mathfrak{a}$ an ideal. If $\mathfrak{a}$ contains a regular sequence of length 2 on M, then every section of $\widetilde{M}$ over $U = \operatorname{Spec} R \setminus V(\mathfrak{a})$ extends to a global section.*

Remark 12.31. The functorial road to sheaves that we've followed above is the most common modern approach. In Godement's influential book [47], however, the order of exposition (open sets $\rightsquigarrow$ stalks) was the reverse. Suppose that $\mathcal{F}$ is a sheaf on some space X, and let E be the topological space with underlying set $\prod_{x \in X} \mathcal{F}_{x,X}$, the product of all stalks of $\mathcal{F}$. Topologize E by (1) defining a map $\pi : E \longrightarrow X$ sending each $\mathcal{F}_{x,X}$ to x and (2) insisting that each stalk have the discrete topology and π be continuous. One can then prove ([47][Théorème I.1.2.1]) that the thus constructed *espace étalé* ("flattened") or *total sheaf space* E has the property that for every open U, $\mathcal{F}(U)$ is naturally identified with the set $C_0(U, E)$ of continuous maps $f : U \longrightarrow E$ such that $f(x) \in \mathcal{F}_{x,X}$. Thus every sheaf *is* a sheaf of functions, as in Lecture 2.

Flasque sheaves and cohomology with supports

It follows immediately from the definition that injective sheaves of $\mathcal{O}_X$-modules are *acyclic* for sheaf cohomology, *i.e.,* $H^j(X, \mathcal{I}) = 0$ for any $j > 0$ and any injective sheaf $\mathcal{I}$. In fact, injective objects in any category are acyclic for any right-derived functors of covariant functors, since we compute such things from injective resolutions. In some sense, injectivity is overkill for our purposes. As part of their extreme acyclicity, injective objects are also extremely complicated in general (for example, over a Noetherian ring R, indecomposable injective R-modules are in one-to-one correspondence with $\operatorname{Spec} R$ itself). For the particular application we have in mind, then, it's worth searching for another class of sheaves which are still acyclic for the global sections functor (that is, for sheaf cohomology), but perhaps more manageable in whatever sense we can manage. If we're lucky, this new class will also be more "intrinsic" to sheaf cohomology, rather than being a generic categorical notion, depending on the whole category of sheaves.

Definition 12.32. A sheaf $\mathcal{F}$ on a topological space X is *flasque* (also "flabby" or "scattered") if for every pair of open sets $V \subseteq U$ in X, the restriction map

$\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective. In particular, sections on open subsets always extend to sections on the whole space.

Here are three examples. The third, of course, is the point.

Example 12.33. Let X be connected and $\mathcal{F}$ a *constant* sheaf over X, so for each open set U, $\mathcal{F}(U) = A$ for some fixed abelian group A. Then $\mathcal{F}$ is flasque.

Example 12.34. *Skyscraper sheaves* are flasque. Fix $x \in X$ and some abelian group A, and define $F(U) = A$ if $x \in U$, $\mathcal{F}(U) = 0$ otherwise. Then every section of $\mathcal{F}$ extends to all of X.

Example 12.35. Injective sheaves are flasque. More specifically, if a sheaf $\mathcal{M}$ of modules over a locally ringed space $(X, \mathcal{O}_X)$ is an injective object in the category of $\mathcal{O}_X$-modules, then $\mathcal{M}$ is flasque. To see this, let $V \subseteq U$ be open sets in X, and let $\mathcal{O}_V$, $\mathcal{O}_U$ be the structure sheaf extended by 0 outside V and U respectively. Then $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ is an embedding of $\mathcal{O}_X$-modules, so $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{M})$ surjects onto $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{M})$. (We haven't talked about Hom of sheaves, but you can pretend it works just like for modules.) Now, $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{M})$ is naturally isomorphic to $\Gamma(U, \mathcal{M})$ (check this!). It follows that $\Gamma(U, \mathcal{M}) \twoheadrightarrow \Gamma(V, \mathcal{M})$, and $\mathcal{M}$ is flasque.

Remark 12.36. Here are some further basic properties of flasque sheaves, most of which are easy, or can be taken for granted at a first pass, or both. We give only first-order approximations to the proofs.

(1) If, in the short exact sequence of sheaves
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0\,,$$
$\mathcal{F}$ is flasque, then
$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow 0$$
is exact. Remember, Γ is left-exact in general, so all we need check is surjectivity. Note that $\mathcal{G} \longrightarrow \mathcal{H}$ is surjective on stalks, and use the definition of stalks as direct limits to get surjectivity for small neighborhoods. Zorn's Lemma provides a maximal extension from such neighborhoods, which by the flasque condition must be all of X.

(2) Quotients of flasque sheaves by flasque subsheaves are flasque. Use the surjectivity from above to extend a section of the quotient to a global section of the large sheaf.

(3) Direct limits of flasque sheaves are flasque over Noetherian topological spaces. (The point here is that $\varinjlim(\mathcal{F}_i(U)) = (\varinjlim \mathcal{F}_i)(U)$ when X is Noetherian; see below). In particular, arbitrary direct sums of flasque sheaves are flasque.

(4) The sheafification of an injective module over a Noetherian ring R is a flasque sheaf. See [63, III.3.4].

Exercise 12.37. This exercise fills in the gaps in (3) above, which makes only approximate sense as it stands. The reader may find a review of Lecture **??** helpful.

(1) Let X be any topological space, I an index poset, and $\{\mathcal{F}_i\}_{i \in I}$ a direct system of presheaves on X. Define the direct limit presheaf $\varinjlim_i \mathcal{F}_i$ by $(\varinjlim_i \mathcal{F}_i)(U) = \varinjlim_i(\mathcal{F}_i(U))$, and check that it is indeed a presheaf.

(2) Using Definition 12.16, identify the stalk of $\varinjlim_i \mathcal{F}_i$ at a point $x \in X$ as the direct limit over $i \in I$ of the stalks of the $\mathcal{F}_i$ at x. You'll want to swap the order of two direct limits, which is legal by Example 4.33.

(3) If the $\mathcal{F}_i$ are *sheaves*, define the direct limit *sheaf* of the system $\{\mathcal{F}_i\}_{i\in I}$ to be the sheafification of the presheaf from part 1. Prove that the direct limit sheaf is the direct limit of the system $\{\mathcal{F}_i\}_{i\in I}$ in the category of sheaves over X.

(4) Assume finally that I is *confluent* and X is a Noetherian space. Prove that the presheaf from part 1 is already a sheaf, as follows.
   (a) (The axiom of local zeroness) Let $s \in (\varinjlim_\alpha \mathcal{F}_\alpha)(U)$ and assume that the restrictions $s|_{V_\alpha} = 0$ are zero for all V_α constituting an open cover of U; replace $\{V_\alpha\}$ by a *finite* subcover, and use the confluence of I to show that $s = 0$.
   (b) (The gluing axiom) Suppose we are given an open cover $\{V_\alpha\}$ of U and sections $s_\alpha \in (\varinjlim_i \mathcal{F}_i)(V_\alpha)$ so that they agree on the overlaps; replace the cover by a finite subcover and use confluence again to patch the sections together into a section in $\mathcal{F}_i(U)$ for some $i \in I$, thence a gluing of the s_α in $(\varinjlim_i \mathcal{F}_i)(U)$.

(5) Let $X = \{x, y\}$ be a space with two points, both open. Then a sheaf of abelian groups on X is just an assignment of three groups G_x, G_y, and G_X so that the sheaf axioms are satisfied; we picture such a sheaf as
$$\begin{tikzcd}[row sep=tiny] & x \\ X \arrow[ur] \arrow[dr] & \\ & y \end{tikzcd}.$$
Let G be any nontrivial abelian group, and define three sheaves of groups on X:
$$\begin{tikzcd}[row sep=tiny] & G \\ \mathcal{F}_1 : G \arrow[ur] \arrow[dr] & \\ & G \end{tikzcd}$$
$$\begin{tikzcd}[row sep=tiny] & G \\ \mathcal{F}_2 : G \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd}$$
$$\begin{tikzcd}[row sep=tiny] & 0 \\ \mathcal{F}_3 : G \arrow[ur] \arrow[dr] & \\ & G \end{tikzcd}$$
with the obvious maps in each case. Check that each of these is a sheaf on X. Define two morphisms of sheaves, $p : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ and $q : \mathcal{F}_1 \longrightarrow \mathcal{F}_3$, each of which is the natural surjection. Prove that the direct limit of the system
$$\begin{tikzcd}[row sep=tiny] & \mathcal{F}_2 \\ \mathcal{F}_1 \arrow[ur] \arrow[dr] & \\ & \mathcal{F}_3 \end{tikzcd}$$
is
$$\begin{tikzcd}[row sep=tiny] & 0 \\ G \arrow[ur] \arrow[dr] & \\ & 0 \end{tikzcd},$$
which is a presheaf, but not a sheaf.

Example 12.38. Let $X = \mathbb{P}^1 = \operatorname{Proj} \mathbb{K}[x, y]$ be the projective line over an algebraically closed field (e.g., the Riemann sphere $\mathbb{C} \cup \{\infty\}$). Let $\mathcal{O}_X$ be the structure sheaf of X, and $\mathcal{K}$ the constant sheaf associated to the function field $\widetilde{\mathbb{K}}$ of X. Then $\mathcal{O}_X$ embeds naturally in $\mathcal{K}$. The quotient sheaf $\mathcal{K}/\mathcal{O}_X$ can be thought of (stalkwise) as the direct sum over all $x \in X$ of the skyscraper sheaf $\mathcal{K}/\mathcal{O}_{X,x}$ at x, so is flasque. Thus

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{O}_X \longrightarrow 0$$

is a *flasque resolution* of $\mathcal{O}_X$. If we take global sections, we get an exact sequence

$$0 \longrightarrow \mathbb{K}[x,y] \longrightarrow \widetilde{\mathbb{K}} \longrightarrow \widetilde{\mathbb{K}}/\mathbb{K}[x,y] \longrightarrow 0\,.$$

As soon as we prove that sheaf cohomology can be computed via flasque resolutions, this will show that $H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for all $j \geq 1$. This example can be souped up beyond recognition and has connections with the Residue Theorem ([63, p. 248]).

If you're willing to accept the assertions above, then you must accept

Proposition 12.39. *A flasque sheaf on a Noetherian locally ringed space $(X, \mathcal{O}_X)$ is acyclic for the global sections functor, i.e., $H^j(X, \mathcal{F}) = 0$ for all $j \geq 1$ if $\mathcal{F}$ is flasque. In particular, resolutions by flasque sheaves may be used to compute sheaf cohomology in place of injective resolutions.*

Proof. Let $\mathcal{F}$ be flasque, and embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$. The quotient $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ is also flasque. Taking global sections gives an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{Q}) \longrightarrow 0\,.$$

The long exact sequence of cohomology thus gives that $H^1(X, \mathcal{F})$ vanishes and $H^j(X, \mathcal{F}) \cong H^{j-1}(X, \mathcal{Q})$ for all $j > 1$. By induction on j, $H^j(X, \mathcal{F}) = 0$ for all $j \geq 1$. $\square$

Remark 12.40. Weirdly enough, sheaf cohomology is trivial for quasi-coherent sheaves over Noetherian affine schemes, that is, spaces of the form $\operatorname{Spec} R$ for Noetherian commutative rings R. Perhaps we should have mentioned this earlier, since this situation would appear to be one of our main motivations. To be specific, let R be such a ring, and let M be an arbitrary R-module. Then an injective resolution of M sheafifies to a flasque resolution of $\widetilde{M}$ (since sheafification is exact), which we can use to compute $H^j(X, \widetilde{M})$. Applying Γ, though, just gets us back the original injective resolution of M! This is exact by design, so $H^j(X, \widetilde{M}) = 0$. In fact, the converse is true as well.

Theorem 12.41 (Serre [142])**.** *Let X be a Noetherian scheme. Then the following are equivalent.*

(1) X is affine;
(2) $H^k(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ on X and all $j > 0$.

Consider again the exact sequence and isomorphisms of Theorem 12.28:

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow H^0(U, \widetilde{M}|_U) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0\,,$$

and $H^{j+1}_{\mathfrak{a}}(M) \cong H^j(U, \widetilde{M}|_U$ for $j \geq 1$. One way of looking at this theorem is that sheaf cohomology of $\widetilde{M}$ on $\operatorname{Spec} R$ away from $V(\mathfrak{a})$ controls the local cohomology of M with support in $\mathfrak{a}$. In other words, only the *support* of $\mathfrak{a}$ matters. Following this idea leads naturally to *cohomology with supports*.

Definition 12.42. Let Z be a close set in some topological space X. For any sheaf $\mathcal{F}$ over X, the group of *sections of $\mathcal{F}$ with support in Z* is the kernel of the restriction map from $\Gamma(X, \mathcal{F})$ to $\Gamma(X \setminus Z, \mathcal{F})$. That is,

$$\Gamma_Z(X, \mathcal{F}) := \ker(\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X \setminus Z, \mathcal{F}))\,.$$

As with global sections, the functor "take sections with support in Z" is left-exact. (Check this! The Snake Lemma should come in handy.)

Definition 12.43. The j^{th} *local cohomology of* $\mathcal{F}$ *with support in* Z is

$$H^j_Z(X,\mathcal{F}) := R^j\Gamma_Z(X,\mathcal{F})\,.$$

In other words, to compute the cohomology of $\mathcal{F}$ with support in Z, take an injective resolution of $\mathcal{F}$, apply $\Gamma_Z = H^0_Z$, and take cohomology.

Remark 12.44. Let X, Z, and $\mathcal{F}$ be as in the definition above, and put $U = X\setminus Z$. Then there is a natural exact sequence

$$0 \longrightarrow H^0_Z(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{F}) \longrightarrow \Gamma(U,\mathcal{F}) \tag{12.44.1}$$

given by the definition of Γ_Z. Assume for the moment that $\mathcal{F}$ is flasque. Then every section of $\mathcal{F}$ over U extends to a global section, so (12.44.1) can be extended to a full short exact sequence.

It follows by taking flasque resolutions that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0_Z(X,\mathcal{F}) & \longrightarrow & H^0(X,\mathcal{F}) & \longrightarrow & H^0(U,\mathcal{F}|_U) & \longrightarrow & \cdots \\ \cdots & \longrightarrow & H^j(X,\mathcal{F}) & \longrightarrow & H^j(U,\mathcal{F}|_U) & \longrightarrow & H^{j+1}_Z(X,\mathcal{F}) & \longrightarrow & \cdots \end{array}$$

Together with Theorem 12.41 and the Snake Lemma, this gives our best connection with local cohomology.

Theorem 12.45. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal, and M an R-module. Set $X = \operatorname{Spec} R$, $Z = V(\mathfrak{a})$, and $U = X \setminus Z$. Then for each $j \geq 1$ we have*

$$H^{j+1}_{\mathfrak{a}}(M) \cong H^j(U,\widetilde{M}|_U) \cong H^j_Z(X,\widetilde{M})\,.$$

Lecture 13. Graded modules and sheaves on the projective space (AL)

In this lecture we explore sheaves on the projective space associated to graded modules. It turns out that considering the same module over same ring, but equipped with various gradings may lead to different sheaves.

Next we show the link between the local and sheaf cohomology established in Lecture 12 in the projective setting. This is followed by the discussion of the sheaf cohomology of a pullback of a coherent sheaf on the projective space.

In the last part, we add '*' to several objects introduced in the non-graded case to get *local rings, *maximal ideals, functors as $^*\operatorname{Ext}$, etc. (See [15] for a much more detailed treatment.) We define *canonical modules* for Cohen-Macaulay local rings and then reiterate the definition to introduce **canonical modules* for the Cohen-Macaulay *local rings.

13.1. **Projective space.** Let $\mathbb{K}$ be a field, recall that the *projective n-space* is defined as $\mathbb{P}^n_{\mathbb{K}} = \operatorname{Proj}(\mathbb{K}[x_0, \cdots, x_n])$, the homogeneous prime ideals in R that do not contain the ideal $(x_0, \cdots, x_n)$.

If $f \in R$ is a homogeneous polynomial, let

$$D_+(f) = \{\mathfrak{p} \in \mathbb{P}^n_{\mathbb{K}} : f \notin \mathfrak{p}\}.$$

Then $\mathbb{P}^n_{\mathbb{K}}$ may be seen as a union of $n+1$ affine patches

$$D_+(x_i) = \operatorname{Spec}(\mathbb{K}[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}]),\ i = 0, ..., n.$$

Note that the topology on these patches defines a topology on $\mathbb{P}^n_{\mathbb{K}}$.

13.2. **Sheaves associated to modules.** The regular functions $X = \operatorname{Spec}(S)$, for any ring S, are elements of S. They can be viewed as functions on the set of maximal ideals that send

$$\mathfrak{m} \mapsto f \mod \mathfrak{m} \in S/\mathfrak{m}$$

for a maximal ideal $\mathfrak{m} \in S$. The sheaf of regular functions $\mathcal{O}_X$ is set up by

$$\mathcal{O}_X(U) = \{\text{elements of } S, \text{ considered as functions, restricted to } U\},$$

for all open $U \subset X$.

In a projective space, this does not work. However, we may consider elements of $\frac{(}{R})$ that are homogeneous and of degree zero; these do give functions.

The same approach works for graded modules M over a graded ring R (throughout this section "graded" = "$\mathbb{Z}$-graded", see Lecture 5 for definitions). We can take homogeneous degree 0 elements in $\frac{(}{R}) \underset{R}{\otimes} M$, i.e.: sums of $\frac{f}{g} \otimes m$ where f, g, m are homogeneous and $\deg(\frac{f}{g}) + \deg(m) = 0$. If M is graded, we form $\widetilde{M}$ — the *sheaf associated to the graded module* M — that, on $D_+(x_i)$, has sections

$$\widetilde{M}(D_+(x_i)) = \left(R[x_i^{-1}] \underset{R}{\otimes} M\right)_0.$$

Example 13.1. On $\mathbb{P}^1$, we shall try to understand what the global sections of $\widetilde{M}$ are for the module $M = R = \mathbb{K}[x_0, x_1]$ with 3 different gradings:

(1) R has the usual degree grading. Then

$$\widetilde{M}(D_+(x_0)) = \mathbb{K}[\frac{x_1}{x_0}] \text{ and } M(D_+(x_1)) = \mathbb{K}[\frac{x_0}{x_1}].$$

Also, $\widetilde{M}(D_+(x_0x_1)) = \mathbb{K}[\frac{x_0}{x_1}, \frac{x_1}{x_0}]$. Given $f \in \mathbb{K}[\frac{x_1}{x_0}]$, we can not write it as a polynomial in $\frac{x_0}{x_1}$ unless $\deg(f) = 0$. Therefore, f can be extended to all of $\mathbb{P}^1$ only if it is a constant; the global sections are

$$\Gamma(\mathbb{P}^1, \widetilde{M}) = H^0(\mathbb{P}^1, R) \cong \mathbb{K}.$$

(2) R has a shifted grading: the degree of an element equals the usual degree plus 1. Call this graded module $M = R(-1)$: we have $R(-1)_m = R_{m-1}$. On one of the standard patches we have

$$\widetilde{M}(D_+(x_0)) = \left(R[x_0^{-1}] \underset{R}{\otimes} R(-1)\right)_0 = R[x_0^{-1}](-1)$$

These elements are linear combinations of monomials $x_0^a x_1^b$ where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $a + b = -1$.

Now consider the other patch, to be a section there, one needs to be a linear combination of $x_0^a x_1^b$ where $a \in \mathbb{N}$, $b \in \mathbb{Z}$, and $a + b = -1$. To be in both patches is impossible, since $-1 \notin \mathbb{N}$, therefore, $H^0(\mathbb{P}^1, \widetilde{R(-1)}) = 0$.

(3) Consider $R(1)$. The same argument as above leads to the conclusion that for a section to live on both patches $D_+(x_0)$ and $D_+(x_1)$ means to be a linear combination of $x_0^a x_1^b$ where $a \in \mathbb{N}$, $b \in \mathbb{N}$, and $a + b = 1$. Hence, the global sections are $H^0(\mathbb{P}^1, \widetilde{R(1)}) = \{\alpha x_0 + \beta x_1 \mid \alpha, \beta \in \mathbb{K}\}$.

The sheaves constructed in the example are usually denoted by $\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(0)$, $\mathcal{O}_{\mathbb{P}^1}(-1)$, and $\mathcal{O}_{\mathbb{P}^1}(1)$, respectively. The latter two are also known as "twisted sheaves". More generally, for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ set $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(m)$.

Exercise 13.2. Show that the global sections of the twisted sheaf $\mathcal{O}_{\mathbb{P}^1}(3)$ form a $\mathbb{K}$-vector space of dimension 4.

13.3. **Functor Γ_*.** Define the functor Γ_* from the category of coherent sheaves to the category of graded modules:

$$\Gamma_*\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{F}(m)).$$

For a graded R-module M there is a homomorphism of graded R-modules $M \longrightarrow \Gamma_*\widetilde{M}$, which is an isomorphism in high degrees.

For a graded R- module M, a sequence related to the one in Theorem 12.28 is exact:

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) = \Gamma_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow \Gamma_*\widetilde{M} \longrightarrow H^1_{\mathfrak{m}}M \longrightarrow 0. \tag{13.2.1}$$

13.4. **Cohomology.** How do you calculate the cohomology on projective space? One can use the Čech approach. That is to say, one must cover $\mathbb{P}^n$ with open sets, evaluate the sheaf on them, form the Čech complex, and then take cohomology. If $n = 1$, then $\mathbb{P}^1 = D_+(x_0) \cup D_+(x_1)$, the resulting Čech complex is

$$0 \longrightarrow \Gamma(D_+(x_0), \mathcal{F}) \bigoplus \Gamma(D_+(x_1), \mathcal{F}) \longrightarrow \Gamma(D_+(x_0x_1), \mathcal{F}) \longrightarrow 0. \tag{13.2.2}$$

If the sheaf $\mathcal{F}$ is the associate sheaf to a module M, then

$$\Gamma(D_+(x_0), \mathcal{F}) = (M[x_0^{-1}])_0,$$
$$\Gamma(D_+(x_1), \mathcal{F}) = (M[x_1^{-1}])_0,$$

$$\Gamma(D_+(x_0x_1),\mathcal{F}) = (M[x_1^{-1}x_0^{-1}])_0.$$

Compare this to the stable Koszul complex of M with respect to $\mathfrak{m}$:

$$0 \longrightarrow M \longrightarrow M[x_0^{-1}] \bigoplus M[x_1^{-1}] \longrightarrow M[x_1^{-1}x_0^{-1}] \longrightarrow 0 \tag{13.2.3}$$

We have almost the same complex, except that the first piece of (13.2.3) is missing in (13.2.2); the latter also restricts only to the degree 0 homogeneous components.

If we sum over all $\widetilde{M}(i)$ of (13.2.2), we will obtain the following isomorphism (this is called Serre-Groethendieck correspondence in [15]), similar to the second part of the Theorem 12.28:

$$H^k_{\mathfrak{m}}(M) \simeq \bigoplus_{i\in\mathbb{Z}} H^{k-1}(\mathbb{P}^n, \widetilde{M}(i))$$

for $k \geq 1$. For $k = 0$, there is the sequence (13.2.1).

Looking at this sequence, we deduce that $M = \Gamma_*\widetilde{M}$ if and only if $\operatorname{depth}(M) \geq 2$; also, $M \subseteq \Gamma_*\widetilde{M}$ if and only if $\operatorname{depth}(M) \geq 1$.

From the above follows (the proof is given as an exercise)

Theorem 13.3. *(1) $H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(t)) = \{$monomials in R of degree $t\}$;*

(2) $H^i(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(t)) = 0$ for $0 < i < n$;

(3) $H^n(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(t)) = \{$monomials in $E_R(\mathbb{K})$ of degree $t\}$.

13.5. **Pullback of a sheaf.** Let $\mathcal{F}$ be a sheaf on a topological space X. Let $Z \subseteq X$ be a closed subspace of X and let $f : Z \longrightarrow X$ be the inclusion map. We define the *pullback of the sheaf* $\mathcal{F}$ to be the sheaf on Z such that for an open set U in Z

$$(f^*\mathcal{F})(U) = \varinjlim \mathcal{F}(V),$$

where the limit is taken over all open subsets $V \subset X$ which contain U.

Proposition 13.4. *If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, then for every closed $Z \subset \mathbb{P}^N$*

(1) $H^i(Z; f^*\mathcal{F})$ is a finite dimensional vector space over $\mathbb{K}$;

(2) $H^i(Z; f^*(\mathcal{F}(t))) = 0$ for all $i \geq 1$, if $t \gg 0$;

(3) $H^i(Z; f^*\mathcal{F}) = 0$ if either $i > dim(Z)$ or $i > \dim \operatorname{Supp}(\mathcal{F})$.

Proof. To compute any of these homologies, begin by using the Čech approach on Z. Consider Z as a subset of $\mathbb{P}^n$ and use a cover of $\mathbb{P}^n$ to cover Z. As a result, one can pretend that $f^*\mathcal{F}$ is a sheaf on $\mathbb{P}^n$: for an open set $U \subset \mathbb{P}^N$

$$(f^*\mathcal{F})(U \cap Z) = \mathcal{F}(U) \otimes_R R/I(Z).$$

To obtain (1) and (2), note that the statements hold for $\mathcal{F} = \widetilde{R}$, because of the well understood properties of $H^i_{\mathfrak{m}}(R)$. Then, see the coherent sheaf $\mathcal{F}$ locally as an associated sheaf to a module $M = R^m/N$ for some m and $N \subset R^m$. Now use the exact sequence $0 \longrightarrow N \longrightarrow R^m \longrightarrow M \longrightarrow 0$ to complete the proof.

To prove part 3, observe that if $\dim(Z) = d$, then $d+1$ generic linear forms $L_1, ..., L_{d+1}$ have an empty intersection with Z. Now, $Z \subset D_+(L_1 \cdot ... \cdot L_{d+1})$ implies that the Čech complex has length at most $d+1$, and the cohomology is 0 beyond d.

The proof is similar for $\dim \operatorname{Supp}(\mathcal{F}) = d$. □

13.6. **Grading local cohomology.** Let R be a graded ring. Then a maximal graded ideal in R is called a **maximal ideal*. R is a **local ring* if it has only one *maximal ideal.

Roughly speaking, '*' preceding any term introduced in the non-graded case modifies this term making it suitable for the graded case. For example, for two graded R-modules M and L

$$^*\operatorname{Ext}^i_R(M, L) = H^i({}^*\operatorname{Hom}_R(M, I^\bullet))$$

defines $^*\operatorname{Ext}$ with the help of *injective resolution $I^\bullet$ and $^*\operatorname{Hom}$ that refers to homogeneous homomorphisms of graded R-modules.

A detailed treatment of the graded case can be found in [15, §§12, 13].

Let M be a graded module over a graded ring R. The main question that interests us at this point is, of course, how to grade the local cohomology modules of M at a graded ideal $\mathfrak{a} \subset R$.

Each of the three definitions of local cohomology provide an answer.

- **Cech complex:** $H^i_{\mathfrak{a}}(M) \cong H^i(C^*(\underline{a}; M))$, where $(\underline{a}) = \mathfrak{a}$. Here it would be natural to induce the grading on $H^i_{\mathfrak{a}}(M)$ via the grading on the components of the Čech complex.

  Is this grading independent of choice of $\underline{a}$?
- **Direct limit:** $^*\operatorname{Ext}$ $H^i_{\mathfrak{a}}(M) \cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$, as $n \longrightarrow \infty$. Since $R/\mathfrak{a}^n \longrightarrow R/\mathfrak{a}^m$ is homogeneous for $n > m$, the induced homomorphism

  $$^*\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M) \longrightarrow {}^*\operatorname{Ext}^i_R(R/\mathfrak{a}^m, M)$$

  is a graded homomorphism. Then, naturally, $\varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ is graded.

  But is this grading the same as above?
- **Torsion functor $\Gamma_{\mathfrak{a}}$ in the category of graded R-modules:** we can get $^*H^i_{\mathfrak{a}}(M)$ using standard homological methods in this category. Forgetting the grading, would we have

  $$^*H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M)?$$

  If so, is the induced grading the same as those that came from the other two approaches?

The answers to all the questions above are positive (see the "reconciliation" section [15, §12.3]), thus establishing the fact that the grading on the local cohomology is intrinsic.

13.7. ***Canonical modules.** The notion of canonical module (Definition 11.34) given in the local Cohen-Macaulay setting can be transplanted to a Cohen-Macaulay *local graded ring. Note that, when $(R, \mathfrak{m})$ is *local, height $\mathfrak{m} = *\dim R$.

A finitely generated graded R-module C is **canonical* iff there are homogeneous isomorphisms

$$\operatorname{Ext}^i_R(R/\mathfrak{m}, C) \cong \begin{cases} 0 & \text{if } i \neq n, \\ R/\mathfrak{m} & \text{if } i = n, \end{cases}$$

where $n = \dim R = \operatorname{height} \mathfrak{m}$.

Lemma 13.5. *With the above assumptions, let C be a finitely generated graded R-module such that $C_{\mathfrak{m}}$ is canonical for the Cohen-Macaulay local ring $R_{\mathfrak{m}}$.*

*Then there is $a \in \mathbb{Z}$ such that $^*E^n(C) \cong {}^*E(R/\mathfrak{m})(-a)$, moreover, the shifted module $C(a)$ is then a *canonical module for R.*

Proof. See the proof of Lemma 13.3.7 in [15]. □

Corollary 13.6. *If* $(R, \mathfrak{m})$ *is a Gorenstein *local graded ring, then there exists* $a \in \mathbb{Z}$ *such that* $R(a)$ *is a *canonical module for* R.

Proof. This follows from Lemma 13.5, since R_m is a canonical module of itself according to Theorem 11.27. □

Example 13.7. Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[x_1, ..., x_n]$ be the ring of polynomials in n variables graded by degree.

Note that R is Gorenstein *local (positively) graded ring, with the unique maximal ideal $\mathfrak{m} = (x_1, ..., x_n)$. According to Corollary 13.6, there is $a \in \mathbb{Z}$ such that $R(a)$ is a *canonical module for R.

To compute a, we notice that there are homogeneous isomorphisms

$$^*E(R/\mathfrak{m}) \cong H^n_{\mathfrak{m}}(R(a)) = H^n_{\mathfrak{m}}(R)(a).$$

We know the last module (see 7.17) before the shift by a:

$$H^n_{\mathfrak{m}}(R) = \mathbb{K}[x_1^{-1}, ..., x_n^{-1}] \cdot \frac{1}{x_1 \cdot ... \cdot x_n}.$$

This module has the usual grading.

On the other hand, the graded submodule $R/\mathfrak{m}$ of $^*E(R/\mathfrak{m})$ is generated by a homogeneous element of degree 0 annihilated by $\mathfrak{m}$. The only candidates in $H^n_{\mathfrak{m}}(R)(a)$ are the constant multiples of the generator $\frac{1}{x_1 \cdot ... \cdot x_n}$, since no other elements have $\mathfrak{m}$ for the annihilator. Since $\deg\left(\frac{1}{x_1 \cdot ... \cdot x_n}\right) = -n$ in $H^n_{\mathfrak{m}}(R)$, the shift a equals $-n$.

For any ideal $\mathfrak{a}$ in a Noetherian ring R, one has $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R$ by Proposition 9.16. This upper bound is sharp: when R is local with maximal ideal $\mathfrak{m}$, Grothendieck's theorem 9.3 implies that $\operatorname{cd}_R(\mathfrak{m}) = \dim R$.

The main result of this lecture, the Hartshorne-Lichtenbaum vanishing theorem [60] stated below, provides a better upper bound on cohomological dimension when $\mathfrak{a}$ is not primary to a maximal ideal.

Theorem 14.1. *Let $\mathfrak{a}$ be an ideal in a complete local domain R. If* $\dim R/\mathfrak{a} \geqslant 1$, *then* $\operatorname{cd}_R(\mathfrak{a}) \leqslant \dim R - 1$.

Remark 14.2. Let X be a scheme of finite type over a field $\mathbb{K}$. The number

$$\operatorname{cd}(X) = \inf\{s \mid H^n(X, \mathcal{F}) = 0 \text{ for all } n \geqslant s+1 \text{ and quasi-coherent sheaves } \mathcal{F}\}$$

is the *cohomological dimension of* X. A theorem of Serre asserts that $\operatorname{cd}(X) = 0$ if and only if X is affine. At the other end of the spectrum, a theorem of Lichtenbaum states that an irreducible scheme X is proper over $\mathbb{K}$ if and only if $\operatorname{cd}(X) = \dim X$.

In [60], Hartshorne uses Theorem 14.1 above to prove that if Y is a closed, connected subset of $\mathbb{P}^d$ of dimension at least 1, then $\operatorname{cd}(\mathbb{P}^d - Y) \leqslant d - 2$.

Proof of Theorem 14.1. First we consider the case where the ideal $\mathfrak{a}$ is a prime ideal, say, $\mathfrak{p}$, with $\dim R/\mathfrak{p} = 1$; then we give the reduction from the general case.

Since $\dim R/\mathfrak{p} = 1$, for each integer t the primary decomposition of $\mathfrak{p}^t$ has the form $\mathfrak{p}^{(t)} \bigcap J_t$ for some $\mathfrak{m}$-primary ideal J_t. The ring R is a domain, so $\bigcap \mathfrak{p}^{(t)} = 0$, since this holds in the localization $R_{\mathfrak{p}}$. Therefore, Chevalley's Theorem 14.3 yields that, for each integer t, there is a k_t such that $J_t \supseteq \mathfrak{p}^{(k_t)}$. Thus any $\mathfrak{p}^t \supseteq \mathfrak{p}^{(\max(t,k_t))}$, and hence the sequences $\{\mathfrak{p}^t\}$ and $\{\mathfrak{p}^{(t)}\}$ are cofinal. Therefore, for any R-module M, and each integer n, Theorem 7.8 implies

$$(*) \qquad \varinjlim_t \operatorname{Ext}^n_R(R/\mathfrak{p}^{(t)}, M) \cong H^n_{\mathfrak{a}}(M).$$

Now assume further that R is Gorenstein.

The depth of the R-module $R/\mathfrak{p}^{(t)}$ is one, because the only prime associated to it is $\mathfrak{p}$, and hence it is Cohen-Macaulay. Using this property, and the local duality theorem 11.32 one deduces that

$$(**) \qquad \operatorname{Ext}^n_R(R/\mathfrak{p}^{(t)}, R) = 0 \qquad \text{for } n \neq \dim R - 1.$$

Then $(*)$ and Theorem 9.6 imply $\operatorname{cd}_R(\mathfrak{p}) = \dim R - 1$.

This completes the proof of the case where $\dim R/\mathfrak{p} = 1$ when R is Gorenstein. Now for the proof when R is not Gorenstein.

By Cohen's Structure Theorem, R has a coefficient ring $\mathbb{K} \hookrightarrow R$ which is either a field or a discrete valuation ring (d.v.r.) with uniformizing parameter q, where q is the characteristic of the residue field.

Note that $\operatorname{height}\mathfrak{p} = d-1$ since R is a complete domain. We claim that we can choose $x_1, \dots, x_{d-1} \in \mathfrak{p}$ such that $\operatorname{height}(x_1, \dots, x_{d-1}) = d-1$. Indeed, one may choose any $x_1 \neq 0$ in $\mathfrak{p}$ and get $\operatorname{height}(x_1) = 1$ since R is a domain. Now suppose that $x_1, \dots, x_i$ have been chosen so that $\operatorname{height}(x_1, \dots, x_i) = i < d-1$. If $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are the minimal primes of $(x_1, \dots, x_i)$ (these have height i by Krull's Principal Ideal Theorem), then $\mathfrak{p} \nsubseteq \bigcup \mathfrak{p}_j$, else $\mathfrak{p} \subseteq \mathfrak{p}_j$ for some j and then $\operatorname{height}\mathfrak{p} \leqslant \operatorname{height}\mathfrak{p}_j < d-1$, a contradiction. Choose an element $x_{i+1} \in \mathfrak{p} \setminus \bigcup \mathfrak{p}_j$; then clearly the height of $(x_1, \ldots, x_{i+1})$ is at least $i+1$ and so exactly $i+1$ by Krull's Principal Ideal Theorem. The claim follows by induction.

Note that $x_1, \ldots, x_{d-1}$ are part of a system of parameters for R: indeed, each x_{i+1} is not any minimal prime $\mathfrak{p}_j$ of $(x_1, \ldots, x_i)$, and so for each i

$$\dim R/(x_1, \ldots, x_i, x_{i+1}) = \dim R/(x_1, \ldots, x_i) - 1.$$

Also note that if $\mathbb{K}$ is a d.v.r. with $q \in \mathfrak{p}$ we may (and do) choose $x_1 = q$.

Next choose an element $x_d \in \mathfrak{p}$ not in any of the other minimal primes (if any) of $x_1, \ldots, x_{d-1}$. Then $\mathfrak{p}$ is the only minimal prime of $x_1, \ldots, x_d$ and so one has

$$\operatorname{rad}(x_1, \ldots, x_d) = \mathfrak{p}.$$

Further note that

a) If $\mathbb{K}$ is a field or if it is a d.v.r. with $q \in \mathfrak{p}$, then choose any $y \notin \mathfrak{p}$. The elements $x_1, \ldots, x_{d-1}, y$ form a system of parameters for R.

b) If $\mathbb{K}$ is a d.v.r. with $q \notin \mathfrak{p}$, then the elements $x_1, \ldots, x_{d-1}, q$ form a system of parameters for R since $\dim R/(x_1, \ldots, x_{d-1}) = 1$, q is prime, and $q \notin (x_1, \ldots, x_{d-1})$ imply that $(x_1, \ldots, x_{d-1}, q)$ is $\mathfrak{m}$-primary. In this case, set $y = 0$.

Now consider the inclusion of rings

$$A = \mathbb{K}[[x_1, \ldots, x_{d-1}, y]] \hookrightarrow B = A[x_d] \hookrightarrow R.$$

Since R is finite over the complete regular local ring A by Cohen's Structure Theorem, so is B and so x_d is integral over A. If $f(x)$ is the minimal polynomial of x_d over A, then $B \cong A[x]/(f(x))$ and so B is a hypersurface and thus Gorenstein.

Let $\mathfrak{q} = \mathfrak{p} \bigcap B$. One can show that $\mathfrak{q} = \operatorname{rad}(x_1, \ldots, x_d)B$ by using the Going Up Theorem applied to the extension $B \hookrightarrow R$. Therefore $\mathfrak{p} = \operatorname{rad}(x_1, \ldots, x_d)R = \operatorname{rad} \mathfrak{q}R$ and so we have that

$$H^d_{\mathfrak{p}}(R) = H^d_{\mathfrak{q}R}(R) = H^d_{\mathfrak{q}}(B) \otimes_B R,$$

where the first equality is by 7.16.2 and the second is by Exercise 9.7. Since B is Gorenstein, we know already that $H^d_{\mathfrak{q}}(B) = 0$, so $H^d_{\mathfrak{p}}(R) = 0$.

This completes the proof in the case that $\mathfrak{a}$ is a prime ideal $\mathfrak{p}$ with $\dim R/\mathfrak{p} = 1$. Now for the reduction to this case from the general case:

We may assume that R is complete by 7.16(3) and the faithful flatness of $R \longrightarrow \widehat{R}$. Take a prime filtration $R = R_0 \supset R_1 \supset \cdots \supset R_n = 0$ of R with $R_i/R_{i+1} \cong R/\mathfrak{p}_i$ for primes ideals $\mathfrak{p}_i$. Since $H^d_{\mathfrak{a}}(-)$ is a half-exact functor, it suffices to show that $H^d_{\mathfrak{a}}(R/\mathfrak{p}_i) = 0$ for each i. Notice that if $\dim R/\mathfrak{p}_i < \dim R$ it is immediate from Corollary 9.16 that $H^d_{\mathfrak{a}}(R/\mathfrak{p}_i) = 0$, leaving the case that $\dim R/\mathfrak{p}_i = \dim R$.

Replacing R by $R/\mathfrak{p}_i$, we may assume that R is a complete local domain of dimension d and show that $\dim R/\mathfrak{a} > 0$ implies that $H^d_{\mathfrak{a}}(R) = 0$. If not, choose a maximal counterexample $\mathfrak{a}$ with $\dim R/\mathfrak{a} > 0$. We claim that $\mathfrak{a}$ is prime and $\dim R/\mathfrak{a} = 1$. If either fails, then there is an $x \notin \mathfrak{a}$ such that $\dim R/(\mathfrak{a}, x) > 0$.

The end of the Brodmann sequence from Exercise 14.4 has the form

$$\cdots \longrightarrow H^d_{(\mathfrak{a},x)}(R) \longrightarrow H^d_{\mathfrak{a}}(R) \longrightarrow H^d_{\mathfrak{a}_x}(R_x) \longrightarrow \cdots$$

Since R is local, $\dim R_x < d$ and so $H^d_{\mathfrak{a}_x}(R_x) = 0$. Therefore, $H^d_{\mathfrak{a}}(R) \neq 0$ implies that $H^d_{(\mathfrak{a},x)}(R) \neq 0$, a contradiction to the maximality of $\mathfrak{a}$. So, indeed $\mathfrak{a}$ is prime and that $\dim R/\mathfrak{a} = 1$; hence Theorem 14.1 yields $H^d_{\mathfrak{a}}(R) = 0$.

□

Here is the theorem due to Chevalley that was used in the preceding argument.

Theorem 14.3. *Let* $(R, \mathfrak{m}, \mathbb{K})$ *be a complete local ring,* M *a finitely generated* R*-module, and let* $\{M_t\}_{t\in\mathbb{Z}}$ *be a non-increasing filtration of* M*. Then* $\bigcap_{t\in\mathbb{Z}} M_t = 0$ *if and only for each integer* n *there exists a* t *such that* $M_t \subseteq \mathfrak{m}^n M$*.*

And here is the long exact sequence from [12] that was used in the same proof.

Exercise 14.4. For any ideal $\mathfrak{a}$ of R and element $x \in R$, there is a long exact sequence

$$\cdots \longrightarrow H^i_{(\mathfrak{a},x)}(R) \longrightarrow H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}_x}(R_x) \xrightarrow{\delta} H^{i+1}_{(\mathfrak{a},x)}(R) \longrightarrow \cdots$$

The non-connecting homomorphisms in the sequence are the natural ones. This sequence is sometimes referred to as the *Brodmann sequence*.

There is an enhancement of the Hartshorne-Lichtenbaum theorem which characterizes ideals $\mathfrak{a}$ with $\mathrm{cd}_R(\mathfrak{a}) \leqslant \dim R - 1$; its derivation from Theorem 14.1 is routine, so the details are left to the reader.

Theorem 14.5. *Let* R *be a local ring and* $\mathfrak{a}$ *an ideal in* R*. Then* $\mathrm{cd}_R(\mathfrak{a}) \leqslant \dim R - 1$ *if and only if* $\mathrm{height}(\mathfrak{a}\widehat{R} + \mathfrak{p}) < \dim R$ *for each* $\mathfrak{p} \in \mathrm{Spec}\, \widehat{R}$ *with* $\dim(\widehat{R}/\mathfrak{p}) = \dim R$*.*

The following theorem characterizes ideals $\mathfrak{a}$ with $\mathrm{cd}_R(\mathfrak{a}) \leqslant \dim R - 2$. It was proved by Hartshorne [60, Theorem 7.5] in the projective case, by Ogus [125, Corollary 2.11] in the characteristic zero case, and by Peskine and Szpiro [127, Chapter III, Theorem 5.5] in the case of positive characteristic. Huneke and Lyubeznik gave a characteristic free proof of this theorem in [80, Theorem 2.9] using a generalization of a result of Faltings, [36, Satz 1].

Theorem 14.6. *Let* $(R, \mathfrak{m})$ *be a complete regular local ring containing a separably closed coefficient field, and let* $\mathfrak{a}$ *be an ideal of* R*. Then* $\mathrm{cd}_R(\mathfrak{a}) \leqslant \dim R - 2$ *if and only if* $\dim R/\mathfrak{a} \geqslant 2$*, and* $\mathrm{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}\}$ *is connected in the Zariski topology.*

Improved bounds for the cohomological dimension, under various additional hypotheses, are obtained by Huneke and Lyubeznik in [80].

This lecture contains various results which illustrate connections between cohomological dimension and connectedness of varieties. An important ingredient in all of this is a local cohomology version of the Mayer-Vietoris theorem encountered in topology, Theorem 15.2 below. For its statement, it is convenient to introduce the following notation:

Notation 15.1. Let $\mathfrak{a}' \supseteq \mathfrak{a}$ be ideals in R and let M be an R-module. The inclusion $\Gamma_{\mathfrak{a}'}(-) \subseteq \Gamma_{\mathfrak{a}}(-)$ induces, for each n, an R-module homomorphism

$$\theta^n_{\mathfrak{a}',\mathfrak{a}}(M)\colon H^n_{\mathfrak{a}'}(M) \longrightarrow H^n_{\mathfrak{a}}(M).$$

This homomorphism is functorial in M; that is to, say, $\theta^n_{\mathfrak{a}',\mathfrak{a}}(-)$ is a natural transformation from $H^n_{\mathfrak{a}'}(-)$ to $H^n_{\mathfrak{a}}(-)$.

Given ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R, for each integer n we set

$$\iota^n_{\mathfrak{a},\mathfrak{b}}(M)\colon H^n_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M), \text{where } z \mapsto (\theta^n_{\mathfrak{a}+\mathfrak{b},\mathfrak{a}}(z), -\theta^n_{\mathfrak{a}+\mathfrak{b},\mathfrak{b}}(z)),$$
$$\pi_{\mathfrak{a},\mathfrak{b}}(M)\colon H^n_{\mathfrak{a}}(M) \oplus H^n_{\mathfrak{b}}(M) \longrightarrow H^n_{\mathfrak{a}\cap\mathfrak{b}}(M), \text{where } (x,y) \mapsto \theta^n_{\mathfrak{a},\mathfrak{a}\cap\mathfrak{b}}(x) + \theta^n_{\mathfrak{b},\mathfrak{a}\cap\mathfrak{b}}(y).$$

It is clear that the homomorphisms $\iota^n_{\mathfrak{a},\mathfrak{b}}(M)$ and $\pi^n_{\mathfrak{a},\mathfrak{b}}(M)$ are also functorial in M.

Theorem 15.2 (Mayer-Vietoris sequence)**.** *Let R be a Noetherian ring and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R. For each R-module M, there is an exact sequence of R-modules*

$$\begin{aligned} 0 \longrightarrow H^0_{\mathfrak{a}+\mathfrak{b}}(M) &\xrightarrow{\iota^0_{\mathfrak{a},\mathfrak{b}}(M)} H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) \xrightarrow{\pi^0_{\mathfrak{a},\mathfrak{b}}(M)} H^0_{\mathfrak{a}\cap\mathfrak{b}}(M) \\ &\longrightarrow H^1_{\mathfrak{a}+\mathfrak{b}}(M) \xrightarrow{\iota^1_{\mathfrak{a},\mathfrak{b}}(M)} H^1_{\mathfrak{a}}(M) \oplus H^1_{\mathfrak{b}}(M) \xrightarrow{\pi^1_{\mathfrak{a},\mathfrak{b}}(M)} H^1_{\mathfrak{a}\cap\mathfrak{b}}(M) \longrightarrow \cdots, \end{aligned}$$

and this sequence is functorial in M.

