Rank Formulas for Hypergeometric Systems

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Outline

Definitions

- Let $A \in \mathbb{Z}^{d \times n}$. Assume $\mathbb{N}A$ is pointed and $\mathbb{Z}A = \mathbb{Z}^d$.
- $D_A = \mathbb{C}[x_A]\langle\partial_A\rangle = \mathbb{C}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle, \;\; R_A = \mathbb{C}[\partial_A]$
- $A = (a_1 \; a_2 \; \cdots \; a_n) : \;\; \deg(\partial_i) = a_i, \; \deg(x_i) = -a_i$
- Toric ideal: $I_A = \langle \partial^u - \partial^v \mid Au = Av, \; u, v \in \mathbb{N}^n \rangle \subseteq R_A$
- $S_A = R_A / I_A \cong \mathbb{C}[\mathbb{N}A]$
- Euler operators: $E_i = \sum_{j=1}^n a_{ij} x_j \partial_j$
- Parameter: $\beta \in \mathbb{C}^d$

Definition: $\mathcal{A}$-hypergeometric system $M_A(\beta)$

$$M_A(\beta) = D_A / D_A \cdot (I_A, \{E_i - \beta_i\}_{i=1}^d)$$

Rank of $\mathcal{A}$-hypergeometric Systems 2

- Work of:
  - Gel'fand-Graev-Kapranov-Zelevinskiĭ
  - Adolphson
  - Sturmfels-Takayama
  - Cattani-D'Andrea-Dickenstein
  - Saito-Sturmfels-Takayama
  - Matusevich-Miller-Walther
- $\operatorname{rank} M_A(\beta) = \operatorname{vol}(A)$ for generic β
- $\operatorname{rank} M_A(\beta) \geq \operatorname{vol}(A)$

Definition: Rank jump of $M_A(\beta)$ at β

$$j_A(\beta) = \operatorname{rank} M_A(\beta) - \operatorname{vol}(A)$$

Structure of the Exceptional Arrangement of A 3

- N: $\mathbb{Z}^d$-graded finitely generated module

$$\mathrm{qdeg}(N) = \text{ Zariski closure of } \deg(N) \subseteq \mathbb{C}^d$$

Theorem [MMW]:

- $$\mathcal{E}_A = \{\beta \in \mathbb{C}^d \mid j_A(\beta) > 0\} \quad \textit{exceptional arrangement of } A$$
$$= \mathrm{qdeg}\left(\bigoplus_{i=0}^{d-1} H^i_{\langle \partial_A \rangle}(S_A)\right)$$
- $j_A(-)$ is upper semi-continuous.

Goal: Stratification result

Determine the structure of

$$\mathcal{E}^i_A = \{\beta \in \mathbb{C}^d \mid j_A(\beta) > i\}.$$

Euler-Koszul Homology

- M: toric module
- $\mathcal{K}_\bullet(M,\beta)$: Euler-Koszul complex in $E-\beta$
- $\mathcal{H}_0(S_A,\beta) = M_A(\beta)$

Theorem [MMW]: Vanishing of Euler-Koszul Homology

- $\mathcal{H}_i(M,\beta) = 0\ \forall i \geq 0 \iff \beta \notin \mathrm{qdeg}(M)$
- $\mathcal{H}_i(M,\beta) = 0\ \forall i > 0 \iff M$ MCM S_A-module

Homological Investigation of Rank Jumps

$$0 \to S_A \to \mathbb{C}[\mathbb{Z}A] \to Q_A \to 0$$

$$\begin{array}{ccccccc}
\mathcal{H}_0(S_A,\beta) & \to & \mathcal{H}_0(\mathbb{C}[\mathbb{Z}A],\beta) & \to & \mathcal{H}_0(Q_A,\beta) & \to & 0 \\
 & & \underbrace{0}_{\mathbb{C}[\mathbb{Z}A]\ \mathrm{CM}} & \to & \mathcal{H}_1(Q_A,\beta) & \to &
\end{array}$$

