Hodge numbers and exponential sums on $\mathbb{P}^n$

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Zeta functions

V = variety over $\mathbb{F}_q$, $N_m(V)$ = cardinality of $V(\mathbb{F}_{q^m})$

$$Z(V,t) = \exp\left(\sum_{m=1}^{\infty} N_m \frac{t^m}{m}\right) \in \mathbb{Q}[[t]]$$

Theorem 1 (Dwork, Grothendieck). *$Z(V,t)$ is a rational function:*

$$Z(V,t) = \frac{\prod_i (1-\alpha_i t)}{\prod_j (1-\beta_j t)},\ \alpha_i, \beta_j \textit{ algebraic integers}$$

i. e., $N_m(V) = \sum_j \beta_j^m - \sum_i \alpha_i^m$

Theorem 2 (Deligne).

$$|\alpha_i|, |\beta_j| \in \{q^{k/2} \mid k = 0, 1, \ldots, 2\dim V\}$$

One also has $|\alpha_i|_\ell = |\beta_j|_\ell = 1$ for all primes $\ell \neq p$. What can one say about $|\alpha_i|_p$ and $|\beta_j|_p$?

p-Adic estimates

$$g(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_n t^n =$$
$$(1 - \alpha_1 t) \cdots (1 - \alpha_n t) \in \bar{\mathbb{Q}}_p[t]$$

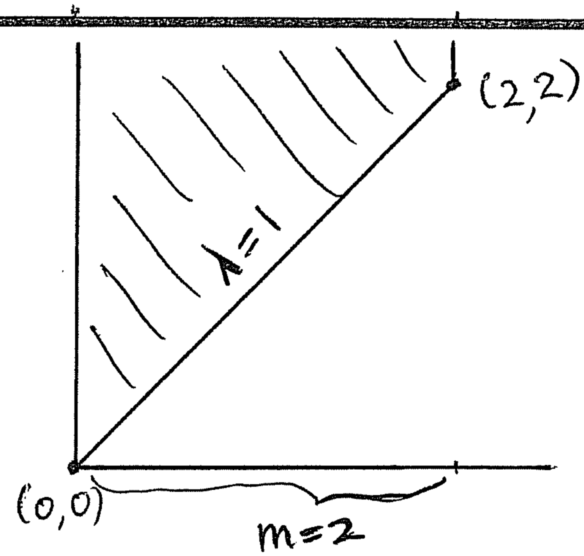
How are the sets $\{\operatorname{ord}_p a_i\}_{i=1}^n$ and $\{\operatorname{ord}_p \alpha_i\}_{i=1}^n$ related? The *Newton polygon* of $g(t)$ is the convex hull of the set

$$\{(i, \operatorname{ord}_p a_i)\}_{i=0}^n \cup \{(0, +\infty)\}.$$

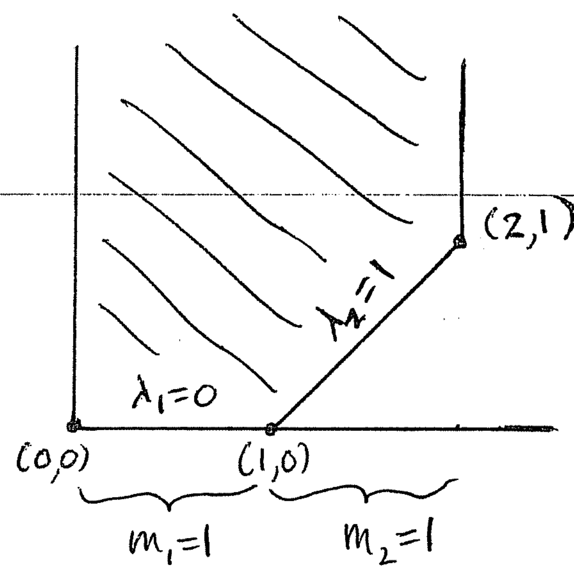
Theorem 3. *Let $\{\lambda_i\}_{j=1}^r$ be the slopes of the nonvertical sides of the Newton polygon of $g(t)$ and let m_j be the length of the projection on the x-axis of the side of slope λ_j. Then exactly m_j of the α_i have p-ordinal equal to λ_j.*

Note: When working over $\mathbb{F}_q$, replace ord_p with ord_q ($\operatorname{ord}_q q = 1$).