Proof. It is an elementary exercise to verify that one has an exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}+\mathfrak{b}}(M) \xrightarrow{\iota^0_{\mathfrak{a},\mathfrak{b}}(M)} \Gamma_{\mathfrak{a}}(M) \oplus \Gamma_{\mathfrak{b}}(M) \xrightarrow{\pi^0_{\mathfrak{a},\mathfrak{b}}(M)} \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(M).$$

We claim that $\pi^0_{\mathfrak{a},\mathfrak{b}}(M)$ is surjective whenever M is an injective R-module. Since $\Gamma_{\mathfrak{a}'}(-)$ commutes with direct sums for any ideal $\mathfrak{a}'$, it suffices to consider the case where $M = E_R(R/\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of R. But then the asserted exactness is immediate from Example 7.5.

Let $I^\bullet$ be an injective resolution of M. In view of the preceding discussion, we have an exact sequence of complexes,

$$0 \longrightarrow \Gamma_{\mathfrak{a}+\mathfrak{b}}(I^\bullet) \longrightarrow \Gamma_{\mathfrak{a}}(I^\bullet) \oplus \Gamma_{\mathfrak{b}}(I^\bullet) \longrightarrow \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(I^\bullet) \longrightarrow 0.$$

The homology exact sequence arising from this is the one we seek. The functoriality in M of the sequence is a consequence of the functoriality of $\iota^n_{\mathfrak{a},\mathfrak{b}}(-)$ and $\pi^n_{\mathfrak{a},\mathfrak{b}}(-)$, and of the connecting homomorphisms in homology long exact sequences. □

Definition 15.3. The *punctured spectrum* of a local ring $(R, \mathfrak{m})$ is the set

$$\operatorname{Spec}^\circ R = \operatorname{Spec} R \setminus \{\mathfrak{m}\},$$

with topology induced by the Zariski topology on $\operatorname{Spec} R$. Similarly, if R is graded with homogeneous maximal ideal $\mathfrak{m}$, then its *punctured spectrum* refers to the topological space $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

Exercise 15.4. Prove that the punctured spectrum of a local domain is connected.

Connectedness of the punctured spectrum can be interpreted entirely in the language of ideals:

Remark 15.5. Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an ideal in R. The punctured spectrum $\operatorname{Spec}^\circ(R/\mathfrak{a})$ of the local ring $R/\mathfrak{a}$ is connected if and only if the following property holds: given ideals $\mathfrak{a}'$ and $\mathfrak{a}''$ in R with

$$\operatorname{rad}(\mathfrak{a}' \cap \mathfrak{a}'') = \operatorname{rad}\mathfrak{a} \quad \text{and} \quad \operatorname{rad}(\mathfrak{a}' + \mathfrak{a}'') = \mathfrak{m},$$

either $\operatorname{rad}\mathfrak{a}'$ or $\operatorname{rad}\mathfrak{a}''$ equals $\mathfrak{m}$; equivalently, either $\operatorname{rad}\mathfrak{a}'$ or $\operatorname{rad}\mathfrak{a}''$ equals $\operatorname{rad}\mathfrak{a}$.

Indeed, this is a direct translation of the definition of connectedness, given that

$$V(\mathfrak{a}') \cup V(\mathfrak{a}'') = V(\mathfrak{a}' \cap \mathfrak{a}'') \quad \text{and} \quad V(\mathfrak{a}') \cap V(\mathfrak{a}'') = V(\mathfrak{a}' + \mathfrak{a}'').$$

Exercise 15.6. Let $R = \mathbb{R}[x, y, ix, iy]$ where $i^2 = -1$. Note that

$$R \cong \mathbb{R}[x, y, u, v]/(u^2 + x^2, v^2 + y^2, xy + uv, xv - uy).$$

Show that the punctured spectrum of R is connected, but that of $R \otimes_{\mathbb{R}} \mathbb{C}$ is not.

The next few results identify conditions under which the punctured spectrum is connected. The first one is a straightforward application of the Mayer-Vietoris sequence 15.2.

Proposition 15.7. *Let R a local ring. If* $\operatorname{depth} R \geqslant 2$*, then* $\operatorname{Spec}^\circ R$ *is connected.*

Proof. Suppose that there are ideals $\mathfrak{a}'$ and $\mathfrak{a}''$ of R with $\operatorname{rad}(\mathfrak{a}' \cap \mathfrak{a}'') = \operatorname{rad}(0)$ and $\operatorname{rad}(\mathfrak{a}' + \mathfrak{a}'') = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. Then Proposition 7.2.2 and the depth sensitivity of local cohomology, Theorem 9.1, imply that

$$H^0_{\mathfrak{a}' \cap \mathfrak{a}''}(R) = R \quad \text{and} \quad H^0_{\mathfrak{a}' + \mathfrak{a}''}(R) = H^1_{\mathfrak{a}' + \mathfrak{a}''}(R) = 0.$$

Given these, the Mayer-Vietoris sequence 15.2 yields an isomorphism of R-modules

$$H^0_{\mathfrak{a}'}(R) \oplus H^0_{\mathfrak{a}''}(R) \cong R.$$

By Exercise 15.8 R is indecomposable as a module over itself, so, without loss of generality, we may assume that $H^0_{\mathfrak{a}'}(R) = R$ and $H^0_{\mathfrak{a}''}(R) = 0$. This implies that $\operatorname{rad}\mathfrak{a}' = \operatorname{rad}(0)$, as desired. □

Exercise 15.8. Let R be a quasi-local ring (i.e., R has a unique maximal ideal but may not be Noetherian). Show that R is indecomposable as a module over itself.

The hypothesis on depth in Theorem 15.7 is optimal in view of Example 15.6 as well as the following example:

Example 15.9. The ring $R = \mathbb{K}[[x, y]]/(xy)$ is local with $\operatorname{depth} R = 1$. Moreover

$$\operatorname{Var}(x) \cup \operatorname{Var}(y) = \operatorname{Spec} R \quad \text{and} \quad \operatorname{Var}(x) \cap \operatorname{Var}(y) = \varnothing.$$

Thus, $\operatorname{Spec}^\circ R$ is not connected.

Here is an amusing application of Proposition 15.7, pointed out by one of the participants at the summer school.

Example 15.10. Let $\mathbb{K}$ be a field and let $R = \mathbb{K}[[x, y, u, v]]/(x, y) \cap (u, v)$. It is not hard to check that $\dim R = 2$. On the other hand, $\operatorname{Spec}^\circ R$ is disconnected (why?), so $\operatorname{depth} R \leqslant 1$ by Proposition 15.7. In particular, R is *not* Cohen-Macaulay!

More sophisticated results on connectedness of punctured spectra are derived from the Hartshorne-Lichtenbaum vanishing theorem, proved in Lecture 14. The one below was originally proved in the equicharacteristic case by Faltings [37, 38] using a different method; the argument we present is due to Brodmann and Rung [14]. The invariant $\operatorname{ara}\mathfrak{a}$, the arithmetic rank of $\mathfrak{a}$, was introduced in Definition 9.10.

Theorem 15.11. *Let R be a complete local domain. If $\mathfrak{a}$ is an ideal of R with $\operatorname{ara}\mathfrak{a} \leqslant \dim R - 2$, then $\operatorname{Spec}^\circ(R/\mathfrak{a})$ is connected.*

Proof. Let $\mathfrak{a}'$ and $\mathfrak{a}''$ be ideals of R with $\operatorname{rad}(\mathfrak{a}' \cap \mathfrak{a}'') = \operatorname{rad}\mathfrak{a}$ and $\operatorname{rad}(\mathfrak{a}' + \mathfrak{a}'') = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. We prove that one of $\operatorname{rad}\mathfrak{a}'$ or $\operatorname{rad}\mathfrak{a}''$ equals $\mathfrak{m}$.

Set $d = \dim R$. Since $\operatorname{ara}\mathfrak{a} \leqslant d-2$, we have

$$H^n_{\mathfrak{a}'\cap\mathfrak{a}''}(R) = H^n_{\mathfrak{a}}(R) = 0 \quad \text{for } n = d-1, d,$$

where the first equality holds by Proposition 7.2.2 and the second by Proposition 9.12. Keeping in mind that $H^n_{\mathfrak{a}'+\mathfrak{a}''}(R) = H^n_{\mathfrak{m}}(R)$ for each n, again by Proposition 7.2.2, the Mayer-Vietoris sequence 15.2 associated to the pair $\mathfrak{a}', \mathfrak{a}''$ reads

$$0 = H^{d-1}_{\mathfrak{a}'\cap\mathfrak{a}''}(R) \longrightarrow H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{a}'}(R) \oplus H^d_{\mathfrak{a}''}(R) \longrightarrow H^d_{\mathfrak{a}'\cap\mathfrak{a}''}(R) = 0.$$

Grothendieck's theorem 9.3 yields that $H^d_{\mathfrak{m}}(R) \neq 0$, so the exact sequence above implies that one of $H^d_{\mathfrak{a}'}(R)$ or $H^d_{\mathfrak{a}''}(R)$ must be nonzero; we may assume without loss of generality that $H^d_{\mathfrak{a}'}(R) \neq 0$. This implies that $\operatorname{cd}_R(\mathfrak{a}') \geqslant d$, and hence, by the Hartshorne-Lichtenbaum Theorem 14.1, that $\operatorname{rad}\mathfrak{a}' = \mathfrak{m}$. □

Faltings' theorem leads to another connectedness result; this one is due to Fulton and Hansen [43]. An interesting feature of the proof is the use of 'reduction to the diagonal' encountered earlier in the proof of Theorem 1.31.

Theorem 15.12. *Let $\mathbb{K}$ be an algebraically closed field, and let X and Y be closed irreducible subschemes of $\mathbb{P}^n_{\mathbb{K}}$. If $\dim X + \dim Y \geqslant n+1$, then $X \cap Y$ is connected.*

Sketch of proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be homogeneous prime ideals in $\mathbb{K}[x_0, \ldots, x_n]$ such that

$$\mathbb{K}[x_0, \ldots, x_n]/\mathfrak{p} \qquad \text{and} \qquad \mathbb{K}[x_0, \ldots, x_n]/\mathfrak{q}$$

are homogeneous coordinate rings of X and Y, respectively. Then

$$\mathbb{K}[x_0, \ldots, x_n]/(\mathfrak{p} + \mathfrak{q})$$

is a homogeneous coordinate ring for the intersection $X \cap Y$. If $X \cap Y$ is disconnected, then so is the punctured spectrum of the local ring

$$\frac{\mathbb{K}[[x_0, \ldots, x_n]]}{(\mathfrak{p} + \mathfrak{q})} \cong \frac{\mathbb{K}[[x_0, \ldots, x_n, y_0, \ldots, y_n]]}{(\mathfrak{p} + \mathfrak{q}' + \Delta)},$$

where $\mathfrak{q}'$ is the ideal generated by polynomials obtained by substituting y_i for x_i in a set of generators for $\mathfrak{q}$, and $\Delta = (x_0 - y_0, \ldots, x_n - y_n)$ is the ideal defining the 'diagonal' in $\mathbb{P}^{2n}_{\mathbb{K}}$. The complete local ring

$$\mathbb{K}[[x_0, \ldots, x_n, y_0, \ldots, y_n]]/(\mathfrak{p} + \mathfrak{q}')$$

is a domain (exercise!) of dimension $\dim X + 1 + \dim Y + 1 \geqslant n+3$. Evidently $\operatorname{ara}\Delta = n + 1 = (n+3) - 2$. But this contradicts Faltings' connectedness theorem 15.11. □

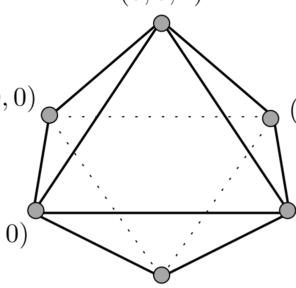
Lecture 16. Polyhedral applications (EM)

Local cohomology and the concepts surrounding it have a lot to say about commutative algebra in quite general settings, as we have seen in previous lectures. This is, of course, not to say that there aren't examples; indeed, we've seen a lot so far, ranging from the arithmetic (over the integers) to the geometric (graded rings). In this lecture, we begin to see interactions with combinatorics: focusing on specific classes of rings can allow deep applications in the context of polyhedral geometry.

16.1. **Polytopes and faces.** To begin, suppose that one has a subset $V \subset \mathbb{R}^d$ in d-dimensional Euclidean space. The smallest convex set in $\mathbb{R}^d$ containing V is called the *convex hull* of V and denoted by $\mathrm{conv}(V)$.

Definition 16.1. The convex hull $P = \mathrm{conv}(V)$ of a finite set $V \subset \mathbb{R}^d$ is called a *polytope*. The *dimension* of P is the dimension of its affine span.

Example 16.2. Consider the six unit vectors in $\mathbb{R}^3$ along the (positive and negative) coordinate axes. The convex hull of these points is a regular octahedron:



More generally, the convex hull of the $2d$ unit vectors along the (positive and negative) axes in $\mathbb{R}^d$ is the regular *cross-polytope* of dimension d.

A *closed halfspace* H^+ is the set of points weakly to one (fixed) side of an affine hyperplane H in $\mathbb{R}^d$. It is a fundamental and nontrivial (but intuitively obvious) theorem that a polytope P can be expressed as an intersection $P = \bigcap_{i=1}^n H_i^+$ of finitely many closed half-spaces H_i^+ [163, Theorem 1.1]. This description makes it much easier to prove the following.

Exercise 16.3. The intersection of a polytope P in $\mathbb{R}^d$ with any affine hyperplane H is another polytope in $\mathbb{R}^d$.

In general, slicing a polytope P with affine hyperplanes yields infinitely many new polytopes, and their dimensions can vary from $\dim(P)$ down to 0. But if we only consider $P \cap H$ for *support hyperplanes* H, meaning that P lies on one side of H, say $P \subset H^+$, then only finitely many new polytopes occur. An intersection $F = P \cap H$ with a support hyperplane is called a *face* of P; that $P = \mathrm{conv}(V)$ has only finitely many faces follows because each face F is the convex hull of $F \cap V$.

Example 16.4. Consider the octahedron from Example 16.2. Any affine plane in $\mathbb{R}^3$ passing through $(0,0,1)$ and whose normal vector is sufficiently close to vertical is a support hyperplane; the corresponding face is the vertex $(0,0,1)$. The plane

$x + y + z = 1$ is a support hyperplane; the corresponding face is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Definition 16.5. Let P be a polytope of dimension d. The *f-vector* of P is the vector $(f_{-1}, f_0, \ldots, f_{d-1}, f_d)$ in which f_i equals the number of dimension i faces of P. Here, $f_{-1} = 1$ counts the empty face of P, and $f_d = 1$ counts P itself (which is also considered to be a face). The numbers f_0, f_1, and f_{d-1} count the number of *vertices*, *edges*, and *facets*, respectively.

The vertices of $P = \mathrm{conv}(V)$ are all elements of the finite set V, but in general V might contain points interior to P that are therefore not vertices of P.

Example 16.6. The octahedron has a total of 28 faces: 6 vertices, 12 edges, 8 facets, plus the whole octahedron itself and one empty face. Hence the f-vector of the octahedron is $(1, 6, 12, 8, 1)$.

Exercise 16.7. What is the f-vector of the cross-polytope in $\mathbb{R}^d$? You might find it easier to calculate the f-vector of the hypercube in $\mathbb{R}^d$ and reverse it; why does this work?

Counting faces raises the following basic issue, our main concern in this lecture.

Question 16.8. How many faces of each dimension could $P \subset \mathbb{R}^d$ possibly have, given that it has (say) a fixed number of vertices?

On the face of it, Question 16.8 has nothing at all to do with commutative algebra, let alone local cohomology; but we shall later see that it does, given the appropriate class of rings.

16.2. **The Upper Bound Theorem.** Fix a polytope $P = \mathrm{conv}(V)$ in $\mathbb{R}^d$ with n vertices, and assume that V equals the vertex set of P. Each face of P is determined by the set of vertices it contains, which is by definition the subset of V minimized by some (perhaps not uniquely determined) linear functional.

Now suppose that we wiggle the vertices V in $\mathbb{R}^d$ a tiny bit to get a set V', the vertex set of a new polytope $P' = \mathrm{conv}(V')$. Each face G of P' has a vertex set $V'_G \subseteq V'$. Unwiggling V'_G yields a subset $V_G \subseteq V$ whose affine span equals the affine span of some face F of P, although V_G might be a proper subset of the vertex set of F. In this way, to each face of P' is associated a well-defined face of P.

If our vertex-wiggling is done at random, what kind of polytope is P'?

Definition 16.9. A polytope of dimension i is a *simplex* if it has precisely $i + 1$ vertices. A polytope P is *simplicial* if every proper face of P is a simplex.

To check that a polytope is simplicial, it enough to know its facets are simplices.

Exercise 16.10. Any simplex of any dimension is itself a simplicial polytope. The octahedron—and more generally, any cross-polytope—is a simplicial polytope.

Exercise 16.11. Define a notion of *generic* for finite point sets in $\mathbb{R}^d$ so that

- the convex hull of every generic set is a simplicial polytope, and
- every finite set V can be made generic by moving each point in V less than any fixed positive distance.

Exercise 16.12. If P' is obtained by generically wiggling the vertices of P, prove that every dimension i face of P is associated to some dimension i face of P'.

[Hint: Let F be a given i-face of P, and call its vertex set V_F. Fix a support hyperplane H for F and a vector ν perpendicular to H. If F' is the orthogonal projection of $\mathrm{conv}(V'_F)$ to the affine span of F, then construct a *regular subdivision* [163, Definition 5.3] of F' using the orthogonal projection of V'_F to the affine span of $F+\nu$. Use a face of maximal volume in this subdivision to get the desired face of P'.]

Exercise 16.12 implies that the collection of subsets of V that are vertex sets of i-faces can only get bigger when V is made generic by wiggling. Consequently, among all polytopes with n vertices, there is a simplicial one that maximizes the number of i-faces. This reduces Question 16.8 to the simplicial case.

In fact, there is a *single* simplicial polytope with n vertices that we shall see maximizes the numbers f_i for all i simultaneously. It is constructed as follows.

Definition 16.13. Consider the *rational normal curve* $(t, t^2, t^3, \ldots, t^d)$ in $\mathbb{R}^d$ parametrized by real numbers t. If $n \geq d+1$, then the convex hull of any n distinct points on the rational normal curve is a *cyclic polytope* $C(n, d)$ of dimension d.

Exercise 16.14. Use the nonvanishing of $(d+1)\times(d+1)$ Vandermonde determinants to show that a cyclic $C(n, d)$ polytope is indeed simplicial of dimension d.

It is nontrivial but elementary [163, Theorem 0.7] that the combinatorial type of $C(n, d)$ is independent of which n points on the rational normal curve are chosen for its vertices. It is similarly elementary [163, Corollary 0.8] that for $i \leq \frac{d}{2}$, every set of i vertices of $C(n, d)$ is the vertex set of an $(i-1)$-face of $C(n, d)$; equivalently, $f_{i-1}(C(n, d)) = \binom{n}{i}$ for $i \leq \frac{d}{2}$. Thus $C(n, d)$ is said to be *neighborly*.

Exercise 16.15. Find a simplicial non-neighborly polytope.

For obvious reasons, $f_{i-1}(P)$ is bounded above by $\binom{n}{i}$ for all polytopes with n vertices. Therefore, no polytope with n vertices has more $(i-1)$-faces than $C(n, d)$ when $i \leq \frac{d}{2}$. What about faces of higher dimension?

Theorem 16.16 (Upper Bound Theorem, polytope f-vector version)**.** *For any polytope $P \subset \mathbb{R}^d$ with n vertices, $f_i(P) \leq f_i(C(n, d))$ for all i.*

Exercise 16.17. Verify that the octahedron appearing in Example 16.2 has the same f-vector as the cyclic polytope $C(6, 3)$. Now, using the result of Exercise 16.7, verify the statement of Theorem 16.16 for the 4-dimensional cross-polytope. By Example 16.6, this amounts to computing the f-vector of the appropriate cyclic polytope (which one is it?).

Theorem 16.16 was proved by McMullen [116]. Following Stanley [149], our present goal is to generalize the statement to simplicial spheres and see why it follows from the Cohen-Macaulay (actually, Gorenstein) property for certain rings.

16.3. **The h-vector of a simplicial complex.** We have assumed for these notes (starting in Lecture 1) that the reader has seen a little bit of algebraic topology, so simplicial complexes should be familiar, though perhaps not as combinatorial objects. For a reminder, recall that a *simplicial complex* with vertices $1, \ldots, n$ is a collection Δ of subsets of $\{1, \ldots, n\}$ that is closed under taking subsets: if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then also $\tau \in \Delta$. For example, the boundary of a simplicial polytope is a simplicial complex.

We can record the numerical data associated to a general simplicial complex the same way we did for polytopes: let $f_i(\Delta)$ be the number of i-faces (in other words, simplices of dimension i, meaning $i+1$ vertices) of Δ. Note, however, that if P is a simplicial polytope with boundary simplicial complex Δ, then we do not count the polytope itself as a face of Δ.

For simplicial complexes there is another, seemingly complicated but in fact often elegant, way to record the face numbers.

Definition 16.18. The *h-vector* $(h_0, h_1, \ldots, h_d)$ of a dimension $d-1$ simplicial complex Δ is defined by

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i},$$

where $f_{i-1} = f_{i-1}(\Delta)$. The left-hand side above is called the *h-polynomial* of Δ. If P is a polytope, write $h_i(P)$ for the i^{th} entry of the h-vector of the boundary of P.

The next section will tell us (i.e., commutative algebraists) conceptual ways to compute h-polynomials, so the octahedron example will appear later, in Example 16.24. For now, we observe that there is a direct translation from the h-vector back to the f-vector.

Lemma 16.19. *For any simplicial complex Δ of dimension $d-1$, each f_i is a positive integer linear combination of $h_0, \ldots, h_d$. More precisely,*

$$f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j.$$

Proof. Dividing both sides of Definition 16.18 by $(1-t)^d$ yields

$$\sum_{i=0}^{d} h_i \frac{t^i}{(1-t)^i} \cdot \frac{1}{(1-t)^{d-i}} = \sum_{i=0}^{d} f_{i-1} \frac{t^i}{(1-t)^i}.$$

Setting $s = \frac{t}{1-t}$, we find that $\frac{1}{1-t} = s+1$, and hence

$$\sum_{i=0}^{d} h_i s^i (s+1)^{d-i} = \sum_{i=0}^{d} f_{i-1} s^i.$$

Now take the coefficient of s^i on both sides. □

As a consequence of Lemma 16.19, if we wish to bound the number of i-faces of a simplicial complex Δ by the number of i-faces of a cyclic polytope, then it is certainly enough to bound the h-vector of Δ entrywise by the h-vector of the cyclic polytope. This is what we shall do in Theorem 16.20.

It might seem from the discussion leading to Theorem 16.16 that its statement is geometric, in the sense that it says something fundamental about convexity. But in fact the statement turns out to be topological, at heart: the *same* result holds for simplicial complexes that are homeomorphic to spheres, without any geometric hypothesis akin to convexity. It should be noted that in a precise sense, most simplicial spheres are not convex [86]. Here is the more general statement.

Theorem 16.20 (Upper Bound Theorem, simplicial sphere h-vector version)**.** *For any dimension $d-1$ simplicial sphere Δ with n vertices, $h_i(\Delta) \le h_i(C(n,d))$ for all i, and consequently $f_i(\Delta) \le f_i(C(n,d))$ for all i.*

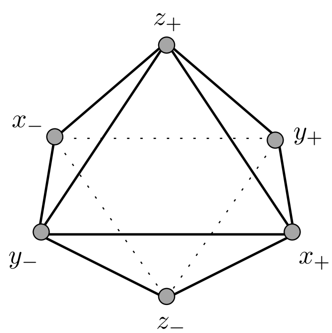
16.4. **Stanley-Reisner rings.** The connection between the Upper Bound Theorem and commutative algebra lies with certain rings constructed from simplicial complexes. For notation, if $\sigma \subseteq \{1, \ldots, n\}$, then write $x^\sigma = \prod_{j \in \sigma} x_j$ for the corresponding squarefree monomial.

Definition 16.21. Let Δ be a simplicial complex vertex set $\{1, \ldots, n\}$. The *Stanley-Reisner ring* of Δ is the quotient

$$\mathbb{K}[\Delta] = \mathbb{K}[x_1, \ldots, x_n]/\langle x^\sigma \mid \sigma \notin \Delta\rangle$$

of the polynomial ring by the *Stanley-Reisner* ideal $I_\Delta = \langle x^\sigma \mid \sigma \notin \Delta\rangle$.

Example 16.22. Let Δ be the boundary of the octahedron from Example 16.2. Let $R = \mathbb{K}[x_-, x_+, y_-, y_+, z_-, z_+]$, with the positively and negatively indexed x, y, and z-variables on the corresponding axes:



The Stanley-Reisner ring of Δ is

$$\mathbb{K}[\Delta] = R/\langle x_- x_+, y_- y_+, z_- z_+\rangle,$$

since every subset of the vertices not lying in Δ contains one of the main diagonals of the octahedron.

Geometrically, each set σ corresponds to a vector subspace of $\mathbb{K}^n$, namely the subspace spanned by the basis vectors $\{e_j \mid j \in \sigma\}$. Hence a simplicial complex Δ corresponds to a configuration of subspaces of $\mathbb{K}^n$, and $\mathbb{K}[\Delta]$ is simply the affine coordinate ring of this configuration.

The easiest way to understand the algebraic structure of $\mathbb{K}[\Delta]$ is to use the $\mathbb{N}^n$-*grading* of the polynomial ring:

$$\mathbb{K}[x_1, \ldots, x_n] = \bigoplus_{a \in \mathbb{N}^n} \mathbb{K} \cdot x^a,$$

where $x^a = x_1^{a_1} \cdots x_n^{a_n}$ for $a = (a_1, \ldots, a_n)$. The Stanley-Reisner ideal I_Δ is an $\mathbb{N}^n$-graded ideal because it is generated by monomials. Let us say that a monomial x^a has *support* $\sigma \subseteq \{1, \ldots, n\}$ if $a_j \neq 0$ precisely for $j \in \sigma$. By Definition 16.21, then, the monomials that remain nonzero in the quotient $\mathbb{K}[\Delta]$ are precisely those supported on faces of Δ.

Proposition 16.23. *The h-polynomial of a dimension $d-1$ simplicial complex Δ equals the numerator of the Hilbert series of its Stanley-Reisner ring $\mathbb{K}[\Delta]$:*

$$P(\mathbb{K}[\Delta]; t) = \frac{h_0(\Delta) + h_1(\Delta)t + h_2(\Delta)t^2 + \cdots + h_d(\Delta)t^d}{(1-t)^d}.$$

Proof. Begin by calculating the $\mathbb{N}^n$-graded Hilbert series $H(\mathbb{K}[\Delta]; x)$, which is by definition the sum of all of the monomials that are nonzero in $\mathbb{K}[\Delta]$:

$$H(\mathbb{K}[\Delta]; x) = \sum_{\sigma \in \Delta} x^\sigma \prod_{j \in \sigma} \frac{1}{1 - x_j}.$$

The summand indexed by σ is simply the sum of all monomials with support exactly σ. Now specialize $x_j = t$ for all j to get the Hilbert-Poincaré series

$$\begin{aligned} P(\mathbb{K}[\Delta]; t) &= \sum_{\sigma \in \Delta} t^{|\sigma|} \cdot \frac{1}{(1-t)^{|\sigma|}} \\ &= \sum_{i=0}^{d} f_{i-1}(\Delta) \frac{t^i}{(1-t)^i} \\ &= \frac{1}{(1-t)^d} \sum_{i=0}^{d} f_{i-1}(\Delta) t^i (1-t)^{d-i}, \end{aligned}$$

whose numerator is the h-polynomial of Δ by definition. □

Example 16.24. The h-vector of octahedron is now easy for us, as commutative algebraists, to compute: the Stanley-Reisner ideal I_Δ is a complete intersection generated by three quadrics, so the Hilbert-Poincaré series of $\mathbb{K}[\Delta]$ is

$$P(\mathbb{K}[\Delta]; t) = \frac{(1-t^2)^3}{(1-t)^6} = \frac{(1+t)^3}{(1-t)^3}.$$

Therefore the h-polynomial of Δ is $1 + 3t + 3t^2 + t^3$, and its h-vector is $(1, 3, 3, 1)$.

Exercise 16.25. Let Δ be a simplicial complex such that the Stanley-Reisner ideal I_Δ is $(de, abe, ace, abcd)$. Find the Betti numbers and Hilbert series of $\mathbb{K}[\Delta]$.

16.5. **Local cohomology of Stanley-Reisner rings.** The salient features of the h-vector in Example 16.24 are that (i) every h_i is positive, and moreover, (ii) the h-vector is symmetric. These combinatorial observations are attributable to homological conditions from commutative algebra: (i) $\mathbb{K}[\Delta]$ is Cohen-Macaulay, and moreover, (ii) $\mathbb{K}[\Delta]$ is Gorenstein. In this section we explain how the proofs of these statements go, although a detail here and there is too nitpicky—and too ubiquitously found in the literature—to warrant including here (our exposition in this section follows [117, Section 13.2], sometimes nearly verbatim).

The key point is a formula, usually attributed as an unpublished result of M. Hochster, for the local cohomology of Stanley-Reisner rings. More precisely, the formula is for the $\mathbb{Z}^n$-graded Hilbert series of the local cohomology. To state it, we need an elementary notion from simplicial topology.

Definition 16.26. The *link* of a face σ inside the simplicial complex Δ is

$$\operatorname{link}_\sigma(\Delta) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \varnothing\},$$

the set of faces that are disjoint from σ but whose unions with σ lie in Δ.

For the reader who has not seen links before, the precise definition is not so important at this stage; one can think of it simply as a way to associate a subcomplex of Δ to each face of Δ. In a precise sense, the link records how Δ behaves near σ; see the proof of Theorem 16.28 for the relevant property of links here.

The $\mathbb{Z}^n$-graded Hilbert series $H(H^i_{\mathfrak{m}}(\mathbb{K}[\Delta]); x)$ in the variables $x = x_1, \ldots, x_n$, where $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, is expressed in terms of reduced cohomology of links in Δ.

Theorem 16.27 (Hochster's formula)**.** *The $\mathbb{Z}^n$-graded Hilbert series of the i^{th} local cohomology module with maximal-support of a Stanley-Reisner ring satisfies*

$$H(H^i_{\mathfrak{m}}(\mathbb{K}[\Delta]);x) = \sum_{\sigma\in\Delta} \dim_{\mathbb{K}} \widetilde{H}^{i-|\sigma|-1}(\mathrm{link}_\sigma(\Delta);\mathbb{K}) \prod_{j\in\sigma} \frac{x_j^{-1}}{1-x_j^{-1}},$$

where $\widetilde{H}$ denotes reduced cohomology and $|\sigma| = \dim(\sigma)+1$ is the cardinality of σ.

Let us parse the statement. The product over $j \in \sigma$ is the sum of all Laurent monomials whose exponent vectors are nonpositive and have support exactly σ. Therefore, the formula for the Hilbert series of $H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])$ is just like the one for $\mathbb{K}[\Delta]$ itself in the first displayed equation in the proof of Proposition 16.23, except that here we consider monomials with negative exponents and additionally take into account the nonnegative coefficients $\dim_{\mathbb{K}} \widetilde{H}^{i-|\sigma|-1}(\mathrm{link}_\sigma(\Delta);\mathbb{K})$ depending on i and σ.

The proof of Theorem 16.27 is accomplished one $\mathbb{Z}^n$-graded degree at a time: the Čech complex of $\mathbb{K}[\Delta]$ in each fixed $\mathbb{Z}^n$-graded degree is (essentially) the cochain complex for the desired link. Details of the proof of Hochster's formula can be found in [150, Chapter II], [16, Chapter 5], or [117, Chapter 13].

Theorem 16.28. *Let Δ be a simplicial sphere. Then $\mathbb{K}[\Delta]$ is a Gorenstein ring. In fact, $\mathbb{K}[\Delta]$ is its own canonical module, even taking into account the $\mathbb{Z}^n$-grading.*

Proof. Set $d = \dim(\Delta)+1$. First of all, we need that $\mathbb{K}[\Delta]$ is Cohen-Macaulay. By Theorem 16.27 and the characterization of the Cohen-Macaulay condition by the vanishing of $H^i_{\mathfrak{m}}$ for $i < d$, the ring $\mathbb{K}[\Delta]$ is Cohen-Macaulay if and only if $\widetilde{H}^{i-|\sigma|-1}(\mathrm{link}_\sigma(\Delta);\mathbb{K})$ is nonzero only for $i = d$. As Δ is a sphere, the link of each face σ is a new sphere of dimension $d-|\sigma|-1$. Since the only nonvanishing reduced cohomology of a sphere is at the top, the desired vanishing holds.

Furthermore, the sole nonvanishing reduced cohomology $\widetilde{H}^{d-|\sigma|-1}(\mathrm{link}_\sigma(\Delta);\mathbb{K})$ is isomorphic to $\mathbb{K}$, since the link is a sphere. Hence the top local cohomology has $\mathbb{Z}^n$-graded Hilbert series

$$H(H^d_{\mathfrak{m}}(\mathbb{K}[\Delta]);x) = \sum_{\sigma\in\Delta}\prod_{j\in\sigma} \frac{x_j^{-1}}{1-x_j^{-1}}.$$

Taking the Matlis dual of $H^d_{\mathfrak{m}}(\mathbb{K}[\Delta])$ results in the replacement $x_j^{-1} \rightsquigarrow x_j$ at the level of Hilbert series. Therefore $H^d_{\mathfrak{m}}(\mathbb{K}[\Delta])^\vee$, which is the canonical module $\omega_{\mathbb{K}[\Delta]}$ in any case, has the same $\mathbb{Z}^n$-graded Hilbert series as $\mathbb{K}[\Delta]$; see the proof of Proposition 16.23. In particular, $\omega_{\mathbb{K}[\Delta]}$ has a $\mathbb{Z}^n$-graded degree zero generator that is unique up to scaling. Since $\omega_{\mathbb{K}[\Delta]}$ is a faithful $\mathbb{K}[\Delta]$-module [16, Proposition 3.3.11], the $\mathbb{K}[\Delta]$-module map sending $1 \in \mathbb{K}[\Delta]$ to a $\mathbb{Z}^n$-graded degree zero generator of $\omega_{\mathbb{K}[\Delta]}$ is injective on each $\mathbb{Z}^n$-degree, and hence $\mathbb{K}[\Delta] \to \omega_{\mathbb{K}[\Delta]}$ is an isomorphism. □

16.6. **Proof of the Upper Bound Theorem.** The key point about the Cohen-Macaulay condition in this combinatorial setting is that it implies positivity in the h-vector: each h_i counts the vector space dimension of a $\mathbb{Z}$-graded piece of a certain finite-dimensional $\mathbb{K}$-algebra.

Proposition 16.29. *Let M be a finitely generated $\mathbb{Z}$-graded Cohen-Macaulay module over $\mathbb{K}[x_1,\ldots,x_n]$, where $\deg(x_j) = 1$ for all j. If Θ is a linear system of parameters for M, then the numerator of $P(M;t)$ equals $P(M/\Theta M;t)$.*

Proof. If $\theta \in \mathbb{K}[x_1, \ldots, x_n]$ is a nonzerodivisor on M, then $P(M/\theta M; t) = P(M; t) - P(M(-1); t) = (1-t)P(M; t)$ by the usual short exact sequence computation. Now repeat for the sequence $\Theta = \theta_1, \theta_2, \ldots$, which is a regular sequence by the Cohen-Macaulay condition on M. $\square$

Remark 16.30. R. Stanley observed that for a graded module M over the ring $\mathbb{K}[x_1, \ldots, x_n]$, where $\deg(x_j) = 1$ for all j, the converse of Proposition 16.29 holds as well: if the numerator of $P(M; t)$ equals $P(M/\Theta M; t)$ for a linear system of parameters Θ, then M is Cohen-Macaulay; see [117, Theorem 13.37.6].

Corollary 16.31. *Let R be a graded Gorenstein ring generated in degree 1. If the numerator of $P(R; t)$ is $\sum_{i=0}^{r} h_i t^i$ with $h_r \neq 0$, then $h_i = h_{r-i}$ for all i.*

Proof. By Proposition 16.29, $P(R; t)$ equals the Hilbert-Poincaré series of a finite-dimensional $\mathbb{K}$-algebra that is isomorphic to its own $\mathbb{K}$-vector space dual. $\square$

Definition 16.32. When the ring R in Corollary 16.31 is the Stanley-Reisner ring $\mathbb{K}[\Delta]$ of a dimension $d-1$ simplicial sphere, the relations $h_i = h_{d-i}$ for all i, which hold by Theorem 16.28, are called the *Dehn-Sommerville equations.*

The Dehn-Sommerville equations are extraordinarily simple when viewed in terms of h-vectors, but they represent quite subtle conditions on f-vectors. Note that if Δ has dimension $d-1$, then indeed $h_d(\Delta)$ is the highest nonzero entry of the h-vector, by Proposition 16.23.

Corollary 16.33. *If Δ is a dimension $d-1$ simplicial complex with n vertices such that $\mathbb{K}[\Delta]$ is Cohen-Macaulay, then $h_i(\Delta) \leq \binom{n-d-1+i}{i}$ for all i.*

Proof. Fix a linear system of parameters Θ in the Stanley-Reisner ring $\mathbb{K}[\Delta]$. Then $h_i(\Delta)$ equals the vector space dimension of the degree i piece $(\mathbb{K}[\Delta]/\Theta)_i$ by Proposition 16.23 and Proposition 16.29. Extending Θ to a basis of the linear polynomials $\mathbb{K}[x_1, \ldots, x_n]_1$, we see that $\mathbb{K}[\Delta]/\Theta$ is isomorphic to a quotient of a polynomial ring in $n-d$ variables. Therefore the dimension of $(\mathbb{K}[\Delta]/\Theta)_i$ is at most the number $\binom{n-d-1+i}{i}$ of degree i monomials in $n-d$ variables. $\square$

Exercise 16.34. Consider any neighborly simplicial polytope P of dimension d with n vertices; for example, the cyclic polytope $C(n,d)$ satisfies this hypothesis. Using the argument of Corollary 16.33, prove that

$$h_i(C(n,d)) = \binom{n-d-1+i}{i} \text{ for } i = 0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor .$$

At this point, we are now equipped to complete the proof of the Upper Bound Theorem for simplicial spheres.

Proof of Theorem 16.20. Assume that a given simplicial sphere Δ has dimension $d-1$ and n vertices. We need only show that $h_i(\Delta) \leq h_i(C(n,d))$ for $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$, because the result for the remaining values of i will then hold by the Dehn-Sommerville equations from Definition 16.32. Using the explicit computation of $h_i(C(n,d))$ in Exercise 16.34 for the desired values of i, the Upper Bound Theorem follows from Corollary 16.33. $\square$

In retrospect, what has happened here?

(1) An explicit local cohomology computation allowed us to conclude that $\mathbb{K}[\Delta]$ is Gorenstein when Δ is a simplicial sphere.

(2) The Cohen-Macaulay property for $\mathbb{K}[\Delta]$ allowed us to write the h-polynomial as the Hilbert-Poincaré series of a finite-dimensional algebra, and to bound the entries of the h-vector by certain binomial coefficients.
(3) The Gorenstein property reduced the Upper Bound Theorem to checking only half of the h-vector.
(4) Finally, an explicit computation for neighborly simplicial polytopes demonstrated that equality with the binomial coefficients indeed occurs for cyclic polytopes, at least on the first half of the h-vector.

Note that equality with the binomial coefficients does not occur, even for cyclic polytopes, on the second half of the h-vector: although $h_i = h_{d-i}$, the same symmetry does not occur with the binomial coefficients, which keep growing with i.

In this lecture we restrict ourselves to the realm of algebraic D-modules as opposed to analytic D-modules; moreover, for computational purposes, we would deal mainly with modules over the *Weyl algebra*, the algebra of the differential operators with polynomial coefficients.

Equipped with the machinery of Gröbner bases — the Weyl algebra turns out to be "Gröbner friendly" — there has been a considerable progress made in the field of computational D-module theory resulting in many striking algorithms, several of which will be discussed in Lecture 23. We will revisit monomial orders from Lecture 5 in order to regear Gröbner bases theory for the Weyl algebra.

The study of D-modules is mainly focused on the class of *holonomic* D-modules, which is a class possessing a lot of nice properties. Not only that, this class is rich in examples coming from other areas (e.g. hypergeometric differential equations; see [136]). We will show how to compute the *characteristic ideal* of a D-module and use it to determine holonomicity algorithmically.

17.1. **Weyl algebra.** Let $\mathbb{K}$ be a field of characteristic 0.

Definition 17.1. The algebra

$$A_n(\mathbb{K}) = \mathbb{K}\langle x, \partial\rangle = \mathbb{K}\langle x_1, \partial_1, \dots, x_n, \partial_n\rangle,\ \partial_j x_i - x_i \partial_j = \delta_{ij}$$

is called the *Weyl algebra* over $\mathbb{K}$ in n variables or simply the n-th Weyl algebra.

The n-th Weyl algebra is isomorphic to the algebra of differential operators (with polynomial coefficients) acting on the affine space $\mathbb{K}^n$; see a proof, for instance, in Coutinho's book [23, Theorem 2.3]. The Weyl algebra A_n acts on the polynomial ring R_n naturally: for $f \in R_n$ the action of the generators is

$$\partial_i \cdot f = \frac{\partial f}{\partial x_i},\ x_i \cdot f = x_i f.$$

It is easy to see that monomials $x^\alpha \partial^\beta \in A_n(\mathbb{K}),\ \alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ form a $\mathbb{K}$-basis of A_n. Thus, every element of $Q \in A_n$ may be presented in the *right normal form*

$$Q = \sum a_{\alpha\beta} x^\alpha \partial^\beta,$$

where all but finitely many of $a_{\alpha\beta}$ are zeros. We refer to the degree of this polynomial expression as the total degree of Q.

The Weyl algebra is simple: the only two-sided ideals are 0 and the whole of A_n. In what follows, all A_n-ideals we mention are left one-sided ideals unless otherwise stated. It has been discovered by Stafford (see [8, Ch.1,§7]) that for every A_n-ideal one can find a generating set of two elements. Such a set may be obtained algorithmically as shown in [100].

In what follows we frequently write D instead of A_n.

Exercise 17.2. For $M = A_1/A_1\partial$, find a generator for $M \oplus M$. Also, find a generator for $M \oplus A_1/A_1 x$.

Exercise 17.3. Show that A_1 has proper left ideals that are not principal. In contrast, let W_1 be the ring of $\mathbb{K}$-linear differential operators on $\mathbb{K}(x)$. (I.e.: W_1 is as $\mathbb{K}$-space identified with $\mathbb{K}(x) \otimes_{\mathbb{K}[x]} A_1$, but the product is twisted: in $(q \otimes P) \cdot (q' \otimes P')$, the operator P acts on $q' \otimes P'$ via the product rule). Show that every proper left ideal of W_1 is principal.

17.2. **Gröbner bases for Weyl algebra.** The right normal form of an element in the Weyl algebra looks as a polynomial, although clearly the multiplication of two such form is not as trivial as in the case of a (commutative) polynomial ring. For example, in $A_1 = \mathbb{K}\langle x, \partial\rangle$ the following holds for all i and j:

$$\partial^l x^m = \sum_{i=0}^{\min\{l,m\}} \frac{l(l-1)...(l-i+1)m(m-1)...(m-i+1)}{i!} x^{m-i}\partial^{l-i}.$$

Nonetheless, there is a way to transplant Gröbner bases techniques to this non-commutative setting.

First of all, we can filter A_n by the total degree: let $\mathcal{F} = \{F_i\}$, where $F_i = \{Q \mid Q \in A_n, \text{ total degree of } Q \text{ is at most } i\}$. The associated graded algebra $\mathrm{gr}_{\mathcal{F}}(A_n)$ with respect to this filtration is commutative; indeed, it is not hard to check that for any $P, Q \in F_i$, $i > 0$, the commutator $[P, Q] = PQ - QP$ belongs to F_{i-1}.

More generally, take a weight vector $(u, v) \in \mathbb{Z}^n \times \mathbb{Z}^n$ such that $u_i + v_i \geq 0$ for all i. Then we can consider $\mathrm{gr}_{(u,v)}(A_n)$ — the associated graded algebra with respect to the filtration by the weighted (u, v)-degree.

Exercise 17.4. We have seen that for $u = v = (1, 1, ..., 1)$ the associated graded algebra of the Weyl algebra is commutative. Prove that

(i) $gr_{(u,v)}(A_n) = \mathbb{K}[x, \xi]$, the ring of polynomials in n variables x and n variables ξ (images of ∂), if and only if $u_i + v_i > 0$ for all i;

(ii) the filtration defined by a weight (u, v) is good — i.e. every component of $gr_{(u,v)}(A_n)$ is finite-dimensional — iff $u_i > 0$ and $v_i > 0$ for all i;

(iii) if $u = -v$ then $gr_{(u,v)}(A_n) = A_n$.

Therefore, where the initial ideal $\mathrm{in}_{(u,v)}(I)$ of a left ideal $I \subset A_n$ lives depends on the weight (u, v). However, no matter whether it is an ideal of $\mathbb{K}[x, \xi]$ or A_n, we need Buchberger's algorithm in the Weyl algebra in order to compute it.

If the order $\geq_{(u,v)}$ — refined with, for instance, lexicographic order — is a term order (this is the case when $u_i > 0$ and $v_i > 0$ for all i), then the usual Buchberger's algorithm (Algorithm 5.30) can be used: the initial (u, v)-forms of the produced Gröbner basis of I would form a generating set of $\mathrm{in}_{(u,v)}(I)$.

But can we do anything if $\geq_{(u,v)}$ is non-term? A positive answer is provided in the form of the following homogenization process.

Definition 17.5. The free associative algebra generated by $h, x_1, ..., x_n, \partial_1, ..., \partial_n$ with the relations $\partial_j x_i - x_i\partial_j = \delta_{ij}h^2$ is called the *homogenized Weyl algebra* $A_n^{(h)}$. We call h the *homogenization variable.* The dehomogenization map $A_n^{(h)} \to A_n$ is defined by substitution $h = 1$, i.e. $Q \mapsto Q|_{h=1}$ for $Q \in A_n^{(h)}$.

If a term order $\succ$ on monomials of $A_n^{(h)}$ is such that $h^2 \prec x_i\partial_i$ for all i, then Algorithm 5.27 for the normal form $NF(f, G)$ terminates provided G consists of homogeneous elements in $A_n^{(h)}$. Also, the Buchberger's algorithm (Algorithm 5.30) terminates on homogeneous inputs in $A_n^{(h)}$. See Proposition 1.2.2 of [136] for proof.