- $j_A(\beta) = \operatorname{rank} \mathcal{H}_1(Q_A,\beta) - \operatorname{rank} \mathcal{H}_0(Q_A,\beta)$
- $\mathcal{E}_A \subseteq \operatorname{qdeg}(Q_A) = \mathcal{R}_A$ *ranking arrangement* of A
- $\beta \in \mathcal{E}_A$: $\mathcal{R}_A(\beta) \subseteq \mathcal{R}_A$ *β-components* of $\mathcal{R}_A$

Simple Case

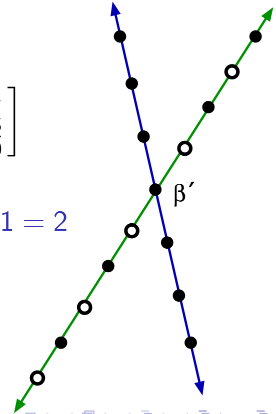
- $\mathcal{R}_A(\beta) = b + \mathbb{C}F, \quad b \in \mathbb{Z}^d$
- F: face of A, $\quad \mathbb{R}F \cap \mathbb{Z}^d = \mathbb{Z}F$
- $\mathcal{H}_\bullet(Q_A, \beta) \cong \mathcal{H}_\bullet(\mathbb{C}[\mathbb{Z}F](b), \beta)$
- $j_A(\beta) = \operatorname{rank} \mathcal{H}_1(\mathbb{C}[\mathbb{Z}F](b), \beta) - \operatorname{rank} \mathcal{H}_0(\mathbb{C}[\mathbb{Z}F](b), \beta)$

Theorem [Okuyama]:

- $\operatorname{rank} \mathcal{H}_q(\mathbb{C}[\mathbb{Z}F](b), \beta) = \binom{\operatorname{codim} F}{q} \cdot \operatorname{vol}(F)$.
- $j_A(\beta) = [\operatorname{codim} F - 1] \cdot \operatorname{vol}(F)$.

Example: Intersecting exceptional lines

- $A = \begin{array}{c} \left[\begin{array}{ccccccc} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \\ \underbrace{\qquad}_{F_1} \qquad\qquad\qquad\qquad \underbrace{\qquad}_{F_2} \end{array}$, $\mathrm{vol}(F_i) = 1$, $\mathrm{codim}(F_i) = 3$
- $\mathcal{E}_A = [\beta' + \mathbb{C}F_1] \cup [\beta' + \mathbb{C}F_2]$, $\beta' = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$
- $\beta \neq \beta'$:
  - $\beta \in [\beta' + \mathbb{C}F_2]$: $j_A(\beta) = [3-1] \cdot 1 = 2$
  - $\beta \in [\beta' + \mathbb{C}F_1]$: $j_A(\beta) = 4$
- $j_A(\beta') = 2 + 4 + 3 - 5 = 4$



Finding a Rank Jump Formula

Main Theorem:

$j_A(\beta)$ is determined by the combinatorics of $\mathcal{R}_A(\beta)$.

Corollaries: Stratification of $\mathcal{R}_A$

- $j_A(-)$ is constant on each stratum of $\mathcal{R}_A$.
- $\mathcal{E}_A^i$ is a union of translates of linear subspaces of $\mathbb{C}^d$.

- Extract essential part of $\mathcal{R}_A(\beta)$: $P^\beta \subseteq Q_A$
  - $\mathcal{H}_\bullet(Q_A, \beta) \cong \mathcal{H}_\bullet(P^\beta, \beta)$
  - Approximate P^β by MCM toric S_F-modules
  - Spectral sequence degenerates
- N MCM toric S_F-module:
  - $\mathcal{H}_\bullet(N, \beta) \cong \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{H}_0^F(N, \beta_F) \otimes_{\mathbb{C}} \left(\bigwedge^{\bullet} \mathbb{Z}F^\perp \right)$

Stratification of the Exceptional Arrangement $\mathcal{E}_A$

Proposition: Structure of $\mathcal{R}_A$

$\mathcal{R}_A = \mathcal{E}_A \cup \mathcal{R}_A^1$, where $\operatorname{codim} \mathcal{R}_A^1 = 1$.