$$1-p^2t^2=(1-pt)(1+pt)$$



$$1-(p+1)t+pt^2$$
$$=(1-t)(1-pt)$$



Smooth complete intersections

$V \subseteq \mathbb{P}^n$ smooth complete intersection defined by $f_1 = \cdots = f_r = 0$, $f_i \in \mathbb{F}_q[x_0, \ldots, x_n]$ homogeneous of degree d_i. Then

$$Z(V,t) = \frac{P(t)^{(-1)^{n-r-1}}}{(1-t)(1-qt)\ldots(1-q^{n-r}t)},$$

$P(t) = \prod_i (1 - \alpha_i t)$ a polynomial ($|\alpha_i| = q^{(n-r)/2}$ $\forall i$ by Deligne)

Question: What is the Newton polygon of $P(t)$?

B. Mazur's Theorem

Let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth complete intersection of multidegree $(d_1, \ldots, d_r)$. Then

$$H^i_{DR}(X, \mathbb{C}) = \bigoplus_{j+k=i} H^{j,k}(X).$$

$h_j := \dim_{\mathbb{C}} H^{j,n-r-j}_{\text{prim}}(X)$ (depends only on the multidegree) The *Hodge polygon* is the Newton polygon of $\prod_{j=0}^{n-r}(1 - q^j t)^{h_j}$.

Theorem 4 (Dwork, Ann. 1964, Mazur, Ann. 1973). *The Newton polygon of $P(t)$ lies on or above the Hodge polygon.*

Ogus and Berthelot (1978) generalized this to arbitrary smooth proper varieties.
Illusie (1990) proved that this lower bound is attained for generic smooth complete intersections.

What is an exponential sum?

Data:

- $\mathbb{F}_q$ = the field with $q = p^a$ elements, p prime
- $\Psi : \mathbb{F}_q \longrightarrow \mathbb{Q}(\zeta_p)^\times$, an additive character ($\Psi(x) = \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(cx)}$, $c \in \mathbb{F}_q$)
- V = algebraic variety over $\mathbb{F}_q$
- f = regular function on V

Associated exponential sum

$$S_1 = S_1(V, f, \Psi) = \sum_{x \in V(\mathbb{F}_q)} \Psi(f(x)) \in \mathbb{Q}(\zeta_p)$$

Associated L-function:

$$S_m = \sum_{x \in V(\mathbb{F}_{q^m})} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f(x)) \in \mathbb{Q}(\zeta_p)$$

$$L(V, f, \Psi; t) = \exp\left(\sum_{m=1}^{\infty} S_m \frac{t^m}{m}\right) \in \mathbb{Q}(\zeta_p)[[t]]$$

Theorem 5 (Dwork, Grothendieck). *$L(t)$ is a rational function:*

$$L(t) = \frac{\prod_i (1 - \alpha_i t)}{\prod_j (1 - \beta_j t)}, \quad \alpha_i, \beta_j \text{ algebraic integers,}$$

i. e., $S_m = \sum_j \beta_j^m - \sum_i \alpha_i^m$

Theorem 6 (Deligne).

$$|\alpha_i|, |\beta_j| \in \{q^{k/2} \mid k = 0, 1, \ldots, 2 \dim V\}$$

One also has $|\alpha_i|_\ell = |\beta_j|_\ell = 1$ for all primes $\ell \neq p$. What can one say about $|\alpha_i|_p$ and $|\beta_j|_p$?

Katz's Theorem

$V \subseteq \mathbb{P}^n$ smooth projective variety

$F = f_0/(f_1^{a_1} \cdots f_r^{a_r})$ rational function on $\mathbb{P}^n$, i. e., f_i homogeneous polynomials with $\deg f_0 = \sum_{j=1}^r a_j \deg f_j$. Assume:

(1) For every nonempty $I \subseteq \{1, \ldots, r\}$

(a) $V \cap (\bigcap_{i \in I} \{f_i = 0\})$ is smooth of codim. $|I|$

(b) $V \cap \{f_0 = 0\} \cap (\bigcap_{i \in I} \{f_i = 0\})$ is smooth of codimension $|I| + 1$

(2) $(a_i, p) = 1$ for $i = 1, \ldots, r$.

Let $U = V - \{f_1 \cdots f_r = 0\}$.