Exercise 17.6. Starting with the weight $(u, v) \in \mathbb{Z}^n \times \mathbb{Z}^n$ (for A_n) show that the weight (t, u, v) (for $A_n^{(h)}$) — subject to the condition $2t \leq u_i + v_i$ for all i — defines the weight order $\geq_{(t,u,v)}$, which — refined with an order where $h < x_i$ and $h < \partial_i$ for all i — gives an order needed in the previous paragraph.

Algorithm 17.7. $G = BuchbergerWA(F)$

Require: $F \subset A_n$, $(u,v) \in \mathbb{Z}^n \times \mathbb{Z}^n$ such that $u_i + v_i \geq 0$ for all i.
Ensure: G is a Gröbner basis of the ideal $A_n \cdot \{F\}$.

(1) Homogenize the generators from F to get $H(F) \subset A_n^{(h)}$.
(2) Compute $G^h := Buchberger(H(F))$ (use the order $\succ$ discussed above).
(3) Dehomogenize: $G := G^h|_{h=1}$.

What follows is a script that shows how to create a Weyl algebra and a homogenized Weyl algebra in Macaulay 2 [**?**]. Then we create an ideal generated by three random elements of total degree 2 and show that it equals the whole ring.

```
i1 : A = QQ[x,y,Dx,Dy, WeylAlgebra=>{x=>Dx,y=>Dy}];

i2 : Dx*x

o2 = x*Dx + 1

o2 : A

i3 : B = QQ[x,y,Dx,Dy,h,WeylAlgebra=>{x=>Dx,y=>Dy,h}];

i4 : Dx*x

                2
o4 = x*Dx + h

o4 : B

i5 : I = ideal (random(2,A), random(2,A), random(2,A))

             5        8  2            2        2          ...
o5 = ideal (-*x*y - -*y  + x*Dx - -*y*Dx - -*y*Dy - ...
             2        9               5        3          ...
o5 : Ideal of A

i6 : gb I

o6 = | 1 |

o6 : GroebnerBasis
```

The package *D-modules for Macaulay 2* [99] broadens the arsenal of functions for the algebraic D-modules. Below, we illustrate Exercise 17.4 with a computation of initial ideals with respect to two weights of different type.

```
i7 : load "D-modules.m2";

i8 : use A;

i9 : I = ideal (x*Dx+2*y*Dy-3, Dx^2-Dy)

                                 2
o9 = ideal (x*Dx + 2y*Dy - 3, Dx  - Dy)

o9 : Ideal of A

i10 : inw(I, {1,3,3,-1})

                     2  2                  2
o10 = ideal (x*Dx, y Dy , y*Dx*Dy, Dx )

o10 : Ideal of QQ [x, y, Dx, Dy]

i11 : inw(I, {-1,-3,1,3})
```

```
o11 = ideal (x*Dx + 2y*Dy - 3, Dy)

o11 : Ideal of A
```

17.3. **Bernstein's inequality.** Let I be a left ideal of A_n and a weight (u, v) such that $u_i + v_i > 0$. We define the *dimension* of I as the dimension of $\mathrm{gr}_{(u,v)}(I)$, a.k.a. $\mathrm{in}_{(u,v)}(I)$, the ideal that lives in the commutative ring $\mathbb{K}[x, \xi]$ as in Exercise 17.4(i). (The definition does not depend on the weight vector.)

This dimension is clearly bounded from above by $2n$, the dimension of the whole ring A_n. The striking fact is that the lower bound is nontrivial:

Theorem 17.8 (Bernstein's inequality)**.** *Let I be a nonzero left A_n-ideal. Then $n \leq \dim(I) \leq 2n$.*

The special case of this theorem for the weight vector $(0, e)$, where $e = (1, ..., 1)$, is known as the weak FTAA (fundamental theorem of algebraic analysis) and first was proved in [7] (see [23] for an elementary proof).

In this special case the initial ideal of I has a special name:

Definition 17.9. For an ideal $I \subset A_n$, the *characteristic ideal* is defined as $\mathrm{in}_{(0,e)}(I)$.

A stronger statement can be made about the characteristic ideal:

Theorem 17.10 (Strong FTAA; Sato, Kawai, Kashiwara [137])**.** *Let I be a nonzero left A_n-ideal. Then $n \leq \dim(J) \leq 2n$, for every minimal prime J of $\mathrm{in}_{(0,e)}(I)$.*

17.4. **Holonomic D-modules.** An ideal $I \subset D = A_n$ is called *holonomic* if its characteristic ideals has dimension n (minimal possible dimension).

The D-module $M = D/I$ is called *holonomic* if I is holonomic.

Alternatively, given a D-module M (not necessarily presented as a quotient of D) we may define holonomicity by developing the dimension theory from scratch. From Lecture 5, recall that a filtration $\mathcal{F}$ on M is *good* if all components of the associated graded module $\mathrm{gr}_{\mathcal{F}}(M)$ are finitely generated.

Proposition 17.11. *A D-module M is finitely generated iff there exists a good filtration on M.*

Proof. See [8, Ch.1: Propositions 2.6,2.7]. □

Let $\mathcal{F} = \{\mathcal{F}_i\}$ be a good filtration on M. Then there is a polynomial $\xi(i) = \dim_{\mathbb{K}}(\mathcal{F}_i)$ (Hilbert polynomial), the degree of which defines the dimension of M. The module M is *holonomic* iff $\dim M = n$.

The above definitions are equivalent for a finitely generated module M, since if it holonomic (in the sense of the latter definition), then it is cyclic (i.e. generated by one element) [8, Ch.1: Theorem 8.18]. Indeed, M is then isomorphic to the quotient of the Weyl algebra A_n by the annihilator of a cyclic generator.

Remark 17.12. From the homological point of view a holonomic D-module M is a D-module for which $\mathrm{Ext}^j_{A_n}(M, A_n)$ vanishes unless $j = n$.

Exercise 17.13. Show that the modules below are holonomic.

- The polynomial ring R_n, which can be viewed as the quotient of A_n by the left ideal $A_n \cdot (\partial_1, ..., \partial_n)$;
- The module $A_1/A_1 \cdot Q$ for any nonconstant $Q \in A_1$;

17.5. **Holonomicity of localization.** For what follows we need

Lemma 17.14. *Let M be a D-module (not necessarily finitely generated).*

If there is a filtration $\{\Gamma_d\}$ on M and $c_1, c_2 \in \mathbb{Z}_{>0}$ such that $\dim_{\mathbb{K}}(\Gamma_d) \leq c_1 d^n + c_2(d+1)^{n-1}$ *for all d, then M is holonomic.*

Proof. See [8, Ch.1, Theorem 5.4] □

For example, $R = R_n$ is a holonomic D-module: we may prove this by showing that $\{\Gamma_d = \{g \in R \mid \deg(g) \leq d\}\}$ is a filtration that satisfies Lemma 17.14.

Localize R by inverting a polynomial $f \in R$ to get $R_f = \mathbb{K}[x, f^{-1}]$, which possesses the natural structure of a D-module:

$$x_i \cdot \frac{g}{f^d} = \frac{xg}{f^d}, \quad \partial_i \cdot \frac{g}{f^d} = \frac{\partial g/\partial x_i}{f^d} - \frac{dg(\partial f/\partial x_i)}{f^{d+1}},$$

for all $1 \leq i \leq n,\ f, g \in R,\ d \in \mathbb{Z}_{>0}$.

Theorem 17.15. *The D-module R_f is a holonomic.*

Proof. Let $d_f = \deg(f)$ and put

$$\Gamma_d = \{\frac{g}{f^d} \mid \deg(g) \leq (d_f+1)d\}.$$

The dimension of Γ_d over $\mathbb{K}$ equals the dimension of the space of polynomial of degree at most $(d_f+1)d$, therefore, for some constant c_2

$$\dim(\Gamma_d) \leq \frac{(d_f+1)^n}{n!} d^n + c_2(d+1)^{n-1},$$

for all d. Now we apply Lemma 17.14 to finish the proof. □

Here is an example a computation in Macaulay 2: the localized module R_f is computed for $R = R_2$ and $f = x^2 + y^2$.

```
i12 : f = x^2+y^2;

i13 : L = Dlocalize(A^1/ideal(Dx,Dy), f)

o13 = cokernel | yDx-xDy xDx+yDy+4 x2Dy+y2Dy+4y |

                             1
o13 : A-module, quotient of A

i14 : isHolonomic L

o14 = true
```

How localization can be computed algorithmically is explained in Lecture 23.

Lecture 18. Local duality revisited, and global duality (SI)

The aim of this lecture is to explain the adjective 'duality' in Grothendieck's duality theorem 11.29 by describing its connection to the classical Poincaré duality theorem for manifolds, and to provide another of perspective on the local duality theorem 11.32, which clarifies its relationship with Serre duality for sheaves on projective spaces. It also discusses global canonical modules, which are required to state and prove Serre duality.

Poincaré duality versus Grothendieck duality

Definition 18.1. Let $\mathbb{K}$ be a field, and $R = \bigoplus_{n=0}^{d} R_i$ a graded $\mathbb{K}$-algebra with $\operatorname{rank}_{\mathbb{K}} R$ finite and $R_0 = \mathbb{K}$. We may as well assume that $R_d \neq 0$. For now, we do not assume that R is commutative.

The product on R provides, for each integer $0 \leqslant n \leqslant d$, a bilinear pairing:

$$R_n \times R_{d-n} \longrightarrow R_d. \tag{18.1.1}$$

We say that R has *Poincaré duality*, or that R is a *Poincaré duality algebra*, if the pairing above is non-degenerate for each $0 \leqslant n \leqslant d$.

The name originates from the prototypical example of such an algebra: the cohomology algebra of a compact, connected, orientable manifold; the non-degeneracy of the pairing 18.1.1 is then Poincaré duality. Perhaps the following consequence of Poincaré duality is more familiar:

Exercise 18.2. Let R be a $\mathbb{K}$-algebra as above. Prove that when R has Poincaré duality, $\operatorname{rank}_{\mathbb{K}}(R_n) = \operatorname{rank}_{\mathbb{K}}(R_{d-n})$ for each $0 \leqslant n \leqslant d$. Give examples that show that the converse does not hold.

Here is why this notion is relevant to these proceedings:

Exercise 18.3. Let R be a graded $\mathbb{K}$-algebra as in 18.1, and assume furthermore that it is commutative. Prove that R is Gorenstein if and only if it is a Poincaré duality algebra.

Claim. Grothendieck duality, encountered in Theorem 11.29, is an extension of Poincaré duality to higher dimensional Gorenstein rings.

In order to explain this claim we re-interpret Poincaré duality as follows.

18.4. The pairing 18.1.1 yields, for each $0 \leqslant n \leqslant d$, a $\mathbb{K}$-linear map

$$R_n \longrightarrow \operatorname{Hom}_{\mathbb{K}}(R_{d-n}, R_d).$$

The non-degeneracy of the pairing is equivalent to this map, and its counterpart $R_{d-n} \longrightarrow \operatorname{Hom}_{\mathbb{K}}(R_n, R_d)$, being bijective. In any case, taking a direct sum of the maps above yields a map of graded $\mathbb{K}$-vector spaces

$$R = \bigoplus_{n=0}^{d} R_n \longrightarrow \bigoplus_{n=0}^{d} \operatorname{Hom}_{\mathbb{K}}(R_{d-n}, R_d) = \operatorname{Hom}_{\mathbb{K}}(R, R_d).$$

Note that as a graded $\mathbb{K}$-vector space, $\operatorname{Hom}_{\mathbb{K}}(R, R_d)$ is situated between degrees $-d$ and 0, and the map above has degree $-d$: it maps R_n to $\operatorname{Hom}_{\mathbb{K}}(R, R_d)_{n-d}$. However, we can make it a map of degree zero by shifting $\operatorname{Hom}_{\mathbb{K}}(R, R_d)$ by d degrees to the left, to get a morphism of graded $\mathbb{K}$-vector spaces:

$$\chi\colon R \longrightarrow \operatorname{Hom}_k(R, R_d)[-d]. \tag{18.4.1}$$

Now the natural R-module structure on R passes to an R-module structure on $\mathrm{Hom}_{\mathbb{K}}(R, R_d)$. Here is the crucial point:

Exercise 18.5. Prove that the morphism χ is one of R-modules, and that the following conditions are equivalent:

(1) The $\mathbb{K}$-algebra R has Poincaré duality;
(2) The homomorphism χ is bijective;
(3) For some $a \in \mathbb{Z}$, the R-modules R and $\mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})[-a]$ are isomorphic.

Perhaps you recognize that the R-module $\mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})$ is the (graded) injective hull of $\mathbb{K}$; if not, then wait for Exercise 18.7 below.

Now we are ready to explain the connection between Poincaré duality and Grothendieck duality.

18.6. In the following paragraphs, a *graded* $\mathbb{K}$*-algebra* will mean a commutative, finitely generated, graded $\mathbb{K}$-algebra $R = \bigoplus_{n \geqslant 0} R_n$, with $R_0 = \mathbb{K}$. Its homogeneous maximal ideal $R_{\geqslant 1}$ will be denoted $\mathfrak{m}$. The Hilbert basis theorem implies that R is Noetherian. For such algebras, the (graded) injective hull of $\mathbb{K}$ is easy to describe:

Exercise 18.7. Let R be a graded $\mathbb{K}$-algebra. Prove that the graded $\mathbb{K}$-dual

$$\mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K}) = \bigoplus_{n \geqslant 0} \mathrm{Hom}_{\mathbb{K}}(R_n, \mathbb{K})$$

situated in degrees $\leqslant 0$, is the (graded) injective hull of $\mathbb{K}$.

The graded version of Grothendieck's theorem 11.29 thus reads:

Theorem 18.8. *Let R be a graded $\mathbb{K}$-algebra. When R is Gorenstein, there exists an integer a such that*

$$H^n_{\mathfrak{m}}(R) = \begin{cases} 0 & \text{if } n \neq \dim R, \\ \mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})[-a] & \text{if } n = \dim R. \end{cases}$$

The integer a that appears in the result is called, well, the *a-invariant* of R. You will find calculations of the a-invariant in Lecture 21.

Exercise 18.9. Assume that $\dim R = 0$; equivalently, that $R_n = 0$ for $n \gg 0$. Prove that $H^n_{\mathfrak{m}}(R) = R$ for $n = 0$ and that $H^n_{\mathfrak{m}}(R) = 0$ for $n \geqslant 0$.

This exercise, and Grothendieck's theorem, yield:

Corollary 18.10. *If R is Gorenstein and $\dim R = 0$, then there exists an $a \in \mathbb{Z}$ such that the R-modules R and $\mathrm{Hom}_{\mathbb{K}}(R, \mathbb{K})[-a]$ are isomorphic.*

Compare this result with Exercise 18.5. In this way does Grothendieck duality extend Poincaré duality.

Remark 18.11. We mentioned that cohomology algebra of compact, connected, orientable manifolds is a Poincaré duality algebra. Thus, they may be seen as analogues in topology of zero-dimensional Gorenstein rings; this analogy is further strengthened when we note that the cohomology algebra of any topological space is commutative, albeit in the graded sense: $a \cdot b = (-1)^{|a||b|} b \cdot a$.

One may then ask if higher dimensional Gorenstein rings have counterparts in topology, and indeed they do: Inspired by commutative algebra, Felix, Halperin, and Thomas [39] introduced and developed a theory 'Gorenstein spaces'. Recently, Dwyer, Greenlees, and Iyengar [30] proved that there is a version of Grothendieck's duality theorem **??** for such spaces.

Local duality revisited

The basic context of this discussion is fairly general: let R be a ring, and let M and N be (left) R-modules. Let $I^\bullet$ and $J^\bullet$ be injective resolutions of M and N respectively. Then one can form the 'Hom complex' $\operatorname{Hom}_R(I^\bullet, J^\bullet)$ with

$$\operatorname{Hom}_R(I^\bullet, J^\bullet)^n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(I^i, J^{i+n}),$$
$$\partial(f) = \partial^{J^\bullet} \circ f - (-1)^{|f|} f \circ \partial^{I^\bullet}.$$

The homology of this complex is a familiar object: for each integer n one has

$$H^n\big(\operatorname{Hom}_R(I^\bullet, J^\bullet)\big) = \operatorname{Ext}^n_R(M, N).$$

For a proof of this result, see, for instance, [158, ??].

18.12. Now suppose that F is an additive, covariant functor on the category of R-modules. For the moment, think of, say, $L \otimes_R -$ or $\operatorname{Hom}_R(L, -)$, for some fixed R-module L; eventually, we would like to think of local cohomology functors, but not yet. Then each $f \in \operatorname{Hom}_R(I^\bullet, J^\bullet)$ induces a homomorphism

$$F(f)\colon F(I^\bullet) \longrightarrow F(J^\bullet)$$

which, in homology, provides for each n, d in $\mathbb{Z}$, a pairing

$$\operatorname{Ext}^{d-n}_R(M, N) \times RF^n(M) \longrightarrow RF^d(N).$$

Here $RF^i(-)$ is the i^{th} right derived functor of F. The slogan here is "Extensions are universal cohomology operations." A natural question then is: Is there a fixed R-module N and integer d, for which this is an isomorphism for each M? Or, at least the induced map

$$RF^n_{\mathfrak{m}}(M) \longrightarrow \operatorname{Hom}_R(\operatorname{Ext}^{d-n}_R(M, N), RF^d(N))$$

is bijective? The point is that if there were such an N and a d, then we would have expressed the derived functors of M, which is presumably the object of our interest, in terms of a more familiar gadget, that is to say, $\operatorname{Ext}^{d-n}_R(M, N)$. In practice, we may want to restrict M to a certain class of modules, for example, the finitely generated ones.

Let us now return to our context.

18.13. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring, and set $d = \dim R$. The preceding discussion specialized to $F = \Gamma_{\mathfrak{m}}(-)$ yields, for each $n \in \mathbb{Z}$, a pairing

$$\operatorname{Ext}^{d-n}_R(M, N) \times H^n_{\mathfrak{m}}(M) \longrightarrow H^d_{\mathfrak{m}}(N).$$

This gives rise to an induced homomorphism of R-modules

$$H^n_{\mathfrak{m}}(M) \longrightarrow \operatorname{Hom}_R(\operatorname{Ext}^{d-n}_R(M, N), H^d_{\mathfrak{m}}(N)).$$

Given this discussion, the formulation of the result below—which extends local duality for Gorenstein rings 11.32—should come as no surprise.

Theorem 18.14. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a Cohen-Macaulay local ring with a canonical module ω, and let M be a finitely generated R-module. Set $d = \dim R$. Then $H^d_{\mathfrak{m}}(\omega) = E_R(\mathbb{K})$, and for each integer n the pairing*

$$\operatorname{Ext}^{d-n}_R(M, \omega) \times H^n_{\mathfrak{m}}(M) \longrightarrow H^d_{\mathfrak{m}}(\omega) = E_R(\mathbb{K})$$

induces a bijection of R-modules

$$H^n_{\mathfrak{m}}(M) \longrightarrow \mathrm{Hom}_R(\mathrm{Ext}^{d-n}_R(M,\omega), E_R(\mathbb{K})).$$

If, in addition, R is complete, then this pairing is perfect.

Sketch of proof. The first part, i.e., that $H^d_{\mathfrak{m}}(\omega) = E_R(\mathbb{K})$, is contained in Theorem 11.43. The bijectivity of the natural homomorphism

$$\chi^n(M)\colon H^n_{\mathfrak{m}}(M) \longrightarrow \mathrm{Hom}_R(\mathrm{Ext}^{d-n}_R(M,\omega), E_R(\mathbb{K}))$$

is proved by a descending induction on n. Its basis is $n = d$ (why?), and in this case Theorem 11.43 yields that $\chi^d(R)$ is bijective. Both functors $H^d_{\mathfrak{m}}(-)$ and $\mathrm{Hom}_R(\mathrm{Hom}_R(-,\omega), E_R(\mathbb{K}))$ are right exact and additive, so the bijectivity of $\chi^d(R)$ implies that of $\chi^d(M)$ for each finitely generated R-module M.

Assuming that $\chi^n(-)$ is bijective for some integer $n \leqslant d$. Using the long exact sequences associated with $\Gamma_{\mathfrak{m}}(-)$ and $\mathrm{Hom}_R(-,\omega)$ one now proves that $\chi^{n-1}(-)$ is bijective as well.

When R is complete, then the homomorphism

$$\mathrm{Ext}^{d-n}_R(M,\omega) \longrightarrow \mathrm{Hom}_R\left(H^n_{\mathfrak{m}}(M), E_R(\mathbb{K})\right)$$

is the Matlis dual of $\chi^n(M)$, and hence bijective. Thus, the pairing is perfect. □

Remark 18.15. The crucial point in the proof of Theorem 18.14 is the existence of *natural* maps, the $\chi^n(M)$; the rest of proof is more or less 'standard' homological algebra, that is to say, 'general nonsense'.

In particular, the argument would work with ω replaced by any complex D, with $H_n(D)$ finitely generated for each n, of finite injective dimension, and with $H^d_{\mathfrak{m}}(D) = E_R(\mathbb{K})$. Such an object exists whenever R is a quotient of a Gorenstein ring (for example, when R is complete) and is called a *dualizing complex* for R. With this on hand, one has a local duality statement even for rings which may not be Cohen-Macaulay; see Hartshorne [59] and Roberts [131].

There are more efficient, if more sophisticated, proofs of local duality; see [59, 131]. The argument given here has the merit that it immediately adapts to schemes.

18.16. Let X be a scheme of dimension d over a field $\mathbb{K}$. The discussion in 18.12 and 18.13 carries over to the context of schemes—for sheaves $\mathcal{F}$ and $\mathcal{G}$, one has, for each integer n, pairings

$$\mathrm{Ext}^{d-n}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \times H^n(X;\mathcal{F}) \longrightarrow H^d(X;\mathcal{G}).$$

Serre's duality theorem for projective spaces then reads

Theorem 18.17. *Let $\mathbb{K}$ be a field, $X = \mathbb{P}^d_{\mathbb{K}}$, and $\mathcal{F}$ a coherent sheaf on X. Let Ω denote the sheaf of differential forms on X, and set $\omega = \wedge^d\Omega$. Then $H^d(X;\omega) \cong \mathbb{K}$, and for each integer n the pairing below is perfect*

$$\mathrm{Ext}^{d-n}_{\mathcal{O}_X}(\mathcal{F},\omega) \times H^n(X;\mathcal{F}) \longrightarrow H^d(X;\omega) \cong \mathbb{K}.$$

Compare this result to Theorem 18.14. The analogy goes deeper than that: the proof of the latter is modelled on Serre's proof of the former. There is more to Serre's theorem than stated above: Serre proves that there is a canonical isomorphism

$$H^d(X;\omega) \longrightarrow \mathbb{K},$$

called the *residue map*, and this is crucial for the proof.

Global duality

Let R be a Cohen-Macaulay ring. It need not be local—what we mean is that $R_{\mathfrak{p}}$ is Cohen-Macaulay for each prime ideal $\mathfrak{p}$ in R, and the Krull dimension of R be finite; this last assumption is in line with the notion of Gorenstein rings adapted in Lecture **??**.

Definition 18.18. A *canonical module* for R is a finitely generated R-module ω, with the property that $\omega_{\mathfrak{m}}$ is a canonical module for $R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R. It follows from Corollary 11.36 that if ω is a canonical module for R, then $\omega_{\mathfrak{p}}$ is a canonical module for $R_{\mathfrak{p}}$ for each prime ideal of R.

For example, when R is Gorenstein, R is itself a canonical module. It turns out that, as in the local case, a ring has a canonical module if and if it is a quotient of a Gorenstein ring. The following exercises lead to that result.

Exercise 18.19. Let R be a ring. Prove the following statements:

(1) $\operatorname{Spec} R = V_1 \sqcup V_2$ if and only if there is an isomorphism of rings $R \cong R_1 \times R_2$, with $\operatorname{Spec} R_i = V_i$.
(2) When R is Noetherian, there exists an isomorphism $R \cong R_1 \times \cdots \times R_d$, where each $\operatorname{Spec} R_i$ is connected.

Exercise 18.20. Let $R_1, \dots, R_d$ be rings, and set $R = R_1 \times \cdots \times R_d$. Prove the following statements:

(1) A R-module M is canonically isomorphic to $M_1 \times \cdots \times M_d$, where M_i is an R_i-module for each i.
(2) The ring R is Cohen-Macaulay if and only if each R_i is Cohen-Macaulay.
(3) Suppose that R is Cohen-Macaulay. Let ω be an R-module and write $\omega = \omega_1 \times \cdots \times \omega_d$, as in (1). Then ω is a canonical module for R if and only if ω_i is a canonical module for R_i, for each i.

Exercise 18.21. Let R be a Noetherian ring. Then $\operatorname{Spec} R$ is connected if and only if given minimal primes $\mathfrak{p}$ and $\mathfrak{p}'$ in R, there is a sequence of ideals

$$\mathfrak{p} = \mathfrak{p}_0, \mathfrak{m}_0, \mathfrak{p}_1, \mathfrak{m}_1, \dots, \mathfrak{p}_n, \mathfrak{m}_n, \mathfrak{p}_{n+1} = \mathfrak{p}',$$

such that $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are minimal primes, and $\mathfrak{m}_i$ contains $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ for $0 \leqslant i \leqslant n$. Evidently, the ideals $\mathfrak{m}_i$ may be chosen to be maximal.

Proposition 18.22. *Let* $Q \longrightarrow R$ *be a surjective homomorphism of Cohen-Macaulay rings. Assume that* $\operatorname{Spec} R$ *is connected. If* $\widetilde{\omega}$ *is a canonical module for* Q*, then* $\operatorname{Ext}^h_Q(R, \widetilde{\omega})$*, where* $h = \dim Q - \dim R$*, is a canonical module for* R*.*

Proof. Suppose that $R = Q/\mathfrak{a}$. We claim that $\operatorname{height}(\mathfrak{a}_{\mathfrak{m}}) = h$ for any prime ideal $\mathfrak{m} \in \operatorname{Spec} Q$ containing $\mathfrak{a}$.

Indeed, if $\mathfrak{p}$ and $\mathfrak{p}'$ are minimal primes of $\mathfrak{a}$ contained in a maximal ideal $\mathfrak{m}$ of Q, then $R_{\mathfrak{m}}$ is Cohen-Macaulay, so $\operatorname{height}(\mathfrak{a}_{\mathfrak{p}}) = \operatorname{height}(\mathfrak{a}_{\mathfrak{p}'})$; see **??**. Therefore, since $\operatorname{Spec} R$ is connected, it follows from Exercise 18.21 that $\operatorname{height}(\mathfrak{a}_{\mathfrak{p}}) = \operatorname{height}(\mathfrak{a}_{\mathfrak{p}'})$ for any pair $\mathfrak{p}, \mathfrak{p}'$ of minimal primes of $\mathfrak{a}$. This settles the assertion.

Set $\omega = \operatorname{Ext}^h_Q(R, \widetilde{\omega})$; this is a finitely generated R-module. For each maximal ideal $\mathfrak{m} \in \operatorname{Spec} Q$ containing $\mathfrak{a}$, since $Q_{\mathfrak{m}}$ is Cohen-Macaulay with canonical module $(\widetilde{\omega})_{\mathfrak{m}}$ and $h = \operatorname{height}(\mathfrak{a}_{\mathfrak{m}})$, it follows from Theorem 11.38 that $\omega_{\mathfrak{m}}$ is a canonical module for $R_{\mathfrak{m}}$. Thus, ω is a canonical module for R. ∎

The following result is a global version of Theorem 11.41.

Proposition 18.23. *Let R be a Cohen-Macaulay ring. Then R has a canonical module if and only if it is a homomorphic image of a Gorenstein ring.*

Proof. Indeed, suppose that R has a canonical module, say ω. Set $Q = R \ltimes \omega$, the trivial extension of R by ω. This is Gorenstein ring: for each $\mathfrak{p} \in \operatorname{Spec} R$, the local ring $Q_\mathfrak{p} = R_\mathfrak{p} \ltimes \omega_\mathfrak{p}$ is Gorenstein because $\omega_\mathfrak{p}$ is a canonical module for $R_\mathfrak{p}$; see Theorem 11.41. It remains to note that the canonical projection $Q \longrightarrow R$ exhibits R as a canonical image of Q.

Suppose that $Q \longrightarrow R$ is a surjective homomorphism of rings with Q Gorenstein. When $\operatorname{Spec} R$ is connected, since Q itself is canonical module for Q, it follows from Proposition 18.22 that $\operatorname{Ext}^h_Q(R, Q)$, where $h = \dim Q - \dim R$, is a canonical module for R. In general, one has an isomorphism $R \cong R_1 \times \cdots \times R_d$, where $\operatorname{Spec} R_i$ connected for each i; see Exercise 18.19. Since R is a quotient of a Gorenstein ring, so is each R_i. Hence, R_i has a canonical module, say ω_i. Now $\omega_1 \times \cdots \times \omega_d$ is a canonical module for R, by Exercise 18.20. □

The preceding result settles the question of existence of canonical modules. The one below deals with uniqueness. One could not have expected a stronger result since, over a Gorenstein ring, any rank one projective module is a canonical module.

Proposition 18.24. *Let R be a Cohen-Macaulay ring with a canonical module ω.*

(1) *If P is a rank one projective, then $P \otimes_R \omega$ is a canonical module for R.*
(2) *If ω' is a canonical module for R, then the R-module $\operatorname{Hom}_R(\omega, \omega')$ is rank one projective, and the following homomorphism is bijective:*

$$\operatorname{Hom}_R(\omega, \omega') \otimes_R \omega \longrightarrow \omega' \qquad \textit{where } f \otimes w \mapsto f(w).$$

Proof. Exercise, using localization. □

Proposition 18.24 is the best one can do for general rings, in that there is no canonical choice of a canonical module; this is a problem when one wants to work with schemes. Fortunately, this problem has a solution for schemes that arise in (classical!) algebraic geometry. For the rest of this section we work in the following context:

Notation 18.25. Let $\mathbb{K}$ be a field and R a $\mathbb{K}$-algebra essentially of finite type.

Definition 18.26. The $\mathbb{K}$-algebra R is said to be *smooth* if for each extension of fields $\mathbb{K} \longrightarrow \mathbb{L}$, the ring $\mathbb{L} \otimes_\mathbb{K} R$ is regular.

Remark 18.27. A number of comments are in order: First, since the $\mathbb{K}$-algebra R is essentially of finite type, the $\mathbb{L}$-algebra $\mathbb{L} \otimes_\mathbb{K} R$ is essentially of finite type, and hence Noetherian. Moreover, the following conditions are equivalent:

(1) the $\mathbb{K}$-algebra R is smooth;
(2) the ring $\mathbb{L} \otimes_\mathbb{K} R$ is regular for each finite inseparable extension $\mathbb{L}$ of $\mathbb{K}$;
(3) the ring $\mathbb{L} \otimes_\mathbb{K} R$ is regular for the smallest perfect field extension $\mathbb{L}$ of $\mathbb{K}$;
(4) the ring $\mathbb{L} \otimes_\mathbb{K} R$ is regular for the algebraic closure $\mathbb{L}$ of $\mathbb{K}$.

In particular, when $\mathbb{K}$ has characteristic zero or is a perfect field, the $\mathbb{K}$-algebra R is smooth if and only if the ring R is regular.

We will not discuss the proof of the equivalence of conditions (1)–(4) in the preceding remark. Instead, we will provide examples and exercises.

Example 18.28. Polynomial rings $\mathbb{K}[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ are indeterminates, are smooth over $\mathbb{K}$, as are localizations of polynomials rings.

Exercise 18.29. Let $\mathbb{K}$ be a field and set

$$R = \mathbb{K}[x, y] / \left(y^3 - x(x-1)(x-2)\right).$$

Prove that R is a smooth $\mathbb{K}$-algebra if and only if the characteristic of $\mathbb{K}$ is not 2.

Evidently, when R is smooth, it is regular. But the converse does not hold:

Example 18.30. Let $\mathbb{K}$ be a field with of characteristic $p > 0$. Assume that $\mathbb{K}$ is not perfect, so that there exists an element $a \in \mathbb{K} \setminus \mathbb{K}^p$. Set $R = \mathbb{K}[x]/(x^p - a)$. Note that R is a field, and in particular, a regular local ring. However, for $\mathbb{L} = R$, the ring $\mathbb{L} \otimes_{\mathbb{K}} R$ is isomorphic to $\mathbb{L}[y]/(y^p)$. Thus, R is not a smooth $\mathbb{K}$-algebra.

The preceding example extends to the following general result; see [114, ??].

Remark 18.31. A finite field extension $\mathbb{K} \longrightarrow \mathbb{L}$ is smooth if and only if $\mathbb{L}$ is separable over $\mathbb{K}$.

Our objective is to construct canonical modules over smooth algebras. This calls for more definitions.

Definition 18.32. Let M be an R-module. A $\mathbb{K}$-*derivation* of R, with coefficients in M, is a $\mathbb{K}$-linear map $\theta \colon R \longrightarrow M$ that satisfies the Leibniz rule:

$$\theta(rs) = \theta(r)s + r\theta(s) \quad \text{for all } r, s \in R.$$

The $\mathbb{K}$-linearity of θ implies that $\theta(1) = 0$.

Definition 18.33. An R-module Ω is said to be a *module of Kähler differentials* of R over $\mathbb{K}$ if there is a $\mathbb{K}$-derivation $\delta \colon R \longrightarrow \Omega$ with the following universal property: Given a $\mathbb{K}$-derivation $\theta \colon R \longrightarrow M$, there exists a unique homomorphism of R-modules $\widetilde{\theta} \colon \Omega \longrightarrow M$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\delta"] \arrow[d, "\theta"'] & \Omega \arrow[dl, dashed, "\widetilde{\theta}"] \\ M & \end{tikzcd}$$

Standard arguments show that a module of Kähler differentials, if it exists, is unique up to a unique isomorphism. We denote it $\Omega_{R|\mathbb{K}}$; the associated derivation $\delta \colon R \longrightarrow \Omega_{R|\mathbb{K}}$ is called the *universal derivation* of R over $\mathbb{K}$.

The module of Kähler differentials exists, and may be constructed as follows.

Construction 18.34. The $\mathbb{K}$-vector space $R \otimes_{\mathbb{K}} R$ is a ring with product given by

$$(r \otimes s) \cdot (r' \otimes s') = (rr' \otimes ss')$$

and this ring is commutative. In the following paragraphs, we view $R \otimes_{\mathbb{K}} R$ as an R-module, with product induced by the left-hand factor: $r' \cdot (r \otimes s) = r'r \otimes s$.

Since R is commutative, the map

$$\mu \colon R \otimes_{\mathbb{K}} R \longrightarrow R \quad \text{where } \mu(r \otimes s) = rs$$

is a homomorphism of rings; set $\mathfrak{a} = \ker(\mu)$. This inherits an R-module structure from $R \otimes_{\mathbb{K}} R$.

As an R-module, $\mathfrak{a}$ is spanned by $\{s \otimes 1 - 1 \otimes s \mid s \in R\}$. Moreover, if the $\mathbb{K}$-algebra R is generated by $s_1, \ldots, s_n$, then the elements

$$\{s_i \otimes 1 - 1 \otimes s_i \mid 1 \leqslant i \leqslant n\}$$

generate the ideal $\mathfrak{a}$ in $R \otimes_{\mathbb{K}} R$.

Since $\mathfrak{a} = \ker(\mu)$, the action of $(R \otimes_{\mathbb{K}} R)$ on $\mathfrak{a}/\mathfrak{a}^2$ factors through R, and hence $\mathfrak{a}/\mathfrak{a}^2$ has a canonical R-module structure. The action of $r \in R$ on the residue class in $\mathfrak{a}/\mathfrak{a}^2$ of the element $s \otimes 1 - 1 \otimes s$ is given by:

$$r \cdot (s \otimes 1 - 1 \otimes s) = (rs \otimes 1 - r \otimes s).$$

Thus, this action coincides with the R-module structure induced by $\mathfrak{a} \subset R \otimes_{\mathbb{K}} R$.

Claim. The map $\delta \colon R \longrightarrow \mathfrak{a}/\mathfrak{a}^2$ with $\delta(s) = s \otimes 1 - 1 \otimes s$ is well defined, and is a $\mathbb{K}$-derivation from R to $\mathfrak{a}/\mathfrak{a}^2$.

Indeed, consider the $\mathbb{K}$-linear maps

$$\begin{aligned} \iota_1 \colon R &\longrightarrow (R \otimes_{\mathbb{K}} R) \qquad \text{where } s \mapsto s \otimes 1 \\ \iota_2 \colon R &\longrightarrow (R \otimes_{\mathbb{K}} R) \qquad \text{where } s \mapsto -1 \otimes s. \end{aligned}$$

One has then a diagram of $\mathbb{K}$-vector spaces

$$R \xrightarrow{\ \Delta\ } (R \oplus R) \xrightarrow{\iota_1 \oplus \iota_2} (R \otimes_{\mathbb{K}} R) \oplus (R \otimes_{\mathbb{K}} R) \xrightarrow{\ \pi\ } R \otimes_k R,$$

where $\Delta(s) = (s, s)$ and $\pi(y \otimes z, y' \otimes z') = y \otimes z + y' \otimes z'$. Let $\widetilde{\delta}$ denote the composed map; this is again $\mathbb{K}$-linear and a direct check shows that $\mu\widetilde{\delta} = 0$, so $\widetilde{\delta}(R) \subseteq \mathfrak{a}$. The map δ is the composition

$$R \xrightarrow{\ \widetilde{\delta}\ } \mathfrak{a} \longrightarrow \mathfrak{a}/\mathfrak{a}^2,$$

and hence a well defined homomorphism of $\mathbb{K}$-vector spaces. For elements r, s of R, one has, in $\mathfrak{a}/\mathfrak{a}^2$, the equation

$$\begin{aligned} 0 &= (r \otimes 1 - 1 \otimes r)(s \otimes 1 - 1 \otimes s) \\ &= rs \otimes 1 - s \otimes r - r \otimes s + 1 \otimes rs \\ &= (rs \otimes 1 - 1 \otimes rs) - (r \otimes 1 - 1 \otimes r)s - r(s \otimes 1 - 1 \otimes s) \end{aligned}$$

Thus, $\delta(rs) = \delta(r)s + r\delta(s)$, that is to say, δ is a derivation.

Claim. The R-module $\mathfrak{a}/\mathfrak{a}^2$ is the module of Kähler differentials, with universal derivation δ.

We need to prove that $\delta : R \longrightarrow \mathfrak{a}/\mathfrak{a}^2$ has the required universal property. Let $\theta \colon R \longrightarrow M$ be a $\mathbb{K}$-derivation. If there exists an R-linear homomorphism $\widetilde{\theta} \colon \mathfrak{a}/\mathfrak{a}^2 \longrightarrow M$ with $\widetilde{\theta} \circ \delta = \theta$, then it must be unique; this is because $\delta \colon R \longrightarrow \mathfrak{a}/\mathfrak{a}^2$ maps onto the R-module generators of $\mathfrak{a}/\mathfrak{a}^2$.

As to the existence of $\widetilde{\theta}$, the map θ induces a homomorphism R-modules

$$R \otimes_{\mathbb{K}} \theta \colon R \otimes_{\mathbb{K}} R \longrightarrow M, \qquad \text{where } r \otimes s \mapsto r\theta(s).$$

Restriction yields a homomorphism of R-modules $\theta' \colon \mathfrak{a} \longrightarrow M$. Now for r, s in R, one has

$$\begin{aligned} \theta'\big((r \otimes 1 - 1 \otimes r)(s \otimes 1 - 1 \otimes s)\big) &= \theta'\big(rs \otimes 1 - s \otimes r - r \otimes s + 1 \otimes rs\big) \\ &= -s\theta(r) - r\theta(s) + \theta(rs) \\ &= 0, \end{aligned}$$

where the second equality holds because $\theta(1) = 0$, owing to its $\mathbb{K}$-linearity, and the third equality holds because θ is a derivation. Thus, $\theta'(\mathfrak{a}^2) = 0$, so θ induces a homomorphism of R-modules $\widetilde{\theta}\colon \mathfrak{a}/\mathfrak{a}^2 \longrightarrow M$. It is clear that $\widetilde{\theta}\circ\delta = \theta$, as desired. This completes the justification of our claim.

Example 18.35. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring with variables $x_1, \dots, x_n$. The module of Kähler differentials is free:

$$\Omega_{R|\mathbb{K}} \cong \bigoplus_{i=1}^{n} R\delta x_i,$$

and the universal derivation is given by

$$\delta(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\delta x_i \qquad \text{for } f \in R.$$

The next exercise can be solved using the preceding one and basic properties of modules of differentials; see [114, ??].

Example 18.36. Let $R = \mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_c)$. The module of Kähler differentials is given by a presentation:

$$R^c \xrightarrow{(\partial f_i/\partial x_j)} R^n \longrightarrow \Omega_{R|\mathbb{K}} \longrightarrow 0.$$

The matrix $(\partial f_i/\partial x_j)$ is called the *Jacobian matrix* of R.

Combining this exercise with the next result, one can "write down" the module of differentials of any $\mathbb{K}$-algebra essentially of finite type.

Lemma 18.37. *Suppose that $R = U^{-1}Q$, where Q is a $\mathbb{K}$-algebra and U is a multiplicatively closed subset of Q. One has a canonical isomorphism of R-modules*

$$U^{-1}\Omega_{Q|k} \cong \Omega_{R|\mathbb{K}}.$$

Proof. Exercise: use universal property of Kähler differentials. $\square$

Here is the result that addresses the question raised in the paragraph after 18.24.

Theorem 18.38. *If the $\mathbb{K}$-algebra R is smooth, then the R-module $\Omega_{R|\mathbb{K}}$ is finitely generated and projective. If, in addition,* $\operatorname{Spec} R$ *is connected, then $\wedge^d\Omega_{R|\mathbb{K}}$, where $d = \operatorname{rank}_R \Omega_{R|\mathbb{K}}$, is a rank one projective, and hence a canonical module for R.* $\square$

Remark 18.39. Suppose that R is a finitely generated $\mathbb{K}$-algebra (rather than a localization of one such), and a domain, then $\operatorname{rank}_R \Omega_{R|\mathbb{K}} = \dim R$.

Theorem 18.40. *If the $\mathbb{K}$-algebra R is d-dimensional Cohen-Macaulay domain, the non-smooth locus of R has codimension at least two, then*

$$\operatorname{Hom}_R(\operatorname{Hom}_R(\wedge^d\Omega_{R|\mathbb{K}}, R), R)$$

is a canonical module for R.

Example 18.41. Let $X = (x_{ij})$ be an $n \times (n+1)$ matrix of indeterminates, and set $R = \mathbb{K}[X]/I_n(X)$, where $I_n(X)$ is the ideal generated by the size n minors of X. Then ω_R is isomorphic to the ideal of size $n-1$ minors of $n-1$ columns of the matrix X.

Some of the most interesting theorems in calculus are those of Green, Stokes and Gauß. In this section we shall start with discussing these theorems and then explore their correctness on open subsets of $\mathbb{R}^n$. In the course of these investigations we shall link local cohomology of complex varieties $X \subseteq \mathbb{C}^n$ to de Rham and singular cohomology of their complements $\mathbb{C}^n \setminus X$, and consider a few examples how this connection can be used.

We deal with this material in subsections that cover the real, complex, and algebraic case respectively and finish with a subsection on the relationship of local and de Rham cohomology. Before that, however, we need some details regarding differentials.

To start with, recall that if R is a $\mathbb{K}$-algebra then the $\mathbb{K}$-*linear differentials* of R form an R-module Ω^1_R and there is a universal derivation

$$d : R \longrightarrow \Omega^1_R$$

on R (i.e., $d(r_1r_2) = r_1\,d(r_2) + r_2\,d(r_1)$). We shall refer to d as the *gradient map* since if $R = \mathbb{K}[x_1, \dots, x_n]$ one may identify Ω^1_R with $R^n = \bigoplus R\,dx_i$ and in this notation d is precisely the classical gradient. If $f \in R$ then any differential on R extends to a differential on $R[f^{-1}]$ by the quotient rule and this identifies $\Omega^1_{R[f^{-1}]}$ with $\Omega^1_R \otimes_R R[f^{-1}]$.

From Ω^1_R one may construct higher order $\mathbb{K}$-linear differentials by setting $\Omega^i_R = \bigwedge^i \Omega^1_R$. In particular, $\Omega^t_{R[f^{-1}]} = \Omega^t_R \otimes_R R[f^{-1}]$. Of course, in this notation, $\Omega^0_R = R$. The gradient map can be used to construct a $\mathbb{K}$-linear map

$$d^t : \Omega^t_R \longrightarrow \Omega^{t+1}_R$$

for all t by $d(f\,dx_{i_1} \wedge \ldots \wedge dx_{i_t}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_t}$ in local coordinates. We call this map the *gradient map* as well. One may verify that due to the sign rule in the wedge product, $d^t \circ d^{t-1} = 0$.

If X is an affine algebraic variety, this construction can be carried out on $\Gamma(X; \mathcal{O}_X)$, and by the localizing property of differentials this sets up a complex with differential $d^\bullet$ on all principal open subsets of X. If X is non-affine, one needs to deal with the principal affine sets directly. In either case, one verifies that on the intersection of two principal open sets the information obtained is identical. This endows each $\mathbb{K}$-scheme with sheaves of differential forms $\Omega^\bullet_X$ and morphisms $d^t : \Omega^t_X \longrightarrow \Omega^{t+1}_X$ which form the *de Rham complex*

$$\Omega^\bullet_X = \left(\Omega^0_X \xrightarrow{d^0} \Omega^1_X \xrightarrow{d^1} \longrightarrow \cdots\right) \tag{19.0.1}$$

If X is algebraic over the field of real or complex numbers then there is a natural continuous map $\iota_X : X^{an} \longrightarrow X$ from the analytic space X^{an} to the algebraic space X. Via (the pullback functor, see [63], of) this map, abusing notation, we view Ω^1_X as a complex of sheaves on X^{an}; there is an induced embedding on the level of all differential forms and de Rham complexes: $\Omega^\bullet_X \hookrightarrow \Omega^\bullet_{X^{an}}$.

♣ should we talk about pullbacks/pushforwards?

19.1. **The real case: de Rham's theorem.** Let us take a look at the

Theorem 19.1 (of Green). *Suppose f and g are C^∞-functions in two variables, $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Assume that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ on all of $\mathbb{R}^2$. Then there is a C^∞-function $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$, called a* potential function *such that $\frac{\partial H}{\partial x} = f$ and $\frac{\partial H}{\partial y} = g$.*

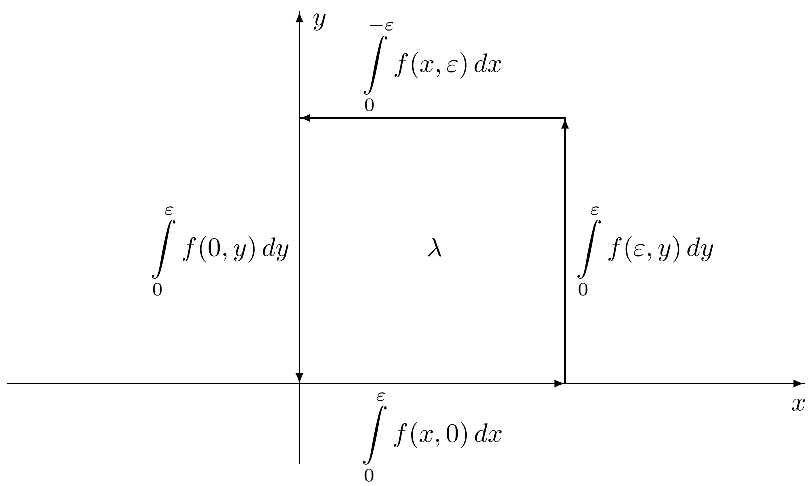
Technically, in fact, f and g just need to have continuous derivatives, but we shall throughout consider smooth functions.