Porism [MMW]:

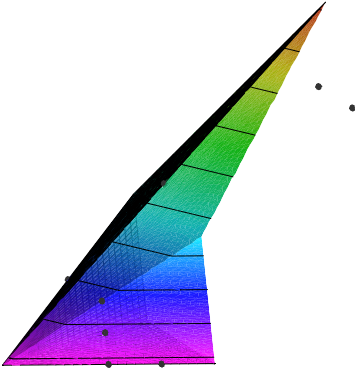
$\mathcal{E}_A$ has codimension ≤ 2.

Conclusions:

- $\mathcal{E}_A$ alone does not carry enough information to compute $j_A(\beta)$.
- An irreducible component of $\mathcal{E}_A$ is the union of strata of $\mathcal{R}_A$.

Example: Nonconstant Rank on a Slab

- $$A = \begin{bmatrix} 2 & 3 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 2 & 3 & 2 & 2 & 2 & 3 & 5 \\ \underbrace{0 \;\; 0 \;\; 1 \;\; 2}_{F_1} & & & & \underbrace{0 \;\; 0 \;\; 1 \;\; 2}_{F_2} & & & & \underbrace{5}_{F_3} & 7 & 7 \end{bmatrix}$$
- $\beta' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- $\mathcal{E}_A = [\beta' + \mathbb{C}F_3] \cup \{6 \text{ points}\}$,



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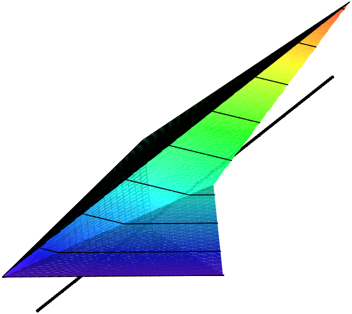
- $\beta' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- $\mathcal{E}_A = [\beta' + \mathbb{C}F_3] \cup \{6 \text{ points}\}$,
- $$j_A(\beta) = \begin{cases} 1 & \text{if } \beta \in [\beta' + \mathbb{C}F_3] \setminus \beta', \\ 2 & \text{if } \beta = \beta'. \end{cases}$$



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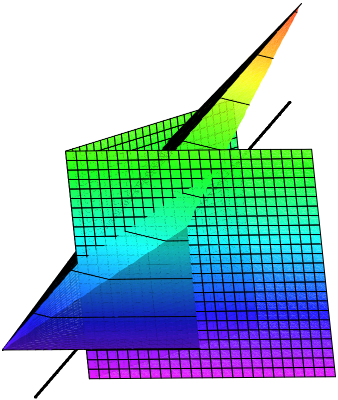
- $j_A(\beta) = \begin{cases} 1 & \text{if } \beta \in [\beta' + \mathbb{C}F_3] \setminus \beta', \\ 2 & \text{if } \beta = \beta'. \end{cases}$
- $\beta \in \mathbb{R}_{\geq 0}A \cap [\beta' + \mathbb{C}F_3] \setminus \beta'$:
  $\mathcal{R}_A(\beta) = \beta' + \mathbb{C}F_3$
- $\mathcal{R}_A(\beta') = \bigcup_{i=1}^{3}[\beta' + \mathbb{C}F_i]$



Looking Ahead

- Isomorphism classes of $M_A(\beta)$
  - [Saito]: $E_\tau(\beta)$ give the isomorphism classes of $M_A(\beta)$ for homogeneous A
  - $\deg(P^\beta)$ is directly related to $E_\tau(\beta)$
- Geometric explanation for $j_A(\beta)$