CHARACTERISTIC VARIETY IN POSITIVE CHARACTERISTIC

1. Rings of diff. operators in char 0

$R = K[x_1, \dots, x_n]$, K field , char $K = 0$

- $D_{R|K} = R\langle \partial_1, \dots, \partial_n \rangle$ Noeth., non comm.

  $\partial_i = \frac{d}{dx_i}$, $\partial_i x_i - x_i \partial_i = 1$

  Ring of diff. operators (Weyl algebra)

- $D_{R|K}$ has a filtration given by the order s.t. $gr(D_{R|K}) = R[a_1, \dots, a_n]$

- M f.g. $D_{R|K}$-mod has a good filtration s.t. $gr(M)$ is a f.g. $gr(D_{R|K})$-mod

- characteristic ideal : $J(M) = rad\left(Ann_{gr(D_{R|K})}(gr(M))\right)$

- characteristic variety : $C(M) = V(J(M)) \subseteq \operatorname{Spec} R[a_1, \dots, a_n] = T^*X$

- Bernstein's inequality : $n \leq \dim C(M) \leq 2n$

- Definition $M \neq 0$ f.g. $D_{R|K}$-mod is holonomic if $\dim C(M) = n$

Goal:

- Better understanding of D-modules in positive characteristic.
- Effective computations

Source:

- P. Berthelot : Theory of arithmetic D-modules

2. Rings of diff. operators in char $p>0$

R regular, F-finite, containing perfect field k, char $k=p>0$

e.g. $R = k[x_1, \dots, x_n]$

- $D_R \equiv$ ring of diff. operators
- $D_{R|k} \equiv$ ring of k-linear diff. operators
- $\bar{D}_R^{(e)} := \mathrm{End}_{R^{p^e}}(R) \equiv$ diff. operators of level e

$$D_{R|k} \underset{\substack{= \\ k \text{ perfect}}}{\subseteq} D_R \underset{\substack{= \\ R \text{ F-finite}}}{\subseteq} \bigcup_e \bar{D}_R^{(e)}$$

- $D_R^{(e)} \equiv$ Berthelot's diff. operators of level e

Denote : $\forall i = 1, \dots, n$ $\quad \partial_i^{[e]} := \dfrac{1}{p^e!}\, \partial_i^{p^e}$

$\partial_i^{[0]} = \partial_i$

We have:

$$\begin{tikzcd}
R \arrow[d, equal] & \frac{R\langle \partial_1,\dots,\partial_n\rangle}{(\partial_1^p,\dots,\partial_n^p)} \arrow[d, equal] & & & & \\
\overline{D}_R^{(-1)} \arrow[r, phantom, "\subseteq"] & \overline{D}_R^{(0)} \arrow[r, phantom, "\subseteq"] & \overline{D}_R^{(1)} \arrow[r, phantom, "\subseteq \;\cdots\; \subseteq"] & \overline{D}_R^{(e)} \arrow[r, phantom, "\subseteq \;\cdots"] & D_R \\
 & D_R^{(0)} \arrow[u, two heads] \arrow[r] \arrow[d, equal] & D_R^{(1)} \arrow[u, two heads] \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & D_R^{(e)} \arrow[ul, two heads] \arrow[r] & \cdots \quad D_R \arrow[ul, equal] \arrow[d, equal] \\
 & R\langle \partial_1,\dots,\partial_n\rangle & R\langle \partial_i^{[j]} \mid i=1,\dots,n,\ j=0,1 \rangle & & & R\langle \partial_i^{[j]} \mid i=1,\dots,n,\ \forall j \rangle
\end{tikzcd}$$

Weyl algebra

not noeth.

<u>Frobenius descent</u> : R regular, F-finite. Then:

$$F^{e*} : R\text{-mod} \longrightarrow D_R^{(e)}\text{-mod}$$

is an equivalence of categories.

3. Characteristic variety at level e

Let $M^{(e)}$ be a f.g. $D_R^{(e)}$ - mod.

We can mimic the construction we have in char 0

- $D_R^{(e)}$ has a filtration given by the order s.t. $gr(D_R^{(e)})$ is a f.g. comm. R-algebra
- $M^{(e)}$ f.g. $D_R^{(e)}$ - mod has a good filtration s.t. $gr(M^{(e)})$ is a f.g. $gr(D_R^{(e)})$ - mod
- characteristic ideal : $J^{(e)}(M^{(e)}) = rad\left(Ann_{gr(D_R^{(e)})}\left(gr(M^{(e)})\right)\right)$
- characteristic variety : $C^{(e)}(M^{(e)}) = V(J^{(e)}(M^{(e)})) \subseteq$

$$\subseteq Spec\,(gr\,D_R^{(e)})_{red} = T^{(e)*}$$

4. Characteristic variety and Frobenius desc.

Let $M^{(0)}$ be a $D_R^{(0)}$ - mod and $M^{(e)} := F^{e*} M^{(0)}$

Then

$$C^{(e)}(M^{(e)}) = C^{(0)}(M^{(0)})$$

Goal : Let M be a D_R- mod . We want to define $C(M)$ descending to level 0.

Finitely generated unit $D_R[F]$- modules

(M, ϕ) , M D_R - mod

$\phi : M \xrightarrow{\cong} F^*M$

such that M is f.g as a module over $D_R[F] := \dfrac{D_R\langle F\rangle}{(r^pF - Fr}$
$r \in D_R$

Example : F - finite F-modules (Lyubeznik '97) are of this type by Frobenius descent.

- How to describe these modules ?

- Lyubeznik : Root N f.g. R- mod
$$N \subseteq F^*N \subseteq \cdots \subseteq \bigcup_e F^{e*}N = M$$

- Berthelot : There is an equivalence $(M, \phi) \longleftrightarrow (M^{(0)}, \phi^{(1)})$

where $M^{(0)}$ $D_R^{(0)}$ - mod
$$\phi^{(1)} : D_R^{(1)} \otimes_{D_R^{(0)}} M^{(0)} \xrightarrow{\cong} F^*M^{(0)}$$

Thm (A. - Blickle - Lyubeznik)

N root of f.g. unit $D_R[F]$ - mod M. Then $M = D_R \cdot N$

Def. Let M be a f.g. unit $D_R[F]$-mod with root N.

$$C(M) := C^{(0)}(D_R^{(0)} \cdot N)$$

Example : $f \in R$, $M = R_f = D_R \cdot \frac{1}{f}$

$$C(M) = C^{(0)}\left(D_R^{(0)} \cdot \frac{1}{f}\right)$$

- Bernstein's inequality : $n \leq C(M) \leq 2n$

Remark : Not true in general, e.g. $M = K[x]/(x^p)$

M $D_R^{(0)}$-mod, does not satisfy Bernstein's inequality.

- Definition : $M \neq 0$ f.g. unit $D_R[F]$-mod is holonomic if $\dim C(M) = n$