Let us take a look at the proof of this theorem. On a closed piecewise differentiable path λ the path integral $\int\limits_{\lambda} f\,dx + g\,dy$ is zero, which means that one can define

$$H(x_0, y_0) = \int\limits_{(0,0)}^{(x_0,y_0)} (f\,dx + g\,dy)$$

where the integral goes along any (piecewise differentiable) path. The interesting part is why the integral is zero along closed loops, and hence only dependent upon the endpoints of the path. In order to define H it is sufficient to consider piecewise linear paths parallel to the coordinate axes. Without loss of generality we may assume the square in question to have corners $(0,0)$ and $(\varepsilon,\varepsilon)$. One then computes

FIGURE 4. $\int_\lambda (f\,dx + g\,dy)$ along a square with vertices $(0,0)$ and $(\varepsilon,\varepsilon)$



$$\begin{aligned}
\int_\lambda f\,dx + g\,dy &= \int_0^\varepsilon (f(x,0) - f(x,\varepsilon))\,dx + \int_0^\varepsilon (g(\varepsilon,y) - g(0,y))\,dy \\
&= \int_0^\varepsilon \left(\int_0^\varepsilon -f_y(x,y)\,dy \right) dx + \int_0^\varepsilon \left(\int_0^\varepsilon g_x(x,y)\,dx \right) dy \\
& \quad \text{(since the Fundamental Theorem of Calculus applies)} \\
&= 0
\end{aligned}$$

as $f_y(x,y) = g_x(x,y)$. This shows that there is a function H with gradient (f,g). In other words, $d(H) = f\,dx + g\,dy$. It follows that the pathintegral is the same for every differentiable path, not just piecewise linear ones.

Let's discuss possible ways of Green's Theorem to fail. If the equation $f_y = g_x$ fails to hold in any point of the domain of definition of f_y, g_x then there is obviously no chance for f, g to be derivatives of a common integral H since by H.A. Schwarz' lemma taking derivatives is independent of the order. It is more interesting to contemplate how matters are affected when $f_x = g_y$ is unbounded somewhere in $\mathbb{R}^2$. From the above calculation it is apparent that if a loop encloses only points on which $f_y = g_x$ is finite then the integral must vanish. However, consider the following example.

Example 19.2. Let $g(x,y) = \frac{x}{x^2+y^2}$ and $f(x,y) = -\frac{y}{x^2+y^2}$. Then $g_x = \frac{y^2-x^2}{x^2+y^2} = f_y(x,y)$. On the other hand, integrating along the circle $\lambda : (x^2+y^2 = \varepsilon)$ with $x = \varepsilon\cos(t)$, $y = \varepsilon\sin(t)$, we find $\int_\lambda f\,dx = -\int_{t=0}^{2\pi} \frac{\varepsilon\sin(t)}{\varepsilon^2}(-\varepsilon\sin(t))\,dt = \pi = \int_\lambda g\,dy$. This means that not only does the method of proof break down, but in fact Green's Theorem must fail. Namely, if there were a function H such that $f = H_x, g = H_y$ then the integral along any closed curve over $f\,dx$ and $g\,dy$ would have to be zero since they are just H evaluated at the start minus H evaluated at the endpoint of the loop.

The example and the preceding discussion show the following. Let f, g be smooth on a punctured disk around $P \in \mathbb{R}^2$ and assume that $f_y = g_x$ holds on the punctured disk. If f, g are smooth on the *entire* disk then the integral $\int(f\,dx + g\,dy)$ is independent of the path and (f, g) is its gradient. On the other hand, there are pairs of smooth functions on the *punctured* disk that satisfy $f_y = g_x$ but not the conclusion of Green's Theorem.

Paraphrasing we may say that $f\,dx + g\,dy$ is locally a gradient if $f_y = g_x$, but the topology of the domain of definition of $f_x = g_y$ may get in the way of a global potential function. More precisely, let U be an open set in the plane, pick a point $P \in U$, and let $\Lambda(U, P)$ be the space of loops in U starting and ending at P. Set $\pi_1(U, P)$ to be the fundamental group of U, planted at $P \in U$. On the other hand, let $\Omega^1(U)$ be the differentials on U with smooth coefficients: $\Omega^1(U) = \{\omega = f\,dx + g\,dy | f, g \in C^\infty(U)\}$. Since U is open, each element of $\pi_1(U, P)$ has a smooth representative. Then integration along a differentiable path $\lambda \in \Lambda(U, P)$ gives a linear map from $\Omega^1(U)$ to $\mathbb{R}$, which we restrict to the differentials $\Omega^1_0(U) = \{\omega = f\,dx + g\,dy | f_y = g_x\}$. Moreover, as potential functions are independent of the integration path, homotopic paths produce the same integral for all forms in $\Omega^1_0(U)$. So we have a pairing $\pi_1(U, P) \times \Omega^1_0(U) \longrightarrow \mathbb{R}$ that is clearly $\mathbb{R}$-linear in the second component and additive in the first. Note that for two loops λ, λ' the integral $\int_{\lambda\lambda'(\lambda)^{-1}(\lambda')^{-1}} \omega$ is zero since it decomposes into the sum of the four separate loop integrals. As the quotient of $\pi_1(U, P)$ by its commutator subgroup is the singular homology group $H_1(U; \mathbb{Z})$, the pairing descends to homology and we can extend coefficients to the reals:

$$\int \quad : \quad H_1(U; \mathbb{R}) \times \Omega^1_0(U) \longrightarrow \mathbb{R}.$$

Now note that $\omega = f\,dx + g\,dy$ is in $\Omega^1_0(U)$ if and only if the second order differential $d(f\,dx + g\,dy) = (f_y - g_x)dx \wedge dy \in \Omega^2(U)$ arising as the gradient of $f\,dx + g\,dy$ is zero; such ω is called a *closed* 1-form. On the other hand one calls *exact* 1-forms those which arise as the gradient $d(H) = H_x\,dx + H_y\,dy$ of a 0-th order differential (that is, a potential function) $H \in \Omega^0(U) = \mathcal{O}(U)$. Necessarily, an exact differential gives a zero integral on all loops. So our pairing descends to an $\mathbb{R}$-linear pairing

$$H_1(U; \mathbb{R}) \times \frac{\ker(d : \Omega^1(U) \longrightarrow \Omega^2(U))}{\operatorname{image}(d : \Omega^0(U) \longrightarrow \Omega^1(U))} \quad \longrightarrow \quad \mathbb{R}. \tag{19.2.1}$$

It follows from Example 19.2 that the pairing is non-degenerate in $H^1(U;\mathbb{R})$, for all nonzero elements $\lambda \in H_1(U;\mathbb{R})$ there exists a closed form $\omega = f\,dx + g\,dy \in \Omega^1_0(U)$ such that $\int_\lambda \omega \neq 0$. On the other hand, pick a closed form $\omega = f\,dx + g\,dy$ on U. If the pairing vanishes identically with this form, then integrating along differentiable paths gives a potential function for ω and in particular then $\omega \in \mathrm{image}(d : \Omega^0(U) \longrightarrow \Omega^1(U))$. So the pairing is also faithful in the second argument. In particular, $\frac{\ker(d:\Omega^1(U)\longrightarrow\Omega^2(U))}{\mathrm{image}(d:\Omega^0(U)\longrightarrow\Omega^1(U))}$ is the vector space dual to $H_1(U;\mathbb{R})$ and hence isomorphic to the singular cohomology group $H^1(U;\mathbb{R})$ since $\mathbb{R}$ is a field.

Somewhat incredibly, a vast generalization of this scenario is true, Theorem 19.3 below. Let M be a differentiable manifold. Recall from (19.0.1) that the differentials on M with the gradient map form a complex $\Omega^\bullet_M$. We denote its i-th cohomology

$$\frac{\ker(d : \Omega^i_M(M) \longrightarrow \Omega^{i+1}_M(M))}{\mathrm{image}(d : \Omega^{i-1}_M(M) \longrightarrow \Omega^i_M(M))} =: H^i_{\mathrm{dR}}(M).$$

As before we call *exact* forms those which are the gradient of another form (the denominator), and *closed* those whose gradient is zero (the numerator), so in this notation $H^i_{\mathrm{dR}}(M)$ is the quotient of the closed i-forms modulo the exact i-forms on M.

A *singular smooth chain* is an element of the real vector space $S_i(M)$ spanned by all smooth maps $\{f : \Delta_i \longrightarrow M\}$ from the standard i-simplex to M. The main theorem relating differential forms and singular cohomology on real smooth manifolds is

Theorem 19.3 (de Rham)**.** *Let M be a real, smooth (that is, C^∞-)manifold. Consider the pairing of integration*

$$\int : S_i(M) \times \Omega^i(M) \longrightarrow \mathbb{R}$$

where if $\sigma : \Delta_i \longrightarrow M$ is smooth and $\omega \in \Omega^i(M)$ then

$$(\sigma, \omega) \mapsto \int_\sigma \omega := \int_{\Delta_i} \sigma^*(\omega),$$

the integral over Δ_i of the pullback of ω along σ. The pairing extends to all $\sigma \in S_i(M)$ by linearity.

If $\sigma \in S_i$ is a boundary then $\int_\sigma \omega = 0$ for all closed differential forms ω. Similarly, if ω is exact then $\int_\sigma \omega = 0$ for all $\sigma \in S_i$ with zero boundary.

Every class in the singular homology $H_i(M;\mathbb{R})$ has a representative in $S_i(M)$, hence integration gives a pairing

$$H_i(M;\mathbb{R}) \otimes_{\mathbb{R}} H^i_{\mathrm{dR}}(M) \longrightarrow \mathbb{R}$$

between singular homology and de Rham cohomology. This pairing is $\mathbb{R}$-linear and perfect and sets up a natural isomorphism

$$H^i_{\mathrm{dR}}(M) \cong \mathrm{Hom}_{\mathbb{R}}(H_i(M;\mathbb{R}), \mathbb{R}) \cong H^i(M;\mathbb{R})$$

that identifies the vector space of closed forms modulo exact forms with the singular cohomology of M (the dual space of the singular homology of M).

Example 19.4. We revisit the circle $\mathbb{S}^1$. In that case, the de Rham Theorem is fancy language for two facts discovered in Lecture 2. Firstly, the de Rham cohomology of $\mathbb{S}^1$ in degree zero is given by the constant functions, since constant functions are closed and have a nonzero integral. These 0-forms are the dual to the singular 0-chain in $\mathbb{S}^1$ that sends the 0-simplex (a point) to any point in $\mathbb{S}^1$. Secondly, the first singular homology class of $\mathbb{S}^1$ is generated by the 1-chain that travels along $\mathbb{S}^1$ precisely once. Its dual de Rham cohomology form in degree one is the form $\omega = d\,t/2\pi$ where t is a local variable on $\mathbb{S}^1$ since its gradient must be zero (there are no 2-forms) and $\int_{\mathbb{S}^1} \omega = 1$.

Example 19.5. Consider real 3-space, $M = \mathbb{R}^3$. Then $H^0_{\mathrm{dR}}(M;\mathbb{R}) \cong \mathbb{R}$ and all higher singular cohomology groups are zero. In particular, all closed forms of order $i > 0$ are exact.

If $i = 1$, this is known as *Stokes' Theorem*: if the rotation of a global smooth vector field is zero (so that the coefficients, when decorated with dx, dy and dz form a closed differential) then the vector field is in fact a gradient.

If $i = 2$, this is Gauß' theorem: if a global smooth vector field has zero divergence (which is to say that the corresponding 2-form is closed) then it is equal to the rotation of some vector field (i.e., the gradient of a 1-form).

As in the plane, the theorems of Stokes and Gauß have a tendency of failing if the vector fields are not defined on all of $\mathbb{R}^3$. Holes arising from the removal of a zero dimensional set may make Gauß' Theorem fail, and those arising from the removal of 1-dimensional sets may make Stokes fail. For example, the form $\frac{1}{x^2+y^2}(x\,dx - y\,dy)$ is closed, but not exact. The reason is the same as in Example 19.2 (with the variable z present but irrelevant): the domain of definition is not simply connected. On the other hand, the form $\omega = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ is closed, and the integral of ω over the 2-sphere (a boundary-free smooth simplicial 2-chain in $\mathbb{R}^3\setminus 0$) is 2π so that ω cannot be a gradient. The point is that ω has an interesting singularity at the origin.

Let us discuss some ideas that lead to a proof of Theorem 19.3. The full story is beyond these notes, but we can give an idea of the techniques involved.

In the first place, one needs to prove that the de Rham Theorem holds for $M = \mathbb{R}^n$. In other words, one needs to show that on $\mathbb{R}^n$ every global i-form ($i > 0$) is closed precisely when it is exact. This statement is known as the *Poincaré Lemma*, which we have already met in Example 2.18. The proof is by explicit calculation generalizing our computation in the proof of Green's Theorem, see for example [11, page 33].

Then observe that since sheaves are of local nature (that is, kernels and cokernels of morphisms between sheaves are defined locally) the Poincaré Lemma assures us that the de Rham complex is locally on any real C^∞-manifold a resolution of the constant sheaf $\mathbb{R}$. Recall that in Example 2.18 we observed precisely that to be the case for the circle.

The next crucial step is to realize that on any real smooth manifold M the sheaves Ω^i_M that show up in the de Rham complex are locally free over the sheaf $\mathcal{O}_M$ of smooth functions and have no higher cohomology. This is because they allow for partitions of unity, compare Example 2.16 (and see the discussion of *soft* sheaves in see [29, Definition 2.1.7]). So the de Rham complex is an acyclic resolution of the constant sheaf $\mathcal{R}$ on M. We discovered some of this in Example 2.18. It follows from the Acyclicity Theorem 2.22 that the cohomology groups of the de Rham complex on M are the derived functor cohomology groups of the constant sheaf $\mathcal{R}$.

It follows that de Rham cohomology enjoys all the properties that derived functor cohomology provides. It is worked out in [84] that derived functor cohomology with coefficients in $\mathcal{R}$ satisfies the the *Eilenberg–Steenrod Axioms* which are characteristic properties of singular cohomology, see [19]. Therefore de Rham cohomology, sheaf cohomology with coefficient sheaf $\mathcal{R}$, and singular cohomology with $\mathbb{R}$-coefficients all give the same result on every manifold M. As $\mathbb{R}$ is a field of characteristic zero, this gives an identification of $H^i(M;\mathcal{R})$ with the dual space of the singular homology $H_i(M;\mathbb{R})$.

In order to see that the isomorphism is induced by the integration pairing, just note that the pairing is by definition functorial: if $f : X \longrightarrow Y$ is a smooth map then

$$\int_\sigma f^*(\omega) = \int_{f\circ\sigma} \omega = \int_{f_*(\sigma)} \omega.$$

A calculation similar to the one in the proof of Green's Theorem shows that integrating a closed form over a boundary chain gives zero, and more generally one has the general *Stokes' Theorem*, : $\int_{\partial(\sigma)} \omega = \int_\sigma d(\omega)$.

Remark 19.6. One can see directly that de Rham cohomology fits into Mayer–Vietoris sequences. Namely, if $= U \cup V$ is an open cover for a smooth manifold M then there is an induced short exact sequence of complexes

$$\Omega^\bullet(M) \longrightarrow \Omega^\bullet(U) \times \Omega^\bullet(V) \longrightarrow \Omega^\bullet(U \cap V)$$

where the last maps sends the pair (ω_U, ω_V) of forms to $\omega_U - \omega_V$ restricted to $U \cap V$. The associated long exact sequence is the Mayer–Vietoris sequence, see [11, Proposition 2.3].

19.2. **Complex manifolds.** We now investigate the case of complex manifolds.

Notation 19.7. Unless indicated otherwise, in this subsection the base field is $\mathbb{C}$. In particular, we abbreviate $\Omega^t_{R;\mathbb{C}}$, the module of $\mathbb{C}$-linear differentials of order t on the $\mathbb{C}$-algebra R, by Ω^t_R. Similarly, $\Omega^t_{M;\mathbb{C}}$, the sheaf of $\mathbb{C}$-linear differentials of order t on the complex analytic manifold M is abbreviated by Ω^t_M.

The sheaf on M that is of main interest in this subsection is $\mathcal{C}$, which attaches to the open set $U \subseteq M$ the locally constant maps $U \longrightarrow \mathbb{C}$ (which are exactly the continuous lifts $f : U \longrightarrow M \times \mathbb{C}$ for the projection $M \times \mathbb{C} \longrightarrow \mathbb{C}$ if $\mathbb{C}$ has the discrete topology). In particular, $\mathcal{C}(U) = \mathcal{Z}(U) \otimes_{\mathbb{Z}} \mathbb{C}$.

There is a substantial difference between the real and complex theory. Namely, if a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is (complex) differentiable (just once) then it is actually holomorphic and can locally be represented by a power series with positive radius of convergence. This is very much unlike the real case: the majority of real one time differentiable functions are not smooth, and the majority of real smooth functions cannot be represented by power series. This difference makes itself noticeable in the absence of partitions of unity over the complex numbers. This in turn has the effect that the sheaf of holomorphic functions on a complex manifold is a lot less flexible than the C^∞-functions on a real manifold. In particular, the structure sheaf and the sheaves of higher holomorphic differentials may have nontrivial cohomology.

Example 19.8. Let E be an elliptic curve defined by the polynomial $f \in R = \mathbb{C}[x, y, z]$. There is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

induced by

$$0 \longrightarrow R(-3) \xrightarrow{f\cdot} R \longrightarrow R/(f) \longrightarrow 0.$$

The corresponding cohomology sequence includes a piece

$$H^1(E; \mathcal{O}_E) \longrightarrow H^2(\mathbb{P}^2_{\mathbb{C}}; \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}})(-3) \xrightarrow{f\cdot} H^2(\mathbb{P}^2_{\mathbb{C}}; \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}})$$

The cohomology group $H^2(\mathbb{P}^2_{\mathbb{C}}; \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}})(t)$ is by Theorem 13.3 equal to the degree t part of $H^3_{(x,y,z)}(R) \cong \bigoplus_{a,b,c \in \mathbb{Z} \cap (-\infty,-1]} \mathbb{C} \cdot \frac{1}{x^a y^b z^c}$. In particular, it is 1-dimensional when $t = -3$ and 0-dimensional when $t = 0$. It follows from the exact cohomology sequence that $H^1(E; \mathcal{O}_E) \neq 0$.

As noted, the sheaves of holomorphic differential forms Ω^t_M on a complex analytic manifold M are locally free. But even when the structure sheaf of M is acyclic, then Ω^i may still have interesting cohomology:

Example 19.9. Let $M = \mathbb{P}^1_{\mathbb{C}^{an}}$ be the projective complex line (homeomorphic to the 2-sphere) endowed with the sheaf of analytic functions $\mathcal{O}_M$. While the structure sheaf $\mathcal{O}_M$ itself has no higher cohomology, the sheaf of holomorphic first differentials Ω^1_M is cohomologically nontrivial. Namely, consider the covering of the algebraic space $\mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj}(\mathbb{C}[x_1, x_2])$ by the two sets $\mathbb{C}_1 = \operatorname{Spec}(\mathbb{C}[x_2/x_1])$ and $\mathbb{C}_2 = \operatorname{Spec}(\mathbb{C}[x_1/x_2])$, each algebraically isomorphic to the complex line. Their intersection $C_{1,2}$ is the (affine) punctured complex line $\operatorname{Spec}(\mathbb{C}[x_1/x_2, x_2/x_1])$. Let $\mathfrak{U} = \{C_1, C_2\}$ and $y = x_2/x_1$. The corresponding Čech complex for the rational complex 1-forms $\Omega^1_{\mathbb{P}^1_{\mathbb{C}}}$ is

$$0 \longrightarrow \mathbb{C}[y]\, dy \oplus \mathbb{C}[1/y]\, d(1/y) \xrightarrow{d^0} \mathbb{C}[y, 1/y]\, dy \longrightarrow 0.$$

Note the conversion rule $d(1/y) = (-1/y^2) \cdot dy$ that describes d^0 on the second summand. It follows that d^0 is injective and that the cokernel is isomorphic to the $\mathbb{C}$-vector space spanned by $(1/y)\, dy$.

This implies that on the analytic space $\mathbb{P}^1_{\mathbb{C}^{an}}$ the sheaf of holomorphic 1-forms has a nonzero first cohomology. Indeed, by Serre's theorems in [141] the cohomology of the analytic coherent sheaf Ω^1_M induced by the algebraic sheaf $\Omega^1_{\mathbb{P}^1_{\mathbb{C}}}$ is isomorphic to the cohomology of $\Omega_{\mathbb{P}^1_{\mathbb{C}}}$ on the algebraic space $\mathbb{P}^1_{\mathbb{C}}$.

Given a complex differentiable manifold M, the complex of holomorphic differential forms $0 \longrightarrow \mathcal{O}_M(M) \longrightarrow \Omega^1_M(M) \longrightarrow \cdots$ endowed with the gradient as differential is the *de Rham complex with complex coefficients*; on complex manifolds we always consider the de Rham complex with complex coefficients, unless expressly indicated otherwise.

A version of the Poincaré Lemma holds also in the complex analytic world: if M is biholomorphic to $\mathbb{C}^n$ then

$$0 \longrightarrow \mathcal{O}_M(M) \longrightarrow \Omega^1_M(M) \longrightarrow \Omega^2_M(M) \longrightarrow \cdots$$

has a unique cohomology group, in degree zero, isomorphic to $\mathbb{C}$. This globalizes to the corresponding statement on sheaves: $\Omega^\bullet_M$ is a resolution for $\mathcal{C}$.

However, Examples 19.9 and 19.8 show that sheaves of differential forms may not be cohomologically trivial on complex analytic manifolds. It follows that on a complex analytic manifold one cannot conclude that the complex of global sections of the de Rham complex $\Omega^\bullet_M$ computes the sheaf cohomology of $\mathcal{C}$ on M. As an example we will discuss below that the projective line $\mathbb{P}^1_\mathbb{C}$ has no second order differential forms and yet the second cohomology group of its de Rham complex is nonzero. In fact, it is *a priori* not even clear that the homology of the de Rham complex is invariant under change of the differential structure. The following is the key concept for coming to grips with this difficulty.

Definition 19.10. Suppose X is an analytic space. Then X is called *Stein* if for every analytic coherent sheaf $\mathcal{F}$ on X one has $H^i(X;\mathcal{F}) = 0$ for all $i > 0$.

To be Stein hence means to be acyclic for the global section functor on the category of analytic coherent sheaves. There is a competing definition in the literature: X is sometimes called Stein if it arises as the common set of zeros of a collection of analytic functions on some $\mathbb{C}^n$. Our definition is slightly more inclusive than this other one: a Stein space in our sense may not be embeddable as a whole in any $\mathbb{C}^n$, but for arbitrary compact subsets $K \subseteq X$ one can find open neighborhoods of K in X that embed in some $\mathbb{C}^n$. (The problem is that for growing compact subsets the n may have to rise.)

Let S be a Stein manifold and consider its de Rham complex. All sheaves appearing in this complex are coherent, hence have no higher cohomology. It follows that the de Rham complex is an acyclic resolution of $\mathcal{C}$ on S with respect to the cover $\{S\}$. Using Theorem 2.22 one sees that singular and de Rham cohomology on S are the same (see for example Serre's works [139] and [138]):

Theorem 19.11. *A Stein manifold S satisfies*

$$H^i(S;\mathbb{C}) = H^i(S;\mathcal{C}) = H^i(\Gamma(\Omega^\bullet_S, S)),$$

the singular cohomology of S with $\mathbb{C}$-coefficients is the same as the cohomology of the global sections of the de Rham complex of $\mathbb{C}$-linear differentials on S, and both agree with the derived functor cohomology of the sheaf $\mathcal{C}$ on S. □

As a particular consequence it follows that the singular cohomology groups of S must be zero beyond $\dim(S)$ since the de Rham complex is zero there. If S has complex dimension n (and hence real dimension $2n$) then it therefore has only "half as much cohomology" as one might have expected. While every non-compact manifold fails Poincaré duality, Stein manifolds fail in particularly grand style.

Remark 19.12. There is a Morse theoretic proof of a much better theorem: every Stein space S is homotopy equivalent to a (real) CW complex of dimension at most $\dim(S)$. This proves the vanishing of all cohomology groups (even with integer coefficients) beyond $\dim(S)$ but of course contains a good bit more information. For the manifold case, see [119]; for more general results [57, 56], and for early vanishing results on Stein spaces, [2].

The case that interests us most is the following. Let $f = 0$ be a holomorphic divisor on the analytic manifold $M = \mathbb{C}^{n,an}$. The complement U_f of the divisor is always Stein since it can be identified with the submanifold of $M \times \mathbb{C}$ defined by the vanishing of $1 - tf$. Hence every coherent sheaf on U_f is generated by its global sections and in particular the complex

$$0 \longrightarrow \mathcal{O}_M(M)[f^{-1}] \longrightarrow \Omega^1_M(M)[f^{-1}] \longrightarrow \cdots \longrightarrow \Omega^n_M(M)[f^{-1}] \longrightarrow 0$$

computes the singular cohomology of U_f. In the algebraic case this is exploited in [123, 155, 124] to give algorithms, implemented in [**?**, 50], for computing these cohomology groups.

Let us imagine now that the complex manifold M has a *Stein cover*; so M is the union of a collection of Stein spaces $\{S_i\}_i = \mathfrak{S}$ and all finite intersections of the S_i are Stein as well. We plan to study the cohomology $H^i(M;\mathcal{C})$ of M. On each intersection $S_I = \bigcap_{i\in I} S_i$, the de Rham complex is a resolution of the sheaf $\mathcal{C}$ by acyclic sheaves. By the Acyclicity Theorem the complex of global section of this resolution computes $H^\bullet(S_I;\mathcal{C})$. The space M itself, however, need not be Stein and so the Acyclicity Theorem may not be applied to M, see Examples 19.9 and 19.8.

For a single sheaf $\mathcal{F}$ the Mayer–Vietoris principle dictates that computing cohomology on M is best done by an acyclic open cover $\mathfrak{U}$ and considering the Čech complex $\check{C}^\bullet(\mathfrak{U};\mathcal{F})$. Computing cohomology of a complex of sheaves is no different in nature, just more involved; we outline the construction in our example.

Note that $\mathbb{C}^1$ *is* Stein, and two copies $\mathbb{C}_1, \mathbb{C}_2$ of $\mathbb{C}^1$ cover M, as in Example 19.9. The intersection $\mathbb{C}_{1,2}$, a punctured line, is also Stein as it is an affine open subset of a Stein space. (Notice how the Stein property is much better to work with than its real counterpart, homotopy equivalence to a point: $\mathbb{C}^1\setminus\{0\}$ is Stein but certainly not contractible.) These three open Stein subspaces of M produce three de Rham complexes, whose complex of global sections gives the singular cohomologies of the three spaces. In the notation of Example 19.9 we have for all $i\in\mathbb{N}$:

$$\begin{array}{rcl} H^i\left(\mathbb{C}\{y,1/y\} \longrightarrow \mathbb{C}\{y,1/y\}\,dy\right) &=& H^i(\mathbb{C}_{1,2};\mathbb{C}) \\ H^i\left(\mathbb{C}\{y\} \longrightarrow \mathbb{C}\{y\}\,dy\right) &=& H^i(\mathbb{C}_1;\mathbb{C}) \\ H^i\left(\mathbb{C}\{1/y\} \longrightarrow \mathbb{C}\{1/y\}\,d(1/y)\right) &=& H^i(\mathbb{C}_2;\mathbb{C}) \end{array}$$

Note that there are natural maps from the lower two complexes into the top one which correspond to restriction of functions and differential forms from $\mathbb{C}$ to the punctured line. This allows to compose a commutative diagram

$$\begin{tikzcd} \mathbb{C}\{y\}\,dy \oplus \mathbb{C}\{1/y\}\,d(1/y) \arrow[r] & \mathbb{C}\{y,1/y\}\,dy \\ \mathbb{C}\{y\} \oplus \mathbb{C}\{1/y\} \arrow[r] \arrow[u] & \mathbb{C}\{y,1/y\} \arrow[u] \end{tikzcd}$$

The lower row of this diagram is the Čech complex for computing the cohomology of the sheaf $\mathcal{O}_M$, the upper row is the Čech complex for computing the cohomology of the sheaf Ω^1_M, both with respect to the cover $M = \mathbb{C}_1 \cup \mathbb{C}_2$. The columns are the de Rham complex on $\mathbb{C}_{1,2}$ (right) and the direct sum of the de Rham complexes on $\mathbb{C}_1$ and $\mathbb{C}_2$ (left). Introducing negative signs in the top row we get the *Čech–de Rham complex* of M relative to the chosen cover. The general construction is the following.

Definition 19.13. The Čech–de Rham complex relative to a Stein cover $\mathfrak{S} = \{S_i\}_{i\in I}$ (with totally ordered index set I) for the analytic manifold M has the form

$$
\begin{tikzcd}
\vdots & \vdots & & \vdots & \\
\prod\limits_{i_0\in I} \Omega^n(S_{i_0}) \arrow[r] & \prod\limits_{i_0<i_1\in I} \Omega^n(S_{i_0}\cap S_{i_1}) \arrow[r] & \cdots \arrow[r] & \prod\limits_{\substack{J\subseteq I\\ |J|=k+1}} \Omega^n(\bigcap\limits_{i_j\in J} S_{i_j}) \arrow[r] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
\prod\limits_{i_0\in I} \Omega^1(S_{i_0}) \arrow[r] \arrow[u] & \prod\limits_{i_0<i_1\in I} \Omega^1(S_{i_0}\cap S_{i_1}) \arrow[r] \arrow[u] & \cdots \arrow[r] & \prod\limits_{\substack{J\subseteq I\\ |J|=k+1}} \Omega^1(\bigcap\limits_{i_j\in J} S_{i_j}) \arrow[r] \arrow[u] & \cdots \\
\prod\limits_{i_0\in I} \mathcal{O}(S_{i_0}) \arrow[r] \arrow[u] & \prod\limits_{i_0<i_1\in I} \mathcal{O}(S_{i_0}\cap S_{i_1}) \arrow[r] \arrow[u] & \cdots \arrow[r] & \prod\limits_{\substack{J\subseteq I\\ |J|=k+1}} \mathcal{O}(\bigcap\limits_{i_j\in J} S_{i_j}) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}
$$

where the n-th row arises from the Čech complex of Ω^n on M (with maps scaled by $(-1)^n$), and the k-th column from the de Rham complexes on the intersections $\bigcap_{i_0<\cdots<i_k} S_{i_j}$ of k elements of $\mathfrak{U}$.

The following theorem is the complex analytic version of the de Rham Theorem:

Theorem 19.14. *The cohomology of the total complex of the Čech–de Rham complex is naturally isomorphic to the singular cohomology of M.*

In view of Theorem 19.11, the proof is an application of the Mayer–Vietoris principle which holds for both sheaf and singular cohomology.

In the case $M = \mathbb{P}^1_{\mathbb{C}}$, we get a total complex

$$\underbrace{\mathbb{C}\{y\}\oplus\mathbb{C}\{1/y\}}_{\text{degree } 0} \longrightarrow \underbrace{\mathbb{C}\{y\}\,dy \oplus \mathbb{C}\{1/y\}\,d(1/y) \oplus \mathbb{C}\{y,1/y\}}_{\text{degree } 1} \longrightarrow \underbrace{\mathbb{C}\{y,1/y\}\,dy}_{\text{degree } 2}$$

It is not hard to see that there is a nontrivial cohomology group in degree zero, the vector space spanned by $(1,1)$ representing our old friend the constant function $1 : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{C}$. Moreover, in degree two there is a (up to complex scaling unique) non-vanishing cohomology class generated by the form $(1/y)\,dy$, the very same form that generates $H^1(\mathbb{P}^1_{\mathbb{C}};\Omega^1)$. Finally, there is no cohomology in degree one. These computations agree with the preconceptions one has about the singular cohomology of $\mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2$ (namely Betti numbers 1, 0, 1).

We close this subsection with an example that shows that open subsets of $\mathbb{C}^n$ need not be Stein:

Example 19.15. Let $M = \mathbb{C}^2 \setminus \{0\}$. As a topological space and real manifold, this is $\mathbb{R}^4 \setminus \{0\}$, and hence diffeomorphic over $\mathbb{R}$ to the product of the 3-sphere $\mathbb{S}^3$

with the real line $\mathbb{R}$. In particular, $H^3(M;\mathbb{C}) \cong \mathbb{C}$. But as a 2-dimensional complex manifold, M cannot possibly have nonzero differentials of order 3. It follows that taking global sections of the complex de Rham complex does not compute the singular cohomology of M and the Acyclicity Theorem must fail for the triple $(\mathfrak{U} = \{M\}, \mathcal{C}, M)$.

Remark 19.16. A fascinating branch of mathematics is concerned with the sheaf cohomology of sheaves that locally look like $\mathcal{C}$ but have different global behavior. For example, imagine that on a complex analytic manifold M the constant sheaf $\mathcal{C}$ is replaced (locally) with the multiples of the function f^λ where f is some meromorphic function and $\lambda \in \mathbb{C}$. This is possible away from the pole and zero locus of f and yields a *locally constant sheaf* there. One may generalize this idea by considering solutions to a differential equation with 1-dimensional solution space. These ideas lead to hypergeometric functions and their cohomology which is discussed for example in [126].

19.3. **The algebraic case.** In this subsection we investigate to what extent the results regarding complex analytic manifolds remain true if we consider an algebraic smooth variety X and compute in the Zariski topology. Let us explain by example what we mean.

Example 19.17. Put $R = \mathbb{C}[x, y]$. We consider the *algebraic de Rham complex* of R

$$0 \longrightarrow (\Omega^0_R = R) \xrightarrow{d^0} \Omega^1_R \xrightarrow{d^1} \Omega^2_R \longrightarrow 0,$$

where $\Omega^i_R \subseteq \Omega^i_X$ is the set of differential forms with polynomial rather than holomorphic coefficients. As in the analytic case, this complex has a kernel in degree zero equal to the space of constant functions. Clearly each polynomial 2-form is the gradient of a polynomial 1-form, in many different ways. If $f\,dx + g\,dy$ is a 1-form in the kernel of d^1, then $f_y = g_x$. Let F, G be polynomials such that $F_x = f$ and $G_y = g$. Since $F_{x,y} = G_{x,y}$, F and G agree on all monomials that are multiples of xy. Hence $F - G$ is a polynomial in $\mathbb{C}[x] + \mathbb{C}[y]$, say $F - G = a(x) - b(y)$. Now let $\widetilde{F} = F + b(y)$ and $\widetilde{G} = G + a(x)$. Note that $\widetilde{F}_x = f$, $\widetilde{G}_y = g$ and $\widetilde{F} - \widetilde{G} = F + b(y) - G - a(x) = 0$. It follows that $f\,dx + g\,dy$ is the gradient of the polynomial $\widetilde{F} = \widetilde{G}$. In particular, the algebraic de Rham complex of R is exact apart from degree zero: R satisfies the Poincaré Lemma, and the complex $\Omega^\bullet_R = \Omega^\bullet_{\mathbb{C}^2}(\mathbb{C}^2)$ computes the singular cohomology of the analytic space $\mathbb{C}^{2,an}$ associated to $\operatorname{Spec}(R)$.

Suppose that in general X is an algebraic manifold over $\mathbb{C}$ (i.e., X is algebraic and has no singularities). In the introduction we noted the embedding

$$\Omega^\bullet_X \hookrightarrow \Omega^\bullet_{X^{an}}$$

of the algebraic de Rham complex of X into the analytic de Rham complex of X^{an}. One might wonder when this map is near to an isomorphism. This would be highly convenient since $\Omega^\bullet_X$ is far smaller (it only contains regular algebraic functions). It is reasonable to look first at affine X since then X is Stein. However, while the Poincaré Lemma works just fine any polynomial ring, there are serious problems with the complex algebraic de Rham complex being a resolution of the constant sheaf on pretty much any other space, affine or otherwise, including proper open subsets of $\mathbb{C}^n$. This is truly bad news for the "small" open sets in the Zariski topology!

Example 19.18. We consider the punctured affine line $X = \mathbb{C}^1 \setminus \{0\}$ and the corresponding ring $\mathbb{C}[x, x^{-1}]$. The 1-form dx/x is not a global derivative, simply because the branches of the local integral $\ln(x)$ do not patch to a global function. The same problem continues to hold in the algebraic case. Now in the analytic situation one is able to make the open set on which one computes so small that it becomes simply connected (a tiny open disk avoiding zero) and on *that* open set the logarithm function $\ln(x)$ is well defined if one picks a particular branch. In the Zariski topology, however, all we are allowed to do is to remove Zariski-closed sets (that is, finitely many points!!!) from X. This, of course, makes the space even worse, since it becomes "less simply connected" with each removed point. In particular, $\ln(x)$ never becomes a function on such set and the algebraic de Rham complex has a nontrivial first cohomology group on all Zariski open subsets of X.

The example shows that on affine smooth X the associated algebraic de Rham complex is *usually not a resolution* for the constant sheaf. At this point we record a true miracle of mathematics:

Theorem 19.19 (Grothendieck–Deligne Comparison Theorem, [54, 63, 123])**.** *Let X be any smooth affine complex algebraic variety (i.e., $X = \operatorname{Spec}(R)$ for some finitely generated $\mathbb{C}$-algebra R with empty singular locus). The cohomology of the algebraic de Rham complex of R is naturally isomorphic to the singular cohomology of X.*

There are several reasons why this statement ought to be a surprise. First off, it says that the highly non-algebraic quantity "singular cohomology" can be computed from algebraic data alone *at all*. The integral pairing returns *periods* $\int_\sigma \omega$ which typically are transcendental, see Examples 19.2 and 19.4. Periods are very interesting and are the subject of intense study, see [18].

Beyond that the Comparison Theorem asserts that even though the algebraic de Rham complex is typically *not a resolution* of the constant sheaf, the *errors* that occur and keep it from being a resolution are *exactly the terms that we were hoping to compute* from a resolution in the first place.

For example, on the affine set $X = \mathbb{C}^1 \setminus \{0\} = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$ the algebraic de Rham complex is

$$0 \longrightarrow \mathbb{C}[x, x^{-1}] \longrightarrow \mathbb{C}[x, x^{-1}]\, dx \longrightarrow 0.$$

Its cohomology we already computed several times, it is precisely the singular cohomology of the punctured (real) plane X^{an}.

Exercise 19.20. Show that the Comparison Theorem fails on varieties over $\mathbb{R}$. Hint: let X is the hyperbola defined by $xy = 1$.

Exercise 19.21. Compute the singular cohomology of the open affine subset of $\mathbb{C}^2$ obtained by removal of the coordinate axes.

Exercise 19.22. Compute the singular cohomology of the open affine subset of $\mathbb{C}^1$ obtained by removal of 0 and 1.

Exercise 19.23. Compute the cohomology of $\mathbb{C}^2$ minus the variety defined by the vanishing of $x * (x - 1) * y$.

Exercise 19.24. Compute the cohomology of $\mathbb{C}^2$ minus the variety defined by the vanishing of $x*(x-y)*y$. Hint: if you can't deal with the Čech–de Rham complex then learn what *Macaulay2* [50] might do for you in this matter.

Remark 19.25 (Hypercohomology)**.** The Grothendieck–Deligne Theorem generalizes to schemes that are not affine. This requires *hypercohomology*, which relates to sheaf cohomology the way complexes relate to modules. It is defined by the following construction that mirrors the computation of sheaf cohomology via acyclic resolutions. Suppose we are given a finite complex $\mathcal{G}^\bullet$ of sheaves. One may produce a complex $\mathcal{F}^\bullet$ consisting entirely of flasque sheaves together with a map of complexes $\varphi : \mathcal{G}^\bullet \longrightarrow \mathcal{F}^\bullet$ such that φ induces isomorphisms between all cohomology sheaves: $H^i(\mathcal{G}^\bullet) = H^i(\mathcal{F}^\bullet)$. There are several ways of making such a complex $\mathcal{G}^\bullet$, one goes under the name of *Cartan–Eilenberg resolutions* [47, 19], another is sketched in [134, 156]. By definition, the i-th hypercohomology group $\mathbb{H}^i(X;\mathcal{F}^\bullet)$ of $\mathcal{F}^\bullet$ is $H^i(\Gamma(X;\mathcal{G}^\bullet))$.

If now X is any smooth scheme over $\mathbb{C}$ then the full version of the Grothendieck–Deligne Comparison Theorem says that the i-th singular cohomology group of X arises naturally as the i-th hypercohomology group of the algebraic de Rham complex on X. This mirrors the theme indicated before: one may either replace the given sheaf (or complex of sheaves) by acyclic ones and compute on X, or stick with the given input and use Čech cohomology. (This latter approach results in a Čech–de Rham complex. Practically the Čech–de Rham approach is typically better than a Cartan–Eilenberg resolution.) For algorithmic considerations in this case consult [155].

Exercise 19.26. Compute the singular cohomology of the complement of $\operatorname{Var}(x(x-1), xy)$.

Remark 19.27. The proper setting for de Rham cohomology, as pointed out already by Grothendieck and worked out in Kashiwara's master thesis, is the theory of $\mathcal{D}$-modules, discussed in sketches Lecture 17. For more on this see [88, 10, 23, 8, 9].

19.4. **Local versus de Rham cohomology, and an involved example.** The Čech–de Rham complex combines analytic information stored in differential forms with combinatorial information contained in the cover that is used to construct the Čech complex. In this subsection we use this interplay to construct an upper bound on the index of the top singular cohomology of a Zariski-open set U in affine space in terms of the local cohomological dimension of the complement of U. We use this estimate in an example of Hochster which in conjunction with Example 21.26 will show that local cohomological dimension of a $\mathbb{Z}$-scheme is not always constant along $\operatorname{Spec}(\mathbb{Z})$. Indeed, in Example **??** we will even see an example $(R, \mathfrak{a})$ where the set $\{p \in \operatorname{Spec}\mathbb{Z} | \operatorname{cd}(R/pR, \mathfrak{a}) \geq 4\}$ is not locally closed in the Zariski topology.

Theorem 19.28. *Let $R = \mathbb{C}[x_1, \dots, x_n]$, suppose $g_1, \dots, g_r \in R$ and let $\mathfrak{a}$ be the ideal generated by $g_1, \dots, g_r$. Put $U = \mathbb{C}^n \setminus \operatorname{Var}(\mathfrak{a})$.*

If $i > \operatorname{cd}(\mathfrak{a}) + n - 1$ then the singular cohomology group $H^i(U;\mathbb{C})$ is zero. In particular, $\operatorname{cd}(\mathfrak{a})$ represents a measure of the topological complexity of U.

Proof. If $\operatorname{cd}(\mathfrak{a}) = 1$ then U is affine and hence Stein. So $H^i(U;\mathbb{C}) = 0$ whenever $i > n$. From the Mayer–Vietoris principle one sees quickly that if U can be covered by r open affine sets (as in the theorem) then $H^i(U;\mathbb{C}) = 0$ whenever $i > r+n-1$.

Of course, in general one only knows the estimate $r \geq \mathrm{cd}(\mathfrak{a})$, so this is not the end of the proof.

Let us suppose that $H^{n+c-1}(U;\mathbb{C}) \neq 0$ and assume that c is maximal with this respect. We shall show that $c > \mathrm{cd}(\mathfrak{a})$ is impossible. In order to do so, we shall look at the (algebraic) Čech–de Rham complex of U corresponding to the cover $U = \bigcup_i D_+(g_i)$ where $D_+(g_i)$ is the open set U_i of $\mathbb{C}^n$ defined by the non-vanishing of g_i. For an integer r, let $[r] := \{1,\ldots,r\}$; then the algebraic Čech–de Rham complex of U with respect to the cover induced by the g_i has the form

$$\begin{tikzcd}
\prod\limits_{i_1\in[r]} \Omega^n(U_{i_1}) \arrow[r] & \prod\limits_{i_1<i_2\in[r]} \Omega^n(U_{i_1}\cap U_{i_2}) \arrow[r] & \cdots \arrow[r] & \prod\limits_{\substack{I\subseteq[r]\\|I|=r}} \Omega^n(\bigcap\limits_{i\in I} U_i) \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] \\
\prod\limits_{i_1\in[r]} \Omega^1(U_{i_1}) \arrow[r] \arrow[u] & \prod\limits_{i_1<i_2\in[r]} \Omega^1(U_{i_1}\cap U_{i_2}) \arrow[r] \arrow[u] & \cdots \arrow[r] & \prod\limits_{\substack{I\subseteq[r]\\|I|=r}} \Omega^1(\bigcap\limits_{i\in I} U_i) \arrow[u] \\
\prod\limits_{i_1\in[r]} \mathcal{O}(U_{i_1}) \arrow[r] \arrow[u] & \prod\limits_{i_1<i_2\in[r]} \mathcal{O}(U_{i_1}\cap U_{i_2}) \arrow[r] \arrow[u] & \cdots \arrow[r] & \prod\limits_{\substack{I\subseteq[r]\\|I|=r}} \mathcal{O}(\bigcap\limits_{i\in I} U_i) \arrow[u]
\end{tikzcd}$$

The dimension of the space U is n, so there are n independent first order differentials. As there are r polynomials g_i, the lower left corner has coordinates $(0,0)$ and the upper right has coordinates $(r-1,n)$.

If $J \subseteq [n]$, let us write dx_J for the wedge $\bigwedge_{j\in J} dx_j$ (in ascending order), and for a subset I of $[r]$ we write $g_I = \prod_{i\in I} g_i$ and $U_I = \bigcap_{i\in I} U_i$. We have $\Omega^t(U_I) = \Omega^t(\mathbb{C}^n) \otimes_R R[g_I^{-1}]$. Note that, for any t, the complex $\Omega^{\bullet,t}$ with differential

$$d^s_{\check{C}} : \Omega^{s,t} = \prod_{I\subseteq[r],|I|=s+1} \Omega^t(U_I) \longrightarrow \prod_{I\subseteq[r],|I|=s+2} \Omega^t(U_I) = \Omega^{s+1,t}$$

computes the local cohomology groups of $\Omega^t_{\mathbb{C}^n}$ with respect to $\mathfrak{a}$:

$$H^i(\Omega^{\bullet,t}) = H^{i+1}_{\mathfrak{a}}(\Omega^t(\mathbb{C}^n)) = H^{i+1}_{\mathfrak{a}}(R) \otimes_{\mathbb{Z}} \bigwedge^t \mathbb{Z}^n.$$

It follows that if $s+1 > \mathrm{cd}(\mathfrak{a})$ then each such horizontal cohomology group is zero.