Theorem 7 (Astérisque, no. 79, chap. 5). *$L(U, F, \Psi; t)^{(-1)^{\dim V - 1}}$ is a polynomial of degree $|\chi(U - \{f_0 = 0\})|$ and all its reciprocal roots α satisfy $|\alpha| = q^{(\dim V)/2}$.*

p-Adic estimates

Question: Find a lower bound for the Newton polygon of $L(U, F, \Psi; t)^{(-1)^{\dim V-1}}$.

For $V = \mathbb{P}^n$, $f_1 = x_0, \ldots, f_r = x_{r-1}$ ($r \leq n+1$), the answer has been worked out by Adolphson-Sperber (Annals, 1989)

For $n = 1$, $V = \mathbb{P}^1$, the answer has been worked out by Hui June Zhu (IMRN, 2004).

The new result

$V = \mathbb{P}^n$, $F = f_0/(f_1 \cdots f_r)$. For $I \subseteq \{1, \ldots, r\}$,
$X_I :=$ smooth complete intersection in $\mathbb{P}^n_{\mathbb{C}}$ of multidegree $(\deg f_i)_{i \in I}$
$Y_I :=$ smooth complete intersection in $\mathbb{P}^n_{\mathbb{C}}$ of multidegree $(\deg f_i)_{i \in I} \cup (\deg f_0)$
$h_s(X_I) := \dim_{\mathbb{C}} H^{s, n-|I|-s}_{\text{prim}}(X_I)$
$h_s(Y_I) := \dim_{\mathbb{C}} H^{s, n-|I|-1-s}_{\text{prim}}(Y_I)$
$Q_I(t) := (1 + t + \cdots + t^{|I|}) \sum_{s=0}^{n-|I|} h_s(X_I) t^s$
$R_I(t) := (t + t^2 + \cdots + t^{|I|}) \sum_{s=0}^{n-|I|-1} h_s(Y_I) t^s$

$$P(t) := \sum_{\emptyset \neq I \subseteq \{1, \ldots, r\}} Q_I(t) + \sum_{\emptyset \neq I \subseteq \{1, \ldots, r\}} R_I(t)$$

Write $P(t) = \sum_{s=0}^{n} h_s t^s$.

Theorem 8 (A.-Sperber). *The Newton polygon of* $L(U, F, \Psi; t)^{(-1)^{n-1}}$ *lies on or above the Newton polygon of* $\prod_{s=0}^{n} (1 - q^s t)^{h_s}$

Proof: Dwork theory gives a finite-dimensional p-adic Banach space W and an endomorphism α such that

$$L(U, F, \Psi; t)^{(-1)^{n-1}} = \det(I - \alpha t \mid W).$$

We estimate p-adically the entries in the matrix representing α relative to a basis of W and use the above formula to get a lower bound for the Newton polygon of $L(t)^{(-1)^{n-1}}$. We then use formulas for Hodge numbers of complete intersections to identify our lower bound with the one stated in the theorem.

Problems: 1. Extend the result to the case of arbitrary $a_1, \ldots, a_r$.

2. Find an intrinsic explanation for the appearance of Hodge numbers in the answer.

Example

$V = \mathbb{P}^2$, $F = f_0/f_1$, where $f_0 = 0$ and $f_1 = 0$ are smooth cubic curves in $\mathbb{P}^2$ intersecting transversally. $I = \{1\}$:

$$Q_I(t) = (1+t)(1+t) = 1 + 2t + t^2$$

$$R_I(t) = (t)(8) = 8t$$

$$P(t) = 1 + 10t + t^2.$$

Newton polygon of $L(t)^{-1}$ is above Newton polygon of $(1-t)(1-qt)^{10}(1-q^2t)$.

Analogue in characteristic 0?

Consider the de Rham-type complex $(\Omega^\bullet_{U/\mathbb{C}}, d + dF\wedge)$. Then $H^i(\Omega^\bullet_{U/\mathbb{C}}) = 0$ for $i \neq n$ and $\dim H^n(\Omega^\bullet_{U/\mathbb{C}}) = |\chi(U - \{f_0 = 0\})|$. Does there exist a filtration $F^\bullet$ on $\Omega^\bullet_{U/\mathbb{C}}$ such that for $s = 0, \ldots, n$

$$\dim_{\mathbb{C}} F^s/F^{s+1}(H^n(\Omega^\bullet_{U/\mathbb{C}})) = h_s?$$