Now consider any form $\omega = \sum\limits_{\substack{t=0,\ldots,n\\s+t=n+c-1}} \omega_{s,t}$ with $\omega_{s,t} \in \Omega^{s,t}$ in the kernel of the total complex of the Čech–de Rham complex, and suppose $c > \mathrm{cd}(\mathfrak{a})$. As $t \leqslant n$, the only possibly nonzero terms in this sum have $s \geq c-1 \geq \mathrm{cd}(\mathfrak{a})$. Consider now just the nonzero term ω_{s_0,t_0} with largest s (located furthest to the right in the above diagram). Since $\omega_{s_0+1,t_0-1} = 0$, we have $d^{s_0}_{\check{C}}(\omega_{s_0,t_0}) = 0$. Hence ω_{s_0,t_0} defines a class in $H^{s_0+1}_{\mathfrak{a}}(\Omega^{t_0})$; this must be the zero class as $s_0 > \mathrm{cd}(\mathfrak{a})$ and hence ω_{s_0,t_0} is a horizontal image, $\omega_{s_0,t_0} = d^{s_0-1}_{\check{C}}(\omega'_{s_0-1,t_0})$. Now subtract from ω the image of

ω'_{s_0,t_0-1} under the differential of the total Čech–de Rham complex. The result is a class that is cohomologous to ω and it has no nonzero components with index $s \geq s_0$.

Repeating this argument for $s_0 - 1, \ldots, \mathrm{cd}(\mathfrak{a})$ we see that the form ω is cohomologous to zero, which proves the theorem. □

Remark 19.29. If the reader believes in the spectral sequence of a double complex, then this proof becomes a triviality.

As an application we shall now provide one half of the proof that local cohomology "depends on the characteristic of the base field". To make sense of this, imagine that R is a $\mathbb{Z}$-algebra and $\mathfrak{a} \subseteq R$ an ideal. One might hope or even expect that if for a prime ideal $p \in \mathrm{Spec}(\mathbb{Z})$ we put $R_p = R \otimes_{\mathbb{Z}} \mathrm{Frac}(\mathbb{Z}/p)$ and $\mathfrak{a}_p = \mathfrak{a} \cdot R_p$ then $\mathrm{cd}(\mathfrak{a}_p, R_p)$ is independent of p. This is, however, not so.

Example 19.30 (Rank one 2×2 matrices)**.** Let $R_{\mathbb{K}} = \mathbb{K}[x_1, x_2, x_3, y_1, y_2, y_3]$ and $A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$ where $\mathbb{K}$ is any field. Define $\mathfrak{a}_{\mathbb{K}}$ to be the ideal generated by the three 2×2-minors of A:

$$\mathfrak{a}_{\mathbb{K}} = (x_1y_2 - x_2y_1, x_1y_3 - x_3y_1, x_2y_3 - x_3y_2)\,.$$

The variety $X_{\mathbb{K}}$ of this ideal in $\mathbb{K}^6$ parameterizes the 2×3-matrices that have no non-vanishing 2×2-minor, and are hence those of rank less than 2. In order to be of rank no more than 1, then one row must be a scalar multiple of the other. This shows that one may choose 4 parameters in such a matrix, one row and the scaling factor. In particular, $\mathfrak{a}_{\mathbb{K}}$ is of height $6 - 4 = 2$ and since $\mathfrak{a}_{\mathbb{K}}$ is 3-generated we infer from **??** and Corollary 7.14 that the only cases for which $H^i_{\mathfrak{a}_{\mathbb{K}}}(R_{\mathbb{K}}) \neq 0$ are $i = 2, 3$.

♣ need a reference here, for ''CM==> height = min-index of local coho''

It will be shown in Example 21.26 that if the characteristic of $\mathbb{K}$ is positive then $H^3_{\mathfrak{a}_{\mathbb{K}}}(R_{\mathbb{K}}) = 0$. In contrast, we shall show here that if $\mathbb{Q} \subseteq \mathbb{K}$ then $H^3_{\mathfrak{a}_{\mathbb{K}}}(R_{\mathbb{K}}) \neq 0$. In particular, this implies that the variety of complex 2×3-matrices with rank defect is defined minimally in $\mathbb{C}^6$ by precisely three equations.

Suppose $\mathbb{Q} \subseteq \mathbb{K}$. Since $\mathbb{K}$ is $\mathbb{Q}$-free, it is enough to show the result when $\mathbb{K} = \mathbb{Q}$. Using this argument one more time we may assume that $\mathbb{K} = \mathbb{C}$. In this case we will prove that the de Rham cohomology $H^8_{\mathrm{dR}}(U; \mathbb{C})$ is nonzero, where $U = \mathbb{C}^6 \setminus X$. In that case Theorem 19.28 yields that with $i = 8$ then $i \leqslant \mathrm{cd}(\mathfrak{a}) + n - 1$, and so $\mathrm{cd}(\mathfrak{a}) \geq 3$. But 3 is the largest possible value for $\mathrm{cd}(\mathfrak{a})$, so $H^3_{\mathfrak{a}}(R) \neq 0$.

We shall hence study the manifold U and its topology. In fact, we shall identify a deformation retract of U and compute its cohomology. Let u_1, u_2 be the two row vectors of a matrix in U. Since U consists of matrices of rank two, $u_1 \neq 0 \neq u_2$.

Note first that we may continuously scale u_1 to a unit vector without affecting its direction, leaving invariant the space of all matrices where u_1 already is a unit vector. Now there is a complex number ω such that u_1 and $u_2 - \omega u_1 \neq 0$ are orthogonal. Replace u_2 by $u_2 - \lambda\omega u_1$, where $\lambda \in \mathbb{R}$ moves in the unit interval. This deformation leaves invariant the matrices where u_1 is a unit vector and orthogonal to u_2. Finally, scale continuously u_2 to a unit vector without affecting its direction. This leaves invariant all matrices with orthogonal rows of length one. Let V be the set consisting of the matrices in U whose rows have unit length and are orthogonal. Then V is a deformation retract of U. In particular, $H^i(U; \mathbb{C}) = H^i(V; \mathbb{C})$ for all i.

Now we'll try to understand V. Let v_1, v_2 be the rows of an element of V. For a given v_1 the vectors v_2 that are perpendicular to v_1 and of unit length sit on a 3-sphere: $\langle v_1, v_2 \rangle = 0$ is one linear complex constraint, with solution space isomorphic to $\mathbb{C}^2 = \mathbb{R}^4$, and to be of unit length forces v_2 to sit on the corresponding 3-sphere in $\mathbb{R}^4$. Hence the map from V to $\mathbb{C}^3$ that sends a matrix to its top row is a fibration with fiber $\mathbb{S}^3$. The image consists of all unit vectors and is hence a 5-sphere. In particular, V is an 8-dimensional real compact manifold.

Note that both $\mathbb{S}^3$ and $\mathbb{S}^5$ inherit an orientation from their natural embedding into $\mathbb{C}^3$ and $\mathbb{C}^2$ respectively. In a neighborhood of a point $p \in \mathbb{S}^5$ the fibration is trivial and one can combine the two orientations to a local orientation of V. Since the base $\mathbb{S}^5$ of the fibration is simply connected this procedure gives a well-defined orientation on all of V. (Well-defined means in this context that for any loop in V along which the orientation is constant, is also the same at start and end point of the loop. Since loops in the base can be contracted, there is actually no problem.) At this point Poincaré duality (see [84]) finishes the proof: on 8-dimensional compact orientable manifolds V one has $H^8_{\mathrm{dR}}(U;\mathbb{C}) \cong H^8(V;\mathbb{C}) \cong \mathbb{C}$, implying the non-vanishing of $H^3_{\mathfrak{a}}(R)$.

Remark 19.31. Suppose $\mathbb{K}$ is algebraically closed, of positive characteristic p, and let ℓ be a prime different from p. The machinery of *étale cohomology* (see [118]) with coefficients in $\mathbb{Z}/\ell\mathbb{Z}$ applies to varieties over fields of characteristic different from ℓ; in particular it applies to $\mathbb{K}$. Étale cohomology enjoys many of the formal properties one is accustomed to from singular cohomology: long exact sequences, functoriality, Mayer–Vietoris sequences, Poincaré duality and others. Formally in exactly the same way above one can show that then $H^8_{et}(U;\mathbb{Z}/\ell\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z}$ where $U_{\mathbb{K}}$ is the complement of $\mathrm{Var}(\mathfrak{a}_{\mathbb{K}})$ in affine space $\mathbb{K}^6$; of course the proofs behind the formalism are of entirely different nature. Despite this, $H^3_{\mathfrak{a}_{\mathbb{K}}}(R_{\mathbb{K}}) = 0$ as shown in Example 21.26.

Exercise 19.32. Let X be the variety of all complex $n \times (n+1)$-matrices whose rank is less than n. Find the number of equations needed to define this variety in $\mathbb{C}^{n\times(n+1)}$.

Semigroup rings are generated by monomials. Geometrically, they give rise to toric varieties. Their combinatorial polyhedral nature makes semigroup rings perhaps the easiest reasonably broad class of algebras over which to compute local cohomology explicitly. On the other hand, the singularities of semigroup rings are sufficiently general for their local cohomology to exhibit a wide range of interesting—and sometimes surprising—phenomena. The purpose of this lecture is to introduce some of the $\mathbb{Z}^d$-graded techniques used to do homological algebra over semigroup rings, including applications to quintessential examples. The key idea is to "resolve modules by polyhedral subsets of $\mathbb{Z}^d$".

20.1. **Semigroup rings.**

Definition 20.1. An *affine semigroup ring* is a subring of a Laurent polynomial ring generated by monomials.

Most of the affine semigroup rings in these lecture notes are *pointed* (see Definition 20.19), which is equivalent to their being subrings of honest polynomial rings instead of Laurent polynomial rings. The simplest example by a long shot is the polynomial ring itself, but we have seen lots of other semigroup rings thus far, too.

Example 20.2. The Extended Example from Lecture 1 is an affine semigroup ring. It appears also in Example 10.21, which draws a relation to invariant theory (this relation is general for affine semigroup rings; see [117, Chapter 10], for instance). Example 9.21 connects it with the Segre embedding $\mathbb{P}^2 \to \mathbb{P}^5$. In Example 19.30, local cohomology is taken with support on its defining ideal.

Example 20.3. The ring $\mathbb{K}[s^4, s^3t, st^3, t^4]$ is an affine semigroup ring by definition. It has appeared in Examples 10.6, 10.11, and 10.18, which illustrate three distinct ways to see the failure of the Cohen-Macaulay property; see Example 20.33 for yet another. The completion of this semigroup ring at its maximal graded ideal comes up in Example 12.6, which is essentially based on the failure of normality.

Example 20.4. Numerous other examples in these lecture notes up to this point treat semigroup rings: all but the last item in Exercise 1.18 are localizations of semigroup rings; the entirety of Exercise 1.23 is about semigroup rings; Example 10.15 treats a semigroup ring (yes, even the parabola is a toric variety); and Example 10.14 is the completion of a semigroup ring.

Example 20.5. The ring $\mathbb{K}[w, x, y, z]/\langle wx - yz\rangle$ whose localization appears in Exercise 1.18(c) is isomorphic to the affine semigroup ring $\mathbb{K}[r, rst, rs, rt] \subset \mathbb{K}[r, s, t]$. Certain local cohomology modules of this ring behave quite badly; in Example 20.49, we will compute one explicitly.

Simply by virtue of being generated by monomials, semigroup rings carry a lot of extra structure. Let us start by writing the monomials in a semigroup ring R using variables $t = t_1, \ldots, t_d$. Thus t^b for $b \in \mathbb{Z}^d$ is shorthand for the (Laurent) monomial $t_1^{b_1} \cdots t_d^{b_d}$. The set

$$Q = \{b \in \mathbb{Z}^d \mid t^b \in R\}$$

forms a subset of $\mathbb{Z}^d$ that is closed under addition and contains $0 \in \mathbb{Z}^d$; hence Q is, by definition, a commutative *semigroup*.[11] Given the extra condition that Q is generated (under addition) by finitely many vectors—namely, the exponents on the generators of R—the semigroup Q is said to be an *affine semigroup*.

Exercise 20.6. Find a subsemigroup of $\mathbb{N}^2$ containing 0 that is not finitely generated. Find uncountably many examples.

An affine semigroup $Q \subseteq \mathbb{Z}^d$ generates a subgroup $\langle Q\rangle$ under addition *and subtraction*. In general, $\langle Q\rangle$ might be a proper subgroup of $\mathbb{Z}^d$, and there are many reasons for wanting to allow $\langle Q\rangle \neq \mathbb{Z}^d$. Often in natural situations, the rank of $\langle Q\rangle$ can even be less than d; see [150, Chapter 1], for example, where the connection with solving linear diophantine equations is detailed (in terms of local cohomology, using techniques based on those in this lecture!). All of that being said, for the purposes of studying the intrinsic properties of R itself,

Notation 20.7. We can and do assume for simplicity that $\langle Q\rangle = \mathbb{Z}^d$.

Basic properties of R can be read directly off of the semigroup Q. The reason is that, as a vector space,

$$R = \mathbb{K}[Q] := \bigoplus_{b\in Q} \mathbb{K}\cdot t^b$$

has a $\mathbb{K}$-basis consisting of the monomials in R.

Lemma 20.8. *If $\langle Q\rangle = \mathbb{Z}^d$ then the semigroup ring $R = \mathbb{K}[Q]$ has dimension d.*

Proof. The main point is that inverting finitely many nonzerodivisors, namely the monomials that generate R, yields a Laurent polynomial ring $\mathbb{K}[\mathbb{Z}^d]$ containing R as a subring. The details are omitted. $\square$

The previously mentioned extra structure induced by the decomposition of R into one-dimensional vector spaces is a "fine grading", as opposed to a "coarse grading" by $\mathbb{Z}$.

Definition 20.9. Let R be a ring and M an R-module.

(1) R is $\mathbb{Z}^d$-*graded* if $R = \bigoplus_{a\in\mathbb{Z}^d} R_a$ and $R_bR_c \subseteq R_{b+c}$ for all $b, c \in \mathbb{Z}^d$.
(2) M is $\mathbb{Z}^d$-*graded* if $M = \bigoplus_{a\in\mathbb{Z}^d} M_a$ and $R_bM_c \subseteq M_{b+c}$ for all $b, c \in \mathbb{Z}^d$.
(3) A homomorphism $M \xrightarrow{\varphi} M'$ of $\mathbb{Z}^d$-graded modules is $\mathbb{Z}^d$-*graded* (of degree zero) if $\varphi(M_a) \subseteq M'_a$ for all $a \in \mathbb{Z}^d$.

Of course, the group $\mathbb{Z}^d$ in this definition could just as easily be replaced by any Abelian group—or any commutative semigroup, such as Q.

Exercise 20.10. Verify that kernels, images, and cokernels of $\mathbb{Z}^d$-graded morphisms are $\mathbb{Z}^d$-graded. Check that tensor products of $\mathbb{Z}^d$-graded modules are naturally $\mathbb{Z}^d$-graded. Prove that if M is finitely presented, then $\mathrm{Hom}(M, N)$ is naturally $\mathbb{Z}^d$-graded whenever M and N are. Why was M assumed to be finitely presented?

[11]The correct term here is really *monoid*, meaning "semigroup with unit element". An affine semigroup is defined to be a monoid; thus, when we say "Q is generated by a set A", we mean that every element in Q is a sum—perhaps with repeated terms and perhaps empty—of elements of A. Allowing the empty sum, which equals the identity element, requires us to be generating Q as a monoid, not just as a semigroup.

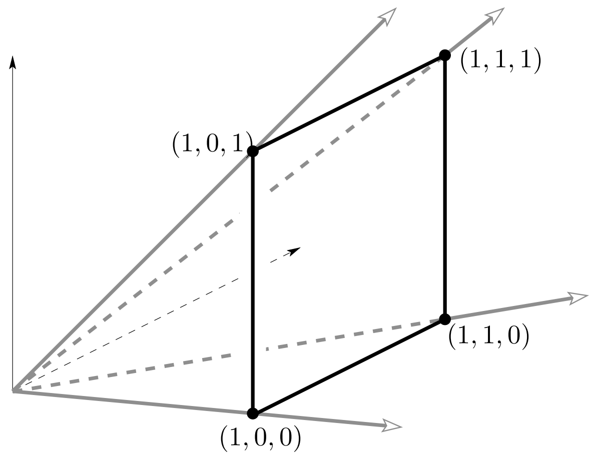


Figure 5. The cone over a square in Example 20.13

Exercise 20.11. Fix a semigroup ring $R = \mathbb{K}[t^{a_1}, \ldots, t^{a_n}]$ and variables $x = x_1, \ldots, x_n$. Prove that the kernel of the map $\mathbb{K}[x] \twoheadrightarrow \mathbb{K}[t]$ sending $x_i \mapsto t^{a_i}$ is

$$I_A := \langle x^u - x^v \mid Au = Av \rangle,$$

where A is the $d \times n$ matrix with columns $a_1, \ldots, a_n$ and $u, v \in \mathbb{Z}^n$ are column vectors of size n. The ideal I_A is called the *toric ideal* for A; find all examples of this notion in these 24 lectures. Hint: Use the $\mathbb{Z}^d$-grading and mimic Exercise 1.34, using the one preceding it.

20.2. **Cones from semigroups.** Suppose that Q is an affine semigroup, again with $\langle Q \rangle = \mathbb{Z}^d$. Taking positive real combinations of elements of Q instead of positive integer combinations yields a *rational polyhedral cone* $C_Q = \mathbb{R}_{\geq 0} Q$. The adjective "rational" means that C_Q is generated as a cone by integer vectors, while "polyhedral" means that C_Q equals the intersection of finitely many closed halfspaces (see Section 16.1).

Example 20.12. Consider the semigroup $\mathbb{K}[s^4, s^3t, st^3, t^4]$ from Example 20.3. By our convention from Notation 20.7, the lattice $\langle Q \rangle = \mathbb{Z}^2$ is *not* the standard lattice in $\mathbb{R}^2$; instead, $\langle Q \rangle$ is generated as an Abelian group by (for instance) $(4, 0)$ and $(-1, 1)$. Under the isomorphism of $\langle Q \rangle$ with $\mathbb{Z}^2$ sending these two generators to the two basis vectors, Q is isomorphic to the semigroup generated by

$$\{(1, 0),\ (1, 1),\ (1, 3),\ (1, 4)\}.$$

The real cone C_Q in this latter representation consists of all (real) points above the horizontal axis and below the line of slope 4 through the origin.

Example 20.13. The ring $\mathbb{K}[r, rst, rs, rt]$ from Example 20.5 is $\mathbb{K}[Q]$ for the affine semigroup Q generated by

$$\{(1, 0, 0),\ (1, 1, 1),\ (1, 1, 0),\ (1, 0, 1)\}.$$

These four vectors are the vertices of a unit square in $\mathbb{R}^3$, and C_Q is the real cone over this square from the origin (Fig. 5). The lattice points in C_Q constitute Q itself.

Theorem 20.14. $\mathbb{K}[Q]$ *is normal if and only if* $Q = C_Q \cap \mathbb{Z}^d$.

Proof. Suppose first that $Q = C_Q \cap \mathbb{Z}^d$, and write $C_Q = \bigcap H_i^+$ as an intersection of closed halfspaces. Then $Q = \bigcap (H_i^+ \cap \mathbb{Z}^d)$ is an intersection of semigroups each of the form $H^+ \cap \mathbb{Z}^d$. Hence $\mathbb{K}[Q]$ is an intersection of semigroup rings $\mathbb{K}[H^+ \cap \mathbb{Z}^d]$ inside the Laurent polynomial ring $\mathbb{K}[\mathbb{Z}^d]$. Each of these subrings is isomorphic to $\mathbb{K}[t_1, \ldots, t_d, t_2^{-1}, \ldots, t_d^{-1}]$, where t_1 has not been inverted. This localization of a polynomial ring is normal, and therefore so is $\mathbb{K}[Q]$.

Next let us assume that $Q \subsetneq C_Q \cap \mathbb{Z}^d$. Exercise 20.15 implies that there is a monomial t^a in the Laurent polynomial ring $\mathbb{K}[\mathbb{Z}^d]$ such that $(t^a)^m \in \mathbb{K}[Q]$ but $t^a \notin \mathbb{K}[Q]$. This monomial is a root of the monic polynomial $y^m - t^{am}$. □

Exercise 20.15. Show that if $a \in C_Q \cap \mathbb{Z}^d$ but $a \notin Q$, then $m \cdot a \in Q$ for all sufficiently large integers m.

Example 20.16. The semigroup ring from Examples 20.3 and 20.12 is not normal, since $s^2t^2 \notin \mathbb{K}[s^4, s^3t, st^3, t^4]$ but

$$s^2t^2 = \frac{s^4 \cdot st^3}{s^3t} \quad \text{and} \quad (s^2t^2)^2 \in \mathbb{K}[s^4, s^3t, st^3, t^4].$$

Example 20.17. The semigroup ring from Example 20.5 is normal, by the last sentence of Example 20.13.

Any rational polyhedral cone C has a unique smallest face (the definition of face is unchanged from the notion for polytopes as in Section 16.1). This smallest face clearly contains the origin, but it also contains any vector $v \in C$ such that $-v$ also lies in C. The cone C is called *pointed* if 0 is the only vector in C whose negative also lies in C. Geometrically, there is a hyperplane H such that C lies on one side of H and intersects H only at 0. Thus C "comes to a point" at the origin.

Thinking of Q instead of C_Q, any vector $a \in Q$ such that $-a \in Q$ corresponds to a monomial $t^a \in \mathbb{K}[Q]$ whose inverse also lies in $\mathbb{K}[Q]$; that is, t^a is a unit in $\mathbb{K}[Q]$. Such a monomial can't lie in any proper ideal of $\mathbb{K}[Q]$. On the other hand, the ideal generated by all nonunit monomials is a proper ideal, the *maximal monomial ideal*. It is the largest ideal of $\mathbb{K}[Q]$ that is $\mathbb{Z}^d$-graded, but it need not be a maximal ideal.

Exercise 20.18. For an affine semigroup Q, the following are equivalent.

(1) The maximal monomial ideal of $\mathbb{K}[Q]$ is a maximal ideal.
(2) The real cone C_Q is pointed.
(3) Q has no nonzero units (that is, $a \in Q$ and $-a \in Q$ implies $a = 0$).

Definition 20.19. Q is *pointed* if the conditions of Exercise 20.18 hold.

A cone C is pointed when there is a hyperplane H intersecting it in exactly one point. Intersecting a (positively translated) parallel hyperplane with C yields a *transverse section* of C, which is a polytope P. The geometry of P can depend on the support hyperplane H, but the combinatorics of P is intrinsic to Q: the poset of faces of P is the same as the poset of faces of Q.

Example 20.20. The square in Example 20.13, namely the convex hull of the four generators of Q, is a transverse section of the cone C_Q there.

In the coming sections, we will see how the homological properties of a semigroup ring $\mathbb{K}[Q]$ are governed by the combinatorics of C_Q.

20.3. **Maximal support: the Ishida complex.** Let Q be a pointed affine semigroup with associated real cone C_Q, and choose a transverse section P_Q. The polytope P_Q is a cell complex, so (after choosing relative orientations for its faces) it has algebraic chain and cochain complexes.

Example 20.21. Let Q be as in Example 20.20, and let P_Q be the square there. The reduced cochain complex of P_Q with coefficients in $\mathbb{K}$ has the form

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{K} & \xrightarrow{\varphi^0} & \mathbb{K}^4 & \xrightarrow{\varphi^1} & \mathbb{K}^4 & \xrightarrow{\varphi^2} & \mathbb{K} & \longrightarrow & 0. \\ & & \varnothing & & \text{vertices} & & \text{edges} & & \text{all of } P_Q & & \end{array}$$

The map φ^0 takes the $\varnothing$-basis vector to the sum of the basis vectors in $\mathbb{K}^4$. The map φ^1 takes the basis vector corresponding to a vertex v to the signed sum of all edges with v as an endpoint; the signs are determined by an (arbitrary) orientation: plus if the edge ends at v, minus if the edge begins at v. Since P_Q is convex—and hence contractible—the cohomology of the above complex is identically zero.

In the same way that the cochain complex of a triangle gives rise to the stable Koszul complex over the polynomial ring, the cochain complex of P_Q gives rise to a complex of localizations of any pointed affine semigroup ring $\mathbb{K}[Q]$. Describing this complex precisely and presenting its role in local cohomology is the goal of this section. Let us first describe the localizations.

Recall that a face of the real cone C_Q is by definition the intersection of C_Q with a support hyperplane. Since C_Q is finitely generated as a cone, it has only finitely many faces, just as the transverse section polytope P_Q does, although most of the faces of C_Q are unbounded, being themselves cones.

Definition 20.22. The intersection of Q with a face of C_Q is called a *face* of Q.

Lemma 20.23. *Let $F \subseteq Q$ be a face. The set of monomials $\{t^b \mid b \notin F\}$ is a prime ideal $\mathfrak{p}_F$ of $\mathbb{K}[Q]$.*

Proof. To check that $\mathfrak{p}_F$ is an ideal it is enough, in view of the ambient $\mathbb{Z}^d$-grading, to check that it is closed under multiplication by monomials from $\mathbb{K}[Q]$. Let ν be a normal vector to a support hyperplane for F such that $\nu(Q) \geq 0$. Thus $\nu(f) = 0$ for some $f \in Q$ if and only if $f \in F$. Assume that $t^b \in \mathfrak{p}_F$ and $t^a \in \mathbb{K}[Q]$. Then $\nu(a+b) \geq \nu(b) > 0$, whence $t^a t^b = t^{a+b}$ lies in $\mathfrak{p}_F$.

The ideal $\mathfrak{p}_F$ is prime because the quotient $\mathbb{K}[Q]/\mathfrak{p}_F$ is isomorphic to the affine semigroup ring $\mathbb{K}[F]$, which is an integral domain. $\square$

Notation 20.24. Let F be a face of Q. Write $\mathbb{K}[Q]_F$ for the localization of $\mathbb{K}[Q]$ by the set of monomials t^f for $f \in F$. For any $\mathbb{K}[Q]$-module M, write $M_F = M \otimes_{\mathbb{K}[Q]} \mathbb{K}[Q]_F$ for the *localization of M along F*.

Exercise 20.25. The localization $\mathbb{K}[Q]_F$ is just the semigroup ring $\mathbb{K}[Q-F]$ for the (non-pointed, if $F \neq \varnothing$) affine semigroup

$$Q - F := \{q - f \mid q \in Q \text{ and } f \in F\}.$$

Here now is the main definition of this lecture.

Definition 20.26. The *Ishida complex* $\mho^{\bullet}_Q$ of the semigroup Q, or of the semigroup ring $\mathbb{K}[Q]$, is the complex

$$0 \to \mathbb{K}[Q] \to \bigoplus_{\text{rays } F} \mathbb{K}[Q]_F \to \cdots \xrightarrow{\delta^{i-1}} \bigoplus_{i\text{-faces } F} \mathbb{K}[Q]_F \xrightarrow{\delta^i} \cdots \to \bigoplus_{\text{facets } F} \mathbb{K}[Q]_F \to \mathbb{K}[\mathbb{Z}^d] \to 0,$$

where an i-face is a face F of Q such that $\dim \mathbb{K}[F] = i$ (so a *ray* is a 1-face and a *facet* is a $(d-1)$-face). The differential δ is composed of natural localization maps $\mathbb{K}[Q]_F \to \mathbb{K}[Q]_G$ with signs as in the algebraic cochain complex of the transverse section P_Q. The terms $\mathbb{K}[Q]$ and $\mathbb{K}[\mathbb{Z}^d]$ sit in cohomological degrees 0 and d.

Exercise 20.27. Write down explicitly the Ishida complex for the cone over the square—the semigroup ring from Example 20.13—using Example 20.21.

Note that when $Q = \mathbb{N}^d$, the Ishida complex is precisely the stable Koszul complex on the variables $t_1, \ldots, t_d$. In general, we still have the following.

Theorem 20.28. *Let $\mathbb{K}[Q]$ be a pointed affine semigroup ring with maximal monomial ideal $\mathfrak{m}$. The local cohomology of any $\mathbb{K}[Q]$-module M supported at $\mathfrak{m}$ is the cohomology of the Ishida complex tensored with M:*

$$H^i_{\mathfrak{m}}(M) \cong H^i(M \otimes \mho^\bullet_Q).$$

The proof of Theorem 20.28 would take an extra lecture (... nah, probably less than that); it is mostly straightforward homological algebra. To give you an idea, it begins with the following.

Exercise 20.29. Check that $H^0(M \otimes \mho^\bullet_Q) \cong H^0_{\mathfrak{m}}(M)$.

What's left is to check that $H^i(M \otimes \mho^\bullet_Q)$ is zero when M is injective and $i > 0$, for then $H^i(- \otimes \mho^\bullet_Q)$ agrees with the derived functors of $\Gamma_{\mathfrak{m}}$. The polyhedral nature of $\mho^\bullet_Q$, in particular the contractibility of certain subcomplexes of P_Q, enters into the proof of higher vanishing for injectives; see [16, Theorem 6.2.5] and its proof.

The natural maps between localizations in the Ishida complex are $\mathbb{Z}^d$-graded of degree zero, so the local cohomology of a $\mathbb{Z}^d$-graded module is naturally $\mathbb{Z}^d$-graded (we could have seen this much from the stable Koszul complex). Sometimes it is the $\mathbb{Z}^d$-graded degrees of the nonzero local cohomology that are interesting, rather than the cohomological degrees or the module structure; see Lecture 24. In any case, the finely graded structure makes local cohomology computations over semigroup rings much more tractable, since they can be done degree-by-degree.

We could, of course, have computed the local cohomology in Theorem 20.28 using a stable Koszul complex, but there is no *natural* choice of elements on which to build one. In contrast, the Ishida complex is based entirely on the polyhedral nature of Q. Combining this with the $\mathbb{Z}^d$-grading provides the truly polyhedral description of the maximal support local cohomology in Corollary 20.32.

Notation 20.30. Write $(\mho^\bullet_Q)_b$ for the complex of $\mathbb{K}$-vector spaces constituting the $\mathbb{Z}^d$-graded degree b piece of $\mho^\bullet_Q$. In addition, let $P_Q(b)$ be the set of faces of P_Q corresponding to faces F of Q with $(\mathbb{K}[Q]_F)_b = 0$, or equivalently, $b \notin Q - F$.

Exercise 20.31. Let Q be a pointed affine semigroup.

(1) Prove that $P_Q(b)$ is a cellular subcomplex of the cell complex P_Q.
(2) Show that $(\mho^\bullet_Q)_b$ is the relative cochain complex for the pair $P_Q(b) \subset P_Q$, up to shifting the cohomological degrees by 1.

Corollary 20.32. *Let Q be a pointed affine semigroup. The degree b part of the maximal support local cohomology of $\mathbb{K}[Q]$ is isomorphic to the relative cohomology of the pair $P_Q(b) \subset P_Q$ with coefficients in $\mathbb{K}$:*

$$H^i_{\mathfrak{m}}(\mathbb{K}[Q])_b \cong H^{i-1}(P_Q, P_Q(b); \mathbb{K}).$$

Example 20.33. We have already seen in Example 20.3 that the semigroup ring $\mathbb{K}[s^4, s^3t, st^3, t^4]$ from Example 20.12 not Cohen-Macaulay. Let us see it yet again, this time by way of its local cohomology. Set $b = (2, 2)$. Then $\mathbb{K}[Q]_b = 0$, so $\varnothing$ is a face of $P_Q(b)$, but every other localization of $\mathbb{K}[Q]$ appearing in the Ishida complex $\mho^\bullet_Q$ is nonzero in degree b. Thus the complex $(\mho^\bullet_Q)_b$ of $\mathbb{K}$-vector spaces is

$$C^\bullet(P_Q, \varnothing; \mathbb{K}) : \; 0 \longrightarrow \mathbb{K} \oplus \mathbb{K} \longrightarrow \mathbb{K}$$

in cohomological degrees 0, 1, and 2. The cohomology is $\mathbb{K}$ in cohomological degree 1 and zero elsewhere. Hence $H^1_{\mathfrak{m}}(\mathbb{K}[Q])$ is nonzero, so $\mathbb{K}[Q]$ is not Cohen-Macaulay. We leave it as an exercise to check that in fact $H^1_{\mathfrak{m}}(\mathbb{K}[Q]) \cong \mathbb{K}$.

The Ishida complex in Example 20.33 turns out to be a stable Koszul complex, as is typical of two-dimensional pointed affine semigroup rings. For higher-dimensional examples, stable Koszul complexes are almost always bigger and less natural than Ishida complexes.

Exercise 20.34. In which $\mathbb{Z}^3$-graded degrees and cohomological degrees is the local cohomology with maximal support of the pointed affine semigroup ring $\mathbb{K}[Q]$ nonzero, if Q is generated by the columns of

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}?$$

Hint: These five vectors have equal last coordinates; plot the first two coordinates of each in the plane. Try also drawing the intersection of Q with the coordinate plane in $\mathbb{R}^3$ spanned by $(1, 0, 0)$ and $(0, 0, 1)$. Which lattice points are "missing"?

The local cohomology of normal affine semigroup rings behaves so uniformly that we can treat them all at once.

Exercise 20.35 (Hochster's Theorem [69])**.** Let $\mathbb{K}[Q]$ be a normal affine semigroup ring. This exercise outlines a proof that $\mathbb{K}[Q]$ is Cohen-Macaulay.

(1) Show that if $P_Q(b)$ equals the boundary of P_Q, consisting of all proper faces of P_Q, then $(\mho^\bullet_Q)_b$ has $\mathbb{K}$ in cohomological degree d and 0 elsewhere.
(2) Prove that if $P_Q(b)$ is properly contained in the boundary of P_Q, then $P_Q(b)$ is contractible. Hint: Show that it has a convex homeomorphic projection.
(3) Deduce that $\mho^\bullet_Q$ has nonzero cohomology only in cohomological degree d.
(4) Conclude that $\mathbb{K}[Q]$ is Cohen-Macaulay. What is its canonical module?

You will need to use normality, of course: by Theorem 20.14, checking whether a $\mathbb{Z}^d$-graded degree b lies in Q (or in $Q - F$ for some face F) amounts to checking that b satisfies a collection of linear inequalities coming from the facets of the real cone C_Q. A detailed solution can be found in [117, Section 12.2], although the arguments there have to be Matlis-dualized to agree precisely with the situation here.

Exercise 20.36. Given an affine semigroup ring, exhibit a finitely generated maximal Cohen-Macaulay module over it.

The intrinsic polyhedral nature of the Ishida complex makes the line of reasoning in the above proof of Hochster's Theorem transparent. It would be more difficult to carry out (though probably still possible) using a stable Koszul complex.

20.4. **Monomial support: $\mathbb{Z}^d$-graded injectives.** In the category of $\mathbb{Z}^d$-graded modules over an affine semigroup ring, the injective objects are particularly uncomplicated. In this section we exploit the polyhedral nature of $\mathbb{Z}^d$-graded injectives to calculate local cohomology supported on monomial ideals. As the motivating example, we'll discover the polyhedral nature of Hartshorne's famous local cohomology module whose socle is not finitely generated.

Notation 20.37. Given a subset $S \subset \mathbb{Z}^d$, write $\mathbb{K}\{S\}$ for the $\mathbb{Z}^d$-graded vector space with basis S, and let $-S = \{-b \mid b \in S\}$.

Example 20.38. Let Q be a pointed affine semigroup.

(1) As a graded vector space, $\mathbb{K}[Q]$ itself is expressed as $\mathbb{K}\{Q\}$.
(2) The injective hull $E_{\mathbb{K}[Q]}$ of $\mathbb{K}$ as a $\mathbb{K}[Q]$-module is $\mathbb{K}\{-Q\}$.
(3) The localization of $\mathbb{K}[Q]$ along a face F of Q is $\mathbb{K}\{Q-F\}$ by Exercise 20.25.
(4) The vector space $\mathbb{K}\{F-Q\}$ is called the *$\mathbb{Z}^d$-graded injective hull of* $\mathbb{K}[F]$.

By $F-Q$ we mean $-(Q-F)$. The justification for the statement in (2) and the nomenclature in (4) are essentially Matlis duality in the $\mathbb{Z}^d$-graded category; see [117, Section 11.2].

Exercise 20.39. $\mathbb{K}\{F-Q\}$ can be endowed with a natural $\mathbb{K}[Q]$-module structure in which multiplication by the monomial t^f is bijective for all $f \in F$.

The module structure you just found makes $\mathbb{K}\{F-Q\}$ injective in the category of $\mathbb{Z}^d$-graded modules. This statement is not difficult [117, Proposition 11.24]; it is more or less equivalent to the statement that the localization $\mathbb{K}[Q-F]$ is flat. As a consequence of injectivity, derived functors of left-exact functors on $\mathbb{Z}^d$-graded modules can be computed using resolutions by such modules. Let us be more precise.

Definition 20.40. Let Q be an affine semigroup. An *indecomposable $\mathbb{Z}^d$-graded injective* is a $\mathbb{Z}^d$-graded translate of $\mathbb{K}\{F-Q\}$ for some face F of Q. A *$\mathbb{Z}^d$-graded injective resolution* of a $\mathbb{Z}^d$-graded $\mathbb{K}[Q]$-module M is a complex

$$I^\bullet : \ 0 \to I^0 \to I^1 \to I^2 \to \cdots$$

of $\mathbb{Z}^d$-graded modules and homomorphisms such that

- each I^j is a $\mathbb{Z}^d$-graded injective;
- $H^0(I^\bullet) \cong M$; and
- $H^j(I^\bullet) = 0$ if $j \geq 1$.

The right derived functors that interest us are, of course, local cohomology. In order to return a $\mathbb{Z}^d$-graded module, the support must be $\mathbb{Z}^d$-graded.

Theorem 20.41. *Let $\mathfrak{a} \subset \mathbb{K}[Q]$ be a monomial ideal. The local cohomology of a $\mathbb{Z}^d$-graded module M supported at $\mathfrak{a}$ can be calculated as*

$$H^i_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(I^\bullet),$$

where $I^\bullet$ is any $\mathbb{Z}^d$-graded injective resolution of M.

What makes this theorem useful is the polyhedral nature of $\mathbb{Z}^d$-graded injectives (in analogy with the polyhedral nature of the Ishida complex) combined with the following extremely easy-to-use characterization of $\Gamma_{\mathfrak{a}}$ on $\mathbb{Z}^d$-graded injectives.

Exercise 20.42. $\Gamma_{\mathfrak{a}}\mathbb{K}\{F-Q\}=0$ unless $\mathfrak{p}_F$ (Lemma 20.23) contains $\mathfrak{a}$, in which case $\Gamma_{\mathfrak{a}}\mathbb{K}\{F-Q\}=\mathbb{K}\{F-Q\}$.

Next we consider the $\mathbb{Z}^d$-*graded Matlis dual* to the Ishida complex.

Definition 20.43. Let Q be a pointed affine semigroup. The *dualizing complex* $\Omega_Q^\bullet$ of the semigroup ring $\mathbb{K}[Q]$ is

$$0\to\mathbb{K}[\mathbb{Z}^d]\to\bigoplus_{\text{facets } F}\mathbb{K}[F-Q]\to\cdots\xrightarrow{\omega^{d-i-1}}\bigoplus_{i\text{-faces } F}\mathbb{K}[F-Q]\xrightarrow{\omega^{d-i}}\cdots\to\mathbb{K}[-Q]\to 0,$$

where the differential ω is composed of natural surjections $\mathbb{K}[F-Q]\to\mathbb{K}[G-Q]$ with signs as in the algebraic chain complex of the transverse section P_Q. The terms $\mathbb{K}[\mathbb{Z}^d]$ and $\mathbb{K}[-Q]$ sit in cohomological degrees 0 and d, respectively.

The reader seeing the $\mathbb{Z}^d$-graded point of view for the first time should make sure to understand the following exercise before continuing.

Exercise 20.44. The complex of $\mathbb{K}$-vector spaces in the $\mathbb{Z}^d$-graded degree b piece of the dualizing complex is exactly the $\mathbb{K}$-dual of the complex in $\mathbb{Z}^d$-graded degree $-b$ of the Ishida complex, up to a cohomological degree shift by d:

$$(\Omega_Q^\bullet)_b[d]\cong((\mho_Q^\bullet)_{-b})^*\text{ for all } b\in\mathbb{Z}^d.$$

Remark 20.45. For readers familiar with dualizing complexes in general [59], Ishida proved that the complex in Definition 20.43 really is one [82, 83]. (The "normalized" dualizing complex would place $\mathbb{K}[\mathbb{Z}^d]$ in cohomological degree $-d$ and $\mathbb{K}[-Q]$ in cohomological degree 0.)

Exercise 20.44 combined with Hochster's Theorem (Exercise 20.35) immediately implies the following.

Corollary 20.46. *If the affine semigroup ring* $\mathbb{K}[Q]$ *is Cohen-Macaulay, then its dualizing complex* $\Omega_Q^\bullet$ *is a* $\mathbb{Z}^d$*-graded injective resolution of some module* ω_Q*. In fact, by local duality, the module* ω_Q *is the canonical module* $\omega_{\mathbb{K}[Q]}$*.*

As a first application, one can compute the Hilbert series of the local cohomology of the canonical module when $\mathbb{K}[Q]$ is normal. Compare the next exercise, where the support ideal $\mathfrak{a}$ is an arbitrary monomial ideal but the module $\omega_{\mathbb{K}[Q]}$ is fixed, with Theorem 16.27, where the support is maximal but the module is the quotient by an arbitrary monomial ideal.

Exercise 20.47 (Yanagawa's formula [161]; see also [117, Theorem 13.14])**.**
Let $\mathbb{K}[Q]$ be a normal affine semigroup ring, and fix a monomial ideal $\mathfrak{a}\subset\mathbb{K}[Q]$.

1. Associate a polyhedral subcomplex $\Delta\subseteq P_Q$ to the (radical of) $\mathfrak{a}$.
2. Write down a polyhedral homological expression in terms of Δ and P_Q for the vector space dimension of the graded piece $H^i_{\mathfrak{a}}(\omega_{\mathbb{K}[Q]})_b$ for $b\in\mathbb{Z}^d$.

Exercise 20.48. Let Δ be the simplicial complex consisting of the isolated point z and the line segment (x,y). Find I_Δ, $H^\bullet_{I_\Delta}(\mathbb{K}[x,y,z])$, and $H^\bullet_{\mathfrak{m}}(\mathbb{K}[\Delta])$.

Our final example is the main example in [61], although the methods here are different, since they rely on the $\mathbb{Z}^d$-grading. Prior to Hartshorne's example, Grothendieck had conjectured that the socle of a local cohomology module should always have finite dimension as a vector space over $\mathbb{K}$.

Example 20.49 (Hartshorne's response to a conjecture of Grothendieck)**.** Let Q be the cone-over-the-square semigroup in Examples 20.5 and 20.13. Retain the notation from those two examples.

The ideal $\mathfrak{a} = \langle rst, rt \rangle$ is the prime ideal $\mathfrak{p}_F$ for the 2-dimensional facet F of Q generated by $(1,0,0)$ and $(1,1,0)$ and hence lying flat in the horizontal plane. Let us compute the local cohomology modules $H^i_{\mathfrak{a}}(\omega_Q)$ of the canonical module using the dualizing complex.

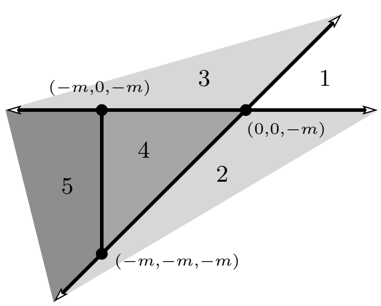
Exercise 20.50. Prove that, ignoring the grading for the time being, $\mathbb{K}[Q] \cong \omega_Q$, so the results below really hold for the local cohomology modules $H^i_{\mathfrak{a}}(\mathbb{K}[Q])$ of the semigroup ring itself.

Let A and B be the rays of Q forming the boundary of F, with A along the axis spanned by $(1,0,0)$ and B cutting diagonally through the horizontal plane. The only monomial prime ideals containing $\mathfrak{a}$ are $\mathfrak{a}$ itself, the ideals $\mathfrak{p}_A$ and $\mathfrak{p}_B$, and the maximal monomial ideal $\mathfrak{m}$. By Exercise 20.42, applying $\Gamma_{\mathfrak{p}}$ to the dualizing complex therefore yields

$$\begin{array}{rccccccccc}
\Gamma_{\mathfrak{a}} I^\bullet : & 0 & \longrightarrow & \mathbb{K}\{F-Q\} & \longrightarrow & \begin{array}{c}\mathbb{K}\{A-Q\}\\ \oplus \\ \mathbb{K}\{B-Q\}\end{array} & \longrightarrow & \mathbb{K}\{-Q\} & \longrightarrow & 0. \\
\text{cohomological} & & & & & & & & & \\
\text{degree:} & 0 & & 1 & & 2 & & 3 & &
\end{array}$$

For lack of a better term, call each of the four indecomposable $\mathbb{Z}^3$-graded injectives a *summand*.

Consider the nonzero contributions of the four summands to a $\mathbb{Z}^3$-graded degree $b = (\alpha, \beta, \gamma)$. If $\gamma > 0$, then none of the four summands contribute, because then $\mathbb{K}\{G-Q\}_b = 0$ whenever G is one of the faces F, A, B, or $\{0\}$ of Q. However, the halfspace beneath the horizontal plane, consisting of vectors b with $\gamma \leq 0$, is partitioned into five *sectors*. For degrees b in a single sector, the subset of the four summands contributing a nonzero vector space to degree b remains constant. The summands contributing to each sector are listed in Fig. 6, which depicts the intersections of the sectors with the plane $\gamma = -m$ as the five regions.



sector 1 : $\mathbb{K}\{F-Q\}$
sector 2 : $\mathbb{K}\{F-Q\}, \mathbb{K}\{A-Q\}$
sector 3 : $\mathbb{K}\{F-Q\}, \qquad \mathbb{K}\{B-Q\}$
sector 4 : $\mathbb{K}\{F-Q\}, \mathbb{K}\{A-Q\}, \mathbb{K}\{B-Q\}$
sector 5 : $\mathbb{K}\{F-Q\}, \mathbb{K}\{A-Q\}, \mathbb{K}\{B-Q\}, \mathbb{K}\{-Q\}$

Figure 6. Intersections of sectors with a horizontal plane

Only in sectors 1 and 4 does $\Gamma_{\mathfrak{a}} I^\bullet$ have any cohomology. The cone of integer points in sector 1 and the cohomology of $\Gamma_{\mathfrak{a}} I^\bullet$ there are as follows:

$$\text{sector } 1: \quad \gamma \leq 0 \text{ and } \alpha > \beta > 0 \quad \Longleftrightarrow \quad H^1_{\mathfrak{a}}(\omega_Q)_b = \mathbb{K}.$$

For sector 4, we get the cone of integer points and cohomology as follows:

$$\text{sector } 4: \quad 0 \geq \beta \geq \alpha > \gamma \quad \Longleftrightarrow \quad H^2_{\mathfrak{a}}(\omega_Q)_{\mathbf{b}} = \mathbb{K}.$$

We claim that sector 4 has infinitely many degrees with socle elements of $H^2_{\mathfrak{a}}(\omega_Q)$: they occupy all degrees $(0, 0, -m)$ for $m > 0$. This conclusion is forced by the polyhedral geometry. To see why, keep in mind that sector 4 is not just the triangle depicted in Fig. 6 (which sits in a horizontal plane below the origin), but the cone from the origin over that triangle. Consider any element $h \in H^2_{\mathfrak{a}}(\omega_Q)$ of degree $(0, 0, -m)$. Multiplication by any nonunit monomial of $\mathbb{K}[Q]$ takes h to an element whose $\mathbb{Z}^3$-graded degree lies outside of sector 4 (this is the polyhedral geometry at work!). Since $H^2_{\mathfrak{a}}(\omega_Q)$ is zero in degrees outside of sector 4, we conclude that h must be annihilated by every nonunit monomial of $\mathbb{K}[Q]$.

Exercise 20.51. What is the annihilator of $H^1_{\mathfrak{a}}(\omega_Q)$? What elements in that local cohomology module have annihilator equal to a prime ideal of $\mathbb{K}[Q]$? Is $H^1_{\mathfrak{a}}(\omega_Q)$ finitely generated? In what $\mathbb{Z}^3$-graded degrees do its generators lie?

Hartshorne's example raises the following basic open problem. All that is known currently is the criterion for $\mathbb{K}[Q]$ to possess a monomial ideal $\mathfrak{a}$ and a finitely generated $\mathbb{Z}^d$-graded module M such that $H^i_{\mathfrak{a}}(M)$ has infinite-dimensional socle for some i: this occurs if and only if P_Q is a simplex [67].

Problem 20.52. Characterize the normal affine semigroup rings $\mathbb{K}[Q]$, monomial ideals $\mathfrak{a} \subset \mathbb{K}[Q]$, and cohomological degrees i such that $H^i_{\mathfrak{a}}(\mathbb{K}[Q])$ has infinite-dimensional socle.

Lecture 21. The Frobenius endomorphism (CM)

Definition 21.1. Let R be a ring containing a field of prime characteristic $p > 0$. The *Frobenius endomorphism* $f : R \longrightarrow R$ is the ring homomorphism $f(r) = r^p$. Its iterates are the maps $f^e : R \longrightarrow R$ with $f^e(r) = r^{p^e}$.

The following theorem of Kunz [92, Theorem 2.1] is a key ingredient in some of the characteristic p methods discussed in this lecture.

Theorem 21.2 (Kunz)**.** *Let R be a Noetherian ring of prime characteristic $p > 0$. The following statements are equivalent:*

(1) *R is regular;*
(2) *f^e is flat for all $e > 0$;*
(3) *f^e is flat for some $e > 0$.*

Proof. We sketch a proof of (1) $\implies$ (2), and refer the reader to [92] for further details. Since a composition of flat maps is flat, it suffices to prove the case $e = 1$. The issue is local, so we may assume that $(R, \mathfrak{m})$ is a regular local ring. It suffices to verify the flatness assertion after replacing R by its $\mathfrak{m}$-adic completion. By Cohen's structure theorem, every complete regular local ring of prime characteristic is a power series ring over a field, so we may assume $R = \mathbb{K}[[x_1, \dots, x_d]]$. The Frobenius map $f : R \longrightarrow R$ may be identified with the composition of the inclusions

$$\mathbb{K}^p[[x_1^p, \dots, x_d^p]] \subset \mathbb{K}[[x_1^p, \dots, x_d^p]] \subset \mathbb{K}[[x_1, \dots, x_d]].$$

The first inclusion can be seen to be flat by the local criterion of flatness: indeed a short computation using the Koszul complex resolution of the first factor yields that

$$\operatorname{Tor}_i^{\mathbb{K}^p[[x_1^p,\dots,x_d^p]]}(\mathbb{K}^p, \mathbb{K}[[x_1^p, \dots, x_d^p]]) = 0 \quad \text{for all } i > 0$$

For the second inclusion, the monomials in the variables x_i in which each exponent is less than p form a basis for $\mathbb{K}[[x_1, \dots, x_d]]$ as a $\mathbb{K}[[x_1^p, \dots, x_d^p]]$-module, hence the inclusion is free, therefore flat. □

Exercise 21.3. Let $R = \mathbb{Z}/2\mathbb{Z}[x^2, xy, y^2]$. Verify that the Frobenius homomorphism $f : R \longrightarrow R$ is not flat.

Exercise 21.4. Find a ring of prime characteristic such that R is flat over $f(R)$, but R is not flat over R via f (i.e., R is not regular).

Definition 21.5. Let M be a module over a ring R of characteristic $p > 0$. Then ${}^{f^e}M$ denotes M with the R-module structure obtained via restriction of scalars along the homomorphism $f^e : R \longrightarrow R$, that is,

$$r \cdot m = f^e(r)m = r^{p^e}m$$

for $r \in R$ and $m \in {}^{f^e}M$. In particular, ${}^{f^e}R$ denotes R viewed as an R-module via $f^e : R \longrightarrow R$.

The following is a generalization of Theorem 21.2. The implication (1) $\implies$ (2) was proved by Peskine and Szpiro [127]; Herzog [68] proved the converse shortly thereafter.

Theorem 21.6 (Peskine-Szpiro, Herzog)**.** *Let R be a Noetherian ring of characteristic $p > 0$ and M a finitely generated R-module. The following are equivalent:*

(1) $\operatorname{pd}_R M < \infty$*;*

(2) $\operatorname{Tor}_i^R(M, {}^{f^e}R) = 0$ *for all* $i > 0$ *and all (equivalently, infinitely many)* $e > 0$.

There are two ways in which the Frobenius is used in conjunction with local cohomology theory. The first is through a natural Frobenius action on local cohomology modules, and the second is via Frobenius powers of ideals. The second approach has been very effective over regular rings in proving vanishing theorems, and for studying associated primes of local cohomology modules. We discuss each approach in turn in the remaining two subsections.

Frobenius action on local cohomology modules

21.7. Let $\varphi : R \longrightarrow S$ be a ring homomorphism. If $\mathfrak{a}$ is an ideal of R, then φ induces a map

$$H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}S}(S).$$

In particular, if R has characteristic $p > 0$, then the Frobenius $f : R \longrightarrow R$ induces a map

$$H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}^{[p]}}(R) = H^i_{\mathfrak{a}}(R),$$

which we call the *Frobenius action* on $H^i_{\mathfrak{a}}(R)$ and, abusing notation, denote by f. Note that $f : H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(R)$ is a map of Abelian groups, but is not an R-module homomorphism in general.

Another way to consider $f : H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(R)$ is as follows. The Frobenius action on R extends to an action on any localization $W^{-1}R$ of R, via the formula

$$f\left(\frac{r}{w}\right) = \frac{r^p}{w^p}, \qquad \text{where } r \in R \text{ and } w \in W.$$

Let $\mathfrak{a} = (x_1, \ldots, x_n)$ be an ideal of R. The local cohomology modules $H^i_{\mathfrak{a}}(R)$ may be computed as the cohomology modules of the stable Koszul complex (7.11)

$$0 \longrightarrow R \longrightarrow \bigoplus_i R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_n} \longrightarrow 0.$$

As the modules in this complex have a natural Frobenius action that commutes with the localisation maps in the complex, so do its cohomology modules $H^i_{\mathfrak{a}}(R)$.

Example 21.8. Let $R = \mathbb{K}[x_1, \ldots, x_d]$ be a polynomial ring over a field $\mathbb{K}$ of characteristic $p > 0$, and let $\mathfrak{m}$ denote the homogeneous maximal ideal of R. Since R is Cohen-Macaulay, the modules $H^i_{\mathfrak{m}}(R)$ are zero for $i < d$. We examine the Frobenius action on $H^d_{\mathfrak{m}}(R)$. By Corollary 7.14 we have

$$H^d_{\mathfrak{m}}(R) = \frac{R_{x_1 \cdots x_d}}{\sum_i R_{x_1 \cdots \widehat{x}_i \cdots x_d}}.$$

As a $\mathbb{K}$-vector space, the module $H^d_{\mathfrak{m}}(R)$ is spanned by the elements

$$\left[\frac{1}{x_1^{n_1} \cdots x_d^{n_d}}\right],$$

where $n_i > 0$ for all $1 \leqslant i \leqslant d$. The Frobenius action f on $H^d_{\mathfrak{m}}(R)$ is a group homomorphism under the additive group structure on $H^d_{\mathfrak{m}}(R)$ and satisfies

$$f : \left[\frac{\lambda}{x_1^{n_1} \cdots x_d^{n_d}}\right] \longmapsto \left[\frac{\lambda^p}{x_1^{pn_1} \cdots x_d^{pn_d}}\right]$$

for $\lambda \in \mathbb{K}$. Note that f is injective in this example, and that it is a $\mathbb{K}$-vector space homomorphism if and only if $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$. The module $H^d_{\mathfrak{m}}(R)$ is supported in degrees $j \leqslant -d$, and

$$f : \left[H^d_{\mathfrak{m}}(R)\right]_j \longrightarrow \left[H^d_{\mathfrak{m}}(R)\right]_{pj}.$$

Example 21.9. Let $R = \mathbb{K}[x_0, \dots, x_d]$ be a polynomial ring over a field $\mathbb{K}$ of characteristic $p > 0$, and $\mathfrak{m}$ its homogeneous maximal ideal. Let $h \in \mathfrak{m}$ be a homogeneous polynomial of degree $n \geqslant d + 2$. Let $S = R/(h)$, which is a hypersurface of dimension d. We shall see that the Frobenius action on $H^d_{\mathfrak{m}}(S)$ is not injective.

The short exact sequence of graded R-modules

$$0 \longrightarrow R(-n) \xrightarrow{\ h\ } R \longrightarrow S \longrightarrow 0$$

induces a long exact sequence of local cohomology modules, the nonzero part of which is

$$0 \longrightarrow H^d_{\mathfrak{m}}(S) \longrightarrow H^{d+1}_{\mathfrak{m}}(R(-n)) \xrightarrow{\ h\ } H^{d+1}_{\mathfrak{m}}(R) \longrightarrow 0.$$

Examining the graded pieces of this exact sequence, we get

$$\left[H^d_{\mathfrak{m}}(S)\right]_{n-d-1} \neq 0 \quad \text{and} \quad \left[H^d_{\mathfrak{m}}(S)\right]_{>n-d-1} = 0.$$

Since $n - d - 1 > 0$, it follows that Frobenius

$$f : \left[H^d_{\mathfrak{m}}(S)\right]_{n-d-1} \longrightarrow \left[H^d_{\mathfrak{m}}(S)\right]_{p(n-d-1)} = 0$$

must be the zero map.

Exercise 21.10. Let $S = \mathbb{K}[x, y]/(xy)$. Determine $\mathbb{K}$-vector space bases for the graded components of $H^1_{\mathfrak{m}}(S)$. If $\mathbb{K}$ has characteristic $p > 0$, describe the Frobenius action on $H^1_{\mathfrak{m}}(S)$.

Exercise 21.11. Let $\mathbb{K}$ be a field of characteristic $p \geqslant 3$, and set

$$R = \mathbb{K}[x, y]/(x^2 + y^2) \qquad \text{and} \qquad S = \mathbb{K}[x, y]/(x^3 + y^3).$$

Show that Frobenius is injective on $H^1_{\mathfrak{m}}(R)$ but not on $H^1_{\mathfrak{m}}(S)$.

The Frobenius action on local cohomology modules was used by Hochster-Roberts [76] in their proof that various rings of invariants are Cohen-Macaulay, and by Smith in her work on F-rational rings [148].

We next use the Frobenius action on local cohomology modules to prove the following theorem:

Theorem 21.12. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and let $x_1, \dots, x_d$ be a system of parameters for R. Then*

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})R$$

for each positive integer t.

Proof. If the assertion is false, there exist $r_i \in R$ and $t \geqslant 1$ such that

$$(x_1 \cdots x_d)^t = r_1 x_1^{t+1} + \cdots + r_d x_d^{t+1}.$$

Using the Čech complex on $x_1, \dots, x_d$ to compute $H^d_{\mathfrak{m}}(R)$, this implies that

$$\eta = \left[\frac{1}{x_1 \cdots x_d}\right] = \left[\sum_i \frac{r_i}{(x_1 \cdots \widehat{x}_i \cdots x_d)^{t+1}}\right] = 0.$$

But then $f^e(\eta) = 0$ for all $e \geqslant 1$. Since every element of $H^d_{\mathfrak{m}}(R)$ has the form

$$\left[\frac{a}{(x_1 \cdots x_d)^{p^e}}\right] = af^e(\eta)$$

for some $a \in R$ and $e \geqslant 1$, this implies that $H^d_{\mathfrak{m}}(R) = 0$, a contradiction. $\square$

The proof of the previous theorem illustrates a basic strategy of positive characteristic methods: start with an equation which is somewhat unlikely ($\eta = 0$ in the situation above); apply Frobenius repeatedly to arrive at infinitely many equations, ($F^e(\eta) = 0$ for all e) which, put together, are downright impossible!

Remark 21.13. Let $x_1, \dots, x_d$ be a system of parameters for a local ring R. Hochster's *monomial conjecture* states that

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})R$$

for all positive integers t. We saw a proof of this in the case that R has positive characteristic. It is also known to be true for rings containing a field of characteristic zero, and Heitmann recently proved it for local rings of mixed characteristic of dimension at most three, [66]. It remains open for mixed characteristic rings of higher dimension, and is equivalent to several other conjectures such as the direct summand conjecture (which states that regular local rings are direct summands of their module-finite extension rings), the canonical element conjecture, and the improved new intersection conjecture, see [72]. Some related conjectures including Auslander's zerodivisor conjecture and Bass' conjecture were proved by Paul Roberts, [133].

Let R be a local ring or an $\mathbb{N}$-graded ring over a field R_0. Hartshorne asked whether there exists a finitely generated Cohen-Macaulay R-module M (graded, in the case R is graded) with $\dim M = \dim R$. Such a module has come to be known as a *small Cohen-Macaulay module*—the word "small" here refers to the finite generation condition on M. For local rings R containing a field, Hochster [70] proved the existence of *big Cohen-Macaulay modules*, i.e., modules M which are Cohen-Macaulay with $\dim M = \dim R$, but are not necessarily finitely generated. In [75] Hochster and Huneke proved that every local ring containing a field has a big Cohen-Macaulay *algebra*.

Exercise 21.14. If $(R, \mathfrak{m})$ is a local ring which has a big Cohen-Macaulay module, prove that the monomial conjecture is true for every system of parameters for R.

If R is an excellent domain of dimension at most two, then the integral closure of R is a small Cohen-Macaulay module. For rings of dimension greater than two, very little is known about the existence of small Cohen-Macaulay modules. The Frobenius action on local cohomology gives us affirmative answers in some cases, Theorem 21.15 and, more generally, Exercise 21.17. This was discovered independently by Hartshorne, by Peskine-Szpiro, and by Hochster.

Theorem 21.15. *Let R be an $\mathbb{N}$-graded domain of dimension three, which is finitely generated over a perfect field $R_0 = \mathbb{K}$ of characteristic $p > 0$. Then R has a small Cohen-Macaulay module.*

We first record a preliminary result:

Proposition 21.16. *Let $(R, \mathfrak{m})$ be a local domain which is a homomorphic image of a Gorenstein ring. Let M be a finitely generated torsion-free R-module such that*

$M_{\mathfrak{p}}$ is Cohen-Macaulay for all prime ideals $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Then $H^i_{\mathfrak{m}}(M)$ has finite length for all $i < \dim M = \dim R$.

Proof. By Matlis duality, specifically Theorem A.38 and Lemma A.30, and local duality, Theorem 18.14, it suffices to show that the R-module

$$H^i_{\mathfrak{m}}(M)^\vee \cong \operatorname{Ext}_R^{\dim R - i}(M, \omega_R)$$

has finite length. Since $\operatorname{Ext}_R^{\dim R-i}(M, \omega_R)$ is a finitely generated R-module, this is equivalent to showing that the $R_{\mathfrak{p}}$-module

$$\operatorname{Ext}_R^{\dim R-i}(M, \omega_R)_{\mathfrak{p}} = \operatorname{Ext}_{R_{\mathfrak{p}}}^{\dim R-i}(M_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}})$$

vanishes for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Another application of local duality, now over $R_{\mathfrak{p}}$, yields

$$\operatorname{Ext}_{R_{\mathfrak{p}}}^{\dim R-i}(M_{\mathfrak{p}}, \omega_{R_{\mathfrak{p}}}) \cong H_{\mathfrak{p}R_{\mathfrak{p}}}^{\dim R_{\mathfrak{p}} - \dim R + i}(M_{\mathfrak{p}})^\vee.$$

The vanishing of the latter module follows from the fact that

$$\dim R_{\mathfrak{p}} - \dim R + i < \dim R_{\mathfrak{p}} = \dim M_{\mathfrak{p}}$$

and the hypothesis that $M_{\mathfrak{p}}$ is Cohen-Macaulay. □

Proof of Theorem 21.15. Since R is a finitely generated domain over a field, the integral closure R' of R is a finitely generated R-module. Replacing R by R', we may assume that R is a normal domain. This implies that $H^0_{\mathfrak{m}}(R) = H^1_{\mathfrak{m}}(R) = 0$ and, using Proposition 21.16, that $H^2_{\mathfrak{m}}(R)$ has finite length.

Since $\mathbb{K}$ is perfect, ${}^{f^e}R$ is a finitely generated R-module for all $e \in \mathbb{N}$. Also, since $\mathbb{K}$ is perfect, the length of a module is unchanged under restriction of scalars and therefore the length of

$${}^{f^e}H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{m}}({}^{f^e}R)$$

does not depend on e. We note how the grading on ${}^{f^e}R$ interacts with restriction of scalars: if

$$m \in \left[{}^{f^e}R\right]_n \quad \text{and} \quad r \in R_j, \quad \text{then} \quad r \cdot m = r^{p^e} m \in \left[{}^{f^e}R\right]_{jp^e+n}.$$

Hence for each e, the R-module ${}^{f^e}R$ is a direct sum of the p^e modules

$$W_{e,i} = R_i + R_{p^e+i} + R_{2p^e+i} + R_{3p^e+i} + \cdots,$$

where $0 \leqslant i \leqslant p^e - 1$. Note that

$$H^2_{\mathfrak{m}}({}^{f^e}R) = H^2_{\mathfrak{m}}(W_{e,0}) \oplus H^2_{\mathfrak{m}}(W_{e,1}) \oplus \cdots \oplus H^2_{\mathfrak{m}}(W_{e,p^e-1}).$$

This module has constant length as e gets large, so there exists $W_{e,i} \neq 0$ with

$$H^2_{\mathfrak{m}}(W_{e,i}) = 0.$$

But then $W_{e,i}$ is a small Cohen-Macaulay module for R. □

Exercise 21.17. Let $(R, \mathfrak{m})$ be an $\mathbb{N}$-graded domain, finitely generated over a perfect field $R_0 = \mathbb{K}$ of characteristic $p > 0$. Let M be a finitely generated graded torsion-free R-module, such that $M_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Prove that R has a small Cohen-Macaulay module.

The Frobenius functor and a vanishing theorem

Another way to use the Frobenius endomorphism to understand $H^i_{\mathfrak{a}}(R)$ is via Frobenius powers of ideals—the Frobenius powers of an ideal are a sequence of ideals cofinal with the (ordinary) powers.

Definition 21.18. Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of a ring R of characteristic $p > 0$. The ideals

$$\mathfrak{a}^{[p^e]} = (x_1^{p^e}, \dots, x_n^{p^e})R$$

are called the *Frobenius powers* (or, informally, the bracket powers) of $\mathfrak{a}$.

Exercise 21.19. Check that the ideals $\mathfrak{a}^{[p^e]}$ do not depend on the choice of generators for $\mathfrak{a}$.

Remark 21.20. If $\mathfrak{a}$ is generated by n elements, then

$$\mathfrak{a}^{np^e} \subseteq \mathfrak{a}^{[p^e]} \subseteq \mathfrak{a}^{p^e}.$$

Therefore, the sequence of bracket powers of $\mathfrak{a}$ is cofinal with the sequence of ordinary powers, and so it can be used to compute local cohomology:

$$H^i_{\mathfrak{a}}(M) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, M),$$

see Remark 7.8.

A natural way to obtain the Frobenius powers of an ideal is to apply a base change along Frobenius to the R-module $R/\mathfrak{a}$, as described next. Recall that for any homomorphism $R \longrightarrow S$ and any R-module M, the base change $S \otimes_R M$ has a natural S-module structure.

Definition 21.21. The *Frobenius functor* (or Peskine-Szpiro functor) is a functor F from the category of R-modules to the category of R-modules, which takes a module M to the module

$$F(M) = {}^f R \otimes_R M,$$

[127, Définition I.1.2]. More generally, we define $F^e(M) = {}^{f^e} R \otimes_R M$.

Note that

$$F^e(R) = {}^{f^e} R \otimes_R R \cong R$$

as R-modules, and if $\varphi : R^s \longrightarrow R^t$ is given by a matrix $[a_{ij}]$ with respect to some choice of bases for R^s and R^t, then $F^e(\varphi) : R^s \longrightarrow R^t$ is given by the matrix $[a_{ij}^{p^e}]$. This implies that if M is finitely generated, so is $F^e(M)$.

Remark 21.22. For an ideal $\mathfrak{a} = (x_1, \dots, x_n)$ of R, consider the exact sequence

$$R^n \xrightarrow{[x_1\ \dots\ x_n]} R \longrightarrow R/\mathfrak{a} \longrightarrow 0.$$

Applying $F^e(-)$, the right exactness of tensor gives us the exact sequence

$$R^n \xrightarrow{[x_1^{p^e}\ \dots\ x_n^{p^e}]} R \longrightarrow F^e(R/\mathfrak{a}) \longrightarrow 0,$$

which shows that $F^e(R/\mathfrak{a}) \cong R/\mathfrak{a}^{[p^e]}$. This can also be seen via the formulas

$$F^e(R/\mathfrak{a}) = {}^{f^e} R \otimes_R R/\mathfrak{a} \cong R/f^e(\mathfrak{a})R = R/\mathfrak{a}^{[p^e]}$$

If R is regular, then F is an exact functor by Theorem 21.2. Even when R is not regular, Theorem 21.6 implies the following:

Proposition 21.23. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let M be a finitely generated R-module of finite projective dimension. Then, for all $e \in \mathbb{N}$,*

(1) $\operatorname{pd}_R F^e(M) = \operatorname{pd}_R M$*; and*
(2) $\operatorname{Ass} F^e(M) = \operatorname{Ass} M$.

Proof. We may assume that R is local. If $P_\bullet$ is a minimal free resolution of M, Theorem 21.6 implies that $F^e(P_\bullet)$ is a free resolution of $F^e(M)$. It is, in fact, a minimal free resolution, since the matrices giving the maps in $F^e(P_\bullet)$ have entries in $\mathfrak{m}^{[p^e]}$, so (1) follows. Localizing at a given prime, (2) reduces to the verification that over a local ring $(R, \mathfrak{m})$ the maximal ideal is an associated prime of M if and only if it is an associated prime of $F^e(M)$. Since M has finite projective dimension, $\mathfrak{m} \in \operatorname{Ass} M$ if and only if $\operatorname{pd}_R M = \operatorname{depth} R$, and likewise for $F^e(M)$. The result now follows from (1). $\square$

This proposition is a key ingredient in the following vanishing theorem of Peskine-Szpiro, [127, Proposition III.4.1], from a paper in which they proved important cases of several conjectures in local algebra.

Theorem 21.24 (Peskine-Szpiro)**.** *Let R be a regular domain of characteristic p. If $\mathfrak{a} \subset R$ is an ideal such that $R/\mathfrak{a}$ is Cohen-Macaulay, then*

$$H^i_{\mathfrak{a}}(R) = 0 \qquad \textit{for } i \neq \operatorname{height} \mathfrak{a}.$$

Proof. We have seen that $H^i_{\mathfrak{a}}(R) = 0$ for all $i < \operatorname{height} \mathfrak{a} = \operatorname{depth}_R(\mathfrak{a}, M)$ cf. Theorem 9.1. There is no loss of generality in assuming that $(R, \mathfrak{m})$ is a regular local ring. Recall that $R/\mathfrak{a}^{[p^e]} \cong F^e(R/\mathfrak{a})$ from Remark 21.20. Using Proposition 21.23(1), the Auslander-Buchsbaum formula, and the assumption that $R/\mathfrak{a}$ is Cohen-Macaulay, we see that

$$\operatorname{pd}_R R/\mathfrak{a}^{[p^e]} = \operatorname{pd}_R R/\mathfrak{a} = \dim R - \operatorname{depth} R/\mathfrak{a} = \dim R - \dim R/\mathfrak{a} = \operatorname{height} \mathfrak{a}.$$

Thus for all $i > \operatorname{height} \mathfrak{a}$, we have $\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) = 0$ and hence

$$H^i_{\mathfrak{a}}(R) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) = 0.$$

$\square$

The following exercise gives an extension of Theorem 21.24. For another extension, due to Lyubeznik, see Theorem 22.1.

Exercise 21.25. Let R be a Noetherian ring of characteristic $p > 0$, and $\mathfrak{a}$ be an ideal such that $R/\mathfrak{a}$ has finite projective dimension. Use Corollary 21.23 to show that

$$H^i_{\mathfrak{a}}(R) = 0 \qquad \text{for all } i > \operatorname{depth} R - \operatorname{depth} R/\mathfrak{a}.$$

The assertion of Theorem 21.24 does not hold if, instead, R is a regular ring of characteristic zero, by the following example due independently to Hartshorne-Speiser [64, Example 5, page 75] and to Hochster.

Example 21.26. Let $R = \mathbb{K}[u, v, w, x, y, z]$ be a polynomial ring over a field $\mathbb{K}$, and $\mathfrak{a}$ be the ideal generated by the size two minors of the matrix

$$X = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix},$$

i.e., $\mathfrak{a} = (\Delta_1, \Delta_2, \Delta_3)$ where

$$\Delta_1 = vz - wy, \quad \Delta_2 = wx - uz, \quad \Delta_3 = uy - vx.$$

Then $\operatorname{height} \mathfrak{a} = 2$, so $H^2_{\mathfrak{a}}(R) \neq 0$. Also, $R/\mathfrak{a}$ is a Cohen-Macaulay ring.

Positive Characteristic: If $\mathbb{K}$ is a field of characteristic $p > 0$, then Theorem 21.24 implies that $H^3_{\mathfrak{a}}(R) = 0$.
Characteristic zero: The group $G = SL_2(\mathbb{K})$ acts on R as in Example 10.30 where $n = 2$ and $d = 3$. The ring of invariants for this action is $R^G = \mathbb{K}[\Delta_1, \Delta_2, \Delta_3]$. If $\mathfrak{n} = (\Delta_1, \Delta_2, \Delta_3)R^G$ denotes the homogeneous maximal ideal of R^G, then $H^3_{\mathfrak{n}}(R^G)$ is nonzero—R^G is a polynomial ring of dimension three, the three minors being algebraically independent over $\mathbb{K}$. Since $\mathbb{K}$ has characteristic zero, G is linearly reductive, and hence R^G is a direct summand of R, i.e., $R \cong R^G \oplus M$ for an R^G-module M. But then

$$H^3_{\mathfrak{n}}(R) \cong H^3_{\mathfrak{n}}(R^G) \oplus H^3_{\mathfrak{n}}(M),$$

so the local cohomology module $H^3_{\mathfrak{a}}(R) = H^3_{\mathfrak{n}}(R)$ is nonzero as well.

Consequently the cohomological dimension $\operatorname{cd}(R, \mathfrak{a})$ depends on the characteristic of the ground field:

$$\operatorname{cd}(R, \mathfrak{a}) = \begin{cases} 3 & \text{if } \mathbb{K} \text{ has characteristic } 0, \\ 2 & \text{if } \mathbb{K} \text{ has characteristic } p > 0. \end{cases}$$

In Example 22.5 we will construct a local cohomology module such that there are infinitely many choices of the prime characteristic for which this module is zero, and infinitely many for which it is nonzero.

Remark 21.27. In general, no algorithm exists to determine whether $H^i_{\mathfrak{a}}(M)$ is zero, even when M is a finitely generated module over a polynomial ring R. However the situation is much better in the case $M = R$. If R is a regular ring of prime characteristic $p > 0$ and $\mathfrak{a}$ is an ideal of R, Lyubeznik [106, Remark 2.4] gave the following algorithm to determine if a local cohomology module $H^i_{\mathfrak{a}}(R)$ is zero:

Recall from Remark 21.20 that

$$H^i_{\mathfrak{a}}(R) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R),$$

where the maps in the direct limit system are induced by the natural surjections

$$R/\mathfrak{a}^{[p^{e+1}]} \longrightarrow R/\mathfrak{a}^{[p^e]}.$$

Compositions of these maps gives us

$$\beta_e : \operatorname{Ext}^i_R(R/\mathfrak{a}, R) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R).$$

Since R is Noetherian, the sequence of ideals

$$\ker \beta_1 \subseteq \ker \beta_2 \subseteq \ker \beta_3 \subseteq \dots$$

stabilizes, and let r be the least integer such that $\ker \beta_r = \ker \beta_{r+1}$. Then $H^i_{\mathfrak{a}}(R)$ is zero if and only if

$$\ker \beta_r = \operatorname{Ext}^i_R(R/\mathfrak{a}, R).$$

For a polynomial ring R over a field of characteristic 0, Walther gave an algorithm to determine if $H^i_{\mathfrak{a}}(R)$ is zero, and more generally, to compute a presentation for $H^i_{\mathfrak{a}}(R)$ as a module over a Weyl algebra, see [154].

Lecture 22. Some curious examples (AS)

In Example 21.26 we saw that local cohomology may behave quite differently in characteristic 0 and in characteristic p. In Example 22.5 we construct a local cohomology module which is zero for infinitely many choices of the prime characteristic p, and also nonzero for infinitely many p. We will use the following theorem of Lyubeznik, [109, Theorem 1.1]:

Theorem 22.1 (Lyubeznik)**.** *Let $(R,\mathfrak{m})$ be a regular local ring of dimension d containing a field of positive characteristic, and let $\mathfrak{a}$ be an ideal of R. Then $H^i_{\mathfrak{a}}(R)=0$ if and only if there exists an integer $e \geqslant 1$ such that the e^{th} Frobenius iteration*

$$f^e : H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}) \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$$

is the zero map.

Sketch of proof. Recall that

$$H^i_{\mathfrak{a}}(R) = \varinjlim_e \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) \cong \varinjlim_e F^e\left(\operatorname{Ext}^i_R(R/\mathfrak{a}, R)\right),$$

so $H^i_{\mathfrak{a}}(R)=0$ if and only if there exists an integer e such that

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, R) \longrightarrow F^e\left(\operatorname{Ext}^i_R(R/\mathfrak{a}, R)\right)$$

is the zero map. Taking Matlis duals, this is equivalent to the following map being zero:

$$F^e\left(H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})\right) \cong H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}) \otimes_R {}^{f^e}R \longrightarrow H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a}).$$

This map sends $\eta \otimes r$ to $rf^e(\eta)$, so it is zero precisely if $f^e(\eta)=0$ for all elements $\eta \in H^{d-i}_{\mathfrak{m}}(R/\mathfrak{a})$. $\square$

Exercise 22.2. Let $R=\mathbb{K}[w,x,y,z]$ where $\mathbb{K}$ is a field of prime characteristic, and let

$$\mathfrak{a} = (x^3-w^2y, x^2z-wy^2, xy-wz, y^3-xz^2).$$

Note that $R/\mathfrak{a} \cong \mathbb{K}[s^4, s^3t, st^3, t^4]$ as in Example 10.18. Use Theorem 22.1 to prove that $H^3_{\mathfrak{a}}(R)=0$.

We next recall some facts about Segre embeddings of products of projective varieties.

Definition 22.3. Let A and B be $\mathbb{N}$-graded rings over a field $A_0 = B_0 = \mathbb{K}$. The *Segre product* of A and B is the ring

$$A\#B = \bigoplus_{n \geqslant 0} A_n \otimes_{\mathbb{K}} B_n$$

which is a subring, in fact a direct summand, of the tensor product $A \otimes_{\mathbb{K}} B$. The ring $A\#B$ has a natural $\mathbb{N}$-grading in which $[A\#B]_n = A_n \otimes_{\mathbb{K}} B_n$. If $U \subseteq \mathbb{P}^r$ and $V \subseteq \mathbb{P}^s$ are projective varieties with homogeneous coordinate rings A and B respectively, then their Segre product $A\#B$ is a homogeneous coordinate ring for the Segre embedding $U \times V \subseteq \mathbb{P}^{rs+r+s}$.

If M and N are $\mathbb{Z}$-graded modules over A and B respectively, their Segre product is the $A\#B$-module

$$M\#N = \bigoplus_{n \in \mathbb{Z}} M_n \otimes_{\mathbb{K}} N_n \quad \text{with} \quad [M\#N]_n = M_n \otimes_{\mathbb{K}} N_n.$$

Remark 22.4. Let A and B be normal $\mathbb{N}$-graded rings over a field $\mathbb{K}$. If the ring $A \otimes_{\mathbb{K}} B$ is normal, which is always the case when $\mathbb{K}$ is algebraically closed, then so is its direct summand $A\#B$. For reflexive $\mathbb{Z}$-graded modules M and N over A and B respectively, we have the *Künneth formula* for local cohomology due to Goto-Watanabe, [48, Theorem 4.1.5]:

$$\begin{aligned} H^k_{m_{A\#B}}(M\#N) \cong \big(M\#H^k_{m_B}(N)\big) &\oplus \big(H^k_{m_A}(M)\#N\big) \\ &\oplus \bigoplus_{i+j=k+1} \big(H^i_{m_A}(M)\#H^j_{m_B}(N)\big) \quad \text{for all } k \geqslant 0. \end{aligned}$$

If $\dim A = r \geqslant 1$ and $\dim B = s \geqslant 1$, the above formula shows that $\dim R = r+s-1$.

Example 22.5. This is due to Hartshorne-Speiser, [64, Example 3, page 75], though we present a different argument based on Theorem 22.1. Let $\mathbb{K}$ be a field of prime characteristic $p \neq 3$ and, as in Example 10.27, let

$$T = \mathbb{K}[x_0, x_1, x_2, y_0, y_1]/(x_0^3 + x_1^3 + x_2^3).$$

Let S be the subring of T which is generated, as a $\mathbb{K}$-algebra, by the six monomials x_iy_j. The ring S is the Segre product of the hypersurface

$$A = \mathbb{K}[x_0, x_1, x_2]/(x_0^3 + x_1^3 + x_2^3)$$

and the polynomial ring $B = \mathbb{K}[y_0, y_1]$. Note that $\operatorname{Proj} A = E$ is an elliptic curve and $\operatorname{Proj} B = \mathbb{P}^1$, so $\operatorname{Proj} S = E \times \mathbb{P}^1$.

Let $R = \mathbb{K}[z_{ij} : 0 \leqslant i \leqslant 2, 0 \leqslant j \leqslant 1]$ be a polynomial ring. Then R has a $\mathbb{K}$-algebra surjection onto S where

$$z_{ij} \longmapsto x_iy_j.$$

Let $\mathfrak{a}$ be the kernel of this surjection, i.e., $R/\mathfrak{a} \cong S$. It is not hard to see that $\mathfrak{a}$ is generated by the seven polynomials

$$\begin{aligned} &z_{10}z_{21} - z_{20}z_{11}, \quad z_{20}z_{01} - z_{00}z_{21}, \quad z_{00}z_{11} - z_{10}z_{01}, \\ &\qquad\qquad z_{10}^{3-k}z_{11}^k + z_{20}^{3-k}z_{21}^k + z_{30}^{3-k}z_{31}^k \quad \text{for } 0 \leqslant k \leqslant 3. \end{aligned}$$

We shall use the Künneth formula to compute $H^2_{\mathfrak{m}_S}(S)$. Note that the Čech complex

$$0 \longrightarrow A \longrightarrow A_{x_0} \oplus A_{x_1} \longrightarrow A_{x_0x_1} \longrightarrow 0$$

may be used to compute $H^2_{\mathfrak{m}_A}(A)$, and shows that $[H^2_{\mathfrak{m}_A}(A)]_0$ is the 1-dimensional $\mathbb{K}$-vector space spanned by

$$\left[\frac{x_2^2}{x_0x_1}\right] \in \frac{A_{x_0x_1}}{A_{x_0} + A_{x_1}}.$$

The Künneth formula now shows that the only nonzero graded component of $H^2_{\mathfrak{m}_S}(S)$ is

$$[H^2_{\mathfrak{m}_S}(S)]_0 \cong [H^2_{\mathfrak{m}_A}(A)]_0\#[B]_0,$$

which is the vector space spanned by $[x_2^2/x_0x_1] \otimes 1$. In particular, S is a normal domain of dimension 3 which is not Cohen-Macaulay. Since $H^2_{\mathfrak{m}_S}(S)$ is a 1-dimensional vector space, an iteration f^e of the Frobenius map

$$f : H^2_{\mathfrak{m}_S}(S) \longrightarrow H^2_{\mathfrak{m}_S}(S)$$

is nonzero if and only if f is nonzero, and this is equivalent to the condition that

$$f\left(\left[\frac{x_2^2}{x_0x_1}\right]\right) = \left[\frac{x_2^{2p}}{x_0^p x_1^p}\right] \neq 0 \quad \text{ in } \frac{A_{x_0x_1}}{A_{x_0}+A_{x_1}}.$$

If $x_2^{2p}/x_0^p x_1^p \in A_{x_0}+A_{x_1}$, then there exist $a, b \in A$ and $N \gg 0$ such that

$$\frac{x_2^{2p}}{x_0^p x_1^p} = \frac{a}{x_0^N} + \frac{b}{x_1^N},$$

so $x_2^{2p}(x_0x_1)^{N-p} \in (x_0^N, x_1^N)A$. Since A is Cohen-Macaulay, this is equivalent to

$$x_2^{2p} \in (x_0^p, x_1^p)A.$$

We determine the primes p for which x_2^{2p} is an element of the ideal $(x_0^p, x_1^p)A$. If $p = 3k+2$, then

$$x_2^{2p} = x_2^{6k+4} = -x_2(x_0^3+x_1^3)^{2k+1} \in (x_0^{3k+3}, x_1^{3k+3})A \subseteq (x_0^p, x_1^p)A.$$

On the other hand, if $p = 3k+1$, then the binomial expansion of

$$x_2^{2p} = x_2^{6k+2} = x_2^2(x_0^3+x_1^3)^{2k},$$

when considered modulo (x_0^p, x_1^p), has a nonzero term

$$\binom{2k}{k} x_2^2 x_0^{3k} x_1^{3k} = \binom{2k}{k} x_2^2 x_0^{p-1} x_1^{p-1},$$

which shows that $x_2^{2p} \notin (x_0^p, x_1^p)A$. We conclude that

$$f : H^2_{\mathfrak{m}_S}(S) \longrightarrow H^2_{\mathfrak{m}_S}(S)$$

is the zero map if $p \equiv 2 \mod 3$, and is nonzero if $p \equiv 1 \mod 3$. Using Theorem 22.1, it follows that

$$H^4_{\mathfrak{a}}(R) \neq 0 \quad \text{if } p \equiv 1 \mod 3, \qquad\qquad H^4_{\mathfrak{a}}(R) = 0 \quad \text{if } p \equiv 2 \mod 3.$$

Exercise 22.6. Let $\mathbb{K}$ be a field, and consider homogeneous polynomials $g \in \mathbb{K}[x_0, \dots, x_m]$ and $h \in \mathbb{K}[y_0, \dots, y_n]$ where $m, n \geqslant 1$. Let

$$A = \mathbb{K}[x_0, \dots, x_m]/(g) \quad \text{ and } \quad B = \mathbb{K}[y_0, \dots, y_n]/(h).$$

Prove that the ring $A\#B$ is Cohen-Macaulay if and only if $\deg g \leqslant m$ and $\deg h \leqslant n$.

Remark 22.7. Let E be a smooth elliptic curve over a field $\mathbb{K}$ of characteristic $p > 0$. There is a Frobenius action

$$f : H^1(E, \mathcal{O}_E) \longrightarrow H^1(E, \mathcal{O}_E)$$

on the 1-dimensional cohomology group $H^1(E, \mathcal{O}_E)$. The elliptic curve E is *supersingular* (or has *Hasse invariant* 0) if f is zero, and is *ordinary* (*Hasse invariant* 1) otherwise. If $E = \operatorname{Proj} A$, then the map f above is precisely the action of the Frobenius on

$$H^1(E, \mathcal{O}_E) = [H^2_{\mathfrak{m}}(A)]_0$$

For example, the cubic polynomial $x_0^3 + x_1^3 + x_2^3$ defines a smooth elliptic curve E in any characteristic $p \neq 3$. Our computation in Example 22.5 says precisely that E is supersingular for primes $p \equiv 2 \mod 3$, and is ordinary if $p \equiv 1 \mod 3$.

Let $g \in \mathbb{Z}[x_0, x_1, x_2]$ be a cubic polynomial defining a smooth elliptic curve $E_{\mathbb{Q}} \subset \mathbb{P}^2_{\mathbb{Q}}$. Then the Jacobian ideal of g in $\mathbb{Q}[x_0, x_1, x_2]$ is primary to the maximal ideal (x_0, x_1, x_2). Hence, after localizing at an appropriate nonzero integer u, the

Jacobian ideal of g in $\mathbb{Z}[u^{-1}][x_0, x_1, x_2]$ contains high powers of x_0, x_1, and x_2. Consequently, for all but finitely many prime integers p, the polynomial $g \mod p$ defines a smooth elliptic curve $E_p \subset \mathbb{P}^2_{\mathbb{Z}/p}$. If the elliptic curve $E_{\mathbb{C}} \subset \mathbb{P}^2_{\mathbb{C}}$ has complex multiplication, then it is a classical result [28] that the *density* of the supersingular prime integers p, i.e.,

$$\lim_{n \longrightarrow \infty} \frac{|\{p \text{ prime} : p \leqslant n \text{ and } E_p \text{ is supersingular}\}|}{|\{p \text{ prime} : p \leqslant n\}|}$$

is $1/2$, and that this density is 0 if $E_{\mathbb{C}}$ does not have complex multiplication. However, even if $E_{\mathbb{C}}$ does not have complex multiplication, the set of supersingular primes is infinite by [34]. It is conjectured that if $E_{\mathbb{C}}$ does not have complex multiplication, then the number of supersingular primes less than n grows asymptotically like $C(\sqrt{n}/\log n)$, where C is a positive constant, [95].

Let $\mathfrak{a} \subset R = \mathbb{Z}[z_{ij} : 0 \leqslant i \leqslant 2,\ 0 \leqslant j \leqslant n]$ be the ideal defining the Segre embedding $E \times \mathbb{P}^n \subset \mathbb{P}^{3n+2}$. Imitating the methods in Example 22.5, we shall see that

$$\operatorname{cd}(R/pR, \mathfrak{a}) = \begin{cases} 2n+1 & \text{if } E_p \text{ is supersingular,} \\ 3n+1 & \text{if } E_p \text{ is ordinary.} \end{cases}$$

The ring $R/(\mathfrak{a} + pR)$ may be identified with the Segre product $A \# B$ where

$$A = \mathbb{Z}/p\mathbb{Z}[x_0, x_1, x_2]/(g) \quad \text{and} \quad B = \mathbb{Z}/p\mathbb{Z}[y_0, \ldots, y_n].$$

Let p be a prime for which E_p is smooth, in which case the ring $A \otimes_{\mathbb{Z}/p\mathbb{Z}} B$, and hence its direct summand $A \# B$, are normal. The Künneth formula shows that

$$H^i_{\mathfrak{m}}(R/(\mathfrak{a} + pR)) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i = 2, \\ 0 & \text{if } 3 \leqslant i \leqslant n+1. \end{cases}$$

The Frobenius action on the one-dimensional vector space $H^2_{\mathfrak{m}}(R/(\mathfrak{a}+pR))$ may be identified with the Frobenius

$$H^1(E_p, \mathcal{O}_{E_p}) \xrightarrow{f} H^1(E_p, \mathcal{O}_{E_p}),$$

which is the zero map precisely when E_p is supersingular. Consequently every element of $H^2_{\mathfrak{m}}(R/(\mathfrak{a} + pR))$ is killed by Frobenius (equivalently, by a Frobenius iteration) if and only if E_p is supersingular. The assertion now follows from Theorem 22.1.

Exercise 22.8. Let $R = \mathbb{Z}/p\mathbb{Z}[x, y, z]/(x^3 + xy^2 + z^3)$ and

$$\eta = \left[\frac{z^2}{xy}\right] \in \frac{R_{xy}}{R_x + R_y} = H^1_{\mathfrak{m}}(R).$$

For which primes p is $f(\eta) = 0$, i.e., for which p is the elliptic curve $\operatorname{Proj} R$ supersingular? (Hint: Consider $p \mod 6$.)

Associated primes of local cohomology modules

As we have seen, local cohomology modules $H^i_{\mathfrak{a}}(R)$ are often not finitely generated as R-modules. However they do possess useful finiteness properties in certain cases, e.g., for a local ring $(R, \mathfrak{m})$, the modules $H^i_{\mathfrak{m}}(R)$ satisfy the descending chain condition. This implies, in particular, that for all $i \geqslant 0$,

$$\operatorname{Hom}_R\left(R/\mathfrak{m}, H^i_{\mathfrak{m}}(R)\right) \cong 0 :_{H^i_{\mathfrak{m}}(R)} \mathfrak{m}$$

is a finitely generated R-module. Grothendieck conjectured that for all ideals $\mathfrak{a} \subset R$, the modules

$$\operatorname{Hom}_R\left(R/\mathfrak{a}, H^i_\mathfrak{a}(R)\right) \cong 0 :_{H^i_\mathfrak{a}(R)} \mathfrak{a}$$

are finitely generated, [55, Exposé XIII, page 173]. In [61, § 3] Hartshorne gave a counterexample to this conjecture, as we saw in Example 20.49. A related question on the torsion in local cohomology modules was raised by Huneke [79] at the Sundance Conference in 1990, and will be our focus for the rest of this lecture.

Question 22.9 (Huneke)**.** Is the number of associated prime ideals of a local cohomology module $H^i_\mathfrak{a}(R)$ always finite?

This issue were discussed briefly in Lecture 9. The first general results were obtained by Huneke and Sharp, [81, Corollary 2.3]:

Theorem 22.10 (Huneke-Sharp)**.** *Let R be a regular ring containing a field of positive characteristic, and $\mathfrak{a} \subset R$ an ideal. Then for all $i \geqslant 0$,*

$$\operatorname{Ass} H^i_\mathfrak{a}(R) \subseteq \operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R).$$

In particular, $\operatorname{Ass} H^i_\mathfrak{a}(R)$ *is a finite set.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass} H^i_\mathfrak{a}(R)$. Localizing at $\mathfrak{p}$, we assume that R is local with maximal ideal $\mathfrak{p}$. The assumption $\mathfrak{p} \in \operatorname{Ass} H^i_\mathfrak{a}(R)$ implies that the socle of $H^i_\mathfrak{a}(R)$ is nonzero. By Remark 21.20

$$H^i_\mathfrak{a}(R) = \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}),$$

so $\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]})$ must have a nonzero socle for some integer e. But then $\mathfrak{p}$ is an associated prime of $\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R)$. Since R is regular,

$$F^e\left(\operatorname{Ext}^i_R(R/\mathfrak{a}, R)\right) \cong \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R).$$

But

$$\operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^e]}, R) = \operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$$

by Proposition 21.23, completing the proof. □

Remark 22.11. The proof of the Huneke-Sharp theorem relies heavily on the flatness of the Frobenius endomorphism, which characterizes regular rings of positive characteristic, Theorem 21.2. The containment

$$\operatorname{Ass} H^i_\mathfrak{a}(R) \subseteq \operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$$

may fail for regular rings of characteristic zero: consider $\mathfrak{a} \subset R$ as in Example 21.26, where R is a polynomial ring over a field of characteristic zero. Then $\operatorname{Ext}^3_R(R/\mathfrak{a}, R) = 0$ since $\operatorname{pd}_R R/\mathfrak{a} = 2$. However, as we saw, $H^3_\mathfrak{a}(R)$ is nonzero.

Though $\operatorname{Ass} H^i_\mathfrak{a}(R)$ may not be a subset of $\operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$, Question 22.9 does have an affirmative answer for all unramified regular local rings by combining the result of Huneke-Sharp with the following two theorems of Lyubeznik, [105, Corollary 3.6 (c)] and [108, Theorem 1];

Theorem 22.12 (Lyubeznik)**.** *Let R be a regular ring containing a field of characteristic zero and $\mathfrak{a}$ be an ideal of R. Then for every maximal ideal $\mathfrak{m}$ of R, the set of associated primes of a local cohomology module $H^i_\mathfrak{a}(R)$ contained in $\mathfrak{m}$ is finite.*

If the regular ring R is finitely generated over a field of characteristic zero, then $\operatorname{Ass} H^i_\mathfrak{a}(R)$ *is a finite set.*

To illustrate the key point here, consider the case where $R = \mathbb{C}[x_1, \ldots, x_n]$, and let D be the ring of $\mathbb{C}$-linear differential operators on R. It turns out that D is left and right Noetherian, that $H^i_{\mathfrak{a}}(R)$ is a finitely generated D-module, and consequently that $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is finite. Lyubeznik's result below also uses D-modules, though the situation in mixed characteristic is more subtle.

Theorem 22.13 (Lyubeznik)**.** *If $\mathfrak{a}$ is an ideal of an unramified regular local ring of mixed characteristic, then $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is a finite set.*

In general Lyubeznik conjectured that if $\mathfrak{a}$ is an ideal of a regular ring R, then $H^i_{\mathfrak{a}}(R)$ has only finitely many associated primes, [105, Remark 3.7 (iii)]. This remains open for ramified regular local rings of mixed characteristic, and also for regular rings such as $\mathbb{Z}[x_1, \ldots, x_d]$; see [145] for some observations regarding this.

If M is a finitely generated R-module, then $H^0_{\mathfrak{a}}(M)$ may be identified with the submodule of M consisting of elements which are killed by a power of the ideal $\mathfrak{a}$, and consequently $H^0_{\mathfrak{a}}(M)$ is a finitely generated R-module. If i is the smallest integer for which $H^i_{\mathfrak{a}}(M)$ is not finitely generated, then the set $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ is also finite, see [13, 90] and Remark 9.2. Other positive answers to Question 22.9 include the following result of Marley, [112, Corollary 2.7]:

Theorem 22.14 (Marley)**.** *Let R be a local ring and M be a finitely generated R-module of dimension at most three. Then $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ is finite for all ideals $\mathfrak{a} \subset R$.*

In general $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ need not be a finite set, as we see from the following example, [144, § 4]:

Example 22.15. Consider the hypersurface

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$$

and the ideal $\mathfrak{a} = (x, y, z)R$. We show that for every prime integer p, the local cohomology module $H^3_{\mathfrak{a}}(R)$ has a p-torsion element; consequently $H^3_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals.

Using the Čech complex on x, y, z to compute $H^i_{\mathfrak{a}}(R)$, we have

$$H^3_{\mathfrak{a}}(R) = \frac{R_{xyz}}{R_{yz} + R_{zx} + R_{xy}}.$$

For a prime integer p, the fraction

$$\lambda_p = \frac{(ux)^p + (vy)^p + (wz)^p}{p}$$

has integer coefficients, and is therefore an element of R. We claim that the element

$$\eta_p = \left[\frac{\lambda_p}{(xyz)^p}\right] \in H^3_{\mathfrak{a}}(R)$$

is nonzero and p-torsion. Note that

$$p \cdot \eta_p = \left[\frac{p\lambda_p}{(xyz)^p}\right] = \left[\frac{u^p}{(yz)^p} + \frac{v^p}{(zx)^p} + \frac{w^p}{(xy)^p}\right] = 0,$$

so all that remains to be checked is that η_p is nonzero. If $\eta_p = 0$, then there exist $c_i \in R$ and an integer $N \gg 0$ such that

$$\frac{\lambda_p}{(xyz)^p} = \frac{c_1}{(yz)^N} + \frac{c_2}{(zx)^N} + \frac{c_3}{(xy)^N}$$

Clearing denominators, this gives the equation

$$\lambda_p(xyz)^{N-p} = c_1x^N + c_2y^N + c_3z^N.$$

We assign weights to the $\mathbb{Z}$-algebra generators of the ring R as follows:

$$\begin{aligned} x &: (1,0,0,0), & u &: (-1,0,0,1), \\ y &: (0,1,0,0), & v &: (0,-1,0,1), \\ z &: (0,0,1,0), & w &: (0,0,-1,1). \end{aligned}$$

With this grading, λ_p is a homogeneous element of degree $(0,0,0,p)$, and there is no loss of generality in assuming that the c_i are homogeneous. Comparing degrees, we see that $\deg(c_1) = (-p, N-p, N-p, p)$, i.e., c_1 must be an integer multiple of the monomial $u^py^{N-p}z^{N-p}$. Similarly c_2 is an integer multiple of $v^pz^{N-p}x^{N-p}$ and c_3 of $w^px^{N-p}y^{N-p}$. Consequently

$$\lambda_p(xyz)^{N-p} \in (xyz)^{N-p}\,(u^px^p, v^py^p, w^pz^p)\,R,$$

and so $\lambda_p \in (u^px^p, v^py^p, w^pz^p)R$. After specializing $u \mapsto 1, v \mapsto 1, w \mapsto 1$, this implies that

$$\frac{x^p + y^p + (-1)^p(x+y)^p}{p} \in (p, x^p, y^p)\,\mathbb{Z}[x,y],$$

which is easily seen to be false.

This example, however, does not shed light on Question 22.9 in the case of local rings or rings containing a field. Katzman [89] constructed the first examples to demonstrate that Huneke's question has a negative answer in these cases as well:

Example 22.16 (Katzman)**.** Let $\mathbb{K}$ be an arbitrary field, and consider the hypersurface

$$R = \mathbb{K}[s,t,u,v,x,y]/\big(sv^2x^2 - (s+t)vxuy + tu^2y^2\big).$$

Katzman showed that the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals. To obtain a local example, one may localize at the homogeneous maximal ideal $\mathfrak{m} = (s,t,u,v,x,y)$.

For $n \in \mathbb{N}$, let

$$\tau_n = s^n + s^{n-1}t + \cdots + t^n$$

and

$$z_n = \left[\frac{sxy^n}{uv^n}\right] \in \frac{R_{uv}}{R_u + R_v} = H^2_{(u,v)}(R).$$

Exercises 22.17–22.19 show that $\operatorname{ann}(z_n) = (u,v,x,y,\tau_n)R$. Consequently for every $n \in \mathbb{N}$, the module $H^2_{(u,v)}(R)$ has an associated prime $\mathfrak{p}_n$ of height 4 with

$$(u,v,x,y,\tau_n) \subseteq \mathfrak{p}_n \subsetneq \mathfrak{m}.$$

Exercise 22.20 shows that the set $\{\mathfrak{p}_n\}_{n\in\mathbb{N}}$ is infinite.

In the following four exercises, we use the notation of Example 22.16.

Exercise 22.17. Show that $(u,v,x,y) \subseteq \operatorname{ann}(z_n)$.

Exercise 22.18. Let A be the subring of the ring R which is generated as a $\mathbb{K}$-algebra by $s, t, vx = a, uy = b$. If $f \in \mathbb{K}[s,t]$, show that $fz_n = 0$ if and only if $fab^n \in (a^{n+1}, b^{n+1})A$. You may find the following multigrading useful:

$$\begin{aligned} \deg s &= (0,0,0), & \deg u &= (1,0,1), & \deg x &= (1,0,0), \\ \deg t &= (0,0,0), & \deg v &= (0,1,1), & \deg y &= (0,1,0). \end{aligned}$$

Exercise 22.19. Let $A = \mathbb{K}[s,t,a,b]/(sa^2 - (s+t)ab + tb^2)$. Show that

$$(a^{n+1}, b^{n+1})A :_{\mathbb{K}[s,t]} ab^n$$

is the ideal of $\mathbb{K}[s,t]$ generated by τ_n. This exercise completes the proof that

$$\operatorname{ann}(z_n) = (u, v, x, y, \tau_n)R.$$

Exercise 22.20. Let $\mathbb{K}[s,t]$ be a polynomial ring over a field $\mathbb{K}$ and

$$\tau_n = s^n + s^{n-1}t + \cdots + t^n \qquad \text{for } n \in \mathbb{N}.$$

If m and n are relatively prime integers, show that

$$\operatorname{rad}(\tau_{m-1}, \tau_{n-1}) = (s,t).$$

Remark 22.21. Since the defining equation of the hypersurface factors as

$$sv^2x^2 - (s+t)vxuy + tu^2y^2 = (svx - tuy)(vx - uy),$$

the ring in Example 22.16 is not an integral domain. In [146] Singh and Swanson generalized Katzman's construction to obtain families of examples which include examples over normal domains and, in fact, over hypersurfaces with mild singularities (e.g., rational singularities). We next record one of the examples from [146].

Example 22.22. Let $\mathbb{K}$ be an arbitrary field, and consider the hypersurface

$$T = \frac{\mathbb{K}[r,s,t,u,v,w,x,y,z]}{\left(su^2x^2 + sv^2y^2 + tuxvy + rw^2z^2\right)}$$

Then T is a unique factorization domain for which the local cohomology module $H^3_{(x,y,z)}(T)$ has infinitely many associated prime ideals. This is preserved if we replace T by the localization at its homogeneous maximal ideal. The hypersurface T has rational singularities if $\mathbb{K}$ has characteristic zero, and is F-regular in the case of positive characteristic.

Lecture 23. Computing localizations and local cohomology using D-modules (AL)

This lecture deals with computational aspects of the theory of D-modules related to local cohomology. (Here we assume $D = A_n(\mathbb{K})$ and all D-modules are left.) One basic observation is that the computation of local cohomology modules is obstructed by the fact that, usually, they are not finitely generated as modules over the ring of polynomials. Departing from classical commutative algebra and introducing "slightly" non-commutative rings, the theory of D-modules makes it possible to describe in a finite way many of the objects that require infinite data for their representation in the commutative world.

The ultimate goal of this lecture is to describe a method for computing local cohomology. This method relies on the algorithm that recovers the localized module R_f of the polynomial ring $R = \mathbb{K}[x_1, ..., x_n]$, where $f \in R \setminus \{0\}$. This algorithm, in turn, depends on having certain information about the roots of the Bernstein-Sato polynomial (a.k.a. b-polynomial) of f. As applications of b-polynomials are not limited to computing local cohomology, they shall make one of the central topics of our discussion.

23.1. **Bernstein-Sato polynomials.**

Theorem 23.1. *For every polynomial $f \in R_n(\mathbb{K})$ there are $b(s) \in \mathbb{K}[s]$ and $Q(x, \partial, s) \in A_n(\mathbb{K})[s]$ such that*

$$b(s)f^s = Q(x, \partial, s) \cdot f^{s+1}. \tag{23.1.1}$$

Proof. Let M be the free $\mathbb{K}(s)[x, f^{-1}]$-module generated by the symbol f^s. It may be viewed as a $A_n(\mathbb{K}(s))$-module, derivation being defined naturally:

$$\partial_i \cdot gf^s = (\partial_i g + sg(\partial_i f)f^{-1})f^s, \ g \in \mathbb{K}(s)[x, f^{-1}].$$

It takes an argument similar to that of Theorem 17.15 to prove that M is a holonomic $A_n(\mathbb{K}(s))$-module.

Consider the sequence of $A_n(\mathbb{K}(s))$-modules $M \supset fM \supset f^2M \supset ....$ Since M is holonomic this sequence stabilizes: assume $f^{d+1}M = f^dM$. Then there is an element $T(s, x, \partial) \in A_n(\mathbb{K}(s))$ such that $f^{s+d} = T(s, x, \partial) \cdot f^{s+d+1}$. Substituting symbol s with $s - d$ we get

$$f^s = T(s - d, x, \partial) \cdot f^{s+1}.$$

Finally, we can clear the denominators, i.e. find $b(s) \in \mathbb{K}[s]$ such that $Q(s, x, \partial) = b(s)T(s - d, x, \partial) \in A_n(\mathbb{K})[s]$ as in the functional equation (23.1.1). $\square$

The polynomials $b(s)$ for which the equation (23.1.1) exists form an ideal in $\mathbb{K}[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the *Bernstein-Sato polynomial* or simply the *b-polynomial* of f.

There is an algorithm for computing the b-polynomial in the general case due to Oaku [122]. However, the complexity of the algorithm is high, since it employs Gröbner bases techniques in the Weyl algebras.

For every nonzero f, the factor $(s+1)$ is always present in its b-polynomial $b_f(s)$. Indeed, the functional equation (23.1.1) should hold when $s = 1$, therefore, forcing $b_f(-1) = 0$. Moreover, it turns out that the b-polynomial can be factored over $\mathbb{Q}$.

Theorem 23.2 (Kashiwara)**.** *For every nonzero $f \in R_n(\mathbb{K})$, the roots of $b_f(s)$ are negative rational numbers.*

This is a less trivial result, whose proof can be looked up in either [88] or [8]. In particular, that proof involves Hironaka's desingularization theorem.

23.2. **Examples of b-polynomials.** Before we prove the existence of the (nonzero) b-polynomial for any f, let us consider several examples that are easy to compute by hand.

Example 23.3. Consider $f = x \in R = R_1 = \mathbb{K}[x]$, then

$$(s+1)x^s = \partial_x \cdot x^{s+1}.$$

Therefore, $b_f(s) = s+1$.

As a matter of fact the b-polynomial obtained in this example is rather common.

Theorem 23.4. *The polynomial $f \in R = R_n$ is regular if and only if $b_f(s) = s+1$.*

Proof. Let f be a regular polynomial, then $R = \langle f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \rangle$. Therefore, there exist $c_i \in R$, $i = 0, ..., n$, such that

$$1 = c_0 f + c_1 \frac{\partial f}{\partial x_1} + ... + c_n \frac{\partial f}{\partial x_n}.$$

One may check that the following functional equation holds:

$$(s+1)f^s = (c_0(s+1) + c_1\partial_1 + ... + c_n\partial_n) \cdot f^{s+1}.$$

The proof in the other direction is less trivial and may be looked up in [9]. □

Example 23.5. Let $f = x_1^2 + ... + x_n^2 \in R = R_n$. Consider the first two derivatives of f^{s+1} with respect to x_i:

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= 2(s+1)x_i f^s, \\ \frac{\partial^2 f}{\partial x_i^2} &= 4(s+1)s x_i^2 f^{s-1} + 2(s+1)f^s. \end{aligned}$$

If $\Delta = \partial_1^2 + ... + \partial_n^2$, then

$$\Delta \cdot f^{s+1} = 4(s+1)sf^s + 2n(s+1)f^s = 4(s+1)(s+\frac{n}{2})f^s.$$

Since f has a singularity at the origin, by Theorem 23.4, $b_f(s) \neq s+1$. However, every b-polynomial is divisible by $s+1$, hence, $b_f(s)$ is equal to $(s+1)(s+n/2)$ that was discovered above for it is the monic polynomial of minimal degree satisfying the functional equation for f.

There are few formulas known for b-polynomials in special cases. For example, a thorough treatment for the case of hyperplane arrangements is given in [157]. A lot is known in the case of isolated singularities; below, we will consider polynomials defining a hypersurface with a single quasi-homogeneous isolated singularity.

Recall that $x_0 \in f^{-1}(0)$ is called an *isolated singularity* if there exists an open neighborhood $U \ni x_0$ such that $\nabla f(x_0) = 0$, but $\nabla f(x) \neq 0$ for all $x \in U \setminus \{x_0\}$. The point x_0 is called a *quasi-homogeneous isolated singularity* , if in addition there exists a vector field $v = \sum v_i(x)\partial f/\partial x_i$ such that $vf = f$.

Theorem 23.6. *For any quasi-homogeneous isolated singularity x_0 there is a local coordinate system with the origin at x_0 such that*

$$\sum a_i x_i \frac{\partial f}{\partial x_i} f = f,$$

where a_i are positive rational numbers.

For proof see [135].

If there is only one singularity on the hypersurface $f = 0$, it is possible to make a global analytic coordinate change such that the conclusion of the Theorem 23.6 holds. As a corollary, then, it follows that, in new coordinates, f is a linear combination of monomials x^α satisfying $\sum a_i \alpha_i = 1$.

It is possible to calculate the b-polynomial for such a linear combination:

Theorem 23.7. *Let $f \in R = R_n$ be a polynomial with a single quasi-homogeneous isolated singularity at the origin and $\sum a_i x_i(\partial f/\partial x_i) = f$. Let λ_j be the eigenvalues of $v = \sum a_i x_i \partial$ on the (finite-dimensional) space $R/R{\cdot}\nabla f = R/(\partial f/\partial x_1, ..., \partial f/\partial x_n)$.*

Then the b-polynomial b_f is equal to

$$(s+1)\prod_j (s + \lambda_j + \sum_{i=1} n a_i),$$

where $\widetilde{\prod}$ stands for the square-free product.

Example 23.8. Let us extend Example 23.5 by considering $f = x_1^{m_i} + ... + x_n^{m_i}$, where $m_i \geq 2$. Then f together with the vector field $v = \sum (1/m_i) x_i \partial_i$ fits the hypothesis of the Theorem 23.7. The quotient ring $R/R \cdot \nabla f$ has the set $\{x_i^\alpha \mid 0 \leq \alpha_i \leq m_i - 2\}$ as a basis of eigenvectors. Hence, the eigenvalues of v are $\sum \alpha_i/m_i$ and, since $\sum \alpha_i/m_i + \sum 1/m_i = \sum(\alpha_i + 1)/m_i$,

$$b_f(s) = (s+1)\widetilde{\prod_\nu}(s + \sum_{i=1}^n \frac{\nu_i}{m_i}),$$

where $1 \leq \nu_i \leq m_i - 1$ for all i and $\widetilde{\prod}$ indicates that the square-free product is taken.

For example, let $n = 2$ and $f = x_1^3 + x_2^4$. Then $v = \frac{1}{3}x_1\partial_1 + \frac{1}{4}x_2\partial_2$ and

$$\begin{aligned} b_f(s) &= (s+1)(s+\frac{1}{3}+\frac{1}{4})(s+\frac{1}{3}+\frac{2}{4})(s+\frac{1}{3}+\frac{3}{4}) \\ &\quad (s+\frac{2}{3}+\frac{1}{4})(s+\frac{2}{3}+\frac{2}{4})(s+\frac{2}{3}+\frac{3}{4}) \\ &= (s+1)(s+\frac{7}{12})(s+\frac{5}{6})(s+\frac{11}{12})(s+\frac{13}{12})(s+\frac{7}{6})(s+\frac{17}{12}). \end{aligned}$$

Macaulay 2 confirms the last calculation:

```
i1 : load "D-modules.m2"

i2 : A = QQ[x_1,x_2,Dx_1,Dx_2,
            WeylAlgebra=>{x_1=>Dx_1,x_2=>Dx_2}];

i3 : bf = globalBFunction (x_1^3+x_2^4)

        7       6   499   5    815   4   227563   3     ...
o3 = $s  + 7$s  + ---*$s  + ---*$s  + ------*$s  + ...
                   24         24        6912           ...

o3 : QQ [$s]
```

```
i4 : factorBFunction bf

               5                    17          13           7   ...
o4 = ($s + -)($s + 1)($s + --)($s + --)($s + --)  ...
               6                    12          12          12   ...

o4 : Product
```

Exercise 23.9. (i) Compute $b_f(s)$ for $f = xy, x^3 + y^3, x^3 + xy$.
(ii) Let $f_m = x_1^m + \ldots + x_n^m \in \mathbb{Q}[x_1, \ldots, x_n]$. Prove that

$$\lim_{m \longrightarrow \infty} \inf(\{\alpha | b_{f_m}(\alpha) = 0\}) = -n.$$

23.3. **Lyubeznik's result.** We know that the roots of b_f are restricted to $\mathbb{Q}_{<0}$, can we make any further statements in case there are additional conditions on the polynomial f?

One of the most basic characteristics of a polynomial is its degree; let $B_d = \{b_f | f \in R_n(\mathbb{K}),\ \deg(f) \leq d,\ \operatorname{char}(\mathbb{K}) = 0\}$ be the set of possible b-polynomials for polynomials of the degree at most d over any field $\mathbb{K}$ of characteristic 0.

Theorem 23.10 (Lyubeznik)**.** *The set B_d is finite for all d.*

Proof. (Sketch) The idea of the proof is to consider the dense polynomial $g = \sum a_\alpha x^\alpha$ of degree d with parametric coefficients a_α, $|\alpha| \leq d$. Viewed as an element of $R_n(\mathbb{Q}(a))$, the polynomial g has the b-polynomial $b_g(s)$ associated with it and satisfies a functional equation

$$b_g(s) g^s = Q \cdot g^{s+1}, \tag{23.10.1}$$

where $Q \in A_n(\mathbb{Q}(a_\alpha))[s]$.

Let the polynomial $E \in \mathbb{Q}[a_\alpha]$ equal the common denominator of the coefficients of Q. Now take a polynomial $f = \sum c_\alpha x^\alpha$ with the specialized coefficients $c_\alpha \in \mathbb{K}$. If $E(c_\alpha) \neq 0$ then the b-polynomial $b_f^{\mathbb{K}}$ divides b_g, since the functional equation 23.10.1 can be specialized for these c_α. The number of the monic divisors of a univariate polynomial is finite.

The exceptional cases, i.e. such that $E(c_\alpha) = 0$, are treated inductively. We consider the zero set $Y = V(E) \subset X = \mathbb{A}_{\mathbb{Q}}^N$, where $\mathbb{A}_{\mathbb{Q}}^N$ is the affine $\mathbb{Q}$-space of dimension N equal to the number of the monomials of degree at most d. A modification of our argument (laid out for X above) applied to each irreducible component of Y (notice: $\dim Y < \dim X$) leads to the proof of the theorem by induction on the dimension. $\square$

This result can be generalized by considering a polynomial $F \in R_n(\mathbb{K}[a])$ with parametric coefficients in the indeterminates a: the set B_F of b-polynomials for all possible specializations of the parameters a is finite. Moreover, it has been conjectured by Lyubeznik [107] and proved in [97] that for a fixed $b(s) \in B_F$ the set of all parameters a that produce $b(s)$ as the b-polynomial is a constructible subset of $\operatorname{Spec}(\mathbb{K}[a])$.

23.4. **Localization.** Let M be a holonomic D-module, for computational purposes it is usually assumed that M is cyclic and is presented as the quotient $M = D/I$, where I is a holonomic ideal in D. Let $f \in R = \mathbb{K}[x_1, ..., x_n]$. We would like to compute $R_f \otimes M$ as a holonomic cyclic D-module, i.e. we would like to find an ideal $J \subset D$ such that $R_f \otimes M \cong D/J$.

There are several algorithms known that find such J, we shall mention two. In case when M is f-saturated, i.e. $f \cdot m = 0$ iff $m = 0$ for all $m \in M$, we refer the reader to Walther's paper [154] for a detailed description of the localization algorithm. Here we point out the main steps of it.

• First of all, one wants to find $J^I(f^s)$, the ideal of operators in $D[s]$ annihilating $f^s \otimes \overline{1} \in R_f[s]f^s \otimes M$, where $\overline{1}$ is the cyclic generator of $M = D/I$ and f^s the generator of $R_f[s]f^s$.

• Another component is the b-polynomial $b_f^I(s)$ relative to the the ideal I — a generalization of the usual b-polynomial — defined as the monic generator of the ideal formed by all $b(s) \in \mathbb{K}[s]$ such that

$$b(s)f^s \otimes \overline{1} = Q(x, \partial, s)(f^{s+1} \otimes \overline{1}),$$

holds in the $D[s]$-module $R_f[s]f^s \otimes M$ for some $Q(x, \partial, s) \in D[s]$. This polynomial exists if M is holonomic.

• The final step consists of finding the smallest integer root a of $b_f^I(s)$ and "plugging in" a for s in the generators of $J^I(f^s)$. The obtained operators generate $J \subset D$ that we started our discussion with.

23.5. **Local cohomology via Čech complex.** Let $X = \mathbb{K}^n$ with the coordinate ring $R = \mathbb{K}[x_1, ..., x_n]$ and $Y = V(I)$, where $I = (f_1, ..., f_d)$. To calculate $H_I^k(R)$ consider the following Čech complex:

$$0 \to C^0 \to C^1 \to ... \to C^d \to 0,$$

where

$$C^k = \bigoplus_{1 \le i_1 < ... < i_k \le d} R_{f_{i_1} \cdots f_{i_k}}$$

and the map $C^k \to C^{k+1}$ is the alternating sum of maps

$$R_{f_{i_1} \cdots f_{i_k}} \to R_{f_{j_1} \cdots f_{j_{k+1}}},$$

which are zero if $\{i_1, ..., i_k\} \not\subseteq \{j_1, ..., j_{k+1}\}$ and are natural, i.e. send $1 \mapsto 1$. The signs in the sum are alternated in such a way that the sequence above is indeed a complex.

The local cohomology groups $H_I^k(R)$ are equal to the cohomology groups $H^k(C^\bullet)$ of the constructed Čech complex.

The complex $C^\bullet$ enables us to compute the local cohomology algorithmically viewing C^k as holonomic D-modules and the maps between them as D-linear maps. An explicit algorithm for local cohomology is written out in [154] and depends on the localization algorithm, which is used to calculate the components $R_{f_{i_1} \cdots f_{i_k}}$ of C^k.

There is an alternative approach to computation of local cohomology using the restriction functor in the D-module category developed in [124].

At the end, let us compute $H_I^i(R)$ for $i = 1, 3$ with $R = R_6$ and the ideal I as in Example 19.30.

```
i5 : W = QQ[x_1..x_6, dx_1..dx_6,
          WeylAlgebra => toList (1..6) / (i -> x_i => dx_i)];

i6 : I = minors(2, matrix{{x_1, x_2, x_3}, {x_4, x_5, x_6}})

o6 = ideal (- x x  + x x , - x x  + x x , - x x  + x x )
               2 4    1 5     3 4    1 6     3 5    2 6
```

```
o6 : Ideal of W

i7 : LC = localCohom ({1,3}, I,
          W^1 / ideal {dx_1, dx_2, dx_3, dx_4, dx_5, dx_6},
          Strategy=>Walther, LocStrategy=>Oaku);

i8 : pruneLocalCohom LC

o8 = HashTable{1 => 0

               3 => | x_4dx_4+x_5dx_5+x_6dx_6+6 ...

o8 : HashTable
```

This direct computation reconfirms the fact that $H^3_I(R) \neq 0$, which was established theoretically before.

Lecture 24. Holonomic ranks in families of hypergeometric systems

In previous lectures, we have seen applications of D-modules to the theory of local cohomology. In this lecture, we shall see an application of local cohomology to certain D-modules. Specifically, associated to any affine semigroup ring is a family of holonomic D-modules, and the $\mathbb{Z}^d$-graded local cohomology of the semigroup ring indicates how the ranks of these D-modules behave in the family. Substantial parts of this lecture are based, sometimes nearly verbatim, on [115].

24.1. **GKZ A-hypergeometric systems.**

"One of the main branches of Mathematics which interacts with D-module theory is doubtlessly the study of hypergeometric functions, both in its classical and generalized senses, in its algebraic, geometric and combinatorial aspects."

– *from the webpage* http://caul.cii.fc.ul.pt/DModHyp/ *for the Workshop on D-modules and Hypergeometric Functions in Lisbon, Portugal, 11 to 14 July 2005*

Generally speaking, hypergeometric functions are power series solutions to certain systems of differential equations. Classical univariate hypergeometric functions go back at least to Gauss, and by now there are various multivariate generalizations. One class was introduced in the late 1980s by Gelfand, Graev, and Zelevinsky [44]. These systems, now called *GKZ systems* or *A-hypergeometric systems*, are closely related to affine semigroup rings (and geometrically, toric varieties). They are constructed as follows.

For the rest of this lecture, fix a $d \times n$ integer matrix $A = (a_{ij})$ of rank d. We do not assume that the columns $a_1, \ldots, a_n$ of A lie in an affine hyperplane, but we do assume that A is *pointed*, meaning that the affine semigroup

$$Q_A = \Big\{ \sum_{i=1}^{n} \gamma_i a_i \mid \gamma_1, \ldots, \gamma_n \in \mathbb{N} \Big\}$$

generated by the column vectors $a_1, \ldots, a_n$ is pointed. To get the appropriate interaction with D-modules, we express the semigroup ring $\mathbb{C}[Q_A]$ as the quotient $\mathbb{C}[\partial_1, \ldots, \partial_n]/I_A$, where

$$I_A = \langle \partial^\mu - \partial^\nu \mid \mu, \nu \in \mathbb{Z}^n, A \cdot \mu = A \cdot \nu \rangle$$

is the toric ideal of A (Exercise 20.11). Notice that $\mathbb{C}[Q_A]$ and $\mathbb{C}[\partial]$ are naturally graded by $\mathbb{Z}^d$ if we define $\deg(\partial_j) = -a_j$, the negative of the j^{th} column of A.

Our choice of signs in the $\mathbb{Z}^d$-grading of $\mathbb{C}[Q_A]$ is compatible with a $\mathbb{Z}^d$-grading on the Weyl algebra D in which $\deg(x_j) = a_j$ and $\deg(\partial_j) = -a_j$. Under this $\mathbb{Z}^d$-grading, the i^{th} *Euler operator*

$$E_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j \in D$$

is homogeneous of degree 0 for $i = 1, \ldots, d$. The terminology arises from the case where A has a row of 1's, in which case the corresponding Euler operator is $x_1\partial_1 + \cdots + x_n\partial_n$. When applied to a homogeneous polynomial $f(x_1, \ldots, x_n)$ of total degree λ, this operator returns $\lambda \cdot f(x_1, \ldots, x_n)$. Therefore, series solutions f to

$$(E_1 - \beta_1) \cdot f = 0, \ldots, (E_d - \beta_d) \cdot f = 0$$

can be thought of as being homogeneous of (multi)degree $\beta \in \mathbb{C}^d$.

Definition 24.1. Given a parameter vector $\beta \in \mathbb{C}^d$, write $E - \beta$ for the sequence $E_1 - \beta_1, \ldots, E_d - \beta_d$. The *A-hypergeometric system* with *parameter* β is the left ideal

$$H_A(\beta) = D \cdot (I_A, E - \beta)$$

in the Weyl algebra D. The *A-hypergeometric D-module* with *parameter* β is

$$\mathcal{M}_\beta^A = D/H_A(\beta).$$

Example 24.2. Letting $d = 2$ and $n = 4$, consider the 2×4 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}.$$

The semigroup ring associated to A is then

$$\mathbb{C}[Q_A] = \mathbb{C}[s, st, st^3, st^4],$$

which is isomorphic to the semigroup ring $\mathbb{C}[s^4, st^3, s^3t, t^4]$ from Example 20.33 (and all the examples mentioned there) via the isomorphism in Example 20.12. As we have seen repeatedly before, in different notation, the toric ideal for A is

$$I_A = \langle \partial_2\partial_3 - \partial_1\partial_4,\ \partial_1^2\partial_3 - \partial_2^3,\ \partial_2\partial_4^2 - \partial_3^3,\ \partial_1\partial_3^2 - \partial_2^2\partial_4 \rangle,$$

these generators corresponding to the equations

$$A\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = A\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \quad A\begin{bmatrix}2\\0\\1\\0\end{bmatrix} = A\begin{bmatrix}0\\3\\0\\0\end{bmatrix}, \quad A\begin{bmatrix}0\\1\\0\\2\end{bmatrix} = A\begin{bmatrix}0\\0\\3\\0\end{bmatrix}, \quad A\begin{bmatrix}1\\0\\2\\0\end{bmatrix} = A\begin{bmatrix}0\\2\\0\\1\end{bmatrix}.$$

Given $\beta = (\beta_1, \beta_2)$, the homogeneities from A are the classical Euler operator

$$x_1\partial_1 + x_2\partial_2 + \ \ x_3\partial_3 + \ \ x_4\partial_4 - \beta_1$$

and the "0134" equation

$$x_2\partial_2 + 3x_3\partial_3 + 4x_4\partial_4 - \beta_2.$$

The left ideal $H_A(\beta)$ is generated by I_A and the above two homogeneities.

The first fundamental results about the systems $H_A(\beta)$ were proved by Gelfand, Graev, Kapranov, and Zelevinsky (the inclusion of Kapranov here explains the 'K' in "GKZ"). These results concerned the case where $\mathbb{C}[Q_A]$ is Cohen-Macaulay and graded in the standard $\mathbb{Z}$-grading [44, 45]. Subsequently, the Cohen-Macaulay and standard $\mathbb{Z}$-graded assumptions were relaxed. For example, results of [45, 1, 78, 136] imply the following nontrivial statement.

Proposition 24.3. *The module $\mathcal{M}_\beta^A$ is holonomic of nonzero rank.*

This observation is what motivates the story here.

Question 24.4. For fixed A, what is the holonomic rank of $\mathcal{M}_\beta^A$ as a function of β?

As we shall see in the coming sections, when $\mathbb{C}[Q_A]$ is Cohen-Macaulay, or when the parameter β is generic, the rank is an easily described constant. However, although we will see that the qualitative change in rank as β varies is partially understood, we still know quite little about the quantitative behavior.

Remark 24.5. Solutions of A-hypergeometric systems appear as toric residues [21], and special cases are mirror transforms of generating functions for intersection numbers on moduli spaces of curves [25]. In the latter case, the A-hypergeometric systems are Picard-Fuchs equations governing the variation of Hodge structures for Calabi-Yau toric hypersurfaces. In general, A-hypergeometric systems constitute an important class of D-modules, playing a role similar to that of toric varieties in algebraic geometry and semigroup rings in commutative algebra: they possess enough combinatorial underpinning to make calculations feasible, but enough diversity of behavior to make them interesting as a test class for conjectures and computer experimentation.

24.2. **Rank vs. volume.** In Question 24.4, one might wonder what role the integer matrix A plays in determining the rank. The answer is pleasantly combinatorial (well, polyhedral). It requires a simple lemma and definition.

Lemma 24.6. *Fix a lattice L of full rank d in $\mathbb{R}^d$. Among all simplices of dimension d having lattice points for vertices, there is one (infinitely many, actually) with minimum Euclidean volume.*

Proof. Assuming that the origin is a vertex, by translating if necessary, the volume of any d-simplex in $\mathbb{R}^d$ (with nonrational vertices allowed) is $\frac{1}{d!}$ times the absolute value of the determinant of the remaining d vertices. The minimal absolute value for the determinant is attained on any basis of L. $\square$

A polytope with its vertices at lattice points is called a *lattice polytope*; if the polytope is a simplex, then it is a *lattice simplex*.

Definition 24.7. Fix a lattice $L \subset \mathbb{R}^d$ of rank d, and let P be a polytope. The *(normalized) volume* of P is the ratio $\mathrm{vol}_L(P)$ between the Euclidean volume of P and the smallest volume of a lattice simplex.

Example 24.8. When $L = \mathbb{Z}^d$ the normalized volume $\mathrm{vol}_{\mathbb{Z}^d}(P)$ of a lattice polytope P is simply $d!$ times the usual volume of P, since the smallest Euclidean volume of a lattice simplex is $\frac{1}{d!}$.

Notation 24.9. We write $\mathrm{vol}(P) = \mathrm{vol}_{\mathbb{Z}^d}(P)$. Given a $d \times n$ integer matrix A, set $\mathrm{vol}(A)$ equal to the normalized volume of the convex hull of the columns of A and the origin $0 \in \mathbb{Z}^d$.

Theorem 24.10 ([44, 45])**.** *If the affine semigroup ring $\mathbb{C}[Q_A]$ is Cohen-Macaulay and can be $\mathbb{Z}$-graded in such a way that it is generated in degree 1, then the A-hypergeometric D-module $\mathcal{M}_\beta^A$ has holonomic rank* $\mathrm{vol}(A)$.

The remarkable fact here is that the rank formula holds independently of the parameter β. What about when $\mathbb{C}[Q_A]$ isn't Cohen-Macaulay, or without the $\mathbb{Z}$-graded hypothesis? Adolphson further proved that, for all "generic" parameters β, the characterization of rank through volume in Theorem 24.10 is still correct.

Theorem 24.11 ([1])**.** *The rank of $\mathcal{M}_\beta^A$ equals* $\mathrm{vol}(A)$ *as long as β lies outside of a certain closed locally finite arrangement of countably many "semi-resonant" affine hyperplanes. If $\mathbb{C}[Q_A]$ is Cohen-Macaulay then* $\mathrm{rank}(\mathcal{M}_\beta^A) = \mathrm{vol}(A)$ *for all β.*

Adolphson made no claim concerning the parameters β lying in the semi-resonant hyperplanes; he did not, in particular, produce (or even prove the existence of) a parameter β where the rank did not equal $\mathrm{vol}(A)$. Hence it seemed natural enough to conjecture that perhaps the rank is actually always constant, even though Adolphson's proof did not show it. It came as quite a surprise when an example was given by Sturmfels and Takayama showing that if $\mathbb{C}[Q_A]$ is not Cohen-Macaulay then not all parameters β have to give the same rank [151]. Which example did they give? Why, 0134, of course!

Example 24.12 ([151])**.** Let A be as in Example 24.2 and set $\beta = (1, 2)$. Then $\mathrm{rank}(\mathcal{M}^A_\beta) = 5$, whereas $\mathrm{vol}(A) = 4$, the latter because the convex hull of A and the origin is a triangle with base length 4 and height 1 in $\mathbb{R}^2$.

Sturmfels and Takayama produced five linearly independent series solutions when $\beta = (1, 2)$. They also showed that $\mathrm{rank}(\mathcal{M}^A_\beta) = \mathrm{vol}(A)$ for all $\beta \neq (1, 2)$, so that $\beta = (1, 2)$ is the only *exceptional parameter*, where the rank changes from its generic value. Nearly at the same time, the case of projective toric curves (the $2 \times n$ case in which the first row is $[1\ 1\ \cdots\ 1\ 1]$) was discussed completely by Cattani, D'Andrea and Dickenstein [20]: the set of exceptional parameters is finite in this case, and empty precisely when $\mathbb{C}[Q_A]$ is Cohen-Macaulay; moreover, at each exceptional parameter β, the rank exceeds the volume by exactly 1. These observations led to the following reasonable surmise.

Conjecture 24.13 (Sturmfels)**.** $\mathcal{M}^A_\beta$ has rank $\mathrm{vol}(A)$ for all $\beta \in \mathbb{C}^d$ precisely when $\mathbb{C}[Q_A]$ is Cohen-Macaulay.

24.3. **Euler-Koszul homology.** The question quickly becomes: what is it about a parameter $\beta \in \mathbb{C}^d$ where the rank jumps that breaks the Cohen-Macaulay condition for $\mathbb{C}[Q_A]$? Stepping back from hypergeometric systems for a little while, what is it about vectors in $\mathbb{C}^d$ in general that witness the failure of the Cohen-Macaulay condition for $\mathbb{C}[Q_A]$?

These being 24 lectures on local cohomology, you might have guessed by now where the answer lies. Recall from Lecture 20 that the local cohomology of $\mathbb{C}[Q_A]$ is $\mathbb{Z}^d$-graded, since the columns of A lie in $\mathbb{Z}^d$.

Definition 24.14. Let $L^i = \{a \in \mathbb{Z}^d \mid H^i_{\mathfrak{m}}(\mathbb{C}[Q_A])_a \neq 0\}$ be the set of $\mathbb{Z}^d$-graded degrees where the i^{th} local cohomology of $\mathbb{C}[Q_A]$ is nonzero. The *exceptional setexceptional set*

$$\mathcal{E}_A = \bigcup_{i=0}^{d-1} -L^i$$

is the negative of the set of $\mathbb{Z}^d$-graded degrees where the local cohomology of $\mathbb{C}[Q_A]$ is nonzero in cohomological degree $< d$. Write $\overline{\mathcal{E}}_A$ for the Zariski closure of $\mathcal{E}_A$.

Warning 24.15. Try not to be confused about the minus sign on L^i in the definition of $\mathcal{E}_A$. Keep in mind that the degree of ∂_j is the *negative* of the j^{th} column of A, and our convention in this lecture is that $\mathbb{C}[Q_A]$ is graded by $-Q_A$ rather than the usual Q_A. These sign conventions are set up so that the $\mathbb{Z}^d$-grading on the Weyl algebra "looks right". The simplest way to think of the sign on the exceptional set is to *pretend* that $\mathbb{C}[Q_A]$ is graded by Q_A, not $-Q_A$; with this pretend convention, $a \in \mathcal{E}_A$ if and only if $H^i_{\mathfrak{m}}(\mathbb{C}[Q_A])_a \neq 0$ for some $i < d$. In particular, the $\mathbb{Z}^d$-graded degrees of nonvanishing local cohomology in Lecture 20 are exceptional degrees; no minus signs need to be introduced.

Example 24.16. For A in Example 24.2, $L^0 = \varnothing$ and $-L^1 = \{(1,2)\} = \mathcal{E}_A = \overline{\mathcal{E}}_A$. This is the "hole" in the semigroup Q_A generated by the columns of A; see Example 20.33, where the degree $(2,2)$ of nonvanishing local cohomology corresponds under the isomorphism of Example 20.12 to our degree $(1,2)$ here.

Anytime $\mathcal{E}_A$ contains infinitely many lattice points along a line, the Zariski closure $\overline{\mathcal{E}}_A$ contains the entire (complex) line through them.

Exercise 24.17. Calculate the exceptional set $\mathcal{E}_A$ for the matrix A displayed in Exercise 20.34. Hint: the Zariski closure of L^2 is a line; which line is it?

Exercise 24.18. What conditions on a set of points in $\mathcal{E}_A$ lying in a plane guarantee that the Zariski closure $\overline{\mathcal{E}}_A$ contains the entire (complexified) plane they lie in?

Exercise 24.19. To get a feel for what the Zariski closure means generally in this context, prove that $\overline{\mathcal{E}}_A$ is a finite union of affine subspaces in $\mathbb{C}^d$, each of which is parallel to one of the faces of Q_A (or of C_{Q_A}). Hint: Use local duality to show that the only associated primes of the Matlis dual $H^i_{\mathfrak{m}}(\mathbb{C}[Q_A])^\vee$ come from faces of Q_A.

Lemma 24.20. *Definition 24.14 associates to the matrix A a finite affine subspace arrangement $\overline{\mathcal{E}}_A$ in $\mathbb{C}^d$ that is empty if and only if $\mathbb{C}[Q_A]$ is Cohen-Macaulay.*

Proof. Exercise 24.19 says that the Zariski closure $\overline{\mathcal{E}}_A$ is a finite subspace arrangement. It is empty if and only if $\mathcal{E}_A$ is itself empty, and this occurs precisely when $H^i_{\mathfrak{m}}(\mathbb{C}[Q_A]) = 0$ for $i < d$. By Theorem 9.3 and Theorem 10.34, this condition is equivalent to the Cohen-Macaulay property for $\mathbb{C}[Q_A]$. $\square$

We had been after rank jumps, but then we took a detour to define an affine subspace arrangement from local cohomology. How does it relate to D-modules?

Inside the Weyl algebra is a commutative polynomial subalgebra $\mathbb{C}[\Theta]$ of D, where

$$\Theta = \theta_1, \ldots, \theta_n \quad \text{and} \quad \theta_j = x_j \partial_j.$$

Each of the Euler operators E_i lies in $\mathbb{C}[\Theta]$, as do the constants. Therefore

$$E_i - \beta_i \in \mathbb{C}[\Theta] \text{ for all } i.$$

Consequently, the Weyl algebra has a commutative subalgebra $\mathbb{C}[E-\beta] \subset D$. The linear independence of the rows of A (we assumed from the outset that the $d \times n$ matrix A has full rank d) implies that $\mathbb{C}[E-\beta]$ is isomorphic to a polynomial ring in d variables.

Recall the $\mathbb{Z}^d$-grading of D from Section 24.1. Suppose that N is a $\mathbb{Z}^d$-graded left D-module. If $y \in N$ is a homogeneous element, then let $\deg_i(y)$ be the i^{th} component in the degree of y, so

$$\deg(y) = \big(\deg_1(y), \ldots, \deg_d(y)\big) \in \mathbb{Z}^d.$$

The $\mathbb{Z}^d$-grading allows us to define a rather nonstandard action of $\mathbb{C}[E-\beta]$ on N.

Notation 24.21. Let N be a $\mathbb{Z}^d$-graded left D-module. For each homogeneous element $y \in N$, set

$$(E_i - \beta_i) \circ y = (E_i - \beta_i - \deg_i(y))y,$$

where the left-hand side uses the left D-module structure.

The funny $\circ$ action is defined on each $\mathbb{Z}^d$-graded piece of N, so N is really just a big direct sum of $\mathbb{C}[E-\beta]$-modules, one for each graded piece of N.

Definition 24.22. Fix a $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module M. Then $D \otimes_{\mathbb{C}[\partial]} M$ is a $\mathbb{Z}^d$-graded left D-module. The *Euler-Koszul complex* is the ordinary Koszul complex

$$\mathcal{K}_\bullet(E-\beta; M) = K_\bullet(E-\beta; D \otimes_{\mathbb{C}[\partial]} M)$$

over $\mathbb{C}[E-\beta]$ using the sequence $E-\beta$ under the $\circ$ action on $D \otimes_{\mathbb{C}[\partial]} M$. The i^{th} *Euler-Koszul homology* of M is $\mathcal{H}_i(E-\beta; M) = H_i(\mathcal{K}_\bullet(E-\beta; M))$.

Why the curly $\mathcal{K}$ instead of the usual Koszul complex K? First of all, we've done more to M than simply placed it in a Koszul complex: we've tensored it with D first. But more importantly, we want to stress that the Euler-Koszul complex is not just a big direct sum (over $\mathbb{Z}^d$) of Koszul complexes in each degree.

Exercise 24.23. Prove that $\mathcal{K}_\bullet(E-\beta; -)$ constitutes a functor from $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-modules to complexes *of D-modules*. In particular, prove that the maps in Definition 24.22 are homomorphisms of D-modules. Hint: See [115, Lemma 4.3].

In a less general form, Euler-Koszul homology was known to Gelfand, Kapranov, and Zelevinsky, as well as to Adolphson, who exploited it in their proofs. In the remainder of this lecture, we shall see why Euler-Koszul homology has been so unreasonably effective for dealing with ranks of hypergeometric systems. A first indication is the following, which GKZ already knew.

Exercise 24.24. For the $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module $\mathbb{C}[Q_A] = \mathbb{C}[\partial]/I_A$, prove that

$$\mathcal{H}_0(E-\beta; \mathbb{C}[Q_A]) = \mathcal{M}^A_\beta.$$

The next indication of the utility of Euler-Koszul homology is that it knows about local cohomology. This result explains why we bothered to take the Zariski closure of the exceptional degrees in Definition 24.14.

Theorem 24.25. *$\mathcal{H}_i(E-\beta; \mathbb{C}[Q_A]) \neq 0$ for some $i \geq 1$ if and only if $\beta \in \overline{\mathcal{E}}_A$.*

A more general and precise statement, in which $\mathbb{C}[Q_A]$ is replaced by an arbitrary finitely generated $\mathbb{Z}^d$-graded $\mathbb{C}[Q_A]$-module, appears in [115, Theorem 6.6]. The proof involves a spectral sequence that combines holonomic duality and local duality; it relies on little (if anything) beyond what is covered in these 24 lectures.

Corollary 24.26. *Euler-Koszul homology detects Cohen-Macaulayness: $\mathbb{C}[Q_A]$ is Cohen-Macaulay if and only if $\mathcal{H}_i(E-\beta; \mathbb{C}[Q_A]) = 0$ for all $i \geq 1$ and all $\beta \in \mathbb{C}^d$.*

Proof. The higher vanishing of Euler-Koszul homology is equivalent to $\overline{\mathcal{E}}_A = \varnothing$ by Theorem 24.25, and $\overline{\mathcal{E}}_A = \varnothing \Leftrightarrow \mathbb{C}[Q_A]$ is Cohen-Macaulay by Lemma 24.20. $\square$

The content of this section has been, more or less, that we have a hypergeometric D-module criterion for the failure of the Cohen-Macaulay condition for semigroup rings. To complete the picture, we need to see what this criterion has to do with changes in the holonomic ranks of hypergeometric systems as β varies.

24.4. **Holonomic families.** The connection from Euler-Koszul homology to rank defects of hypergeometric systems proceeds by characterizing rank defects in general families of holonomic modules. This, in turn, begins with Kashiwara's fundamental algebraic characterization of holonomic rank.

Theorem 24.27 (Kashiwara)**.** *For any module M over $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$, denote by $M(x)$ the localization $M \otimes_{\mathbb{C}[x]} \mathbb{C}(x)$. Then $\operatorname{rank}(M) = \dim_{\mathbb{C}(x)} M(x)$ whenever M is a holonomic D-module.*

Now suppose that $\mathcal{M}$ is an *algebraic family* of D-modules over $\mathbb{C}^d$. By definition, we mean to say that $\mathcal{M}$ is a finitely generated left $D[b]$-module, where $b = b_1, \ldots, b_d$ is a collection of commuting variables.

Definition 24.28. $\mathcal{M}$ is a *holonomic family* if

(1) the fiber $\mathcal{M}_\beta = \mathcal{M}/\langle b - \beta\rangle\mathcal{M}$ is a holonomic D-module for all $\beta \in \mathbb{C}^d$, where $b - \beta$ is the sequence $b_1 - \beta_1, \ldots, b_d - \beta_d$ in $D[b]$; and
(2) $\mathcal{M}(x)$ is a finitely generated module over $\mathbb{C}[b](x)$.

Condition (2) here is a subtle coherence requirement. It is quite obvious that $\mathcal{M}(x)$ is finitely generated as a left module over $D[b](x)$, and it follows from condition (1) along with Theorem 24.27 that the fibers of $\mathcal{M}(x)$ over $\mathbb{C}^d$ are finite-dimensional $\mathbb{C}(x)$-vector spaces, but this does not guarantee that $\mathcal{M}(x)$ will be finitely generated over $\mathbb{C}[b](x)$.

Exercise 24.29. Take $\mathcal{M} = D[b]/\langle bx\partial - 1\rangle$. When $\beta \neq (b-0)$, the fiber over β is the rank 1 holonomic module corresponding to the solution $x^{1/\beta}$. But when $\beta = (b-0)$ the fiber of $\mathcal{M}$ is zero. We leave it as an exercise to prove that $\mathcal{M}(x)$ is not finitely generated as a module over $\mathbb{C}[b](x)$.

The point of looking at $\mathcal{M}(x)$ is that its fiber over $\beta \in \mathbb{C}^d$ is a $\mathbb{C}(x)$-vector space of dimension $\mathrm{rank}(\mathcal{M}_\beta)$. Condition (2) in Definition 24.28 implies that constancy of holonomic rank in a neighborhood of β is detected by ordinary Koszul homology.

Proposition 24.30. *The* rank function $\beta \mapsto \mathrm{rank}(\mathcal{M}_\beta)$ *of any holonomic family $\mathcal{M}$ is upper-semicontinuous on $\mathbb{C}^d$, and it is is constant near $\beta \in \mathbb{C}^d$ if and only if the ordinary Koszul homology $H_i(b - \beta; \mathcal{M}(x))$ is zero for all $i \geq 1$.*

Proof. Upper-semicontinuity follows from the coherence condition; there are details omitted here. The fiber dimension of $\mathcal{M}(x)$ is constant near β if and only if (by the coherence condition again) $\mathcal{M}(x)$ is flat near $\beta \in \mathbb{C}^d$, and this occurs if and only if $H_i(b - \beta; \mathcal{M}(x)) = 0$ for all $i \geq 1$. □

Upper-semicontinuity means that the holonomic ranks of the fibers in a holonomic family can only jump up on closed sets. In particular, for any holonomic family there is a well-defined "generic" rank taken on by the fibers over a Zariski open subset of $\mathbb{C}^d$.

Example 24.31. In Exercise 24.29, the holonomic rank of $\mathcal{M}_\beta$ equals 1 if $\beta \neq 0$, but the rank drops to zero when $\beta = 0$. This actually constitutes a solution to Exercise 24.29, if one is willing to accept Proposition 24.30!

The motivating application for holonomic families is the hypergeometric case.

Definition 24.32. Set $\mathcal{M}^A = D[b]/D[b]\langle I_A, E - b\rangle$.

The definition of $\mathcal{M}^A$ is obtained from that of $\mathcal{M}^A_\beta$ by replacing the constants $\beta_1, \ldots, \beta_d \in \mathbb{C}^d$ with the commuting variables $b_1, \ldots, b_d$.

Proposition 24.33 ([115])**.** *$\mathcal{M}^A$ is a holonomic family whose fiber over each parameter vector $\beta \in \mathbb{C}^d$ is the A-hypergeometric D-module $\mathcal{M}^A_\beta$.*

The proof of the first part of this proposition requires some criteria for when an algebraic family of D-modules is a holonomic family. It is not particularly difficult, but we will not go into it here. In contrast, we leave it as an easy exercise to check the second part: namely, that the fiber $(\mathcal{M}^A)_\beta$ is $\mathcal{M}^A_\beta$.

Example 24.34. Proposition 24.33 implies that the upper-semicontinuity dictated by Proposition 24.30 should hold for the algebraic family $\mathcal{M}^A$ constructed from the matrix A in Example 24.2. And indeed it does, by Example 24.12.

Now what we have is a D-module homological theory (Euler-Koszul homology on $E-\beta$) for detecting failures of Cohen-Macaulayness in hypergeometric systems, and a commutative algebraic homological theory (ordinary Koszul homology on $b-\beta$) for detecting jumps of holonomic ranks in hypergeometric families. The final point, then, is that these two homological theories coincide.

Theorem 24.35 ([115]). *$H_i(b-\beta;\mathcal{M}^A) \cong \mathcal{H}_i(E-\beta;\mathbb{C}[Q_A])$.*

Corollary 24.36. *Conjecture 24.13 is true:* $\operatorname{rank}(\mathcal{M}^A_\beta) = \operatorname{vol}(A)$ *for all $\beta \in \mathbb{C}^d$ if and only if $\mathbb{C}[Q_A]$ is Cohen-Macaulay. In general,* $\operatorname{rank}(\mathcal{M}^A_\beta) \geq \operatorname{vol}(A)$*, and*

$$\operatorname{rank}(\mathcal{M}^A_\beta) > \operatorname{vol}(A) \;\Leftrightarrow\; \beta \in \overline{\mathcal{E}}_A.$$

Proof. The left side of Theorem 24.35 detects when β yields $\operatorname{rank}(\mathcal{M}^A_\beta) > \operatorname{vol}(A)$, while the right side detects when β violates the Cohen-Macaulayness of $\mathbb{C}[Q_A]$, by way of the exceptional set (Theorem 24.25 and Corollary 24.26). □

Example 24.37. For the 0134 matrix in Example 24.2, the coincidence of the jump in rank of $\mathcal{M}^A_\beta$ at $\beta = (1,2)$ in Example 24.12 and the inclusion of $\beta = (1,2)$ in the exceptional set $\overline{\mathcal{E}}_A$ from Example 24.16 is a consequence of Corollary 24.36.

Exercise 24.38. Prove that the set of exceptional parameters for a GKZ hypergeometric system $\mathcal{M}^A_\beta$ has codimension at least 2 in $\mathbb{C}^d$.

Appendix A. Injective Modules and Matlis Duality

These notes are intended to give the reader an idea what injective modules are, where they show up, and, to a small extent, what one can do with them. Let R be a commutative Noetherian ring with an identity element. An R-module E is injective if $\operatorname{Hom}_R(-, E)$ is an exact functor. The main messages of these notes are

- Every R-module M has an *injective hull* or *injective envelope*, denoted by $E_R(M)$, which is an injective module containing M, and has the property that any injective module containing M contains an isomorphic copy of $E_R(M)$.
- A nonzero injective module is *indecomposable* if it is not the direct sum of nonzero injective modules. Every injective R-module is a direct sum of indecomposable injective R-modules.
- Indecomposable injective R-modules are in bijective correspondence with the prime ideals of R; in fact every indecomposable injective R-module is isomorphic to an injective hull $E_R(R/\mathfrak{p})$, for some prime ideal $\mathfrak{p}$ of R.
- The number of isomorphic copies of $E_R(R/\mathfrak{p})$ occurring in any direct sum decomposition of a given injective module into indecomposable injectives is independent of the decomposition.
- Let $(R, \mathfrak{m})$ be a complete local ring and $E = E_R(R/\mathfrak{m})$ be the injective hull of the residue field of R. The functor $(-)^\vee = \operatorname{Hom}_R(-, E)$ has the following properties, known as *Matlis duality*:
  (1) If M is an R-module which is Noetherian or Artinian, then $M^{\vee\vee} \cong M$.
  (2) If M is Noetherian, then $M^\vee$ is Artinian.
  (3) If M is Artinian, then $M^\vee$ is Noetherian.

Any unexplained terminology or notation can be found in [4] or [114]. Matlis' theory of injective modules was developed in the paper [113], and may also be found in [114, § 18] and [16, § 3].

A.1. **Injective Modules.** Throughout, R is a commutative ring with an identity element $1 \in R$. All R-modules M are assumed to be unitary, i.e., $1 \cdot m = m$ for all $m \in M$.

Definition A.1. An R-module E is *injective* if for all R-module homomorphisms $\varphi : M \longrightarrow N$ and $\psi : M \longrightarrow E$ where φ is injective, there exists an R-linear homomorphism $\theta : N \longrightarrow E$ such that $\theta \circ \varphi = \psi$.

Exercise A.2. Show that E is an injective R-module E if and only if $\operatorname{Hom}_R(-, E)$ is an exact functor, i.e., applying $\operatorname{Hom}_R(-, E)$ takes short exact sequences to short exact sequences.

Theorem A.3 (Baer's Criterion)**.** *An R-module E is injective if and only if every R-module homomorphism $\mathfrak{a} \longrightarrow E$, where $\mathfrak{a}$ is an ideal, extends to a homomorphism $R \longrightarrow E$.*

Proof. One direction is obvious. For the other, if $M \subseteq N$ are R-modules and $\varphi : M \longrightarrow E$, we need to show that φ extends to a homomorphism $N \longrightarrow E$. By Zorn's lemma, there is a module N' with $M \subseteq N' \subseteq N$, which is maximal with respect to the property that φ extends to a homomorphism $\varphi' : N' \longrightarrow E$. If $N' \neq N$, take an element $n \in N \setminus N'$ and consider the ideal $\mathfrak{a} = N' :_R n$. By hypothesis, the composite homomorphism $\mathfrak{a} \xrightarrow{n} N' \xrightarrow{\varphi'} E$ extends to a homomorphism $\psi : R \longrightarrow$

E. Define $\varphi'' : N' + Rn \longrightarrow E$ by $\varphi''(n' + rn) = \varphi'(n') + \psi(r)$. This contradicts the maximality of φ', so we must have $N' = N$. $\square$

Exercise A.4. Let R be an integral domain. An R-module M is *divisible* if $rM = M$ for every nonzero element $r \in R$.

(1) Prove that an injective R-module is divisible.
(2) If R is a principal ideal domain, prove that an R-module is divisible if and only if it is injective.
(3) Conclude that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.
(4) Prove that any nonzero Abelian group has a nonzero homomorphism to $\mathbb{Q}/\mathbb{Z}$.
(5) If $(-)^\vee = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ and M is any $\mathbb{Z}$-module, prove that the natural map $M \longrightarrow M^{\vee\vee}$ is injective.

Exercise A.5. Let R be an A-algebra.

(1) Use the adjointness of $\otimes$ and Hom to prove that if E is an injective A-module, and F is a flat R-module, then $\mathrm{Hom}_A(F, E)$ is an injective R-module.
(2) Prove that every R-module can be embedded in an injective R-module. Hint: If M is the R-module, take a free R-module F with a surjection $F \twoheadrightarrow \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Apply $(-)^\vee = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. See Lecture 4 for more on adjointness.

Proposition A.6. *Let $M \neq 0$ and N be R-modules, and let $\theta : M \hookrightarrow N$ be a monomorphism. Then the following are equivalent:*

(1) Every nonzero submodule of N has a nonzero intersection with $\theta(M)$.
(2) Every nonzero element of N has a nonzero multiple in $\theta(M)$.
(3) If $\varphi \circ \theta$ is injective for a homomorphism $\varphi : N \longrightarrow Q$, then φ is injective.

Proof. (1) $\implies$ (2) If n is a nonzero element of N, then the cyclic module Rn has a nonzero intersection with $\theta(M)$.

(2) $\implies$ (3) If (3) fails then $\ker\varphi$ has a nonzero intersection with $\theta(M)$, contradicting the assumption that $\varphi \circ \theta$ is injective.

(3) $\implies$ (1) Let N' be a nonzero submodule of N, and consider the canonical surjection $\varphi : N \longrightarrow N/N'$. Then φ is not injective, hence the composition $\varphi \circ \theta : M \longrightarrow N/N'$ is not injective, i.e., N' contains a nonzero element of $\theta(M)$. $\square$

Definition A.7. If $\theta : M \hookrightarrow N$ satisfies the equivalent conditions of the previous proposition, we say that N is an *essential extension* of M.

Example A.8. If R is a domain and $\mathrm{Frac}(R)$ is its field of fractions, then $R \subseteq \mathrm{Frac}(R)$ is an essential extension. More generally, if $S \subseteq R$ is the set of nonzerodivisors in R, then $S^{-1}R$ is an essential extension of R.

Example A.9. Let $(R, \mathfrak{m})$ be a local ring and N be an R-module such that every element of N is killed by a power of $\mathfrak{m}$. The *socle* of N is the submodule $\mathrm{soc}(N) = 0 :_N \mathfrak{m}$. Then $\mathrm{soc}(N) \subseteq N$ is an essential extension: if $n \in N$ is a nonzero element, let t be the smallest integer such that $\mathfrak{m}^t n = 0$. Then $\mathfrak{m}^{t-1}n \subseteq \mathrm{soc}(N)$, and $\mathfrak{m}^{t-1}n$ contains a nonzero multiple of n.

Exercise A.10. Let I be an index set. Then $M_i \subseteq N_i$ is essential for all $i \in I$ if and only if $\bigoplus_{i \in I} M_i \subseteq \bigoplus_{i \in I} N_i$ is essential.

Example A.11. Let $R = \mathbb{C}[[x]]$ which is a local ring with maximal ideal (x), and take $N = R_x/R$. Every element of N is killed by a power of the maximal ideal, and $\operatorname{soc}(N)$ is the 1-dimensional $\mathbb{C}$-vector space generated by $[1/x]$, i.e., the image of $1/x$ in N. By Example A.9, $\operatorname{soc}(N) \subseteq N$ is an essential extension. However, prove that $\prod_{\mathbb{N}} \operatorname{soc}(N) \subseteq \prod_{\mathbb{N}} N$ is not an essential extensionby studying the element

$$\big([1/x], [1/x^2], [1/x^3], \dots\big) \in \prod_{\mathbb{N}} N$$

Proposition A.12. *Let L, M, N be nonzero R-modules.*

(1) $M \subseteq M$ is an essential extension.
(2) Suppose $L \subseteq M \subseteq N$. Then $L \subseteq N$ is an essential extension if and only if both $L \subseteq M$ and $M \subseteq N$ are essential extensions.
(3) Suppose $M \subseteq N$ and $M \subseteq N_i \subseteq N$ with $N = \bigcup_i N_i$. Then $M \subseteq N$ is an essential extension if and only if $M \subseteq N_i$ is an essential extension for every i.
(4) Suppose $M \subseteq N$. Then there exists a module N' with $M \subseteq N' \subseteq N$, which is maximal with respect to the property that $M \subseteq N'$ is an essential extension.

Proof. The assertions (1), (2), and (3) elementary. For (4), let

$$\mathcal{F} = \{N' \mid M \subseteq N' \subseteq N \text{ and } N' \text{ is an essential extension of } M\}.$$

Then $M \in \mathcal{F}$ so $\mathcal{F}$ is nonempty. If $N_1' \subseteq N_2' \subseteq N_3' \subseteq \dots$ is a chain in $\mathcal{F}$, then $\bigcup_i N_i' \in \mathcal{F}$ is an upper bound. By Zorn's Lemma, the set $\mathcal{F}$ has maximal elements. $\square$

Definition A.13. The module N' in Proposition A.12 (4) is a *maximal essential extension of M in N*. If $M \subseteq N$ is essential and N has no proper essential extensions, we say that N is a *maximal essential extension of M*.

Proposition A.14. *Let M be an R-module. The following conditions are equivalent:*

(1) M is injective;
(2) M is a direct summand of every module containing M;
(3) M has no proper essential extensions.

Proof. (1) $\implies$ (2) $\implies$ (3) is left as an exercise, and we prove the implication (3) $\implies$ (2). Consider an embedding $M \hookrightarrow E$ where E is injective. By Zorn's lemma, there exists a submodule $N \subseteq E$ which is maximal with respect to the property that $N \cap M = 0$. This implies that $M \hookrightarrow E/N$ is an essential extension, and hence that it is an isomorphism. But then $E = M + N$ implies $E = M \oplus N$. Since M is a direct summand of an injective module, it must be injective. $\square$

Proposition A.15. *Let M and E be R-modules.*

(1) If E is injective and $M \subseteq E$, then any maximal essential extension of M in E is an injective module, hence is a direct summand of E.
(2) Any two maximal essential extensions of M are isomorphic.

Proof. (1) Let E' be a maximal essential extension of M in E and let $E' \subseteq Q$ be an essential extension. Since E is injective, the identity map $E' \longrightarrow E$ lifts to a homomorphism $\varphi : Q \longrightarrow E$. Since Q is an essential extension of E', it follows that φ must be injective. This gives us $M \subseteq E' \subseteq Q \hookrightarrow E$, and the maximality of E'

implies that $Q = E'$. Hence E' has no proper essential extensions, and so it is an injective module by Proposition A.14.

(2) Let $M \subseteq E$ and $M \subseteq E'$ be maximal essential extensions of M. Then E' is injective, so $M \subseteq E'$ extends to a homomorphism $\varphi : E \longrightarrow E'$. The inclusion $M \subseteq E$ is an essential extension, so φ is injective. But then $\varphi(E)$ is an injective module, and hence a direct summand of E'. Since $M \subseteq \varphi(E) \subseteq E'$ is an essential extension, we must have $\varphi(E) = E'$. $\square$

Definition A.16. The *injective hull* or *injective envelope* of an R-module M is a maximal essential extension of M, and is denoted by $E_R(M)$.

Definition A.17. Let M be an R-module. A *minimal injective resolution* of M is a complex

$$0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \dots$$

such that $E^0 = E_R(M)$, $E^1 = E_R(E^0/M)$, and

$$E^{i+1} = E_R(E^i/\operatorname{image}(E^{i-1})) \quad \text{for all } i \geqslant 2.$$

Note that the modules E^i are injective, and that $\operatorname{image}(E^i) \subseteq E^{i+1}$ is an essential extension for all $i \geqslant 0$.

A.2. **Injectives over a Noetherian Ring.**

Proposition A.18 (Bass)**.** *A ring R is Noetherian if and only if every direct sum of injective R-modules is injective.*

Proof. We show first that if M is a finitely generated R-module, then

$$\operatorname{Hom}_R(M, \bigoplus_i N_i) \cong \bigoplus_i \operatorname{Hom}_R(M, N_i).$$

Independently of the finite generation of M, there is a natural injective homomorphism $\varphi : \bigoplus_i \operatorname{Hom}_R(M, N_i) \longrightarrow \operatorname{Hom}_R(M, \bigoplus_i N_i)$. If M is finitely generated, the image of a homomorphism from M to $\bigoplus_i N_i$ is contained in the direct sum of finitely many N_i. Since Hom commutes with forming finite direct sums, φ is surjective as well.

Let R be a Noetherian ring, and E_i be injective R-modules. Then for an ideal $\mathfrak{a}$ of R, the natural map $\operatorname{Hom}_R(R, E_i) \longrightarrow \operatorname{Hom}_R(\mathfrak{a}, E_i)$ is surjective. Since $\mathfrak{a}$ is finitely generated, the above isomorphism implies that $\operatorname{Hom}_R(R, \bigoplus E_i) \longrightarrow \operatorname{Hom}_R(\mathfrak{a}, \bigoplus E_i)$ is surjective as well. Baer's criterion now implies that $\bigoplus E_i$ is injective.

If R is not Noetherian, it contains a strictly ascending chain of ideals

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq \dots.$$

Let $\mathfrak{a} = \bigcup_i \mathfrak{a}_i$. The natural maps $\mathfrak{a} \hookrightarrow R \longrightarrow\!\!\!\!\!\rightarrow R/\mathfrak{a}_i \hookrightarrow E_R(R/\mathfrak{a}_i)$ give us a homomorphism $\mathfrak{a} \longrightarrow \prod_i E_R(R/\mathfrak{a}_i)$. The image lies in the submodule $\bigoplus_i E_R(R/\mathfrak{a}_i)$, (check!) so we have a homomorphism $\varphi : \mathfrak{a} \longrightarrow \bigoplus_i E_R(R/\mathfrak{a}_i)$. Lastly, check that φ does not extend to a homomorphism $R \longrightarrow \bigoplus_i E_R(R/\mathfrak{a}_i)$. $\square$

Theorem A.19. *Let E be an injective module over a Noetherian ring R. Then*

$$E \cong \bigoplus_i E_R(R/\mathfrak{p}_i),$$

where $\mathfrak{p}_i$ are prime ideals of R. Moreover, any such direct sum is an injective R-module.

Proof. The last statement follows from Proposition A.18. Let E be an injective R-module. By Zorn's Lemma, there exists a maximal family $\{E_i\}$ of injective submodules of E such that $E_i \cong E_R(R/\mathfrak{p}_i)$, and their sum in E is a direct sum. Let $E' = \bigoplus E_i$, which is an injective module, and hence is a direct summand of E. There exists an R-module E'' such that $E = E' \oplus E''$. If $E'' \neq 0$, pick a nonzero element $x \in E''$. Let $\mathfrak{p}$ be an associated prime of Rx. Then $R/\mathfrak{p} \hookrightarrow Rx \subseteq E''$, so there is a copy of $E_R(R/\mathfrak{p})$ contained in E'' and $E'' = E_R(R/\mathfrak{p}) \oplus E'''$, contradicting the maximality of family $\{E_i\}$. □

Definition A.20. Let $\mathfrak{a}$ be an ideal of a ring R, and M be an R-module. We say M is $\mathfrak{a}$-*torsion* if every element of M is killed by some power of $\mathfrak{a}$.

Theorem A.21. *Let $\mathfrak{p}$ be a prime ideal of a Noetherian ring R, and let $E = E_R(R/\mathfrak{p})$ and $\kappa = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, which is the fraction field of $R/\mathfrak{p}$. Then*

(1) if $x \in R \setminus \mathfrak{p}$, then $E \xrightarrow{x} E$ is an isomorphism, and so $E = E_\mathfrak{p}$;
(2) $0 :_E \mathfrak{p} = \kappa$;
(3) $\kappa \subseteq E$ is an essential extension of $R_\mathfrak{p}$-modules and $E = E_{R_\mathfrak{p}}(\kappa)$;
(4) E is $\mathfrak{p}$-torsion and $\operatorname{Ass}(E) = \{\mathfrak{p}\}$;
(5) $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E) = \kappa$ and $\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E_R(R/\mathfrak{q})_\mathfrak{p}) = 0$ for primes $\mathfrak{q} \neq \mathfrak{p}$.

Proof. (1) κ is an essential extension of $R/\mathfrak{p}$ by Example A.8, so E contains a copy of κ and we may assume $R/\mathfrak{p} \subseteq \kappa \subseteq E$. Multiplication by $x \in R \setminus \mathfrak{p}$ is injective on κ, and hence also on its essential extension E. The submodule xE is injective, so it is a direct summand of E. But $\kappa \subseteq xE \subseteq E$ are essential extensions, so $xE = E$.

(2) $0 :_E \mathfrak{p} = 0 :_E \mathfrak{p}R_\mathfrak{p}$ is a vector space over the field κ, and hence the inclusion $\kappa \subseteq 0 :_E \mathfrak{p}$ splits. But $\kappa \subseteq 0 :_E \mathfrak{p} \subseteq E$ is an essential extension, so $0 :_E \mathfrak{p} = \kappa$.

(3) The containment $\kappa \subseteq E$ is an essential extension of R-modules, hence also of $R_\mathfrak{p}$-modules. Suppose $E \subseteq M$ is an essential extension of $R_\mathfrak{p}$-modules, pick $m \in M$. Then m has a nonzero multiple $(r/s)m \in E$, where $s \in R \setminus \mathfrak{p}$. But then rm is a nonzero multiple of m in E, so $E \subseteq M$ is an essential extension of R-modules, and therefore $M = E$.

(4) Let $\mathfrak{q} \in \operatorname{Ass}(E)$. Then there exists $x \in E$ such that $Rx \subseteq E$ and $0 :_R x = \mathfrak{q}$. Since $R/\mathfrak{p} \subseteq E$ is essential, x has a nonzero multiple y in $R/\mathfrak{p}$. But then the annihilator of y is $\mathfrak{p}$, so $\mathfrak{q} = \mathfrak{p}$ and $\operatorname{Ass}(E) = \{\mathfrak{p}\}$.

If $\mathfrak{a}$ is the annihilator of a nonzero element of E, then $\mathfrak{p}$ is the only associated prime of $R/\mathfrak{a} \hookrightarrow E$, so E is $\mathfrak{p}$-torsion.

(5) For the first assertion,

$$\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E) = \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, E) \cong 0 :_{\mathfrak{p}R_\mathfrak{p}} E = \kappa.$$

Since elements of $R \setminus \mathfrak{q}$ act invertibly on $E_R(R/\mathfrak{q})$, we see that $E_R(R/\mathfrak{q})_\mathfrak{p} = 0$ if $\mathfrak{q} \nsubseteq \mathfrak{p}$. In the case $\mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\operatorname{Hom}_{R_\mathfrak{p}}(\kappa, E_R(R/\mathfrak{q})_\mathfrak{p}) \cong 0 :_{\mathfrak{p}R_\mathfrak{p}} E_R(R/\mathfrak{q})_\mathfrak{p} = 0 :_{\mathfrak{p}R_\mathfrak{p}} E_R(R/\mathfrak{q}).$$

If this is nonzero, then there is a nonzero element of $E_R(R/\mathfrak{q})$ killed by $\mathfrak{p}$, which forces $\mathfrak{q} = \mathfrak{p}$ since $\operatorname{Ass} E_R(R/\mathfrak{q}) = \{\mathfrak{q}\}$. □

We are now able to strengthen Theorem A.19 to obtain the following structure theorem for injective modules over Noetherian rings.

Theorem A.22. *Let E be an injective over a Noetherian ring R. Then*

$$E = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\alpha_\mathfrak{p}},$$

and the numbers $\alpha_\mathfrak{p}$ are independent of the direct sum decomposition.

Proof. Theorem A.19 implies that a direct sum decomposition exists. By Theorem A.21 (5), $\alpha_\mathfrak{p}$ is the dimension of the $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$-vector space

$$\operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, E_\mathfrak{p}),$$

which does not depend on the decomposition. $\square$

The following proposition can be proved along the lines of Theorem A.21, and we leave the proof as an exercise.

Proposition A.23. *Let $S \subset R$ be a multiplicative set.*

(1) If E is an injective R-module, then $S^{-1}E$ is an injective module over the ring $S^{-1}R$.
(2) If $M \hookrightarrow N$ is an essential extension (or a maximal essential extension) of R-modules, then the same is true for $S^{-1}M \hookrightarrow S^{-1}N$ over $S^{-1}R$.
(3) The indecomposable injectives over $S^{-1}R$ are the modules $E_R(R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset$.

Definition A.24. Let M be an R-module, and let $E^\bullet$ be a minimal injective resolution of R where

$$E^i = \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}.$$

Then $\mu_i(\mathfrak{p}, M)$ is the i-th *Bass number of M with respect to* $\mathfrak{p}$. The following theorem shows that these numbers are well-defined.

Theorem A.25. *Let $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Then*

$$\mu_i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}).$$

Proof. Let $E^\bullet$ be a minimal injective resolution of M where the i-th module is $E^i = \oplus E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)}$. Localizing at $\mathfrak{p}$, Proposition A.23 implies that $E^\bullet_\mathfrak{p}$ is a minimal injective resolution of $M_\mathfrak{p}$ over the ring $R_\mathfrak{p}$. Moreover, the number of copies of $E_R(R/\mathfrak{p})$ occurring in E^i is the same as the number of copies of $E_R(R/\mathfrak{p})$ in $E^i_\mathfrak{p}$. By definition, $\operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p})$ is the i-th cohomology module of the complex

$$0 \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^0_\mathfrak{p}) \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^1_\mathfrak{p}) \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^2_\mathfrak{p}) \longrightarrow \dots$$

and we claim all maps in this complex are zero. If $\varphi \in \operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^i_\mathfrak{p})$, we need to show that the composition

$$\kappa(\mathfrak{p}) \xrightarrow{\varphi} E^i_\mathfrak{p} \xrightarrow{\delta} E^{i+1}_\mathfrak{p}$$

is the zero map. If $\varphi(x) \neq 0$ for $x \in \kappa(\mathfrak{p})$, then $\varphi(x)$ has a nonzero multiple in $\operatorname{image}(E^{i-1}_\mathfrak{p} \longrightarrow E^i_\mathfrak{p})$. Since $\kappa(\mathfrak{p})$ is a field, it follows that

$$\varphi(\kappa(\mathfrak{p})) \subseteq \operatorname{image}(E^{i-1}_\mathfrak{p} \longrightarrow E^i_\mathfrak{p}),$$

and hence that $\delta \circ \varphi = 0$. By Theorem A.21 (5)

$$\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), E^i_\mathfrak{p}) \cong \kappa(\mathfrak{p})^{\mu_i(\mathfrak{p},M)},$$

so $\operatorname{Ext}^i_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}})$ is the i-th cohomology module of the complex

$$0 \longrightarrow \kappa(\mathfrak{p})^{\mu_0(\mathfrak{p},M)} \longrightarrow \kappa(\mathfrak{p})^{\mu_1(\mathfrak{p},M)} \longrightarrow \kappa(\mathfrak{p})^{\mu_2(\mathfrak{p},M)} \longrightarrow \dots .$$

where all maps are zero, and the required result follows. □

Remark A.26. We next want to consider the special case in which $(R, \mathfrak{m}, K)$ is a Noetherian local ring. Recall that we have natural surjections

$$\dots \longrightarrow R/\mathfrak{m}^3 \longrightarrow R/\mathfrak{m}^2 \longrightarrow R/\mathfrak{m} \longrightarrow 0,$$

and that the $\mathfrak{m}$-*adic completion* $\widehat{R}$ of R is the inverse limit of this system, i.e.,

$$\varprojlim_k (R/\mathfrak{m}^k) = \left\{ (r_0, r_1, r_2, \dots) \in \prod_k R/\mathfrak{m}^k \mid r_k - r_{k-1} \in \mathfrak{m}^{k-1} \right\}.$$

Morally, elements of the $\mathfrak{a}$-adic completion of R are power series of elements of R where "higher terms" are those contained in higher powers of the ideal $\mathfrak{a}$. There is no reason to restrict to local rings or maximal ideals—for topological purposes, completions at other ideals can be very interesting; see, for example, [62].

Note that $\widehat{R}/\mathfrak{m}^k\widehat{R} \cong R/\mathfrak{m}^k$. Consequently if M is $\mathfrak{m}$-torsion, then the R-module structure on M makes it an $\widehat{R}$-module. In particular, $E_R(R/\mathfrak{m})$ is an $\widehat{R}$-module.

Theorem A.27. *Let $(R, \mathfrak{m}, K)$ be a local ring. Then $E_R(K) = E_{\widehat{R}}(K)$.*

Proof. The containment $K \subseteq E_R(K)$ is an essential extension of R-modules, hence also of $\widehat{R}$-modules. If $E_R(K) \subseteq M$ is an essential extension of $\widehat{R}$-modules, then M is $\mathfrak{m}$-torsion. (Prove!) If $m \in M$ is a nonzero element, then $\widehat{R}m \cap E_R(K) \neq 0$. But $\widehat{R}m = Rm$, so $E_R(K) \subseteq M$ is an essential extension of R-modules, which implies $M = E_R(K)$. It follows that $E_R(K)$ is a maximal essential extension of K as an $\widehat{R}$-module. □

Theorem A.28. *Let $\varphi : (R, \mathfrak{m}, K) \longrightarrow (S, \mathfrak{n}, L)$ be a homomorphism of local rings such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, the ideal $\varphi(\mathfrak{m})S$ is $\mathfrak{n}$-primary, and S is module-finite over R. Then*

$$\operatorname{Hom}_R(S, E_R(K)) = E_S(L).$$

Proof. By Exercise A.5, $\operatorname{Hom}_R(S, E_R(K))$ is an injective S-module. Every element of $\operatorname{Hom}_R(S, E_R(K))$ is killed by a power of $\mathfrak{m}$ and hence by a power of $\mathfrak{n}$. It follows that $\operatorname{Hom}_R(S, E_R(K))$ is a direct sum of copies of $E_S(L)$, say $\operatorname{Hom}_R(S, E_R(K)) \cong E_S(L)^\mu$. To determine μ, consider

$$\operatorname{Hom}_S(L, \operatorname{Hom}_R(S, E_R(K))) \cong \operatorname{Hom}_R(L \otimes_S S, E_R(K)) \cong \operatorname{Hom}_R(L, E_R(K)).$$

The image of any element of $\operatorname{Hom}_R(L, E_R(K))$ is killed by $\mathfrak{n}$, hence

$$\operatorname{Hom}_R(L, E_R(K)) \cong \operatorname{Hom}_R(L, K) \cong \operatorname{Hom}_K(L, K)$$

and $L^\mu \cong \operatorname{Hom}_K(L, K)$. Considering vector space dimensions over K, this implies $\mu \dim_K L = \dim_K L$, so $\mu = 1$. □

Corollary A.29. *Let $(R, \mathfrak{m}, K)$ be a local ring and let $S = R/\mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R. Then the injective hull of the residue field of S is*

$$\operatorname{Hom}_R(R/\mathfrak{a}, E_R(K)) \cong 0 :_{E_R(K)} \mathfrak{a}.$$

Since every element of $E_R(K)$ is killed by a power of $\mathfrak{m}$, we have

$$E_R(K) = \bigcup_{t\in\mathbb{N}} (0 :_{E_R(K)} \mathfrak{m}^t) = \bigcup_{t\in\mathbb{N}} E_{R/\mathfrak{m}^t}(K).$$

This motivates the study of $E_R(K)$ for Artinian local rings.

A.3. **The Artinian case.** Recall that the *length* of a module M is the length of a composition series for M, and is denoted $\ell(M)$. The length is additive over short exact sequences. If $(R, \mathfrak{m}, K)$ is an Artinian local ring, then every finitely generated R-module has a composition series with factors isomorphic to $R/\mathfrak{m}$.

Lemma A.30. *Let $(R, \mathfrak{m}, K)$ be a local ring. Then $(-)^\vee = \operatorname{Hom}_R(-, E_R(K))$ is a faithful functor, and $\ell(M^\vee) = \ell(M)$ for every R-module M of finite length.*

Proof. Note that $(R/\mathfrak{m})^\vee = \operatorname{Hom}_R(R/\mathfrak{m}, E_R(K)) \cong K$. If M is a nonzero R-module, we need to show that $M^\vee$ is nonzero. Taking a nonzero cyclic submodule $R/\mathfrak{a} \hookrightarrow M$, we have $M^\vee \twoheadrightarrow (R/\mathfrak{a})^\vee$, so it suffices to show that $(R/\mathfrak{a})^\vee$ is nonzero. The surjection $R/\mathfrak{a} \twoheadrightarrow R/\mathfrak{m}$ yields $(R/\mathfrak{m})^\vee \hookrightarrow (R/\mathfrak{a})^\vee$, and hence $(-)^\vee$ is faithful.

For M of finite length, we use induction on $\ell(M)$ to prove $\ell(M^\vee) = \ell(M)$. The result is true for modules of length 1 since $(R/\mathfrak{m})^\vee \cong K$. For a module M of finite length, consider $m \in \operatorname{soc}(M)$ and the exact sequence

$$0 \longrightarrow Rm \longrightarrow M \longrightarrow M/Rm \longrightarrow 0.$$

Applying $(-)^\vee$, we obtain an exact sequence

$$0 \longrightarrow (M/Rm)^\vee \longrightarrow M^\vee \longrightarrow (Rm)^\vee \longrightarrow 0.$$

Since $Rm \cong K$ and $\ell(M/Rm) = \ell(M) - 1$, we are done by induction. $\square$

Corollary A.31. *Let $(R, \mathfrak{m}, K)$ be an Artinian local ring. Then $E_R(K)$ is a finite length module and $\ell(E_R(K)) = \ell(R)$.*

Theorem A.32. *Let $(R, \mathfrak{m}, K)$ be a Artinian local ring and $E = E_R(K)$. Then the map $R \longrightarrow \operatorname{Hom}_R(E, E)$, which takes a ring element r to the homomorphism "multiplication by r," is an isomorphism.*

Proof. By the previous results, $\ell(R) = \ell(E) = \ell(E^\vee)$, so R and $\operatorname{Hom}_R(E, E)$ have the same length, and it suffices to show the map is injective. If $rE = 0$, then $E = \operatorname{ann}_E(r) = E_{R/Rr}(K)$ so $\ell(R) = \ell(R/Rr)$, forcing $r = 0$. $\square$

Theorem A.33. *Let $(R, \mathfrak{m}, K)$ be a local ring. Then R is an injective R-module if and only if the following two conditions are satisfied:*

(1) R is Artinian, and
(2) $\operatorname{soc}(R)$ is 1-dimensional vector space over K.

Proof. If $R = M \oplus N$ then $K \cong (M \otimes_R K) \oplus (N \otimes_R K)$, so one of the two summands must be zero, say $M \otimes_R K = 0$. But then Nakayama's lemma implies that $M = 0$. It follows that a local ring in indecomposable as a module over itself. Hence if R is injective, then $R \cong E_R(R/\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} R$. This implies R that is $\mathfrak{p}$-torsion and it follows that $\mathfrak{p}$ is the only prime ideal of R and hence that R is Artinian. Furthermore, $\operatorname{soc}(R)$ is isomorphic to $\operatorname{soc}(E_R(K))$, which is 1-dimensional.

Conversely, if R is Artinian with $\operatorname{soc}(R) = K$, then R is an essential extension of its socle. The essential extension $K \subseteq R$ can be enlarged to a maximal essential extension $K \subseteq E_R(K)$. Since $\ell(E_R(K)) = \ell(R)$, we must have $E_R(K) = R$. $\square$

A.4. **Matlis duality.**

Theorem A.34. *Let $(R, \mathfrak{m}, K)$ be a local ring and let $E = E_R(K)$. Then E is also an $\widehat{R}$-module, and the map $\widehat{R} \longrightarrow \operatorname{Hom}_R(E, E)$, which takes an element $r \in \widehat{R}$ to the homomorphism "multiplication by r," is an isomorphism.*

Proof. Since $E = E_{\widehat{R}}(K)$, there is no loss of generality in assuming that R is complete. For integers $t \geqslant 1$, consider the rings $R_t = R/\mathfrak{m}^t$. Then $E_t = 0 :_E \mathfrak{m}^t$ is the injective hull of the residue field of R_t. If $\varphi \in \operatorname{Hom}_R(E, E)$, then $\varphi(E_t) \subseteq E_t$, so φ restricts to an element of $\operatorname{Hom}_{R_t}(E_t, E_t)$, which equals R_t by Theorem A.32. The homomorphism φ, when restricted to E_t, is multiplication by an element $r_t \in R_t$. Moreover $E = \bigcup_t E_t$ and the elements r_t are compatible under restriction, i.e., $r_{t+1} - r_t \in \mathfrak{m}^t$. Thus φ is precisely multiplication by the element $(r_1 - r_2) + (r_2 - r_3) + \cdots \in R$. $\square$

Corollary A.35. *For a local ring $(R, \mathfrak{m}, K)$, the module $E_R(K)$ satisfies the descending chain condition (DCC).*

Proof. Consider a descending chain of submodules

$$E_R(K) = E \supseteq E_1 \supseteq E_2 \supseteq \dots.$$

Applying the functor $(-)^\vee = \operatorname{Hom}_R(-, E)$ gives us surjections

$$\widehat{R} \cong E^\vee \longrightarrow\!\!\!\!\!\rightarrow E_1^\vee \longrightarrow\!\!\!\!\!\rightarrow E_2^\vee \longrightarrow\!\!\!\!\!\rightarrow \dots.$$

Since $\widehat{R}$ is Noetherian, the ideals $\ker(\widehat{R} \longrightarrow\!\!\!\!\!\rightarrow E_t^\vee)$ stabilize for large t, and hence $E_t^\vee \longrightarrow\!\!\!\!\!\rightarrow E_{t+1}^\vee$ is an isomorphism for $t \gg 0$. Since $(-)^\vee$ is faithful, it follows that $E_t = E_{t+1}$ for $t \gg 0$. $\square$

Theorem A.36. *Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring. The following conditions are equivalent for an R-module M.*

(1) M is $\mathfrak{m}$-torsion and $\operatorname{soc}(M)$ is a finite-dimensional K-vector space;
(2) M is an essential extension of a finite-dimensional K-vector space;
(3) M can be embedded in a direct sum of finitely many copies of $E_R(K)$;
(4) M satisfies the descending chain condition.

Proof. The implications $(1) \implies (2) \implies (3) \implies (4)$ follow from earlier results, so we focus on $(4) \implies (1)$. Let $x \in M$. The descending chain

$$Rx \supseteq \mathfrak{m}x \supseteq \mathfrak{m}^2 x \supseteq \dots$$

stabilizes, so $\mathfrak{m}^{t+1}x = \mathfrak{m}^t x$ for some t. But then Nakayama's lemma implies $\mathfrak{m}^t x = 0$, and it follows that M is $\mathfrak{m}$-torsion. Since $\operatorname{soc}(M)$ is a vector space with DCC, it must be finite-dimensional. $\square$

Example A.37. Let $(R, \mathfrak{m}, K)$ be a discrete valuation ring with maximal ideal $\mathfrak{m} = Rx$. (For example, R may be a power series ring $K[[x]]$ or the ring of p-adic integers $\widehat{\mathbb{Z}}_p$.) We claim that $E_R(K) \cong R_x/R$. To see this, note that $\operatorname{soc}(R_x/R)$ is a 1-dimensional K-vector space generated by the image of $1/x \in R_x$, and that every element of R_x/R is killed by a power of x. It follows tht R_x/R is an essential extension of K. ♣ and now???

The next result explains the notion of duality in the current context.

Theorem A.38. *Let $(R, \mathfrak{m}, K)$ be a complete Noetherian local ring, and use $(-)^\vee$ to denote the functor $\operatorname{Hom}_R(-, E_R(K))$.*

(1) If M has ACC then $M^\vee$ has DCC, and if M has DCC then $M^\vee$ has ACC. Hence the category of R-modules with DCC is anti-equivalent to the category of R-modules with ACC.

(2) If M has ACC or DCC, then $M^{\vee\vee} \cong M$.

Proof. Let $E = E_R(K)$. If M has ACC, consider a presentation

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

Applying $(-)^\vee$, we get an exact sequence $0 \longrightarrow M^\vee \longrightarrow (R^n)^\vee \longrightarrow (R^m)^\vee$. Since $(R^n)^\vee \cong E^n$ has DCC, so does its submodule $M^\vee$. Applying $(-)^\vee$ again, we get the commutative diagram with exact rows

$$\begin{tikzcd}
(R^m)^{\vee\vee} \arrow[r] & (R^n)^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & 0 \\
R^m \arrow[r] \arrow[u] & R^n \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Since $R \longrightarrow R^{\vee\vee}$ is an isomorphism, it follows that $M \longrightarrow M^{\vee\vee}$ is an isomorphism as well.

If M has DCC, we embed it in E^m and obtain an exact sequence

$$0 \longrightarrow M \longrightarrow E^m \longrightarrow E^n.$$

Applying $(-)^\vee$ gives an exact sequence $(E^n)^\vee \longrightarrow (E^m)^\vee \longrightarrow M^\vee \longrightarrow 0$. The surjection $R^n \cong (E^m)^\vee \twoheadrightarrow M^\vee$ shows that M has ACC, while a similar commutative diagram gives the isomorphism $M^{\vee\vee} \cong M$. $\square$

Remark A.39. Let M be a finitely generated module over a complete local ring $(R, \mathfrak{m}, K)$. Then

$$\begin{aligned}
\operatorname{Hom}_R(K, M^\vee) \cong \operatorname{Hom}_R(K \otimes_R M, E_R(K)) &\cong \operatorname{Hom}_R(M/\mathfrak{m}M, E_R(K)) \\
&\cong \operatorname{Hom}_K(M/\mathfrak{m}M, K),
\end{aligned}$$

so the number of generators of M as an R-module equals the vector space dimension of $\operatorname{soc}(M^\vee)$.

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